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FORMAL NORMAL FORMS FOR HOLOMORPHIC MAPS TANGENT TO THE IDENTITY

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Abstract. We describe a procedure for constructing formal normal forms of holomorphic maps with a hypersurface of fixed points, and we apply it to obtain a complete list of formal normal forms for 2-dimensional holomorphic maps tangential to a curve of fixed points.

1. **Introduction.** When studying a class of holomorphic dynamical systems, one of the main goals often is the classification under topological, holomorphic or formal conjugation; one would like to have a complete list of all the possible topological (respectively, holomorphic or formal) normal forms.

For discrete holomorphic local dynamical systems tangent to the identity, that is germs f about the origin of holomorphic self-maps of $\mathbb{C}^n$ such that $f(O) = O$ and $df_O = \mathrm{id}$, this problem has been extensively studied in dimension one. The computation of the formal normal forms is elementary: if

$$f(z) = z + a_{\nu+1}z^{\nu+1} + O(z^{\nu+2}), \tag{1}$$

is a germ of holomorphic self-map of $\mathbb{C}$ tangent to the identity at the origin, then it is easy to see that f is formally conjugated to

$$z \mapsto z + z^{\nu+1} + \beta z^{2\nu+1}, \tag{2}$$

where $\nu + 1$ is the multiplicity and β the index of the origin as a fixed point of f.

The topological normal forms have been obtained by Camacho [4] and Shcherbakov [8]: they proved that a germ of the form (1) is topologically conjugated to

$$z \mapsto z + z^{\nu+1}.$$

The holomorphic classification, obtained by Écalle [5] and Voronin [9], is also known, though much more complicated; in particular, the normal forms depend on an uncountable number of functional invariants.

When $n \geq 2$, as far as we know almost nothing is known for the topological and holomorphic classification. Our note is a contribution to the formal classification, at least in dimension 2. To explain our approach, let us recall what is already known.

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In his monumental work [6] (see also [7] for a short survey) Écalle studied the formal classifications of discrete holomorphic local dynamical systems tangent to the identity in dimension $n \geq 2$, giving a complete set of formal invariants for maps satisfying a generic condition. To be more precise, write the germ f in the form

$$f(z) = z + P_\nu(z) + O(\|z\|^{\nu+1}), \tag{3}$$

where $P_\nu \not\equiv O$ is an n-uple of homogeneous polynomials of total degree ν in the variables $z = (z_1, \ldots, z_n)$. A *characteristic direction* (or *eigenradius,* in Écalle's terminology) for f is a direction $v \in \mathbb{C}^n \setminus \{O\}$ such that $P_\nu(v) = \lambda v$ for some $\lambda \in \mathbb{C}$; the characteristic direction v is said *non-degenerate* if $\lambda \neq 0$, *degenerate* otherwise. Then Écalle studied maps f with at least one non-degenerate characteristic direction. This is a generic hypothesis, but not always satisfied: for instance, in the classification we shall discuss in Section 2 of this note, maps of the classes $(\star_1^0)$, (J_1) and (J_0) have *no* non-degenerate characteristic directions, and so Écalle's methods cannot be applied to them.

We encountered a similar situation in the past studying the generalization of the Fatou flower theorem to dimension two (see [1]). One way of dealing with maps having no non-degenerate characteristic directions is by blowing up the origin. In this way we replace $\mathbb{C}^2$ by a manifold M, the origin by an exceptional divisor S, and the original f by a germ $\tilde{f}$ of self-map of M fixing pointwise the exceptional divisor S. Choosing, as we may, coordinates in which the exceptional divisor is the line $\{w_1 = 0\}$ we can then write

$$\tilde{f}(w) = w + w_1^\nu f^o(w), \tag{4}$$

where $\nu \geq 1$ and f^o is a (germ of) holomorphic map not divisible by w_1. Now, if these coordinates are centered at a non-degenerate characteristic direction, then necessarily $f^o(O) = O$. However, there might be other points of the exceptional divisor where this happens; such points are called *singular points* of $\tilde{f}$ (or *singular directions* of f). It turns out that singular directions always exist, and that they are the generalization of non-degenerate characteristic directions needed to get a Fatou flower theorem in dimension 2.

This suggests to study the formal classification of all maps of the form (4) with $f^o(O) = O$, and this is the aim of this note. Brochero Martinez has followed a similar approach in [3], but he was looking for normal forms with respect to semi-formal conjugations (that is conjugations which depends holomorphically on the w_2 variable and formally on the w_1 variable), and so his results are not really comparable to ours. His paper is however relevant to our study because he also discusses how to deduce information about the formal classification of the original map f ([3], Theorem 6.2).

In Section 2 of this note we shall describe a general procedure (inspired by the classical construction of Poincaré-Dulac normal forms) producing normal forms of maps of the form (4) with $f^o(O) = O$ in dimension 2 (actually, the same procedure works in any dimension; we restrict ourselves to dimension 2 for simplicity).

