Diagonalization of non-diagonalizable discrete holomorphic dynamical systems

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Abstract. We shall describe a canonical procedure to associate to any (germ of) holomorphic self-map F of $\mathbb{C}^n$ fixing the origin so that dF_O is invertible and non-diagonalizable an n-dimensional complex manifold M, a holomorphic map $\pi\colon M \to \mathbb{C}^n$, a point $\mathbf{e} \in M$ and a (germ of) holomorphic self-map $\tilde{F}$ of M such that: π restricted to $M \setminus \pi^{-1}(O)$ is a biholomorphism between $M \setminus \pi^{-1}(O)$ and $\mathbb{C}^n \setminus \{O\}$; $\pi \circ \tilde{F} = F \circ \pi$; and $\mathbf{e}$ is a fixed point of $\tilde{F}$ such that $d\tilde{F}_{\mathbf{e}}$ is diagonalizable. Furthermore, we shall use this construction to describe the local dynamics of such an F nearby the origin when $\mathrm{sp}(dF_O) = \{1\}$.

0. Introduction

In passing from one to several variables, possibly the first new phenomenon one has to deal with is the existence of non-diagonalizable linear maps. Roughly speaking, one can think of them as some sort of singularity in the space of all linear maps; indeed, a generic linear endomorphism is diagonalizable. It would be interesting to have a device to "resolve" the singularity, similarly to what happens in algebraic geometry for singularities of complex spaces.

In this paper we shall describe exactly such a device, in a more general holomorphic setting. Let $F \in \mathrm{End}(\mathbb{C}^n, O)$ be a (germ of) holomorphic self-map of $\mathbb{C}^n$ keeping the origin fixed and such that dF_O is invertible and non-diagonalizable. We shall build in a canonical way (depending only on the block structure of the Jordan form of dF_O) a new holomorphic map $\tilde{F}$ semi-conjugate to F (and actually conjugate to F outside the origin) with a canonical fixed point $\mathbf{e}$ such that $d\tilde{F}_{\mathbf{e}}$ is diagonalizable; the price to pay is that we have to change the base manifold. We shall in fact prove the following result (see Theorem 2.4):

Theorem 0.1: (Diagonalization Theorem) *Let $F \in \mathrm{End}(\mathbb{C}^n, O)$ be such that dF_O is invertible and non-diagonalizable. Then there exist a complex n-dimensional manifold M, a holomorphic projection $\pi\colon M \to \mathbb{C}^n$, a canonical point $\mathbf{e} \in M$ and a (germ at $\pi^{-1}(O)$ of) holomorphic self-map $\tilde{F}\colon M \to M$ such that:*

(i) *π restricted to $M \setminus \pi^{-1}(O)$ is a biholomorphism between $M \setminus \pi^{-1}(O)$ and $\mathbb{C}^n \setminus \{O\}$;*
(ii) *$\pi \circ \tilde{F} = F \circ \pi$;*
(iii) *$\mathbf{e}$ is a fixed point of $\tilde{F}$, and $d\tilde{F}_{\mathbf{e}}$ is diagonalizable.*

More precisely, if the Jordan canonical form of dF_O contains $\rho \ge 1$ blocks of length $\mu_1 \ge \mu_2 \ge \cdots \ge \mu_\rho \ge 1$ corresponding respectively to eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_\rho \in \mathbb{C}$, then $d\tilde{F}_{\mathbf{e}}$ has eigenvalues $\tilde{\lambda}_1, 1, \lambda_2/\lambda_1, \ldots, \lambda_\rho/\lambda_1$ of multiplicity respectively 1, $\mu_1 - 1$, $\mu_2, \ldots, \mu_\rho$, where $\tilde{\lambda}_1 = \lambda_1$ if $\mu_1 > \mu_2$, and $\tilde{\lambda}_1 = \lambda_1^2/\lambda_2$ if $\mu_1 = \mu_2$.

One subtle point must be stressed here. If the only aim is to diagonalize the differential, one can choose among several different constructions; but most of them are useless for the dynamical applications we have in mind. For instance, the standard way to resolve singularities in algebraic geometry is by blowing up points. One could do the same here: M could be obtained by $\mathbb{C}^n$ blowing up a suitable sequence of points, and then there is a unique way to lift F to a self-map $\tilde{F}$ of M enjoying some of the properties we are looking for. Unfortunately, this naive approach is too rough: the manifold M constructed in this way is so large that many properties of the original map F will be hidden inside the singular divisor $\pi^{-1}(O)$.

To give an idea why this is the case (see Remark 3.3 for a more precise explanation), let us discuss what is known about the local dynamics of F nearby the fixed point O. In the hyperbolic case (that is, when dF_O

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1991 Mathematical Subjects Classification: Primary 32H50, 32H02, 58F23 has no eigenvalues of modulus one) the stable manifold theorem (see, e.g., [Wu] for the statement in the complex case; see also [S] and [R1, 2] for the attracting case) describes completely the situation: there are two local F-invariant manifolds, the stable one W^s and the unstable one W^u, intersecting transversally at the origin, such that $(F|_{W^s})^k \to O$ and $(F|_{W^u})^{-k} \to O$ as $k \to +\infty$, uniformly on compact sets. More generally, the local dynamics is topologically conjugated to the dynamics induced by the differential dF_O, with W^s corresponding to the direct sum of the generalized eigenspaces associated to eigenvalues with modulus less than one, and W^u corresponding to the direct sum of the generalized eigenspaces associated to eigenvalues with modulus greater than one.

In the non-hyperbolic case, the theory at present is far less complete. One can recover a good generalization of the classical one-variable Fatou-Leau theorem in the semi-attractive case, when dF_O has 1 as eigenvalue of multiplicity one, and the others eigenvalues have absolute value less than 1. In this case (studied first by Fatou [F], and later by Ueda [U1, 2] and Hakim [H1]) either F admits a holomorphic curve of fixed points passing through the origin or there exists a basin of attraction to the origin, formed by $k-1$ petals, where $k \geq 2$ is the multiplicity of the origin as fixed point of F; furthermore, Nishimura [N] has a description of the dynamics when there is a curve of fixed points.

