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Backward iteration in strongly convex domains

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Abstract

We prove that a backward orbit with bounded Kobayashi step for a hyperbolic, parabolic or strongly elliptic holomorphic self-map of a bounded strongly convex C^2 domain in $\mathbb{C}^d$ necessarily converges to a repelling or parabolic boundary fixed point, generalizing previous results obtained by Poggi-Corradini in the unit disk and by Ostapyuk in the unit ball of $\mathbb{C}^d$.

MSC: 32H50; 37F99

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0. Introduction

The theory of non-invertible discrete dynamical systems (that is, the iteration theory of a non-invertible self-map $f : X \to X$ of a set X) is usually devoted to the study of the behavior of forward orbits of the system (that is, of sequences of the form $\{f^n(x)\}_{n\in\mathbb{N}}$, where $x \in X$ and f^n denotes the composition of f with itself n times). In this paper we shall instead study the behavior of *backward orbits,* that is of sequences $\{x_n\}_{n\in\mathbb{N}}$ such that $f(x_{n+1}) = x_n$ for all $n \in \mathbb{N}$, in the context of holomorphic self-maps of bounded strongly convex domains.

Backward orbits (also called *backward iteration sequences*) for holomorphic self-maps of the unit disk $\Delta \subset \mathbb{C}$ have been studied by Poggi-Corradini in [11]. He proved that (unless f is a

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non-Euclidean rotation of Δ) a backward orbit must converge to a point in the boundary of Δ, which is (in the sense of non-tangential limits) a repelling or parabolic fixed point of the map f. Ostapyuk [9] generalized Poggi-Corradini's results to backward orbits in the unit ball $B^d \subset \mathbb{C}^d$. The aim of this paper is to extend Poggi-Corradini's results to backward orbits in general bounded strongly convex C^2 domains in $\mathbb{C}^n$. To do so, we shall systematically use the geometric properties of the Kobayashi distance of strongly convex domains; and it is interesting to notice that the better geometric understanding given by this tool (and the impossibility of using the kind of explicit computations done in [9] for the ball) yields proofs that are both simpler and clearer than the previous ones, even for the ball and the unit disk.

To state precisely, and put in the right context, our results, let us first describe what is known about holomorphic discrete dynamical systems in strongly convex domain. As proved several years ago by one of us (see [1–3]), the fundamental dichotomy for holomorphic dynamics in complex taut manifolds is between self-maps whose sequence of iterates is compactly divergent and self-maps whose sequence of iterates is relatively compact in the space of all holomorphic self-maps of the manifold (endowed with the compact-open topology, which is equivalent to the topology of pointwise convergence). In a convex domain D, it turns out that the sequence of iterates of a holomorphic self-map $f \in \mathrm{Hol}(D, D)$ is compactly divergent if and only if f has no fixed points inside D; so the dichotomy is between self-maps without fixed points and maps with fixed points.

Following the usual one-variable terminology, we shall call *elliptic* a holomorphic self-map of a bounded convex domain $D \subset \mathbb{C}^n$ with a not empty fixed point set. If $f \in \mathrm{Hol}(D, D)$ is elliptic, then the dynamics of f is concentrated along a, possibly lower-dimensional, submanifold D_0, the *limit manifold* of f, in the sense that all limits of subsequences of iterates of f are given by the composition of a holomorphic retraction of D onto D_0 with an automorphism of D_0. Clearly, D_0 contains the fixed point set of f, but in general can be strictly larger; furthermore, $f|_{D_0}$ is an automorphism of D_0, generating a group whose closure is the product of a torus with a finite cyclic group (see [3]). In particular, backward orbits in D_0 are just forward orbits for the inverse of $f|_{D_0}$, and so their behavior is known; for this reason here we shall instead study backward orbits for maps, called *strongly elliptic*, whose limit manifold reduces to a point, necessarily fixed. In particular, f is strongly elliptic if and only if the sequence of iterates of any $x \in D$ converges to a point $p \in D$, which is thus an *attracting* fixed point.

When $f \in \mathrm{Hol}(D, D)$ has no fixed points, and $D \subset \mathbb{C}^d$ is a bounded strongly convex C^2 domain, the main dynamical fact is the generalization [1] of the classical Wolff–Denjoy theorem, saying that the sequence of iterates converges to a point $\tau \in \partial D$, the *Wolff point* of f. The Wolff point is a *boundary fixed point,* in the sense that f has K-limit τ at τ (see Section 1 for the precise definition of *K-limit,* also known as *admissible limit;* here it just suffices to say that the existence of the K-limit implies the existence of the non-tangential limit, and thus our f has non-tangential limit τ at τ). Furthermore, it is possible to define the *boundary dilation* β_τ at τ, which, roughly speaking, is the derivative of the normal component of f along the normal direction to ∂D at τ (and is the natural generalization of the one-variable angular derivative); and the fact that forward orbits converge to τ implies that $0 < \beta_\tau \leqslant 1$. Again following the classical one-variable terminology, we shall say that f is *hyperbolic* and τ is *attracting* if $0 < \beta_\tau < 1$; and that f and τ are *parabolic* if $\beta_\tau = 1$.

Before turning our attention to backward orbits, a final remark is needed. Forward orbits always have *bounded Kobayashi step,* that is the Kobayashi distance $k_D(f^{n+1}(z), f^n(z))$ between two consecutive elements of the orbit is bounded by a constant independent of n (but depending on the orbit): indeed, $k_D(f^{n+1}(z), f^n(z)) \leqslant k_D(f(z), z)$, because the Kobayashi distance k_D is weakly contracted by holomorphic maps.

Summing up, if $f \in \mathrm{Hol}(D, D)$ is strongly elliptic, hyperbolic or parabolic, then all forward orbits (have bounded Kobayashi step and) converge to the Wolff point $\tau \in \overline{D}$ (for the sake of uniformity, we are calling *Wolff point* the unique fixed point of a strongly elliptic map too), which is an attracting or parabolic (possibly boundary) fixed point. Our main result states that, analogously, backward orbits with bounded Kobayashi step for a strongly elliptic, hyperbolic or parabolic map always converge to a repelling or parabolic boundary fixed point, where a *boundary fixed point* is a point $\sigma \in \partial D$ such that f has K-limit σ at σ, and σ is *repelling* if the boundary dilation β_σ of f at σ is larger than 1.

More precisely, in Section 2 we shall prove the following

Theorem 0.1. *Let $D \Subset \mathbb{C}^d$ be a bounded strongly convex C^2 domain. Let $f \in \mathrm{Hol}(D, D)$ be either hyperbolic, parabolic, or strongly elliptic, with Wolff point $\tau \in \overline{D}$. Let $\{z_n\} \subset D$ be a backward orbit for f with bounded Kobayashi step. Then*:

(i) *the sequence $\{z_n\}$ converges to a repelling or parabolic boundary fixed point $\sigma \in \partial D$*;
(ii) *if f is strongly elliptic or hyperbolic then σ is repelling*;
(iii) *if $\sigma = \tau$, then f is parabolic*;
(iv) *$\{z_n\}$ goes to σ inside a K-region, that is, there exists $M > 0$ so that $z_n \in K_p(\sigma, M)$ eventually, where p is any point in D.*

See Section 1 for (preliminaries and in particular) the definition of K-region; going to the boundary inside a K-region is the natural several variables generalization of the one-variable notion of non-tangential approach.

To show that our theorem is not empty we must prove the existence of backward orbits with bounded Kobayashi step. This is done in Section 3 where, slightly adapting an argument due to Poggi-Corradini ([10]; see also [9]), we shall prove that if $\sigma \in \partial D \setminus \{\tau\}$ is an isolated repelling boundary point for a self-map $f \in \mathrm{Hol}(D, D)$ strongly elliptic, hyperbolic or parabolic, then there always exist a backward orbit with bounded Kobayashi step converging to σ.

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1. Preliminaries

In this section we shall collect a few facts about the geometry of the Kobayashi distance and the dynamics of holomorphic self-maps of bounded strongly convex domains needed in the rest of the paper.