In Section 3 we shall apply this procedure to obtain the formal normal forms of maps satisfying two additional hypotheses. The first one is just technical: we shall assume that the linear part of f^o does not vanish, a hypothesis that can be always satisfied after a finite number of blow-ups (see [1]). Furthermore, we shall also assume that the linear part of f^o preserves the exceptional line $\{w_1 = 0\}$. The reason behind this hypothesis is more conceptual: if $\tilde{f}$ is the blow-up of a map f, then the linear part of f^o (actually, the map f^o itself) always preserves the exceptional line $\{w_1 = 0\}$ *unless* f is dicritical, that is, writing f as in (3), unless if $P_\nu(z) = q(z)z$ for a suitable homogeneous polynomial q of total degree $\nu - 1$, a very special case. Since the dynamics of maps such that f^o does not preserve the exceptional line is much easier to study than the dynamics of maps which do (see [2]), and moreover Brochero Martinez ([3], Theorem 6.3) has formal normal forms for maps which are the blow-up of dicritic maps, we shall restrict ourselves to maps satisfying this additional hypothesis. So our main result Theorem 1 shall list the formal normal forms of 2-dimensional maps of the form (4) with $f^o(O) = O$ and such that the linear part of f^o (is not zero and) preserves the line $\{w_1 = 0\}$.

2. **The procedure.** We shall denote by $\mathrm{End}(\mathbb{C}^2, O)$ the set of germs at the origin of holomorphic self-maps of $\mathbb{C}^2$ fixing the origin, and we shall say that $f \in \mathrm{End}(\mathbb{C}^2, O)$ is *tangent to the identity* if $df_O = \mathrm{id}$. We are interested in finding formal normal forms for maps $f \in \mathrm{End}(\mathbb{C}^2, O)$, $f \neq \mathrm{id}_{\mathbb{C}^2}$, tangent to the identity and satisfying a few additional hypotheses. The first hypothesis is

($\mathbf{H}_1$) there exists a curve $S \subset \mathbb{C}^2$ of fixed points such that $O \in S$ is a regular (i.e., smooth) point of S.

This can always be achieved by blowing-up the origin and taking as S the exceptional divisor; as discussed in the introduction, this is an often useful procedure to study the dynamics of maps tangent to the identity.

Up to a (convergent) change of coordinates, we can assume that $S = \{z_1 = 0\}$ near the origin. In particular, all the changes of coordinates we shall use from now on will preserve the curve $\{z_1 = 0\}$.

Since f is tangent to the identity, there exists a maximal number $\nu \in \mathbb{N}^*$ (the *order of contact* of f with S) so that we can write

$$f(z) = z + z_1^\nu f^o(z)$$

for a suitable map f^o not divisible by z_1. Our second hypothesis then is

($\mathbf{H}_2$) $f^o(O) = O$, that is the origin is a *(dynamically) singular* (see [1]) point of S.

The reason for this hypothesis is that, as proved in [1], the dynamics of f is concentrated around its singular points, and we are eventually interested in using formal normal forms to study the dynamics.

We can also expand f^o in series of homogeneous polynomials, writing

$$f^o(z) = P^1(z) + P^2(z) + \cdots ,$$

where each $P^j(z)$ is a pair of homogeneous polynomials in z_1, z_2 of (total) degree j. The least $j \geq 1$ such that $P^j \not\equiv O$ is the *pure order* of f at the origin.

The third hypothesis we shall (starting from next section) assume is

($\mathbf{H}_3$) f has pure order 1 at the origin, that is $P^1 \not\equiv O$.

This is a technical hypothesis, used to simplify computations. However, in [1] it is shown that with a finite number of blow-ups every f can be transformed in a map with pure order 1 at all its singular points.

Our last hypothesis will be

($\mathbf{H}_4$) P^1 sends the curve $S = \{z_1 = 0\}$ into itself.

As shown in [1] and [2], from a dynamical point of view the most interesting maps tangent to the identity are the ones *tangential* to their fixed point set. We refer to [2] for the general definition of tangential maps; in our setting, f is tangential to S

if and only if f^o sends S into itself, that is, writing $f^o = (f_1^o, f_2^o)$, if z_1 divides f_1^o. So our hypothesis $(\mathbf{H}_4)$ is a weak version of tangentiality. We shall also discuss what happens when f is actually tangential to S.

A few more notations: we shall write $P^j = (P_1^j, P_2^j)$ and

$$P_i^j(z) = \sum_{k=0}^{j} a_{i,k}^j z_1^k z_2^{j-k};$$

in particular, $(\mathbf{H}_4)$ holds if and only if $a_{1,0}^1 = 0$. Finally, we shall use the symbol O_d to indicate any formal series whose expansion in homogeneous polynomials has no terms of degree less than d.

Any formal change of coordinates can be written as the composition of a linear change of coordinates A followed by a formal change of coordinates χ tangent to the identity, that is whose expansion in homogeneous polynomials is of the form

$$\chi(z) = z + H^d(z) + H^{d+1}(z) + \cdots$$

with $d \geq 2$ and $H^d \not\equiv O$, where again $H^j = (H_1^j, H_2^j)$ is a pair of homogeneous polynomials of degree j. Furthermore, since in our setting we are interested only in changes of coordinates preserving the line $\{z_1 = 0\}$, we should be able to write $H_1^j = z_1 \check{H}_1^j$ for each $j \geq d$, where $\check{H}_1^j$ is a homogeneous polynomial of degree $j - 1$.