Another situation that has been studied is when $dF_O = \mathrm{id}$, that is when F is tangent to the identity. In this case Hakim [H2, 3] (see also Weickert [W]) has proved that for F generic there exists an F-invariant stable (i.e., attracted to the origin) holomorphic curve with the origin *in its boundary;* furthermore, there are estimates on the rate of approach of stable orbits to the origin (see Section 3 for a precise statement of Hakim's results). Notice that, in general, it is not possible to extend such a stable curve holomorphically through the origin. It should also be mentioned that Rivi [Ri] combined Hakim's results on maps tangent to the identity with results on the semiattractive case to obtain a description of the dynamics when there is a dF_O-invariant decomposition $\mathbb{C}^n = V_1 \oplus V_2$, with $dF_O|_{V_1} = \mathrm{id}$ and $\mathrm{sp}(dF_O|_{V_2}) \subset \{|\lambda| < 1\}$.

One feature that Hakim's and Weickert's works made clear is that one has to study orbits converging to the origin tangentially to a given direction $v \in \mathbb{C}^n$. It is easy to see that such a v must be an eigenvector of dF_O. Of course, not all the eigenvectors are tangent to an orbit; but nevertheless this observation points out that, from a dynamical point of view, the eigenvectors of dF_O should be treated differently from the non-eigenvectors.

Now we can go back to our discussion of the manifold M in Theorem 0.1. Blowing up points one deals with all the tangent directions in the same way; and the previous discussion suggests that this should not be the case. The correct replacement is blowing up *submanifolds;* in this way we are able to keep track of the different status of the different tangent directions — and we shall then be able to recover easily informations about the local dynamics of F from informations about the dynamics of $\tilde{F}$ (see, e.g., Corollary 3.2).

In Section 1 we describe the canonical procedure for building the manifold M. It depends only on the Jordan block structure of the differential dF_O, and is obtained by blowing up a sequence of at most $\mu_1 + 1$ submanifolds, where μ_1 is the dimension of the largest Jordan block in dF_O. In Section 2 we describe how to lift the map F to the blow-ups, and we give the proof of Theorem 0.1. It should be remarked that the construction is completely explicit; for instance, it is possible to compute the local power series expansion of the lifted map $\tilde{F}$ in terms of the local power series expansion of F, and this is essential for the applications.

In Section 3 we apply the Diagonalization Theorem to dynamics. Since the eigenvalues of $d\tilde{F}_{\mathbf{e}}$ are quotients of the eigenvalues of dF_O, this is really meaningful only when all the eigenvalues of dF_O have modulus one. We shall concentrate on the case $\mathrm{sp}(dF_O) = \{1\}$, because then $\tilde{F}$ is tangent to the identity. It turns out that, for generic F, one and exactly one of the $\tilde{F}$-stable holomorphic curves whose existence is guaranteed by Hakim's results is contained in $M \setminus \pi^{-1}(O)$; its projection under π is then an F-stable holomorphic curve, with the origin in its boundary (Corollary 3.2).

Thus we can apply Hakim's theory to generic maps F whose differential is non-diagonalizable and such that $\mathrm{sp}(dF_O) = \{1\}$. Actually, our technique is flexible enough to be used even for some classes of non-generic maps (see Section 3 for the definition of "generic" in this context). For instance, we have fairly complete results in the bi-dimensional case (Corollary 3.3), showing among other things that the dynamics might depend strongly on the third degree terms of the map F even when the quadratic part is not identically zero. Furthermore, we get yet another version of the Fatou-Bieberbach phenomenon (Remark 3.7).

A priori, one might suspect that other $\tilde{F}$-stable holomorphic curves might give rise at least to some other F-orbits converging to the origin, if not to F-stable holomorphic curves. In the last section of this paper we shall show that, under some mild assumption on the rate of convergence to zero of the orbit, if dF_O is the canonical Jordan block J_n of order n associated to 1 then this is not the case: roughly speaking, then, for such maps the stable dynamics nearby the origin is described by Corollary 3.2.

I would like to end this introduction quoting a few lines from [F, p. 135–137]: “Ce cas [that is, $n=2$ and $\mathrm{sp}(dF_O)=\{1\}$], très important au point de vue des applications aux équations de la dynamique, exigerait de longues et difficiles recherches pour être élucidé complètement. (...) Prenons par example (...) le cas limite

$$\begin{cases} x_1 = x + \alpha y, \\ y_1 = y + a' x^2, \end{cases}$$

substitution birationnelle que nous étudierons plus en detail dans la second partie de ce Mémoire”. Unfortunately, the promised second part never appeared; but now, after seventy-five years, we are at last able to describe the dynamics of Fatou’s example.

1. The blow-up sequence

As described in the introduction, to diagonalize a non-diagonalizable dynamical system we shall replace $\mathbb{C}^n$ by a suitable complex manifold obtained blowing-up a specific sequence of submanifolds, depending on the Jordan block structure of the differential of the map generating the dynamical system. In this section we introduce the general machinery needed.

First of all we fix a number of notations. Given $0 \le r < n$, a *splitting* $\mathcal{P}$ *of weight* r of n is a subdivision of $\{1,\ldots,n\}$ as a disjoint union $\{1,\ldots,n\} = \mathcal{P}' \cup \mathcal{P}''$, where $\mathrm{card}\,\mathcal{P}' = r$ e $\mathrm{card}\,\mathcal{P}'' = n-r$. The *standard splitting* of weight r is $\{1,\ldots,r\} \cup \{r+1,\ldots,n\}$. If $z=(z_1,\ldots,z_n)\in\mathbb{C}^n$ and $\mathcal{P}$ is a splitting of weight $r>0$ with $\mathcal{P}'=\{i_i,\ldots,i_r\}$ and $\mathcal{P}''=\{i_{r+1},\ldots,i_n\}$ (where $i_1<\cdots<i_r$ and $i_{r+1}<\cdots<i_n$), we shall write $z'=(z_{i_1},\ldots,z_{i_r})$ and $z''=(z_{i_{r+1}},\ldots,z_{i_n})$; if $r=0$ we set $z''=z$, and z' is empty. Finally, if V is any vector space and $v\in V\setminus\{O\}$, we denote by $[v]$ the projection of v in $\mathbb{P}(V)$.