Let us briefly recall the definition and the main properties of the Kobayashi distance; we refer to [2,6,7] for details and much more. Let k_Δ denote the Poincaré distance on the unit disk $\Delta \subset \mathbb{C}$. If X is a complex manifold, the *Lempert function* $\delta_X : X \times X \to \mathbb{R}^+$ of X is defined by

$$\delta_X(z, w) = \inf\bigl\{k_\Delta(\zeta, \eta) \bigm| \exists \phi : \Delta \to X \text{ holomorphic, with } \phi(\zeta) = z \text{ and } \phi(\eta) = w\bigr\}$$

for all $z, w \in X$. The *Kobayashi pseudodistance* $k_X : X \times X \to \mathbb{R}^+$ of X is the largest pseudodistance on X bounded above by δ_X. We say that X is (*Kobayashi*) *hyperbolic* if k_X is a true distance — and in that case it is known that the metric topology induced by k_X coincides with the manifold topology of X (see, e.g., [2, Proposition 2.3.10]). For instance, all bounded domains are hyperbolic (see, e.g., [2, Theorem 2.3.14]).

The main property of the Kobayashi (pseudo)distance is that it is contracted by holomorphic maps: if $f : X \to Y$ is a holomorphic map then

$$\forall z, w \in X \quad k_Y\big(f(z), f(w)\big) \leqslant k_X(z, w).$$

In particular, the Kobayashi distance is invariant under biholomorphisms.

It is easy to see that the Kobayashi distance of the unit disk coincides with the Poincaré distance. Furthermore, the Kobayashi distance of the unit ball $B^d \subset \mathbb{C}^d$ coincides with the Bergman distance (see, e.g., [2, Corollary 2.3.6]); and the Kobayashi distance of a bounded convex domain coincides with the Lempert function (see, e.g., [2, Proposition 2.3.44]). Moreover, the Kobayashi distance of a bounded convex domain D is complete [2, Proposition 2.3.45], and thus for each $p \in D$ we have that $k_D(p, z) \to +\infty$ if and only if $z \to \partial D$.

A *complex geodesic* in a hyperbolic manifold X is a holomorphic map $\varphi : \Delta \to X$ which is an isometry with respect to the Kobayashi distance of Δ and the Kobayashi distance of X. Lempert's theory (see [8] and [2, Chapter 2.6]) of complex geodesics in strongly convex domains is one of the main tools for the study of the geometric function theory of strongly convex domains. In particular, we shall need the following facts, summarizing Lempert's and Royden–Wong's theory, valid for any bounded convex domain $D \Subset \mathbb{C}^d$:

(a) [2, Theorem 2.6.19 and Corollary 2.6.30] for every pair of distinct points $z, w \in D$ there exists a complex geodesic $\varphi : \Delta \to D$ such that $\varphi(0) = z$ and $\varphi(r) = w$, where $0 < r < 1$ is such that $k_\Delta(0, r) = k_D(z, w)$; furthermore, if D is strongly convex then φ is unique;
(b) [2, Theorem 2.6.19] a holomorphic map $\varphi \in \mathrm{Hol}(\Delta, D)$ is a complex geodesic if and only if $k_D(\varphi(\zeta_1), \varphi(\zeta_2)) = k_\Delta(\zeta_1, \zeta_2)$ for a pair of distinct points $\zeta_1, \zeta_2 \in \Delta$;
(c) [2, Proposition 2.6.22] every complex geodesic $\varphi \in \mathrm{Hol}(\Delta, D)$ admits a *left-inverse,* that is a holomorphic map $\tilde{p}_\varphi : D \to \Delta$ such that $\tilde{p}_\varphi \circ \varphi = \mathrm{id}_\Delta$; the map $p_\varphi = \varphi \circ \tilde{p}_\varphi$ is then a holomorphic retraction of D onto the image of φ;
(d) [2, Theorem 2.6.29] if D is strongly convex of class C^2, then every complex geodesics extend continuously (actually, $\frac{1}{2}$-Hölder) to the boundary of Δ, and the image of φ is transversal to ∂D;
(e) [2, Theorem 2.6.45] if D is strongly convex and of class C^2, then for every $z \in D$ and $\tau \in \partial D$ there is a complex geodesic $\varphi \in \mathrm{Hol}(\Delta, D)$ with $\varphi(0) = z$ and $\varphi(1) = \tau$; and for every pair of distinct points $\sigma, \tau \in \partial D$ there is a complex geodesic $\varphi \in \mathrm{Hol}(\Delta, D)$ such that $\varphi(-1) = \sigma$ and $\varphi(1) = \tau$. (The statement of [2, Theorem 2.6.45] requires D of class C^3, but the proof of the existence works assuming just C^2 smoothness.)

Now let $D \Subset \mathbb{C}^d$ be a bounded strongly convex C^2 domain, and $f \in \mathrm{Hol}(D, D)$ a holomorphic self-map of D. As mentioned in the Introduction, if the set $\mathrm{Fix}(f)$ of fixed points of f in D is not empty, then (see [1–3]) the sequence $\{f^n\}$ of iterates of f is relatively compact in $\mathrm{Hol}(D, D)$, and there exists a submanifold $D_0 \subseteq D$, the *limit manifold* of f, such that every limit of a subsequence of iterates is of the form $\gamma \circ \rho$, where $\rho : D \to D_0$ is a holomorphic retraction, and γ is a biholomorphism of D_0; furthermore, $f|_{D_0}$ is a biholomorphism of D_0, and $\mathrm{Fix}(f) \subseteq D_0$.

Definition 1.1. Let $D \Subset \mathbb{C}^d$ be a bounded strongly convex C^2 domain. A holomorphic map $f \in \mathrm{Hol}(D, D)$ is *elliptic* if $\mathrm{Fix}(f) \neq \emptyset$; and *strongly elliptic* if its limit manifold reduces to a point (called the *Wolff point* of the strongly elliptic map). We shall say that a point $p \in \mathrm{Fix}(f)$ is *attracting* if all the eigenvalues of df_p have modulus less than 1.

Later on we shall need an equivalent characterization of strongly elliptic maps:

Lemma 1.1. *Let $D \Subset \mathbb{C}^d$ be a bounded strongly convex C^2 domain, and $f \in \mathrm{Hol}(D, D)$. Then the following facts are equivalent*:

(i) *f is strongly elliptic*;
(ii) *the sequence of iterates of f converges to a point $p \in D$*;
(iii) *f has an attracting fixed point $p \in D$*;
(iv) *there exists $p \in \mathrm{Fix}(f)$ such that $k_D(p, f(z)) < k_D(p, z)$ for all $z \in D \setminus \{p\}$.*

Proof. The equivalence of (i), (ii) and (iii) is well known, and more generally valid in taut manifolds (see, e.g., [2, Corollary 2.4.2]).

Now, if f is not strongly elliptic, the limit manifold D_0 has positive dimension. Being a holomorphic retract of D, the Kobayashi distance of D_0 coincides with the restriction of Kobayashi distance of D; hence $k_D(f(z), f(w)) = k_D(z, w)$ for all $z, w \in D_0$, because $f|_{D_0}$ is a biholomorphism of D_0 (and thus an isometry for the Kobayashi distance). Since $\mathrm{Fix}(f) \subseteq D_0$, this shows that (iv) implies (i).