A standard computation shows how a change of coordinates acts on a map satisfying $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$.

Proposition 1. *Let $f \in \mathrm{End}(\mathbb{C}^2, O)$ satisfy hypotheses $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$, so that it can be written as*

$$f(z) = z + z_1^\nu \big(P^\mu(z) + O_{\mu+1}\big),$$

where $\nu \geq 1$ is the order of contact of f with $\{z_1 = 0\}$, and $\mu \geq 1$ is the pure order of f at the origin. If

$$A = \begin{pmatrix} \alpha_{11} & 0 \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \tag{5}$$

is a linear change of coordinates preserving the line $\{z_1 = 0\}$, then

$$A^{-1} \circ f \circ A(z) = z + z_1^\nu \big(\alpha_{11}^\nu A^{-1} P^\mu(Az) + O_{\mu+1}\big).$$

Furthermore, if

$$\chi(z) = z + H^d(z) + O_{d+1}$$

is a formal change of coordinates tangent to the identity and preserving the line $\{z_1 = 0\}$, with $d \geq 2$, then

$$\begin{aligned} \chi^{-1} \circ f \circ \chi(z) \;&=\; z + z_1^\nu \big[P^\mu(z) + \cdots + P^{\mu+d-2}(z) \\ &\quad + P^{\mu+d-1}(z) + \mathrm{Jac}(P^\mu) \cdot H^d(z) - \mathrm{Jac}(H^d) \cdot P^\mu(z) + \nu \check{H}_1^d(z) P^\mu(z) \\ &\quad + O_{\mu+d}\big], \end{aligned} \tag{6}$$

where $\mathrm{Jac}(H) \cdot v$ denotes the multiplication of the vector $v \in \mathbb{C}^2$ by the Jacobian matrix of the map H (computed at z).

Formula (6) suggests the introduction of a family of linear maps. Let $\mathbb{C}_d[z_1, z_2]$ denote the space of homogeneous polynomials of degree d in two variables, and set $\tilde{\mathcal{V}}_d = \mathbb{C}_d[z_1, z_2]^2$ (that is, $\tilde{\mathcal{V}}_d$ is the space of pairs of homogeneous polynomials of degree d in two variables) and

$$\mathcal{V}_d = \{H \in \tilde{\mathcal{V}}_d \mid H_1(0, \cdot) \equiv 0\},$$

so that $H \in \mathcal{V}_d$ implies that there is $\check{H}_1 \in \mathbb{C}_{d-1}[z_1, z_2]$ such that $H_1 \equiv z_1 \check{H}_1$.

Given $P \in \tilde{\mathcal{V}}_\mu$, for any $d \geq 2$ we can then define a linear map $L_{P,d} \colon \mathcal{V}_d \to \tilde{\mathcal{V}}_{d+\mu-1}$ by setting

$$L_{P,d}(H) = \mathrm{Jac}(P) \cdot H - \mathrm{Jac}(H) \cdot P + \nu \check{H}_1 P.$$

We explicitely remark that if $P \in \mathcal{V}_\mu$ then $L_{P,d}$ sends $\mathcal{V}_d$ into $\mathcal{V}_{d+\mu-1}$.

From formula (6) we then deduce the following:

Corollary 1. *Let $f \in \mathrm{End}(\mathbb{C}^2, O)$ of the form*

$$f(z) = z + z_1^\nu \big(P^\mu(z) + \cdots + P^{\mu+d-1}(z) + O_{d+\mu}\big),$$

and take $Q \in \tilde{\mathcal{V}}_{\mu+d-1}$. Then we can find a change of coordinates of the form $\chi(z) = z + H^d(z)$ with $H^d \in \mathcal{V}_d$ such that

$$\chi^{-1} \circ f \circ \chi(z) = z + z_1^\nu \big(P^\mu(z) + \cdots + P^{\mu+d-1}(z) + Q(z) + O_{d+\mu}\big)$$

if and only if Q belongs to the image of $L_{P^\mu,d}$. In particular, we can find a change of coordinates of the form $\chi(z) = z + H^d(z)$ with $H^d \in \mathcal{V}_d$ such that

$$\chi^{-1} \circ f \circ \chi(z) = z + z_1^\nu \big(P^\mu(z) + \cdots + P^{\mu+d-2}(z) + O_{d+\mu}\big)$$

if and only if $P^{\mu+d-1}$ belongs to the image of $L_{P^\mu,d}$.

This corollary suggests the announced procedure for finding formal normal forms. First of all, one uses a linear change of variables to put P^μ in normal form. Then one uses a change of variables of the form $\chi(z) = z + H^2(z)$ to put $P^{\mu+1}$ in normal form, subtracting elements of the image of $L_{P^\mu,2}$. Then one proceeds by induction: with a change of variables of the form $\chi(z) = z + H^d(z)$, subtracting elements of the image of $L_{P^\mu,d}$ one can put $P^{\mu+d-1}$ in normal form without modifying the preceding terms. Composing all these changes of variables for $d \to \infty$ we get a formal change of variables putting the map in formal normal form.