Let M be a complex manifold of dimension $n\ge 2$, and $X\subset M$ a closed complex submanifold of dimension $r\ge 0$. Let $N_{X/M}$ denote the normal bundle of X in M, and let $E_X=\mathbb{P}(N_{X/M})$ be the projective normal bundle, whose fiber over $p\in X$ is $E_p=\mathbb{P}(T_pM/T_pX)$. The *blow-up of* M *along* X is the set

$$\tilde{M}_X = (M\setminus X)\cup E_X\ ,$$

endowed with the manifold structure we shall presently describe, together with the projection $\sigma\colon \tilde{M}_X\to M$ given by $\sigma|_{M\setminus X}=\mathrm{id}_{M\setminus X}$ and $\sigma|_{E_p}\equiv\{p\}$ for $p\in X$. The set $E_X=\sigma^{-1}(X)$ is the *exceptional divisor* of the blow-up.

A chart $\varphi=(z_1,\ldots,z_n)\colon V\to\mathbb{C}^n$ is *adapted to* X if there is a splitting $\mathcal{P}$ of weight $r=\dim X$ such that $V\cap X=\{z''=0\}$. Choose a chart (V,φ) adapted to X, and for $j\in\mathcal{P}''$ and $q\in V\cap X$ set $X_j=\{z_j=0\}\subset V$, $L_{j,q}=\mathbb{P}\big(\mathrm{Ker}(dz_j)_q/T_qX\big)\subset E_q$, $L_j=\bigcup_{q\in V\cap X}L_{j,q}$, $E_{V\cap X}=\sigma^{-1}(V\cap X)$ and $V_j=(V\setminus X_j)\cup(E_{V\cap X}\setminus L_j)$. Define $\chi_j\colon V_j\to\mathbb{C}^n$ by

$$\chi_j(q)_h = \begin{cases} \varphi(q)_h & \text{if } h\in\mathcal{P}', \\ z_h(q)/z_j(q) & \text{if } h\in\mathcal{P}''\setminus\{j\}, \\ z_j(q) & \text{if } h=j, \end{cases}$$

if $q\in V\setminus X_j$, and by

$$\chi_j([v])_h = \begin{cases} \varphi\big(\sigma([v])\big)_h & \text{if } h\in\mathcal{P}', \\ d(z_h)_{\sigma([v])}(v)/d(z_j)_{\sigma([v])}(v) & \text{if } h\in\mathcal{P}''\setminus\{j\}, \\ 0 & \text{if } h=j, \end{cases}$$

if $[v]\in E_{V\cap X}\setminus L_j$. Then it is not difficult to check that the charts (V_j,χ_j), together with an atlas of $M\setminus X$, endow $\tilde{M}_X$ with a structure of n-dimensional complex manifold, as claimed, such that the projection σ is holomorphic everywhere. For future reference, we record here that

$$\varphi\circ\sigma\circ\chi_j^{-1}(w)_h = \begin{cases} w_h & \text{if } h\in\mathcal{P}'\cup\{j\}, \\ w_j w_h & \text{if } h\in\mathcal{P}''\setminus\{j\}. \end{cases} \tag{1.1}$$

The fiber E_p of the exceptional divisor over a point $p \in X$ is a projective space; so the choice of an adapted chart yields an explicit isomorphism with $\mathbb{P}^{n-r-1}(\mathbb{C})$ that we shall denote by $\iota_{p,\varphi}: E_p \to \mathbb{P}^{n-r-1}(\mathbb{C})$. Finally, if $Y \subseteq M$ is a submanifold of M, then the *proper transform* of Y is $\tilde{Y} = \overline{\sigma^{-1}(Y \setminus X)} \subset \tilde{M}_X$.

To describe the sequence of blow-ups we need some more notations. Given $\rho \geq 1$, a *ρ-partition of n* is a set $\mathcal{M} = \{\mu_1, \ldots, \mu_\rho\} \subset \mathbb{N}$ with $\mu_1 \geq \cdots \geq \mu_\rho \geq 1$ and $\mu_1 + \cdots + \mu_\rho = n$. The *length* $\ell(\mathcal{M})$ of $\mathcal{M}$ is μ_1 if $\mu_2 < \mu_1$, and $\mu_1 + 1$ if $\mu_2 = \mu_1$.

To a ρ-partition $\mathcal{M}$ we can associate several objects. First of all, we define $\nu_1, \ldots, \nu_\rho \in \mathbb{N}$ by setting $\nu_1 = 0$ and $\nu_j = \nu_{j-1} + \mu_{j-1}$ for $j = 2, \ldots, \rho$. Then we define sets $\mathcal{P}'_{kl} \subset \{1, \ldots, n\}$ for $0 \leq k \leq \mu_1 - 1$ and $1 \leq l \leq \rho$ by setting

$$\mathcal{P}'_{kl} = \begin{cases} \varnothing & \text{if } k = 0, \\ \{\nu_l + 1, \ldots, \nu_l + \min(k, \mu_l)\} & \text{if } 1 \leq k \leq \mu_1 - 1. \end{cases}$$

If $\mu_2 = \mu_1$, we also define $\mathcal{P}'_{\mu_1,l}$ for $1 \leq l \leq \rho$ by

$$\mathcal{P}'_{\mu_1,l} = \begin{cases} \{\nu_l + 1, \ldots, \nu_l + \mu_l\} & \text{if } l \neq 2, \\ \{\nu_2 + 1, \ldots, \nu_2 + \mu_2 - 1\} & \text{if } l = 2; \end{cases}$$

we also set $\mathcal{P}'_{\mu_1+1,1} = \{1, \ldots, \mu_1, \nu_2 + \mu_2\}$.