Finally, assume that (iv) does not hold, and thus there are $p \in \mathrm{Fix}(f)$ and $z_0 \in D \setminus \{p\}$ with $k_D(p, f(z_0)) = k_D(p, z_0)$. Let $\varphi \in \mathrm{Hol}(\Delta, D)$ be a complex geodesic with $\varphi(0) = p$ and $\varphi(r) = z_0$, for a suitable $0 < r < 1$. Then

$$k_D\bigl(p, f\bigl(\varphi(r)\bigr)\bigr) = k_D(p, z_0) = k_\Delta(0, r);$$

since $f(p) = p$ this implies that $f \circ \varphi$ is still a complex geodesic. Since complex geodesics are also infinitesimal isometries with respect to the Kobayashi metric (see [2, Corollary 2.6.20]), the Kobayashi length of $\varphi'(0)$ must be equal to the Kobayashi length of $(f \circ \varphi)'(0) = df_p(\varphi'(0))$. In particular, p cannot be an attracting fixed point, and thus f cannot be strongly elliptic. $\square$

In the study of the dynamics of self-maps without fixed points, a crucial role is played by the horospheres, a generalization (introduced in [1]) of the classical notion of horocycle. Let $D \Subset \mathbb{C}^d$ be a bounded strongly convex C^2 domain. For every $\tau \in \partial D$ and $p \in D$ let $h_{\tau,p} : D \to \mathbb{R}^+$ be given by

$$\frac{1}{2} \log h_{\tau,p}(z) = \lim_{w \to \tau} \bigl[k_D(z, w) - k_D(p, w)\bigr];$$

notice that the existence of the limit is a non-trivial fact (see [1, Theorem 2.6.47] or [5]). Then the *horosphere* of *center* $\tau \in \partial D$, *radius* $R > 0$ and *pole* $p \in D$ is the set

$$E_p(\tau, R) = \bigl\{z \in D \bigm| h_{\tau,p}(z) < R\bigr\}.$$

It is well known (see, again, [1,2,5]) that the horospheres with pole at the origin in the unit disk $\Delta \subset \mathbb{C}$ coincide with the classical horocycles, that the horospheres with pole at the origin in the unit ball $B^n \subset \mathbb{C}^n$ again coincide with the usual horospheres, and that the horospheres in strongly convex domains are convex. Furthermore, the closure of a horosphere intersects the boundary of D exactly in the center of the horosphere; and the shape of a horosphere near the boundary is comparable to the shape of the horospheres in the ball, that is, they are close to be ellipsoids. An easy observation we shall need later on is that changing the pole amounts to multiplying the radius by a fixed constant:

Lemma 1.2. *Let $D \Subset \mathbb{C}^d$ be a bounded strongly convex C^2 domain, and $\tau \in \partial D$. Then*

$$h_{\tau,q} = \frac{1}{h_{\tau,p}(q)} h_{\tau,p}$$

for all $p, q \in D$. In particular,

$$\forall R > 0 \quad E_q(\tau, R) = E_p\bigl(\tau, h_{\tau,p}(q)R\bigr).$$

Proof. It suffices to write

$$k_D(z,w) - k_D(q,w) = \bigl[k_D(z,w) - k_D(p,w)\bigr] - \bigl[k_D(q,w) - k_D(p,w)\bigr],$$

and let $w \to \tau$. □

In a similar way we can introduce K-regions. Let $D \Subset \mathbb{C}^d$ be a bounded strongly convex C^2 domain. The *K-region* $K_p(\tau, M)$ of *center* $\tau \in \partial D$, *amplitude* $M > 0$ and *pole* $p \in D$ is the set

$$K_p(\tau, M) = \left\{ z \in D \;\middle|\; \frac{1}{2}\log h_{\tau,p}(z) + k_D(p,z) < \log M \right\}.$$

It is well known (see [2,3]) that the K-regions with pole at the origin in the unit disk coincide with the classical Stolz regions, and that the K-regions with pole at the origin in the unit ball $B^n \subset \mathbb{C}^n$ coincide with the usual Korányi approach regions. Furthermore, in strongly convex domains K-regions are comparable to Stein admissible regions; and changing the pole does not change much the K-regions, because [2, Lemma 2.7.2] for each $p, q \in D$ there is $L > 0$ such that

$$K_p(\tau, M/L) \subseteq K_q(\tau, M) \subseteq K_p(\tau, ML) \tag{1.1}$$

for every $M > 0$. Given $\tau \in \partial D$, we shall say that a function $F : D \to \mathbb{C}^n$ has *K-limit* $\ell \in \mathbb{C}^n$ at τ if $F(z) \to \ell$ as $z \to \tau$ inside any K-region centered at τ; notice that the choice of the pole is immaterial because of (1.1). Since K-regions in strongly convex domains are comparable to Stein admissible regions, the notion of K-limit is equivalent to Stein admissible limit, and thus it is the right generalization to several variables of the one-dimensional notion of non-tangential limit (in particular, the existence of a K-limit always implies the existence of a non-tangential limit). Finally, the intersection of a horosphere (or K-region) of center $\tau \in \partial D$ and pole $p \in D$ with the image of a complex geodesic φ with $\varphi(0) = p$ and $\varphi(1) = \tau$ is the image via φ of the horosphere (or K-region) of center 1 and pole 0 in the unit disk [2, Proposition 2.7.8 and Lemma 2.7.16].

The correct generalization of the one-variable notion of angular derivative is given by the dilation coefficient (see [2, Section 1.2.1 and Theorem 2.7.14]):

Definition 1.2. Take $f \in \mathrm{Hol}(D, D)$, where again $D \Subset \mathbb{C}^d$ is a bounded strongly convex C^2 domain, and let $\sigma \in \partial D$. The *dilation coefficient* $\beta_{\sigma,p} \in (0, +\infty]$ of f at $\sigma \in \partial D$ with pole $p \in D$ is given by

$$\frac{1}{2}\log \beta_{\sigma,p} = \liminf_{z\to\sigma}\bigl[k_D(p,z) - k_D\bigl(p, f(z)\bigr)\bigr].$$

Furthermore, $\sigma \in \partial D$ is a *boundary fixed point* of f if f has K-limit σ at σ.

Since

$$k_D(p,z) - k_D\bigl(p, f(z)\bigr) \geqslant k_D\bigl(f(p), f(z)\bigr) - k_D\bigl(p, f(z)\bigr) \geqslant -k_D\bigl(p, f(p)\bigr),$$

the dilation coefficient cannot be zero. We also recall the following useful formulas for computing the dilation coefficient [2, Lemma 2.7.22]:

$$\begin{aligned}\frac{1}{2}\log \beta_{\sigma,p} &= \lim_{t\to 1}\bigl[k_D\bigl(p, \varphi(t)\bigr) - k_D\bigl(p, f\bigl(\varphi(t)\bigr)\bigr)\bigr]\\ &= \lim_{t\to 1}\bigl[k_D\bigl(p, \varphi(t)\bigr) - k_D\bigl(p, p_\varphi \circ f\bigl(\varphi(t)\bigr)\bigr)\bigr],\end{aligned}\tag{1.2}$$

where $\varphi \in \mathrm{Hol}(\Delta, D)$ is a complex geodesic with $\varphi(0) = p$ and $\varphi(1) = \sigma$, and $p_\varphi = \varphi \circ \tilde{p}_\varphi$ is the holomorphic retraction associated to φ.

When σ is a boundary fixed point then the dilation coefficient does not depend on the pole:

Lemma 1.3. *Let $D \Subset \mathbb{C}^d$ be a bounded strongly convex C^2 domain, $f \in \mathrm{Hol}(D, D)$ and $\sigma \in \partial D$ a boundary fixed point of f. Then $\beta_{\sigma,p} = \beta_{\sigma,q}$ for all $p, q \in D$.*

Proof. If the dilation coefficient is infinite for all poles we are done. Assume there is $p \in D$ such that the dilation coefficient $\beta_{\sigma,p}$ is finite. Given $q \in D$, write

$$\begin{aligned}k_D(q,z) - k_D\bigl(q, f(z)\bigr) = k_D(p,z) - k_D\bigl(p, f(z)\bigr) &+ \bigl[k_D(q,z) - k_D(p,z)\bigr]\\ &+ \bigl[k_D\bigl(p, f(z)\bigr) - k_D\bigl(q, f(z)\bigr)\bigr].\end{aligned}\tag{1.3}$$

The first term inside square brackets converges to $\frac{1}{2}\log h_{\sigma,p}(q)$ when $z \to \sigma$. Now, let $\varphi \in \mathrm{Hol}(\Delta, D)$ be a complex geodesic with $\varphi(0) = p$ and $\varphi(1) = \sigma$. Since $\varphi(t) \to \sigma$ non-tangentially as $t \to 1^-$, we have $f(\sigma(t)) \to \sigma$. Therefore if we put $z = \varphi(t)$ in (1.3), letting $t \to 1^-$ and recalling (1.2) we get

$$\frac{1}{2}\log \beta_{\sigma,q} \leqslant \frac{1}{2}\log \beta_{\sigma,p} + \frac{1}{2}\log h_{\sigma,p}(q) - \frac{1}{2}\log h_{\sigma,p}(q) = \frac{1}{2}\log \beta_{\sigma,p}.$$

Thus $\beta_{\sigma,q}$ is finite too, and reversing the roles of p and q we get the assertion. □

In particular, we shall simply denote by β_σ the dilation coefficient at a boundary fixed point.