3. **Normal forms.** From now on we shall assume hypotheses $(\mathbf{H}_1)$–$(\mathbf{H}_4)$; so let $f \in \mathrm{End}(\mathbb{C}^2, O)$ be of the form

$$f(z) = z + z_1^\nu \big(P^1(z) + P^2(z) + \cdots\big)$$

with $P^1 \not\equiv O$ and $a^1_{1,0} = 0$. The first step in our procedure consists in putting P^1 in normal form using a linear change of coordinates A of the form (5). Using the notations introduced in the previous section, Proposition 1 says that P^1 is sent by A in

$$\alpha_{11}^\nu A^{-1} \circ P^1 \circ A = \alpha_{11}^\nu \left| \begin{matrix} a^1_{1,1} & 0 \\ \frac{\alpha_{21}}{\alpha_{22}}(a^1_{2,0} - a^1_{1,1}) + \frac{\alpha_{11}}{\alpha_{22}} a^1_{2,1} & a^1_{2,0} \end{matrix} \right|;$$

We have several cases to consider.

– If $a^1_{2,0} \neq 0$, we can choose A so that $\alpha_{11}^\nu a^1_{2,0} = 1$, and we have three subcases:

$(\star_1^\lambda)$ If $a^1_{1,1} \neq a^1_{2,0}$ we can choose A so that $\frac{\alpha_{21}}{\alpha_{22}}(a^1_{2,0} - a^1_{1,1}) + \frac{\alpha_{11}}{\alpha_{22}} a^1_{2,1} = 0$, and we have reduced P^1 to

$$\left| \begin{matrix} \lambda & 0 \\ 0 & 1 \end{matrix} \right|,$$

where $\lambda = a^1_{1,1}/a^1_{2,0} \neq 1$ is the *residual index* of f at the origin along $\{z_1 = 0\}$ (see [1] for the definition of residual index).

(J_1) If instead $a^1_{1,1} = a^1_{2,0}$ but $a^1_{2,1} \neq 0$ we can choose A so that $\frac{\alpha_{11}^{\nu+1}}{\alpha_{22}} a^1_{2,1} = 1$, and we have reduced P^1 to

$$\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}.$$

$(\star_1^1)$ Finally, if $a^1_{1,1} = a^1_{2,0}$ and $a^1_{2,1} = 0$ we have reduced P^1 to

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}.$$

– If instead $a^1_{2,0} = 0$ we have two subcases to consider:

$(\star_2)$ If $a^1_{1,1} \neq 0$ we can choose A so that $\alpha_{11}^{\nu} a^1_{1,1} = 1$ and $\frac{\alpha_{11}}{\alpha_{22}} a^1_{2,1} - \frac{\alpha_{21}}{\alpha_{22}} a^1_{1,1} = 0$, and we have reduced P^1 to

$$\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}.$$

(J_0) Finally, if $a^1_{1,1} = 0$ also, necessarily $a^1_{2,1} \neq 0$ and we can choose A so that $\frac{\alpha_{11}^{\nu+1}}{\alpha_{22}} a^1_{2,1} = 1$. Thus we have reduced P^1 to

$$\begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix}.$$

Hence there are five possible normal forms for the linear part of a map satisfying $(\mathbf{H}_1)$–$(\mathbf{H}_4)$.

To apply the procedure described at the end of the previous section, we must study the linear maps $L_{P^\mu,d}\colon \mathcal{V}_d \to \tilde{\mathcal{V}}_{d+\mu-1}$. For the sake of generality, for a while we shall not assume $(\mathbf{H}_3)$ or $(\mathbf{H}_4)$. We shall use monomials as a basis of $\tilde{\mathcal{V}}_d$: setting

$$v^h_d = \begin{cases} (z_1^h z_2^{d-h}, 0) & \text{for } h = 0, \dots, d, \\ (0, z_1^k z_2^{d-k}) & \text{for } h = d+1, \dots, 2d+1, \text{ where } k = h - (d+1), \end{cases}$$

then $\{v^0_d, \dots, v^{2d+1}_d\}$ is a basis of $\tilde{\mathcal{V}}_d$, while $\{v^1_d, \dots, v^{2d+1}_d\}$ is a basis of $\mathcal{V}_d$.