Then we get $\ell(\mathcal{M})$ splittings $\mathcal{P}_k$ of n by setting $\mathcal{P}'_k = \bigcup_{l=1}^{\rho} \mathcal{P}_{kl}$ and $\mathcal{P}''_k = \{1, \ldots, n\} \setminus \mathcal{P}'_k$. Furthermore, we also get a sequence of linear subspaces $\varnothing = Y^0 \subset Y^1 \subset \cdots \subset Y^{\ell(\mathcal{M})-1} \subset \mathbb{P}^{n-1}(\mathbb{C})$ by letting Y^k to be the subspace generated by $\{[e_h] \mid h \in \mathcal{P}'_k\}$, where $\{e_1, \ldots, e_n\}$ is the canonical basis of $\mathbb{C}^n$.

We are now ready to associate a sequence of $\ell(\mathcal{M})$ blow-ups to any ρ-partition $\mathcal{M}$ of n. Set $M^0 = \mathbb{C}^n$, $\chi_0 = \mathrm{id}_{\mathbb{C}^n}$, $\mathbf{e}_0 = O$ and $X^0 = \{O\}$. We start by blowing up the origin, taking $M^1 = \tilde{M}^0_{X^0}$ and $\pi_1 = \sigma_1: M^1 \to M^0$. Since $M^0 = \mathbb{C}^n$ has a canonical chart adapted to X^0 (that is, centered at the origin), the exceptional divisor $E^1 = \pi_1^{-1}(X^0)$ is canonically isomorphic to $\mathbb{P}^{n-1}(\mathbb{C})$. This allows us to define a distinguished point $\mathbf{e}_1 \in E^1$, corresponding to $[e_1] \in \mathbb{P}^{n-1}(\mathbb{C})$, and also distinguished linear subspaces $Y^k \subset E^1$ for $k = 1, \ldots, \ell(\mathcal{M}) - 1$, corresponding to the previously defined linear subspaces of $\mathbb{P}^{n-1}(\mathbb{C})$ associated to $\mathcal{M}$.

Now put $X^1 = Y^1$ and set $M^2 = \tilde{M}^1_{X^1}$. Let $X^2 \subset M^2$ be the proper transform of Y^2, and set $M^3 = \tilde{M}^2_{X^2}$. Next, let $X^3 \subset M^3$ be the proper transform (with respect to $\sigma_3: M^3 \to M^2$) of the proper transform (with respect to $\sigma_2: M^2 \to M^1$) of Y^3, and put $M^4 = \tilde{M}^3_{X^3}$. Proceeding in this way, we define for $k = 2, \ldots, \ell(\mathcal{M}) - 1$ the manifold M^{k+1} as the blow-up of M^k along the iterated proper transform X^k of Y^k; we denote by $\sigma_{k+1}: M^{k+1} \to M^k$ the associated projection, and by $E^{k+1} = \sigma_{k+1}^{-1}(X^k) \subset M^{k+1}$ the exceptional divisor. For $k = 1, \ldots, \ell(\mathcal{M})$ we also put $\pi_k = \sigma_1 \circ \cdots \circ \sigma_k: M^k \to M^0$; the set $\pi_k^{-1}(X^0)$ will be called the *singular divisor* of M^k.

At each stage of this construction there are canonical charts adapted to the submanifolds involved:

Lemma 1.1: *For $1 \leq k \leq \ell(\mathcal{M})$ we can find a distinguished point $\mathbf{e}_k \in M^k$ and a canonical chart (V_k, χ_k) centered in $\mathbf{e}_k$ such that:*

$$V_k \cap X^k = \chi_k^{-1}\left(\{w_1 = 0\} \cap \bigcap_{h \in \mathcal{P}''_k} \{w_h = 0\}\right); \tag{1.2}$$

$$V_k \cap \pi_k^{-1}(X^0) = \chi_k^{-1}\left(\bigcup_{h \in \mathcal{P}'_{k1}} \{w_h = 0\}\right) \supset V_k \cap X^k; \tag{1.3}$$

and such that for $h = k+1, \ldots, \ell(\mathcal{M}) - 1$ the intersection of V_k with the iterated proper transform of Y^h is

$$\chi_k^{-1}\left(\{w_1 = 0\} \cap \bigcap_{h \in \mathcal{P}''_h} \{w_h = 0\}\right).$$

Furthermore, $\chi_0 \circ \sigma_1 \circ \chi_1^{-1}(w) = (w_1, w_1 w_2, \ldots, w_1 w_n)$, $\chi_{\mu_1} \circ \sigma_{\mu_1+1} \circ \chi_{\mu_1+1}^{-1}(w) = (w_1 w_{\nu_2+\mu_2}, w_2, \ldots, w_n)$, and for $2 \leq k \leq \mu_1$

$$\chi_{k-1} \circ \sigma_k \circ \chi_k^{-1}(w)_h = \begin{cases} w_h & \text{if } h \in (\mathcal{P}'_{k-1} \setminus \{1\}) \cup \{k\}, \\ w_k w_h & \text{if } h \in \{1\} \cup (\mathcal{P}''_{k-1} \setminus \{k\}). \end{cases} \tag{1.4}$$

Proof: For $k=1$, the existence of a canonical chart adapted to X^0 yields a canonical chart (V_1,χ_1) centered at $\mathbf{e}_1$ and adapted to X^1; in turn this yields a canonical basis $\{\partial/\partial w_1,\ldots,\partial/\partial w_n\}$ of $T_{\mathbf{e}_1}M^1$. Furthermore, it is easy to check that

$$V_1\cap E^1=\chi_1^{-1}(\{w_1=0\})=V_1\cap\pi_1^{-1}(X^0)\supset V_1\cap X^1=\chi_1^{-1}\left(\{w_1=0\}\cap\bigcap_{h\in\mathcal{P}_1''}\{w_h=0\}\right),$$

and that

$$\chi_0\circ\sigma_1\circ\chi_1^{-1}(w)=(w_1,w_1w_2,\ldots,w_1w_n).$$

So the lemma is proved for $k=1$.