Definition 1.3. Let $\sigma \in \partial D$ be a boundary fixed point for a self-map $f \in \mathrm{Hol}(D, D)$ of a bounded strongly convex C^2 domain $D \Subset \mathbb{C}^d$. We shall say that σ is *attracting* if $0 < \beta_\sigma < 1$, *parabolic* if $\beta_\sigma = 1$ and *repelling* if $\beta_\sigma > 1$.

We can now quote the general versions of Julia's lemma proved in [1,3] (see [2, Theorem 2.4.16 and Proposition 2.7.15]) that we shall need in this paper:

Proposition 1.4. *(See Abate [1].) Let $D \Subset \mathbb{C}^n$ be a bounded strongly convex C^2 domain, and $f \in \mathrm{Hol}(D, D)$. Let $\sigma \in \partial D$ and $p \in D$ be such that the dilation coefficient $\beta_{\sigma,p}$ is finite. Then there exists a unique $\tau \in \partial D$ such that*

$$\forall R > 0 \quad f\bigl(E_p(\sigma, R)\bigr) \subseteq E_p(\tau, \beta_{\sigma,p} R),$$

and f has K-limit τ at σ.

Finally, we recall the several variable version of the Wolff–Denjoy theorem given in [1] (see [2, Theorems 2.4.19 and 2.4.23]):

Theorem 1.5. *(See Abate [1].) Let $D \Subset \mathbb{C}^n$ be a bounded strongly convex C^2 domain, and $f \in \mathrm{Hol}(D, D)$ without fixed points. Then there exists a unique $\tau \in \partial D$ such that the sequence of iterates of f converges to τ.*

Definition 1.4. Let $D \Subset \mathbb{C}^n$ be a bounded strongly convex C^2 domain, and $f \in \mathrm{Hol}(D, D)$ without fixed points. The point $\tau \in \partial D$ introduced in the previous theorem is the *Wolff point* of f.

The Wolff point can be characterized by the dilation coefficient:

Proposition 1.6. *Let $D \Subset \mathbb{C}^d$ be a bounded strongly convex C^2 domain, and $f \in \mathrm{Hol}(D, D)$ without fixed points in D. Then the following assertions are equivalent for a point $\tau \in \partial D$:*

(i) *τ is a boundary fixed point with $0 < \beta_\tau \leqslant 1$;*
(ii) *$f(E_p(\tau, R)) \subseteq E_p(\tau, R)$ for all $R > 0$ and any (and hence all) $p \in D$;*
(iii) *τ is the Wolff point of f.*

Proof. (i) $\Longrightarrow$ (ii): it follows immediately from Proposition 1.4.

(ii) $\Longrightarrow$ (iii): it follows, as in the proof of [2, Theorem 2.4.23], from the facts that the sequence of iterates $\{f^n\}$ is compactly divergent and that $\overline{E_p(\tau, R)} \cap \partial D = \{\tau\}$ for all $R > 0$ and $p \in D$.

(iii) $\Longrightarrow$ (i): since f has no fixed points, by [2, Theorem 2.4.19] there is a $\tau' \in \partial D$ such that $f(E_p(\tau', R)) \subseteq E_p(\tau', R)$ for all $R > 0$ and $p \in D$. Since $\overline{E_p(\tau', R)} \cap \partial D = \{\tau'\}$ we must have $\tau' = \tau$. Now fix $p \in D$ and let $\varphi \in \mathrm{Hol}(\Delta, D)$ be a complex geodesic with $\varphi(0) = p$ and $\varphi(1) = \tau$. Let $\tilde{p}_\varphi \in \mathrm{Hol}(D, \Delta)$ be the left-inverse of φ, and put $\tilde{f} = \tilde{p}_\varphi \circ f \circ \varphi \in \mathrm{Hol}(\Delta, \Delta)$. Since, as observed before, complex geodesics and left-inverses preserve the horospheres (see [2, Proposition 2.7.8 and Lemma 2.7.16]), we have $\tilde{f}(E_0(1, R)) \subseteq E_0(1, R)$ for all $R > 0$. This easily implies that either $\tilde{f}$ has no fixed points or it is the identity. In the latter case (1.2) implies that $\beta_\tau = 1$, and we are done.

If instead $\tilde{f}$ has no fixed points, by the one-variable Wolff's lemma, $1 \in \partial\Delta$ is its Wolff point, and [2, Corollary 1.2.16] the dilation coefficient β of $\tilde{f}$ at 1 belongs to $(0, 1]$. But, again by (1.2), $\beta = \beta_\tau$, and we are done. □

Definition 1.5. Let $D \Subset \mathbb{C}^n$ be a bounded strongly convex C^2 domain, and $f \in \mathrm{Hol}(D, D)$ without fixed points and with Wolff point $\tau \in \partial D$. We shall say that f is *hyperbolic* if $0 < \beta_\tau < 1$ and *parabolic* if $\beta_\tau = 1$.

2. Convergence of backward orbits

In this section we shall prove our main Theorem 0.1. This will be accomplished by a sequence of lemmas, but first we recall a couple of definitions:

Definition 2.1. Let $f : X \to X$ be a self-map of a set X. A *backward orbit* (or *backward iteration sequence*) for f is a sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ so that $f(x_{n+1}) = x_n$ for all $n \in \mathbb{N}$.

Definition 2.2. Let X be a (Kobayashi) hyperbolic manifold. We say that a sequence $\{z_n\} \subset X$ has *bounded Kobayashi step* if

$$a = \sup_n k_X(z_{n+1}, z_n) < +\infty.$$

The number a is the *Kobayashi step* of the sequence.

We shall first deal with the hyperbolic and parabolic cases.

Lemma 2.1. *Let $D \Subset \mathbb{C}^d$ be a bounded strongly convex C^2 domain. Let $\{z_n\} \subset D$ be a backward orbit for a parabolic or hyperbolic self-map $f \in \mathrm{Hol}(D, D)$. Then $z_n \to \partial D$ as $n \to +\infty$.*

Proof. Assume, by contradiction, that the sequence does not converge to ∂D. Then there exists a subsequence $\{z_{n_k}\}$ converging to $w_0 \in D$, that is, such that

$$k_D(w_0, z_{n_k}) \to 0 \quad \text{as } k \to +\infty.$$

Therefore

$$k_D\bigl(f^{n_k}(w_0), f^{n_k}(z_{n_k})\bigr) \leqslant k_D(w_0, z_{n_k}) \to 0 \quad \text{as } k \to +\infty.$$

But, on the other hand, $f^{n_k}(z_{n_k}) = z_0$ for all k; moreover, $f^{n_k}(w_0) \to \tau$ as $k \to +\infty$, where $\tau \in \partial D$ is the Wolff point of f, and so

$$\lim_{k\to\infty} k_D\bigl(f^{n_k}(w_0), f^{n_k}(z_{n_k})\bigr) = +\infty,$$

because k_D is complete, a contradiction. □

Lemma 2.2. *Let $D \Subset \mathbb{C}^d$ be a bounded strongly convex C^2 domain. Let $\{z_n\} \subset D$ be a sequence with bounded Kobayashi step $a > 0$ converging toward the boundary of D. Then there exists $\sigma \in \partial D$ such that $z_n \to \sigma$ as $n \to +\infty$.*

Proof. Since $z_n \to \partial D$, we can find a subsequence $\{z_{n_j}\}$ converging to a point $\sigma \in \partial D$; we claim that the whole sequence converges to σ.