Using the notations introduced in the previous section, we can write

$$P^\mu(z) = \left(\sum_{i=0}^{\mu} a^\mu_{1,i} z_1^i z_2^{\mu-i}, \sum_{i=0}^{\mu} a^\mu_{2,i} z_1^i z_2^{\mu-i} \right) = \sum_{i=0}^{\mu} a^\mu_{1,i} v^i_\mu + \sum_{i=0}^{\mu} a^\mu_{2,i} v^{d+1+i}_\mu.$$

Then the action of $L_{P^\mu,d}$ on the elements of the basis of $\mathcal{V}_d$ is given by:

$$L_{P^\mu,d}(v^h_d) = \sum_{i=0}^{\mu+1} [(\nu+i-h) a^\mu_{1,i} - (d-h) a^\mu_{2,i-1}]\, v^{h+i-1}_{d+\mu-1} + \sum_{i=0}^{\mu} (\nu+i) a^\mu_{2,i}\, v^{h+i+d}_{d+\mu-1} \quad (7)$$

for $h = 1, \dots, d$, where we have put $a^\mu_{1,\mu+1} = 0 = a^\mu_{2,-1}$, and

$$\begin{aligned} L_{P^\mu,d}(v^{k+d+1}_d) &= \sum_{i=0}^{\mu-1} (\mu - i) a^\mu_{1,i}\, v^{k+i}_{d+\mu-1} \\ &\quad + \sum_{i=0}^{\mu+1} [(\mu+k+1-d-i) a^\mu_{2,i-1} - k a^\mu_{1,i}]\, v^{k+i+d}_{d+\mu-1} \end{aligned} \quad (8)$$

for $k = 0, \dots, d$, where again we have put $a^\mu_{1,\mu+1} = 0 = a^\mu_{2,-1}$.

When $(\mathbf{H}_3)$ and $(\mathbf{H}_4)$ hold we have $\mu = 1$ and $a^1_{1,0} = 0$ in (7) and (8); in particular, the image of $L_{P^1,d}$ is always contained in $\mathcal{V}_d$. This implies that we can never remove from the normal form monomials proportionals to $(z_2^d, 0)$; however, these monomials appear if and only if f is not tangential.

Assume now that P^1 is in normal form, and apply our procedure in the five cases.

$(\star_1^\lambda)$. We have $a_{1,1}^1 = \lambda$, $a_{2,0}^1 = 1$ and $a_{1,0}^1 = a_{2,1}^1 = 0$. The matrix representing $L_{P^1,d}$ with respect to the basis $\{v_d^1, \ldots, v_d^{2d+1}\}$ of $\mathcal{V}_d$ is lower triangular; in particular, $L_{P^1,d}$ is surjective if and only if all the elements on the diagonal are different from zero. Now, these elements are of two forms:

$$\sigma_{d,h} = (\nu + 1 - h)\lambda - (d - h)$$

with $h = 1, \ldots, d$, and

$$\tau_{d,k} = (k + 1 - d) - k\lambda$$

with $k = 0, \ldots, d$. In particular, $\sigma_{\nu+1,\nu+1} = 0$ always; this implies that we can never remove from the normal form monomials proportional to $(z_1^{\nu+1}, 0)$.

Let $E \subset \mathbb{C}$ denote the set of $\lambda \in \mathbb{C}$ such that $\sigma_{d,h} = 0$ for some pair $(d, h) \neq (\nu + 1, \nu + 1)$, or such that $\tau_{d,k} = 0$ for some pair (d, k). Clearly, if $\lambda \notin E$ we have that $L_{P^1,d}$ is surjective for all $d \neq \nu + 1$, and that

$$\mathrm{Im}(L_{P^1,\nu+1}) = \mathrm{Span}(v_{\nu+1}^1, \ldots, v_{\nu+1}^\nu, v_{\nu+1}^{\nu+2}, \ldots, v_{\nu+1}^{2\nu+3});$$

so we can remove from the normal form all the monomials but the ones proportional to $(z_1^{\nu+1}, 0)$ or to $(z_2^d, 0)$. Thus if $\lambda \notin E$ the formal normal form of f is

$$\hat{f}(z) = \bigl(z_1 + z_1^\nu[\lambda z_1 + a z_1^{\nu+1} + z_2^2 t(z_2)], z_2 + z_1^\nu z_2\bigr)$$

with $a \in \mathbb{C}$ and $t \in \mathbb{C}[\![\zeta]\!]$, where $\mathbb{C}[\![\zeta]\!]$ is the space of formal power series in one variable, with $t \equiv 0$ if and only if f is tangential.

So it becomes important to determine E. First of all, $\tau_{d,k} = 0$ if and only if $\lambda = -(d - k - 1)/k$ with $1 \le k \le d$ and $d \ge 2$, and it is easy to see that

$$\left\{ -\frac{d-k-1}{k} \;\middle|\; 1 \le k \le d, d \ge 2 \right\} = \left\{ \frac{1}{d} \;\middle|\; d \ge 2 \right\} \cup \{0\} \cup \mathbb{Q}^-.$$

On the other hand, $\sigma_{d,h} = 0$ if and only if $\lambda = (d - h)/(\nu + 1 - h)$ with $1 \le h \le d$, $h \neq \nu + 1$, $d \ge 2$. If $h > \nu + 1$ then λ is a negative rational number; if $h = d$ we get $\lambda = 0$. Assume then $1 \le h < \min\{\nu + 1, d\}$, so that λ is a positive rational number. Setting $q = \nu + 1 - h$ and $p = d - h = d + q - (\nu + 1)$, we see that we must have $\max\{0, \nu + 1 - d\} < q \le \nu$ and $\max\{d - (\nu + 1), 0\} < p \le d - 1$. Since $d \ge 2$ is generic, we get

$$\left\{ \frac{d-h}{\nu+1-h} \;\middle|\; 1 \le h \le d, h \neq \nu + 1, d \ge 2 \right\} \cap \mathbb{Q}^+ = \bigcup_{q=1}^{\nu} \frac{1}{q}\mathbb{N},$$

and so

$$E = \bigcup_{q=1}^{\nu} \frac{1}{q}\mathbb{N} \cup \left\{ \frac{1}{d} \;\middle|\; d \ge 2 \right\} \cup \{0\} \cup \mathbb{Q}^-.$$

We now study $L_{P^1,d}$ when $\lambda \in E$.