Assume, by induction, that the lemma holds for $k-1$. In particular, we have a distinguished point $\mathbf{e}_{k-1}$ and a canonical chart (V_{k-1},χ_{k-1}) centered at $\mathbf{e}_{k-1}$ and adapted to X^{k-1}. We thus have a canonical basis $\{\partial/\partial w_1,\ldots,\partial/\partial w_n\}$ of $T_{\mathbf{e}_{k-1}}M^{k-1}$ such that $\{\partial/\partial w_h\mid h\in\mathcal{P}'_{k-1}\setminus\{1\}\}$ spans $T_{\mathbf{e}_{k-1}}X^{k-1}$. Put

$$\mathbf{e}_k=\left[\frac{\partial}{\partial w_k}+T_{\mathbf{e}_{k-1}}X^{k-1}\right]\in\sigma_k^{-1}(\mathbf{e}_{k-1})\ ,$$

(or $\mathbf{e}_k=\left[\frac{\partial}{\partial w_{\nu_2+\mu_2}}+T_{\mathbf{e}_{\mu_1}}X^{\mu_1}\right]\in\sigma_{\mu_1}^{-1}(\mathbf{e}_{\mu_1})$ if $k=\mu_1+1$), and let (V_k,χ_k) be the canonical chart centered in $\mathbf{e}_k$ constructed, as before, via (V_{k-1},χ_{k-1}). Then it is not too difficult to check using the inductive hypothesis that (V_k,χ_k) is as desired. □

We end this section by remarking that it is easy to prove by induction that if we fix $1\le k\le\ell(\mathcal{M})$ and write $z=\chi_0\circ\pi_k\circ\chi_k^{-1}(w)$ then

$$\begin{aligned}
z_j&=\begin{cases} w_1\prod\limits_{h=2}^{j}(w_h)^2\prod\limits_{h=j+1}^{k}w_h & \text{if } j\in\mathcal{P}'_{k1},\\ w_1\prod\limits_{h=2}^{j-\nu_l}(w_h)^2\left(\prod\limits_{h=j-\nu_l+1}^{k}w_h\right)w_j & \text{if } j\in\mathcal{P}'_{kl},\ 2\le l\le\rho;\\ w_1\prod\limits_{h=2}^{k}(w_h)^2\,w_j & \text{if } j\in\mathcal{P}''_k;\end{cases}\qquad \text{if } 1\le k\le\mu_1;\\
z_j&=\begin{cases} w_1\prod\limits_{h=2}^{j}(w_h)^2\left(\prod\limits_{h=j+1}^{\mu_1}w_h\right)w_{\nu_2+\mu_2} & \text{if } j\in\mathcal{P}'_{\mu_1,1},\\ w_1\prod\limits_{h=2}^{j-\nu_l}(w_h)^2\left(\prod\limits_{h=j-\nu_l+1}^{\mu_1}w_h\right)w_jw_{\nu_2+\mu_2} & \text{if } j\in\mathcal{P}'_{\mu_1,l},\ 2\le l\le\rho;\\ w_1\prod\limits_{h=2}^{\mu_1}(w_h)^2\,(w_{\mu_2+\mu_2})^2 & \text{if } j\in\mathcal{P}''_{\mu_1};\end{cases}\qquad \text{if } k=\mu_1+1.
\end{aligned}\tag{1.5}$$

Furthermore, if $z_1,\ldots,z_k\neq0$ then

$$\begin{aligned}
w_j&=\begin{cases}(z_1)^2/z_k & \text{if } j=1,\\ z_j/z_{j-1} & \text{if } j\in\mathcal{P}'_{k1}\setminus\{1\},\\ z_j/z_{j-\nu_l} & \text{if } j\in\mathcal{P}'_{kl},\ 2\le l\le\rho,\\ z_j/z_k & \text{if } j\in\mathcal{P}''_k;\end{cases}\qquad \text{if } 1\le k\le\mu_1;\\
w_j&=\begin{cases}(z_1)^2/z_{\nu_2+\mu_2} & \text{if } j=1,\\ z_j/z_{j-1} & \text{if } j\in(\mathcal{P}'_{\mu_1,1}\setminus\{1\}),\\ z_j/z_{j-\nu_l} & \text{if } j\in\mathcal{P}'_{\mu_1,l},\ 2\le l\le\rho,\\ z_j/z_{\mu_1} & \text{if } j\in\mathcal{P}''_{\mu_1}.\end{cases}\qquad \text{if } k=\mu_1+1.
\end{aligned}\tag{1.6}$$

2. The diagonalization theorem

We shall denote by $\mathrm{End}(\mathbb{C}^n, O)$ the set of germs of holomorphic self-maps of $\mathbb{C}^n$ sending the origin O to itself; more generally, if X is a closed set of a complex manifold M, we shall denote by $\mathrm{End}(M, X)$ the set of germs at X of holomorphic self-maps of M sending X into itself. Every germ $F \in \mathrm{End}(\mathbb{C}^n, O)$ has a homogeneous expansion of the form

$$F(z) = \sum_{j=1}^{\infty} P_j(z),$$

where $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, and the P_j's are n-uples of homogeneous polynomials of degree j in $z_1, \dots, z_n$.

Let M be a complex manifold of dimension n, and X a closed submanifold of dimension $r \geq 0$. We are interested to see when a germ $F \in \mathrm{End}(M, X)$ can be lifted to the blow-up $\tilde{M}_X$ as a germ $\tilde{F} \in \mathrm{End}(M_X, E_X)$. Take $p \in X$, and choose charts (V, φ) and $(\tilde{V}, \tilde{\varphi})$ adapted to X so that $p \in V$ and $F(p) \in \tilde{V}$. In a neighbourhood of p we can write the homogeneous expansion of $G = \tilde{\varphi} \circ F \circ \varphi^{-1}$ as

$$G(z) = \sum_{l \geq 0} P_{l,z'}(z'') ,$$

where $P_{l,z'}$ is a n-uple of l-homogeneous polynomials with coefficients holomorphic in z'. The condition $F(X) \subseteq X$ then translates to

$$(P_{0,z'})'' \equiv 0 .$$

The *order of F at p along X* is

$$\nu_X(F, p) = \min\{l \mid (P_{l,\varphi(p)'})'' \not\equiv 0\} \geq 1 ;$$

it is easily checked that $\nu_X(F, p)$ does not depend on the adapted charts chosen. The *order of F along X* is then given by

$$\nu_X(F) = \min\{\nu_X(F, p) \mid p \in X\} .$$

Clearly the set $\{p \in X \mid \nu_X(F, p) = \nu_X(F)\}$ is open in X.