If for every $k \in \mathbb{N}$ the subsequence $\{z_{n_j+k}\}$ converges to σ, then clearly the whole sequence converges to σ and we are done. Otherwise, there exists a minimum $k > 0$ such that the sequence $\{z_{n_j+k-1}\}$ converges to σ but $\{z_{n_j+k}\}$ does not. Up to extracting a subsequence in both and renaming, we may then assume that $\{z_{n_j}\}$ converges to σ while $\{z_{n_j+1}\}$ converges to $\tilde{\sigma} \in \partial D$ different from σ.

Then [2, Corollary 2.3.55] yields $\varepsilon > 0$ and $K > 0$ such that, as soon as $\|z_{n_j} - \sigma\| < \varepsilon$, and $\|z_{n_j+1} - \tilde{\sigma}\| < \varepsilon$, we have

$$K - \frac{1}{2}\log d(z_{n_j}, \partial D) - \frac{1}{2}\log d(z_{n_j+1}, \partial D) \leqslant k_D(z_{n_j}, z_{n_j+1}) \leqslant a.$$

Letting $j \to +\infty$ we get a contradiction. □

Lemma 2.3. *Let $D \Subset \mathbb{C}^d$ be a bounded strongly convex C^2 domain, and fix $p \in D$. Let $f \in \mathrm{Hol}(D, D)$, and $\{z_n\} \subset D$ be a backward orbit for f with bounded Kobayashi step $a = \frac{1}{2}\log\alpha$ converging to $\sigma \in \partial D$. Then σ is a boundary fixed point of f and $\beta_\sigma \leqslant \alpha$.*

Proof. Fix $p \in D$. First of all we have

$$\begin{aligned}\frac{1}{2}\log\beta_{\sigma,p} = \liminf_{w\to\sigma}\bigl[k_D(w,p) - k_D\bigl(f(w),p\bigr)\bigr] &\leqslant \liminf_{n\to+\infty}\bigl[k_D(z_{n+1},p) - k_D(z_n,p)\bigr]\\ &\leqslant \liminf_{n\to+\infty} k_D(z_{n+1}, z_n)\\ &\leqslant a = \frac{1}{2}\log\alpha.\end{aligned}$$

Since $z_n \to \sigma$ and $f(z_n) = z_{n-1} \to \sigma$ as $n \to +\infty$, using [2, Proposition 2.4.15] we get that $f(E_p(\sigma, R)) \subseteq E_p(\sigma, \alpha R)$ for all $R > 0$. Then Proposition 1.4 implies that f has K-limit σ at σ, and we are done. □

Lemma 2.4. *Let $D \Subset \mathbb{C}^d$ be a bounded strongly convex C^2 domain. Let $f \in \mathrm{Hol}(D, D)$ be hyperbolic or parabolic with Wolff point $\tau \in \partial D$ and dilation coefficient $0 < \beta_\tau \leqslant 1$. Let $\{z_n\} \subset D$ be a backward orbit for f with bounded Kobayashi step converging to $\sigma \in \partial D \setminus \{\tau\}$. Then*

$$\beta_\sigma \geqslant \frac{1}{\beta_\tau} \geqslant 1.$$

Proof. Let $\varphi : \overline{\Delta} \to \overline{D}$ be a complex geodesic such that $\varphi(-1) = \sigma$ and $\varphi(1) = \tau$, and set $p = \varphi(0)$. Proposition 1.4 yields

$$p \in \overline{E_p(\sigma, 1)} \quad\Longrightarrow\quad f(p) \in \overline{E_p(\sigma, \beta_\sigma)}$$

and

$$p \in \overline{E_p(\tau, 1)} \quad\Longrightarrow\quad f(p) \in \overline{E_p(\tau, \beta_\tau)}.$$

Hence $\overline{E_p(\sigma, \beta_\sigma)} \cap \overline{E_p(\tau, \beta_\tau)} \neq \emptyset$.

Let $\tilde{p}_\varphi : D \to \Delta$ be the left-inverse of φ. Then

$$\begin{aligned}\emptyset \neq \tilde{p}_\varphi\big(\overline{E_p(\sigma,\beta_\sigma)} \cap \overline{E_p(\tau,\beta_\tau)}\big) &\subseteq \tilde{p}_\varphi\big(\overline{E_p(\sigma,\beta_\sigma)}\big) \cap \tilde{p}_\varphi\big(\overline{E_p(\tau,\beta_\tau)}\big)\\ &= \overline{E_0(-1,\beta_\sigma)} \cap \overline{E_0(1,\beta_\tau)}.\end{aligned}$$

Now, $E_0(1,\beta_\tau)$ is a Euclidean disk of radius $\beta_\tau/(\beta_\tau+1)$ tangent to $\partial\Delta$ in 1, and $E_0(-1,\beta_\sigma)$ is a Euclidean disk of radius $\beta_\sigma/(\beta_\sigma+1)$ tangent to $\partial\Delta$ in -1. So these disks intersect if and only if

$$1 - \frac{2\beta_\tau}{\beta_\tau+1} \leqslant -1 + \frac{2\beta_\sigma}{\beta_\sigma+1},$$

which is equivalent to $\beta_\sigma\beta_\tau \geqslant 1$, as claimed. □

In this way we have proved Theorem 0.1(i) for hyperbolic and parabolic maps. Now we prove Theorem 0.1(iv):

Lemma 2.5. *Let $D \Subset \mathbb{C}^d$ be a bounded strongly convex C^2 domain, and fix $p \in D$. Let $f \in \mathrm{Hol}(D,D)$ be hyperbolic or parabolic with Wolff point $\tau \in \partial D$, and let $\{z_n\} \subset D$ be a backward orbit for f with bounded Kobayashi step $a = \frac{1}{2}\log\alpha$ converging to $\sigma \in \partial D$. Then for every $p \in D$ there exists $M > 0$ such that $z_n \in K_p(\sigma, M)$ eventually.*

Proof. Choose $p \in M$. We clearly have

$$\liminf_{n\to\infty}\big[k_D(p,z_{n+1}) - k_D(p,z_n)\big] \geqslant \frac{1}{2}\log\beta_\sigma;$$

since, by the previous lemma, $\beta_\sigma \geqslant 1$, there thus exists $n_0 \geqslant 0$ such that

$$k_D(p,z_{n+1}) - k_D(p,z_n) \geqslant \frac{1}{2}\log\beta_\sigma^{1/2}$$

for all $n \geqslant n_0$. Therefore

$$k_D(p,z_{n+1}) - k_D(p,z_n) - k_D(z_{n+1},z_n) \geqslant \frac{1}{2}\log\beta_\sigma^{1/2} - \frac{1}{2}\log\alpha > -\infty,$$

and hence

$$\begin{aligned}&k_D(p,z_{n+2}) - k_D(p,z_n) - k_D(z_{n+2},z_n)\\ &\quad\geqslant k_D(p,z_{n+1}) - k_D(z_{n+2},z_{n+1}) - k_D(p,z_n) - k_D(z_{n+2},z_n)\\ &\quad\geqslant k_D(p,z_{n+1}) - k_D(p,z_n) - k_D(z_{n+1},z_n)\\ &\quad\geqslant \frac{1}{2}\log\frac{\beta_\sigma^{1/2}}{\alpha}.\end{aligned}$$

By induction, for any $m > n \geqslant n_0$ we thus have

$$k_D(p,z_m) - k_D(p,z_n) - k_D(z_m,z_n) \geqslant \frac{1}{2}\log\frac{\beta_\sigma^{1/2}}{\alpha},$$

i.e.,

$$k_D(z_m, z_n) - k_D(p, z_m) + k_D(p, z_n) \leqslant \frac{1}{2}\log\bigl(\alpha\beta_\sigma^{-1/2}\bigr).$$