- $\lambda = 0$. In this case $\tau_{d,k} = 0$ if and only if $k = d - 1$, and $\sigma_{d,h} = 0$ if and only if $h = d$. So $v_d^{k+d+1} \in \mathrm{Im}\, L_{P^1,d}$ for $k = 0, \ldots, d - 2, d$, and thus $v_d^h \in \mathrm{Im}\, L_{P^1,d}$ for $h = 1, \ldots, d - 1$. Furthermore, $L_{P^1,d}(v_d^d) = \nu v_d^{2d}$, and so $\mathrm{Im}\, L_{P^1,d} = \mathrm{Span}(v_d^1, \ldots, v_d^{d-1}, v_d^{d+1}, \ldots, v_d^{2d+1})$. Therefore the formal normal form of f in this case is

$$\hat{f}(z) = \bigl(z_1 + z_1^\nu[z_1^2 g(z_1) + z_2^2 t(z_2)], z_2 + z_1^\nu z_2\bigr),$$

where g, $t \in \mathbb{C}[\![\zeta]\!]$, with $t \equiv 0$ if and only if f is tangential.

– $\lambda = -p/q \in \mathbb{Q}^-$, with $(p,q) = 1$. In this case we have $\tau_{d,k} = 0$ if and only if $d - 1 - k = kp/q$; in particular, since p and q are relatively prime, we must have $k = ql$ and $d - 1 - k = pl$ for some $l \geq 1$.

If $\tau_{d,k} \neq 0$ then $v_d^{k+d+1} \in \operatorname{Im} L_{P^1,d}$. If $\tau_{d,k} = 0$ with $1 \leq k \leq d-2$, we have

$$L_{P^1,d}(v_d^{k+1}) = -\lambda\nu v_d^{k+1} + \nu v_d^{k+d+1};$$

hence ($\sigma_{d,k+1} \neq 0$ and) we are able to remove from the normal form monomials proportional to v_d^{k+1+d} if we retain monomials proportional to $v_d^{k+1} = (z_1^{k+1} z_2^{d-k-1}, 0)$. Notice that we can write

$$z_1^{k+1} z_2^{d-k-1} = z_1 (z_1^q z_2^p)^l.$$

On the other hand, $\sigma_{d,h} = 0$ if and only if $d - h = (h - \nu - 1)p/q$; in particular we must have $h - \nu - 1 = lq$ and $d - h = lp$ for some $l \geq 1$. This means that $v_d^h \in \operatorname{Im} L_{P^1,d}$, except in the case when we can write

$$z_1^h z_2^{d-h} = z_1^{\nu+1} (z_1^q z_2^p)^l.$$

Summing up, we have shown that the formal normal form in this case is

$$\hat{f}(z) = \bigl(z_1 + z_1^\nu [z_1 g_1(z_1^q z_2^p) + z_1^{\nu+1} g_2(z_1^q z_2^p) + z_2^2 t(z_2)], z_2 + z_1^\nu z_2\bigr)$$

where g_1, g_2, $t \in \mathbb{C}[\![\zeta]\!]$ with $g_1(0) = \lambda$, and $t \equiv 0$ if and only if f is tangential.

– $\lambda = 1$. In this case we have $\tau_{d,k} \neq 0$ always, and $\sigma_{d,h} = 0$ if and only if $d = \nu + 1$. This means that $\operatorname{Im} L_{P^1,d} = \mathcal{V}_d$ if $d \neq \nu + 1$, and $\operatorname{Im} L_{P^1,\nu+1} = \operatorname{Span}(v_{\nu+1}^{\nu+2}, \ldots, v_{\nu+1}^{2\nu+3})$, and so the normal form in this case is

$$\hat{f}(z) = \bigl(z_1 + z_1^\nu [z_1 + z_1 p_\nu(z_1, z_2) + z_2^2 t(z_2)], z_2 + z_1^\nu z_2\bigr)$$

where $p_\nu \in \mathbb{C}_\nu[z_1, z_2]$ and $t \in \mathbb{C}[\![\zeta]\!]$, with $t \equiv 0$ if and only if f is tangential.