We shall say that F is *non-degenerate at p along X* if

(i) $F^{-1}(p) \subseteq X$,
(ii) $\nu_X(F, p) = \nu_X(F)$, and
(iii) $\big(P_{l_0,\varphi(p)'}(v)\big)'' = 0$ iff $v = O \in \mathbb{C}^{n-r}$, where $l_0 = \nu_X(F)$.

If F is non-degenerate along X at all points of X we shall say that F is *non-degenerate along X*.

Proposition 2.1: *Let M be a complex manifold of dimension n, and $X \subset M$ a closed submanifold of dimension $r \geq 0$. Let $F \in \mathrm{End}(M, X)$ be non-degenerate along X. Then there exists a unique $\tilde{F} \in \mathrm{End}(\tilde{M}_X, E_X)$ such that $F \circ \sigma = \sigma \circ \tilde{F}$. Furthermore, if $p \in X$ and (V, φ), $(\tilde{V}, \tilde{\varphi})$ are charts adapted to X with $p \in V$ and $F(p) \in \tilde{V}$, then*

$$\tilde{F}\big([v]\big) = (\iota_{F(p),\tilde{\varphi}})^{-1} \left(\Big[P_{l_0,\varphi(p)'}\big(\iota_{p,\varphi}([v])\big)'' \Big] \right) \tag{2.1}$$

for all $[v] \in E_p$, where $l_0 = \nu_X(F)$.

Proof: Since $F^{-1}(X) \subseteq X$, if q does not belong to X we can safely set $\tilde{F}(q) = F(q)$; we are left to define $\tilde{F}$ on the exceptional divisor.

Choose $p \in X$, and the charts as in the statement of the theorem; without loss of generality, we can assume that for both charts the associated splitting is the standard one. For $[v] \in E_p$ choose $r + 1 \leq j \leq n$ so that $[v] \in V_j$; if $\tilde{F}$ exists, we must have

$$F \circ \sigma \circ \chi_j^{-1} = \sigma \circ \tilde{F} \circ \chi_j^{-1} .$$

If $[v] = (\iota_{p,\varphi})^{-1}[v_{r+1} : \ldots : v_n]$, we have

$$[v] = \lim_{\zeta \to 0} \chi_j^{-1} \left(\varphi(p)', \frac{v_{r+1}}{v_j}, \dots, \zeta, \dots, \frac{v_n}{v_j} \right) ,$$

and so, setting again $G = \tilde{\varphi} \circ F \circ \varphi^{-1}$,

$$\tilde{F}([v]) = \lim_{\zeta \to 0} \sigma^{-1} \left(\tilde{\varphi}^{-1} \left(G \left(\varphi(p)', \frac{\zeta}{v_j} v \right) \right) \right) ,$$

where with a slight abuse of notation we have put $v = (v_{r+1}, \ldots, v_n) \in \mathbb{C}^{n-r}$.

Now, given a sequence $\{q_k\} \subset M \setminus X$ converging to $q \in X$, the sequence $\{\sigma^{-1}(q_k)\}$ converges in $\tilde{M} \setminus X$ iff $\{[\tilde{\varphi}(q_k)'']\}$ converges in $\mathbb{P}^{n-r-1}(\mathbb{C})$, and then

$$\lim_{k \to \infty} \sigma^{-1}(q_k) = \iota_{q,\tilde{\varphi}}^{-1} \left(\lim_{k \to \infty} [\tilde{\varphi}(q_k)''] \right) .$$

In our case we have

$$G \left(\varphi(p)', \frac{\zeta}{v_j} v \right)'' = \sum_{l \geq l_0} P_{l,\varphi(p)'} \left(\frac{\zeta}{v_j} v \right)'' = \left(\frac{\zeta}{v_j} \right)^{l_0} \left(P_{l_0,\varphi(p)'}(v)'' + \zeta Q(\zeta) \right) ,$$

for a suitable holomorphic map Q. Therefore $[G(\varphi(p)', \zeta v/v_j)''] \to [P_{l_0,\varphi(p)'}(v)'']$, and thus if $\tilde{F}$ exists it is given by (2.1) on the exceptional divisor.

To finish the proof we must show that an $\tilde{F}$ defined by (2.1) on the exceptional divisor and by F elsewhere is holomorphic. Take $[v] \in E_p$, and choose $r+1 \leq h, k \leq n$ so that $[v] \in V_h$ and $\tilde{F}([v]) \in \tilde{V}_k$; we must show that $\chi_k \circ \tilde{F} \circ \chi_h^{-1}$ is holomorphic. We know that

$$G \circ (\varphi \circ \sigma \circ \chi_h^{-1}) = (\varphi \circ \sigma \circ \chi_k^{-1}) \circ (\chi_k \circ \tilde{F} \circ \chi_h^{-1}) ;$$

so putting $\chi_k \circ \tilde{F} \circ \chi_h^{-1} = (\tilde{f}_1, \ldots, \tilde{f}_n)$ and recalling (1.1) we must have

$$G(w', w_h w_{r+1}, \ldots, w_h, \ldots, w_h w_n) = \left(\tilde{f}_1(w), \ldots, \tilde{f}_r(w), \tilde{f}_k(w)\tilde{f}_{r+1}(w), \ldots, \tilde{f}_k(w), \ldots, \tilde{f}_k(w)\tilde{f}_n(w) \right) .$$

Writing $G = (g_1, \ldots, g_n)$ we find that if $w_h \neq 0$ then

$$\tilde{f}_i(w) = \begin{cases} g_i(w', w_h w_{r+1}, \ldots, w_h, \ldots, w_h w_n) & \text{if } 1 \leq i \leq r \text{ or } i = k, \\ \dfrac{g_i(w', w_h w_{r+1}, \ldots, w_h, \ldots, w_h w_n)}{g_k(w', w_h w_{r+1}, \ldots, w_h, \ldots, w_h w_n)} & \text{if } r+1 \leq i \neq k \leq n. \end{cases} \tag{2.2}$$

Since the g_i's are holomorphic and $\{w_h = 0\}$ has codimension 1 in $\chi_h(V_h)$, to end the proof it suffices to show that the quotients in (2.2) have a limit when $w \to \chi_h([v])$.