Then

$$\begin{aligned}\lim_{w\to\sigma}\bigl[k_D(z_n, w) - k_D(p, w)\bigr] + k_D(p, z_n) &= \lim_{m\to\infty}\bigl[k_D(z_n, z_m) - k_D(p, z_m)\bigr] + k_D(p, z_n)\\ &\leqslant \frac{1}{2}\log\bigl(\alpha\beta_\sigma^{-1/2}\bigr) < +\infty,\end{aligned}$$

for all $n \geqslant n_0$, and we are done. □

To prove Theorem 0.1(iii) we need another lemma:

Lemma 2.6. *Let $D \Subset \mathbb{C}^d$ be a bounded strongly convex C^2 domain, and fix $p \in D$. Let $f \in \mathrm{Hol}(D, D)$ be hyperbolic or parabolic with Wolff point $\tau \in \partial D$ and dilation coefficient $0 < \beta_\tau \leqslant 1$. Let $\{z_n\} \subset D$ be a backward orbit for f. Then*

$$\forall n \in \mathbb{N} \quad h_{\tau,p}(z_n) \geqslant \left(\frac{1}{\beta_\tau}\right)^n h_{\tau,p}(z_0).$$

Proof. Put $t_n = h_{\tau,p}(z_n)$. By definition, $z_n \in \partial E_p(\tau, t_n)$. By Proposition 1.4, if $z_{n+1} \in E_p(\tau, R)$ then $z_n \in E_p(\tau, \beta_\tau R)$. Since $z_n \notin E_p(\tau, t_n)$, we have that $z_{n+1} \notin E_p(\tau, \beta_\tau^{-1} t_n)$, that is

$$t_{n+1} \geqslant \frac{1}{\beta_\tau} t_n, \tag{2.1}$$

and the assertion follows by induction. □

Corollary 2.7. *Let $D \Subset \mathbb{C}^d$ be a bounded strongly convex C^2 domain. Let $f \in \mathrm{Hol}(D, D)$ be hyperbolic with Wolff point $\tau \in \overline{D}$. Let $\{z_n\} \subset D$ be a backward orbit for f with bounded Kobayashi step $a > 0$ converging to $\sigma \in \partial D$. Then $\sigma \neq \tau$.*

Proof. By Lemma 2.5, the sequence $\{z_n\}$ converges to σ inside a Korányi region with center σ. But, by Lemma 2.6, z_n is eventually outside any horosphere centered in τ, and this clearly implies $\tau \neq \sigma$ as claimed. □

So we have Theorem 0.1(iii), and together with Lemma 2.4 we have also proved Theorem 0.1(ii) for the hyperbolic case.

Remark 2.1. Lemma 2.6 can be used to give another proof of the convergence of a backward orbit of bounded Kobayashi step for hyperbolic maps. First of all [4, Remark 3] yields a constant $C_1 > 0$ such that

$$\|z_n - z_{n+1}\| \leqslant \frac{C_1}{\sqrt{1-\hat{a}^2}}\sqrt{d(z_n, \partial D)} \leqslant \frac{C_1}{1-\hat{a}}\sqrt{d(z_n, \partial D)}, \tag{2.2}$$

where $\hat{a} = \tanh a \in (0, 1)$ and a is the Kobayashi step of the backward orbit $\{z_n\}$. On the other hand, given $p \in D$ the triangular inequality and the upper estimate [2, Theorem 2.3.51] on the boundary behavior of the Kobayashi distance yield a constant $C_2 > 0$ such that

$$\frac{1}{2}\log h_{\tau,p}(z_n) \leqslant k_D(p, z_n) \leqslant C_2 - \frac{1}{2}\log d(z_n, \partial D),$$

and thus

$$\|z_n - z_{n+1}\| \leqslant \frac{C}{1-\hat{a}}\sqrt{\frac{1}{h_{\tau,p}(z_n)}}, \tag{2.3}$$

for a suitable $C > 0$. Therefore using Lemma 2.6 we get that for every $m \geqslant n \geqslant 0$ we have

$$\|z_m - z_n\| \leqslant \sum_{j=n}^{m-1}\|z_{j+1} - z_j\| \leqslant \frac{C}{1-\hat{a}}\frac{1}{\sqrt{h_{\tau,p}(z_0)}}\sum_{j=n}^{m-1}\beta_\tau^{j/2} \leqslant \frac{C}{1-\hat{a}}\frac{1}{1-\beta_\tau^{1/2}}\frac{\beta_\tau^{n/2}}{\sqrt{h_{\tau,p}(z_0)}},$$

and so $\{z_n\}$ is a Cauchy sequence in $\mathbb{C}^d$, converging to a point $\sigma \in \partial D$ by Lemma 2.1.

Let us now deal with strongly elliptic maps. We need a preliminary lemma:

Lemma 2.8. *Let $D \Subset \mathbb{C}^d$ be a bounded strongly convex C^2 domain. Let $f \in \mathrm{Hol}(D, D)$ be strongly elliptic with Wolff point $p \in D$. Then for every $R_0 > 0$ there exists $0 < c = c(R_0) < 1$ such that*

$$k_D\big(f(z), p\big) - k_D(z, p) \leqslant \frac{1}{2}\log c < 0$$

for all $z \in D$ with $k_D(z, p) \geqslant R_0$.

Proof. By contradiction, assume that for every $c < 1$ there exists $z(c) \in D$ with $k_D(z(c), p) \geqslant R_0$ so that

$$k_D\big(f(z(c)), p\big) - k_D\big(z(c), p\big) > \frac{1}{2}\log c.$$

Let $z_\infty \in \overline{D}$ be a limit point of the sequence $\{z(1 - 1/n)\}$. If $z_\infty \in D$ then

$$k_D\big(f(z_\infty), p\big) - k_D(z_\infty, p) \geqslant \frac{1}{2}\log 1 = 0$$

against Lemma 1.1. Thus $z_\infty \in \partial D$. But then

$$\liminf_{z \to z_\infty}\big[k_D(z, p) - k_D\big(f(z), p\big)\big] \leqslant 0.$$

By Proposition 1.4 we then have $f(E_p(z_\infty, R)) \subseteq E_p(z_\infty, R)$ for every $R > 0$. Choose $R < 1$ so that $p \notin \overline{E_p(z_\infty, R)}$, and let $w \in \overline{E_p(z_\infty, R)}$ be the point closest to p with respect to the

Kobayashi distance. Since $f(w) \in \overline{E_p(z_\infty, R)}$, it follows that $k_D(f(w), p) \geqslant k_D(w, p)$, which is again impossible, because $w \neq p$ and f is strongly elliptic. $\square$

Lemma 2.9. *Let $D \Subset \mathbb{C}^d$ be a bounded strongly convex C^2 domain. Let $f \in \mathrm{Hol}(D, D)$ be strongly elliptic with Wolff point $p \in D$, and let $\{z_n\} \subset D$ be a backward orbit with Kobayashi bounded step $a = \frac{1}{2}\log\alpha$. Then $z_n \to \sigma \in \partial D$, and σ is a boundary fixed point of f with $\beta_\sigma \leqslant \alpha$.*

Proof. Let define $s_n > 0$ by setting $-\frac{1}{2}\log s_n = k_D(z_n, p)$. Without loss of generality, we can assume that $z_0 \neq p$; let $R_0 = k_D(z_0, p)$, and $c = c(R_0) < 1$ given by Lemma 2.8. Arguing by induction we have

$$k_D(z_n, p) - k_D(z_{n+1}, p) \leqslant \frac{1}{2}\log c < 0;$$

in particular, $k_D(z_{n+1}, p) > k_D(z_n, p) \geqslant R_0$ always. Hence

$$-\frac{1}{2}\log s_n + \frac{1}{2}\log s_{n+1} \leqslant \frac{1}{2}\log c,$$

that is

$$s_{n+1} \leqslant c s_n. \tag{2.4}$$

Therefore $s_{n+k} \leqslant c^k s_n$ for every $n, k \in \mathbb{N}$. So $s_n \to 0$ as $n \to +\infty$, that is $z_n \to \partial D$, and the assertion follows from Lemmas 2.2 and 2.3. $\square$

Remark 2.2. We can give another proof of the convergence of a backward orbit $\{z_n\}$ with bounded Kobayashi step $a > 0$ for strongly elliptic maps along the lines of Remark 2.1. Indeed, using (2.2) and [2, Theorem 2.3.51] we get

$$\|z_n - z_{n+1}\| \leqslant \frac{C}{1 - \hat{a}}\sqrt{s_n}$$

for a suitable $C > 0$, where $\hat{a} = \tanh a$ and $-\frac{1}{2}\log s_n = k_D(p, z_n)$. Since (2.4) yields $s_n \leqslant c^n s_0$, arguing as in Remark 2.1 we see that $\{z_n\}$ is a Cauchy sequence in $\mathbb{C}^d$ converging to a point $\sigma \in \partial D$.