– $\lambda = 1/q$, with $q \geq 2$. In this case $\tau_{d,k} = 0$ if and only if $d = k = q$. On the other hand, $\sigma_{d,h} = 0$ if and only if $\nu + 1 - h = (d - h)q$; in particular, if we set $l = d - h$ we must have $h = \nu + 1 - lq$ with $1 \leq l \leq \nu/q$, and thus we can write

$$z_1^h z_2^{d-h} = z_1^{\nu+1} (z_1^{-q} z_2)^l.$$

Arguing as before we then see that the formal normal form in this case is

$$\hat{f}(z) = \bigl(z_1 + z_1^\nu [\lambda z_1 + z_1^{\nu+1} p_o(z_1^{-q} z_2) + z_2^2 t(z_2)], z_2 + z_1^\nu (z_2 + a z_1^q)\bigr),$$

where $a \in \mathbb{C}$, $p_o \in \mathbb{C}[\zeta]$ is a polynomial of degree at most ν/q (and it is just a constant if $q \geq \nu + 1$), and $t \in \mathbb{C}[\![\zeta]\!]$, with $t \equiv 0$ if and only if f is tangential.

– $\lambda = p/q$, with $1 \leq q \leq \nu$, $p \geq 2$ and $(p,q) = 1$. In this case $\tau_{d,k} \neq 0$ always, and $\sigma_{d,h} = 0$ if and only if $d - h = (\nu + 1 - h)p/q$. Therefore we can write $\nu + 1 - h = lq$, $d - h = lp$ and

$$z_1^h z_2^{d-h} = z_1^{\nu+1} (z_1^{-q} z_2^p)^l,$$

with $1 \leq l \leq \nu/q$. Hence arguing as before we see that the formal normal form in this case is

$$\hat{f}(z) = \bigl(z_1 + z_1^\nu [\lambda z_1 + z_1^{\nu+1} p_o(z_1^{-q} z_2^p) + z_2^2 t(z_2)], z_2 + z_1^\nu z_2\bigr),$$

where $p_o \in \mathbb{C}[\zeta]$ is a polynomial of degree at most ν/q (and it is just a constant if $q \geq \nu + 1$), and $t \in \mathbb{C}[\![\zeta]\!]$, with $t \equiv 0$ if and only if f is tangential.

This ends the discussion of the case $(\star_1^\lambda)$. We now deal with the remaining cases:

(J_1). In this case we have $a^1_{1,1} = a^1_{2,0} = a^1_{2,1} = 1$ and $a^1_{1,0} = 0$. The matrix representing $L_{P^1,d}$ with respect to the basis $\{v^1_d, \ldots, v^{2d+1}_d\}$ of $\mathcal{V}_d$ is lower triangular; in particular, $L_{P^1,d}$ is surjective if and only if $d \neq \nu + 1$. Furthermore, it is not difficult to check that $\operatorname{Im} L_{P^1,\nu+1} = \operatorname{Span}(v^2_{\mu+1}, \ldots, v^{2\mu+3}_{\mu+1})$, and thus the formal normal form in this case is

$$\hat{f}(z) = \big(z_1 + z_1^\nu[z_1 + az_1z_2^\nu + z_2^2t(z_2)], z_2 + z_1^\nu(z_1 + z_2)\big),$$

where $a \in \mathbb{C}$ and $t \in \mathbb{C}[\![\zeta]\!]$, with $t \equiv 0$ if and only if f is tangential.

$(\star_2)$. In this case we have $a^1_{1,1} = 1$ and $a^1_{2,0} = a^1_{2,1} = a^1_{1,0} = 0$. The matrix representing $L_{P^1,d}$ with respect to the basis $\{v^1_d, \ldots, v^{2d+1}_d\}$ of $\mathcal{V}_d$ is diagonal; in particular, $L_{P^1,d}(v^h_d) = O$ if and only if $h = \nu+1$ (and $d \geq \nu+1$), and $L_{P^1,d}(v^{k+d+1}_d) = O$ if and only if $k = 0$. Therefore the formal normal form in this case is

$$\hat{f}(z) = \big(z_1 + z_1^\nu[z_1 + z_1^{\nu+1}g_1(z_2) + z_2^2t(z_2)], z_2 + z_1^\nu z_2^2 g_2(z_2)\big),$$

where g_1, g_2, $t \in \mathbb{C}[\![\zeta]\!]$, where $t \equiv 0$ if and only if f is tangential.

(J_0). In this case we have $a^1_{2,1} = 1$ and $a^1_{2,0} = a^1_{1,1} = a^1_{1,0} = 0$. The matrix representing $L_{P^1,d}$ with respect to the basis $\{v^1_d, \ldots, v^{2d+1}_d\}$ of $\mathcal{V}_d$ is strictly lower triangular, but with no zeroes on the diagonal just below the main diagonal. It follows easily that the image of $L_{P^1,d}$ is generated by

$$\{v^2_d, \ldots, v^d_d, v^{d+2}_d, \ldots, v^{2d+1}_d\},$$

and thus the formal normal form in this case is

$$\hat{f}(z) = \big(z_1 + z_1^\nu[z_1z_2g_1(z_2) + z_2^2t(z_2)], z_2 + z_1^\nu[z_1 + z_2^2g_2(z_2)]\big),$$

where g_1, g_2, $t \in \mathbb{C}[\![\zeta]\!]$, where $t \equiv 0$ if and only if f is tangential.