Write again $\iota_{p,\varphi}([v]) = [v_{r+1} : \ldots : v_n]$ and $v = (v_{r+1}, \ldots, v_n)$, and assume then that $w \to \chi_h([v])$. This means that $w' \to \varphi(p)'$, $w_h \to 0$ and $(w_{r+1}, \ldots, 1, \ldots, w_n) \to v_h^{-1} v$. Now,

$$G(w', w_h w_{r+1}, \ldots, w_h, \ldots, w_h w_n)'' = \sum_{l \geq l_0} \left(\frac{w_h}{v_h} \right)^l P_{l,w'}(w_{r+1} v_h, \ldots, v_h, \ldots, w_n v_h)'' .$$

Since $\tilde{F}([v]) \in \tilde{V}_k$ we have $P_{l_0,\varphi(p)'}(v)_k \neq 0$; therefore

$$\frac{g_i(w', w_h w_{r+1}, \ldots, w_h, \ldots, w_h w_n)}{g_k(w', w_h w_{r+1}, \ldots, w_h, \ldots, w_h w_n)} \to \frac{P_{l_0,\varphi(p)'}(v)_i}{P_{l_0,\varphi(p)'}(v)_k} ,$$

and we are done. □

Now, our construction involves iterated blow-ups; thus we are interested to know when the map $\tilde{F}$ is still non-degenerate along suitable submanifolds of $\tilde{M}_X$. We shall limit ourselves to two special cases, which are enough for our aims.

Proposition 2.2: *Let M be a complex manifold of dimension n, and $X \subset M$ a closed submanifold of dimension $r \geq 0$. Let $F \in \mathrm{End}(M, X)$ be non-degenerate along X, and $\tilde{F} \in \mathrm{End}(\tilde{M}_X, E_X)$ its lifting. Let $Y \subseteq M$ be a submanifold of M of dimension $r + s$ (with $s \geq 1$), and $\tilde{Y} \subseteq \tilde{M}$ its proper transform. Assume that*

(i) *Y contains properly X;*
(ii) *$F(Y) \subseteq Y$ and $F^{-1}(Y) \subseteq Y$;*
(iii) *dF_q is invertible for all $q \in Y$.*

Then $\tilde{F}$ is non-degenerate along $\tilde{Y}$, and $d\tilde{F}_{\tilde{q}}$ is invertible for all $\tilde{q} \in \tilde{Y}$.

Proof: First of all, notice that if $p \in X$ then $\tilde{Y} \cap E_p = \mathbb{P}(T_pY/T_pX)$, and that $\tilde{F}|_{E_p}$ is induced by dF_p. Since, by construction, $\tilde{F}(\tilde{Y}) \subseteq \tilde{Y}$ and $\tilde{F}^{-1}(\tilde{Y} \setminus E_X) \subseteq \tilde{Y} \setminus E_X$, it suffices to prove that $d\tilde{F}_{[v]}$ is invertible for all $[v] \in \tilde{Y} \cap E_X$.

Fix $p \in X$ and $[v] \in \tilde{Y} \cap E_p$, and choose two charts (V, φ) and $(\tilde{V}, \tilde{\varphi})$, centered in p, respectively in $F(p)$, such that $V \cap X = \{z_{r+1} = \cdots = z_n = 0\}$, $V \cap Y = \{z_{r+s+1} = \cdots = z_n = 0\}$, and analogously for $\tilde{V}$. In particular,

$$\iota_{p,\varphi}(\tilde{Y} \cap E_p) = \iota_{F(p),\tilde{\varphi}}(\tilde{Y} \cap E_{F(p)}) = \{v_{r+s+1} = \cdots = v_n = 0\} ,$$

and we can also assume that $\iota_{p,\varphi}([v]) = \iota_{F(p),\tilde{\varphi}}\big(\tilde{F}([v])\big) = [1 : 0 : \cdots : 0]$. Then the charts (V_{r+1}, χ_{r+1}) and $(\tilde{V}_{r+1}, \tilde{\chi}_{r+1})$ are centered in $[v]$, respectively in $\tilde{F}([v])$, and adapted to $\tilde{Y}$.

Set $G = \tilde{\varphi} \circ F \circ \varphi^{-1} = (g_1, \ldots, g_n)$ and $\tilde{G} = \tilde{\chi}_{r+1} \circ \tilde{F} \circ \chi_{r+1}^{-1} = (\tilde{f}_1, \ldots, \tilde{f}_n)$; the relation between the g_i's and the $\tilde{f}_j$'s is given by (2.2). Since $F(X) \subseteq X$ and $F(Y) \subseteq Y$, the jacobian matrix of G at the origin is of the form

$$\mathcal{A} = \left|\begin{array}{c|c} A & * \\ \hline O & \begin{array}{c|c} B & * \\ \hline O & C \end{array} \end{array}\right| ,$$

with $A \in M_{r,r}(\mathbb{C})$, $B \in M_{s,s}(\mathbb{C})$ and $C \in M_{n-r-s,n-r-s}(\mathbb{C})$. Since, by assumption, dF_p is invertible, we have

$$\det(\mathcal{A}) = \det(A)\det(B)\det(C) \neq 0 .$$

Finally, $\tilde{F}([v]) \in \tilde{V}_{r+1}$ translates in

$$\lambda = \frac{\partial g_{r+1}}{\partial z_{r+1}}(O) \neq 0 .$$

Our aim is to compute $\partial \tilde{f}_i / \partial w_j$ at $w = O$. This is easy when $1 \leq i \leq r+1$; in fact, (2.2) with $h = k = r+1$ yields

$$\frac{\partial \tilde{f}_i}{\partial w_j}(O) = \begin{cases} \dfrac{\partial g_i}{\partial z_j}(O) & \text{for } 1 \leq i \leq r+1,\ 1 \leq j \leq r+1, \\ 0 & \text{for } 1 \leq i \leq r+1,\ r+2 \leq j \leq n. \end{cases}$$