Lemma 2.10. *Let $D \Subset \mathbb{C}^d$ be a bounded strongly convex C^2 domain. Let $f \in \mathrm{Hol}(D, D)$ be strongly elliptic with Wolff point $p \in D$. If $\sigma \in \partial D$ is a boundary fixed point then $\beta_\sigma > 1$.*

Proof. Indeed, Lemma 2.8 yields $0 < c < 1$ such that

$$\frac{1}{2}\log\beta_\sigma = \liminf_{z \to \sigma}\bigl[k_D(z, p) - k_D\bigl(f(z), p\bigr)\bigr] \geqslant -\frac{1}{2}\log c > 0,$$

and we are done. $\square$

So we have proven Theorem 0.1(i) and (ii); (iii) follows from the obvious fact that $p \in D$ whereas $\sigma \in \partial D$. We now conclude the proof of Theorem 0.1 with

Lemma 2.11. *Let $D \Subset \mathbb{C}^d$ be a bounded strongly convex C^2 domain. Let $f \in \mathrm{Hol}(D, D)$ be strongly elliptic, with Wolff point $p \in D$. Let $\{z_n\} \subset D$ be a backward orbit for f with bounded Kobayashi step converging to $\sigma \in \partial D$. Then for every $q \in D$ there exists $M > 0$ such that $z_n \in K_q(\sigma, M)$ eventually.*

Proof. As usual, it suffices to prove the statement for $q = p$. Lemma 2.8 yields $0 < c < 1$ such that

$$\liminf_{n\to\infty}\bigl[k_D(p, z_{n+1}) - k_D(p, z_n)\bigr] \geqslant \frac{1}{2}\log\frac{1}{c} > 0,$$

and then the assertion follows arguing as in the proof of Lemma 2.5. □

3. Construction of backward orbits with bounded Kobayashi step

In this section we shall construct backward orbits with bounded Kobayashi step converging to isolated boundary fixed points. To do so we need a definition and two lemmas.

Definition 3.1. Let $D \Subset \mathbb{C}^d$ be a bounded strongly convex C^2 domain, and $f \in \mathrm{Hol}(D, D)$. A boundary fixed point $\sigma \in \partial D$ with dilation coefficient β_σ is *isolated* if there is a neighborhood $U \subset \mathbb{C}^d$ of σ in $\mathbb{C}^d$ such that $U \cap \partial D$ contains no other boundary fixed point of f with dilation coefficient at most β_σ.

Lemma 3.1. *Let $D \Subset \mathbb{C}^d$ be a bounded strongly convex C^2 domain, and $f \in \mathrm{Hol}(D, D)$. Let $\sigma \in \partial D$ be a boundary fixed point of f with finite dilation coefficient β_σ, and choose a complex geodesic $\varphi \in \mathrm{Hol}(\Delta, D)$ with $\varphi(1) = \sigma$. Then*

$$\lim_{t\to 1^-} k_D\bigl(\varphi(t), f\bigl(\varphi(t)\bigr)\bigr) = \frac{1}{2}|\log\beta_\sigma|.$$

Proof. We shall first prove the statement when $D = \Delta$ and $\varphi = \mathrm{id}_\Delta$. In this case

$$k_\Delta\bigl(t, f(t)\bigr) = \frac{1}{2}\log\frac{1 + \bigl|\frac{t-f(t)}{1-tf(t)}\bigr|}{1 - \bigl|\frac{t-f(t)}{1-tf(t)}\bigr|}.$$

Now, the classical Julia–Wolff–Carathéodory theorem yields

$$\frac{1-tf(t)}{1-t} = 1 + t\frac{1-f(t)}{1-t} \to 1 + \beta_1, \qquad \frac{1-tf(t)}{1-f(t)} = 1 + f(t)\frac{1-t}{1-f(t)} \to 1 + \frac{1}{\beta_1}$$

as $t \to 1^-$; therefore

$$\frac{t-f(t)}{1-tf(t)} = \frac{1-f(t)}{1-tf(t)} - \frac{1-t}{1-tf(t)} \to \frac{1}{1+(1/\beta_1)} - \frac{1}{1+\beta_1} = \frac{\beta_1 - 1}{\beta_1 + 1},$$

and the assertion follows.

In the general case, let $\tilde{p}_\varphi \in \mathrm{Hol}(D, \Delta)$ be the left-inverse of φ, and $p_\varphi = \varphi \circ \tilde{p}_\varphi$. Put $f_\varphi = p_\varphi \circ f \in \mathrm{Hol}(D, D)$ and $\tilde{f} = \tilde{p}_\varphi \circ f \circ \varphi \in \mathrm{Hol}(\Delta, \Delta)$. First of all

$$\begin{aligned} k_D\big(\varphi(t), f\big(\varphi(t)\big)\big) &= k_D\big(\varphi(t), f_\varphi\big(\varphi(t)\big)\big) + k_D\big(\varphi(t), f\big(\varphi(t)\big)\big) - k_D\big(\varphi(t), f_\varphi\big(\varphi(t)\big)\big) \\ &= k_\Delta\big(t, \tilde{f}(t)\big) + k_D\big(\varphi(t), f\big(\varphi(t)\big)\big) - k_D\big(\varphi(t), f_\varphi\big(\varphi(t)\big)\big). \end{aligned}$$

Since σ is a boundary fixed point of f it immediately follows that 1 is a boundary fixed point of $\tilde{f}$. Furthermore, (1.2) implies that the dilation coefficient of $\tilde{f}$ at 1 is β_σ; hence $k_\Delta(t, \tilde{f}(t)) \to \frac{1}{2}|\log\beta_\sigma|$ as $t \to 1^-$. Now,

$$\big|k_D\big(\varphi(t), f\big(\varphi(t)\big)\big) - k_D\big(\varphi(t), f_\varphi\big(\varphi(t)\big)\big)\big| \leqslant k_D\big(f\big(\varphi(t)\big), f_\varphi\big(\varphi(t)\big)\big);$$

so to conclude the proof it suffices to show that $k_D(f(\varphi(t)), f_\varphi(\varphi(t))) \to 0$ as $t \to 1^-$.

Set $\gamma(t) = f(\varphi(t))$. By [2, Proposition 2.7.11] it suffices to prove

- that $p_\varphi \circ \gamma(t) \to \sigma$ non-tangentially;
- that $\gamma(t)$ is eventually inside a Euclidean ball contained in D and tangent to ∂D in σ;
- and that

$$\lim_{t\to 1^-} \frac{\|\gamma(t) - p_\varphi(\gamma(t))\|^2}{d(p_\varphi(\gamma(t)), \partial D)} = 0. \tag{3.1}$$

Since φ is transversal to ∂D, to prove that $p_\varphi \circ \gamma(t) \to \sigma$ non-tangentially it suffices to show that $\tilde{p}_\varphi \circ \gamma(t) = \tilde{f}(t) \to 1$ non-tangentially. But the classical Julia–Wolff–Carathéodory theorem yields

$$\frac{|1 - \tilde{f}(t)|}{1 - |\tilde{f}(t)|} = \left|\frac{1 - \tilde{f}(t)}{1 - t}\right| \frac{1 - t}{1 - |\tilde{f}(t)|} \to \beta_\sigma \cdot \frac{1}{\beta_\sigma} = 1, \tag{3.2}$$

and this is done.