Summing up, we have proved the following

Theorem 1. *Let $f \in \operatorname{End}(\mathbb{C}^2, O)$, $f \neq \mathrm{id}_{\mathbb{C}^2}$, be a map tangent to the identity fixing pointwise the line $S = \{z_1 = 0\}$. Assume moreover that the origin is a singular point for f on S, that f has order of contact $\nu \geq 1$ with S, pure order 1 at the origin, and that $(\mathbf{H}_4)$ holds. Set*

$$E = \bigcup_{q=1}^{\nu} \frac{1}{q}\mathbb{N} \cup \left\{\frac{1}{d} \;\middle|\; d \geq 2\right\} \cup \{0\} \cup \mathbb{Q}^-.$$

Then f is formally conjugated to one (and only one) of the following normal forms:

$(\star_1^\lambda)$ *with $\lambda \in \mathbb{C} \setminus E$:*

$$\hat{f}(z) = \big(z_1 + z_1^\nu[\lambda z_1 + az_1^{\nu+1} + z_2^2t(z_2)], z_2 + z_1^\nu z_2\big)$$

where $a \in \mathbb{C}$ and $t \in \mathbb{C}[\![\zeta]\!]$ is a generic power series;

$(\star_1^\lambda)$ *with $\lambda = 0$:*

$$\hat{f}(z) = \big(z_1 + z_1^\nu[z_1^2g(z_1) + z_2^2t(z_2)], z_2 + z_1^\nu z_2\big),$$

where g, $t \in \mathbb{C}[\![\zeta]\!]$ are generic power series;

$(\star_1^\lambda)$ *with $\lambda = -p/q \in \mathbb{Q}^-$ and $(p, q) = 1$:*

$$\hat{f}(z) = \big(z_1 + z_1^\nu[z_1g_1(z_1^qz_2^p) + z_1^{\nu+1}g_2(z_1^qz_2^p) + z_2^2t(z_2)], z_2 + z_1^\nu z_2\big)$$

where g_1, g_2, $t \in \mathbb{C}[\![\zeta]\!]$ are generic power series with $g_1(0) = \lambda$;

($\star_1^\lambda$) *with* $\lambda = 1/q$ *and* $q \in \mathbb{N}$, $q \geq 2$:
$$\hat f(z) = \big(z_1 + z_1^\nu[\lambda z_1 + z_1^{\nu+1} p_o(z_1^{-q} z_2) + z_2^2 t(z_2)], z_2 + z_1^\nu(z_2 + a z_1^q)\big),$$
where $a \in \mathbb{C}$, $p_o \in \mathbb{C}[\zeta]$ *is a polynomial of degree at most* ν/q, *and* $t \in \mathbb{C}[\![\zeta]\!]$ *is a generic power series;*

($\star_1^\lambda$) *with* $\lambda = p/q \in \mathbb{Q}$, $1 \leq q \leq \nu$, $p \geq 2$ *and* $(p,q) = 1$:
$$\hat f(z) = \big(z_1 + z_1^\nu[\lambda z_1 + z_1^{\nu+1} p_o(z_1^{-q} z_2^p) + z_2^2 t(z_2)], z_2 + z_1^\nu z_2\big),$$
where $p_o \in \mathbb{C}[\zeta]$ *is a polynomial of degree at most* ν/q, *and* $t \in \mathbb{C}[\![\zeta]\!]$ *is a generic power series;*

($\star_1^\lambda$) *with* $\lambda = 1$:
$$\hat f(z) = \big(z_1 + z_1^\nu[z_1 + z_1 p_\nu(z_1, z_2) + z_2^2 t(z_2)], z_2 + z_1^\nu z_2\big)$$
where $p_\nu \in \mathbb{C}_\nu[z_1, z_2]$ *is a generic homogeneous polynomial of degree* ν, *and* $t \in \mathbb{C}[\![\zeta]\!]$ *is a generic power series;*

(J_1)
$$\hat f(z) = \big(z_1 + z_1^\nu[z_1 + a z_1 z_2^\nu + z_2^2 t(z_2)], z_2 + z_1^\nu(z_1 + z_2)\big),$$
where $a \in \mathbb{C}$ *and* $t \in \mathbb{C}[\![\zeta]\!]$ *is a generic power series;*

($\star_2$)
$$\hat f(z) = \big(z_1 + z_1^\nu[z_1 + z_1^{\nu+1} g_1(z_2) + z_2^2 t(z_2)], z_2 + z_1^\nu z_2^2 g_2(z_2)\big),$$
where g_1, g_2, $t \in \mathbb{C}[\![\zeta]\!]$ *are generic power series;*

(J_0)
$$\hat f(z) = \big(z_1 + z_1^\nu[z_1 z_2 g_1(z_2) + z_2^2 t(z_2)], z_2 + z_1^\nu[z_1 + z_2^2 g_2(z_2)]\big),$$
where g_1, g_2, $t \in \mathbb{C}[\![\zeta]\!]$ *are generic power series.*

In all these cases, $t \equiv 0$ *if and only if* f *is tangential.*

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