To prove (3.1), we first recall that [2, Proposition 2.7.23] yields

$$\lim_{t\to 1^-} \frac{\|\gamma(t) - p_\varphi(\gamma(t))\|^2}{1 - t} = 0. \tag{3.3}$$

Furthermore, we already noticed that

$$\lim_{t\to 1^-} \frac{1 - t}{1 - |\tilde{f}(t)|} = \frac{1}{\beta_\sigma} > 0. \tag{3.4}$$

Finally, the lower estimate [2, Theorem 2.3.52] on the boundary behavior of the Kobayashi distance yields $c_1 \in \mathbb{R}$ such that

$$\frac{1}{2}\log\frac{1 + |\tilde{f}(t)|}{1 - |\tilde{f}(t)|} = k_\Delta\big(0, \tilde{f}(t)\big) = k_D\big(\varphi(0), f_\varphi\big(\varphi(t)\big)\big) \geqslant c_1 - \frac{1}{2}\log d\big(f_\varphi\big(\varphi(t)\big), \partial D\big),$$

that is

$$\frac{1-|\tilde{f}(t)|}{d(f_\varphi(\varphi(t)),\partial D)} \leqslant 2e^{-2c_1}. \tag{3.5}$$

Putting together (3.3), (3.4) and (3.5) we get (3.1).

More precisely, (3.2) says that the curve $t \mapsto \tilde{f}(t)$ converges to 1 radially (that is, tangent to the radius ending in 1); therefore the curve $p_\varphi \circ \gamma$ goes to σ tangentially to the transversal curve $t \mapsto \varphi(t)$. Furthermore, the upper estimate [2, Theorem 2.3.51] yields $c_2 \in \mathbb{R}$ such that

$$\frac{1}{2}\log\frac{1+|\tilde{f}(t)|}{1-|\tilde{f}(t)|} \leqslant c_2 - \frac{1}{2}\log d\bigl(f_\varphi\bigl(\varphi(t)\bigr),\partial D\bigr);$$

hence recalling (3.4) and (3.5) we see that $d(f_\varphi(\varphi(t)),\partial D)$ is comparable to $1-t$. Recalling (3.3) we then obtain that $\gamma(t)$ is eventually contained in Euclidean balls internally tangent to ∂D in σ of arbitrarily small radius, and we are done. $\square$

Lemma 3.2. *Let $D \Subset \mathbb{C}^d$ be a bounded strongly convex C^2 domain, and $f \in \mathrm{Hol}(D,D)$. Let $\{z_n\} \subset D$ be a sequence converging to $\sigma \in \partial D$ such that $\limsup_{n\to+\infty} k_D(z_n, f(z_n)) = \frac{1}{2}\log\alpha < +\infty$. Then σ is a boundary fixed point with dilation coefficient at most α.*

Proof. The lower estimate [2, Corollary 2.3.55] immediately implies that $f(z_n) \to \sigma$ as well. Fix $p \in D$; then

$$\frac{1}{2}\log\beta_{\sigma,p} \leqslant \liminf_{n\to+\infty}\bigl[k_D(z_n,p) - k_D\bigl(f(z_n),p\bigr)\bigr] \leqslant \limsup_{n\to+\infty} k_D\bigl(z_n, f(z_n)\bigr) = \frac{1}{2}\log\alpha.$$

The assertion then follows arguing as in the proof of Lemma 2.3. $\square$

And now we can prove the announced

Theorem 3.3. *Let $D \Subset \mathbb{C}^d$ be a bounded strongly convex C^2 domain, and take $f \in \mathrm{Hol}(D,D)$ hyperbolic, parabolic or strongly elliptic with Wolff point $\tau \in \overline{D}$. Let $\sigma \in \partial D \setminus \{\tau\}$ be an isolated repelling boundary fixed point for f with dilation coefficient $\beta_\sigma > 1$. Then there is a backward orbit with Kobayashi step bounded by $\frac{1}{2}\log\beta_\sigma$ converging to σ.*

Proof. We follow closely the proof of [10, Lemma 1.4].

Let $U \subseteq \mathbb{C}^d$ be a small ball centered at σ in $\mathbb{C}^d$ such that $\overline{U} \cap \overline{D}$ contains neither τ nor other boundary fixed points with dilation coefficient at most β_σ, and put $J = \partial U \cap D$.

Let $\varphi \in \mathrm{Hol}(\Delta, D)$ be a complex geodesic with $\varphi(1) = \sigma$, and put $p = \varphi(0)$. Furthermore, let $n_0 \geqslant 0$ be such that $E_k = E_p(\sigma, \beta_\sigma^{-n_0-k}) \subset U$ for all $k \geqslant 0$; set $r_k = \varphi(t_k)$, where $t_k \in (0,1)$ is such that $r_k \in \partial E_k \cap \varphi(\Delta)$.

For each k, let γ_k be the line segment connecting r_k and $f(r_k)$. Since $f^n(r_k) \to \tau \notin \overline{U}$, and $\bigcup_{j=0}^{n-1} f^j(\gamma_k)$ is a path connecting r_k with $f^n(r_k)$, there is a smallest integer n_k such that $f^{n_k}(\gamma_k)$ intersects J. Since, by Proposition 1.4, $f(E_{k+1}) \subseteq E_k$, and the horospheres are convex, we necessarily have $n_k > k$.

Put $z_k = f^{n_k}(r_k) \in U \cap D$; we claim that the sequence $\{z_k\}$ is relatively compact in D. If not, we can extract a subsequence $\{z_{k_j}\}$ converging to a point $\eta \in \partial D$. By Lemma 3.1, $k_D(z_{k_j}, f(z_{k_j})) \to \frac{1}{2}\log\beta_\sigma$. It follows, by Lemma 3.2, that η is a boundary fixed point with dilation coefficient at most β_σ; since $\eta \in \overline{U} \cap \partial D$, this contradicts the choice of U.

So there is an infinite set $I_0 \subseteq \mathbb{N}$ such that $\{z_k\}_{k\in I_0}$ converges to $w_0 \in D$. Fix $j \geqslant 1$, and assume that we can extract from I_{j-1} an infinite set I_j such that $\{f^{n_k-j}(r_k)\}$ converges to some $w_j \in D$. Let $S = \{f^{n_k-(j+1)}(r_k)\}_{k\in I_j}$; since

$$k_D\bigl(f^{n_k-(j+1)}(r_k), f^{n_k-j}(r_k)\bigr) \leqslant k_D\bigl(r_k, f(r_k)\bigr) \to \frac{1}{2}\log\beta_\sigma,$$

the sequence S is still relatively compact in D, and thus we can extract from I_j an infinite set I_{j+1} such that $\{f^{n_k-(j+1)}(r_k)\}_{k\in I_{j+1}}$ converges to a point $w_{j+1} \in D$. Notice that, by construction, $f(w_{j+1}) = w_j$; therefore $\{w_j\}$ is a backward orbit.

Since points of the form $f^{n_k-j}(r_k)$ with $n_k > j$ are contained in U, we have that $w_j \in \overline{U} \cap D$ for all j. Furthermore,

$$k_D(w_{j+1}, w_j) = \lim_{k\in I_{j+1}} k_D\bigl(f^{n_k-(j+1)}(r_k), f^{n_k-j}(r_k)\bigr) \leqslant \lim_{k\in I_{j+1}} k_D\bigl(r_k, f(r_k)\bigr) = \frac{1}{2}\log\beta_\sigma;$$

so we are left to prove that $w_j \to \sigma$.

Assume, by contradiction, that there is a subsequence $\{w_{j_h}\}$ converging to $q \in \overline{D} \setminus \{\sigma\}$. If $q \in D$, then the sequence $K = \{w_{j_h}\}$ is relatively compact in D; so there is an $n > 0$ such that $f^n(K) \cap U = \emptyset$. But K is a subsequence of a backward orbit contained in U, and so $f^n(K) \cap U \neq \emptyset$ for all $n \geqslant 0$.

Finally, if $q \in \partial D$, then, again by Lemma 3.2, q is a boundary fixed point with dilation coefficient at most β_σ; since $q \in \overline{U}$, this contradicts the choice of U, and we are done. □

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