In particular,

$$\frac{\partial \tilde{f}_{r+1}}{\partial w_j}(O) = \begin{cases} 0 & \text{if } j \neq r+1, \\ \lambda \neq 0 & \text{if } j = r+1. \end{cases}$$

Now set $\tilde{g}_i(w) = g_i(w', w_{r+1}, w_{r+1}w_{r+2}, \ldots, w_{r+1}w_n)$, and write again

$$G(z) = \sum_{l \geq 0} P_{l,z'}(z'') ,$$

recalling that $(P_{0,z'})'' \equiv O$. For $r+2 \leq i \leq n$ we have

$$\frac{\partial \tilde{f}_i}{\partial w_j}(O) = \lim_{w \to O} \frac{1}{\tilde{g}_{r+1}(w)} \left[\frac{\partial \tilde{g}_i}{\partial w_j}(w) - \frac{\tilde{g}_i(w)}{\tilde{g}_{r+1}(w)} \frac{\partial \tilde{g}_{r+1}}{\partial w_j}(w) \right] . \tag{2.3}$$

Since

$$\tilde{g}_i(w) = \sum_{l\geq 0} (w_{r+1})^l P_{l,w'}(1, w_{r+2}, \ldots, w_n)_i \ ,$$

(2.3) yields

$$\frac{\partial \tilde{f}_i}{\partial w_j}(O) = \begin{cases} \dfrac{1}{\lambda}\left[\dfrac{\partial^2 g_i}{\partial z_j \partial z_{r+1}}(O) - \dfrac{1}{\lambda}\dfrac{\partial^2 g_{r+1}}{\partial z_j \partial z_{r+1}}(O)\dfrac{\partial g_i}{\partial z_{r+1}}(O)\right] & \text{for } r+2 \leq i \leq n \text{ and } 1 \leq j \leq r+1, \\ \dfrac{1}{\lambda}\left[\dfrac{\partial g_i}{\partial z_j}(O) - \dfrac{1}{\lambda}\dfrac{\partial g_{r+1}}{\partial z_j}(O)\dfrac{\partial g_i}{\partial z_{r+1}}(O)\right] & \text{for } r+2 \leq i, j \leq n. \end{cases}$$

In particular, we find

$$\frac{\partial \tilde{f}_i}{\partial w_j}(O) = \frac{1}{\lambda}\frac{\partial g_i}{\partial z_j}(O) \quad \text{for } r+s+1 \leq i \leq n,\ r+2 \leq j \leq n.$$

Summing up, we have proved that the Jacobian matrix of $\tilde{G}$ at the origin is

$$\tilde{\mathcal{A}} = \left| \begin{array}{c|c|c} A & * & O \\ \hline O & \lambda & O \\ \hline * & * & \begin{array}{c|c} \tilde{B} & * \\ \hline O & \frac{1}{\lambda}C \end{array} \end{array} \right| \ , \tag{2.4}$$

where $\tilde{B} \in M_{s-1,s-1}(\mathbb{C})$. Now, if we subtract to the j-th column of B (for $j = 2, \ldots, s$) the first column of B multiplied by $\lambda^{-1}\partial g_{r+1}/\partial z_{r+j}(O)$ we get

$$\left| \begin{array}{c|c} \lambda & O \\ \hline * & \lambda\tilde{B} \end{array} \right| \ .$$

Since these elementary operations do not change the determinant, we obtain $\det(B) = \lambda^s \det(\tilde{B})$. Therefore

$$\det(\tilde{\mathcal{A}}) = \frac{1}{\lambda^{n-r-1}} \det(\mathcal{A}) \neq 0 \ ,$$

and we are done. □

A similar argument yields:

Proposition 2.3: *Let M be a complex manifold of dimension n, and $X \subset M$ a closed submanifold of dimension $r \geq 0$. Let $F \in \mathrm{End}(M, X)$ be non-degenerate along X, and $\tilde{F} \in \mathrm{End}(\tilde{M}_X, E_X)$ its lifting. Take $p \in X$ and a linear subspace $L \subseteq E_p$ of dimension $s-1$ (with $s \geq 1$). Assume that*

(i) *$\tilde{F}(L) \subseteq L$, and*
(ii) *dF_p is invertible.*

Then $\tilde{F}$ is non-degenerate along L, and $d\tilde{F}_{[v]}$ is invertible for all $[v] \in L$.

Proof: Condition (i) implies that p is a fixed point of F, and condition (ii) implies that $\nu_X(F) = 1$. In particular, $\tilde{F}|_{E_p}$ is induced by the differential of F at p; thus $\tilde{F}|_L$ is injective, and the invertibility of $d\tilde{F}_{[v]}$ for all $[v] \in L$ will imply that $\tilde{F}$ is non-degenerate along L.

Fix $[v] \in L$, and choose two charts (V, φ), $(\tilde{V}, \tilde{\varphi})$ centered in p adapted to X such that

$$\iota_{p,\varphi}([v]) = \iota_{p,\tilde{\varphi}}\big(\tilde{F}([v])\big) = [1 : 0 : \ldots : 0]$$

and

$$\iota_{p,\varphi}(L) = \iota_{p,\tilde{\varphi}}(L) = \{v_{r+s+1} = \cdots = v_n = 0\} \ .$$

Then the charts (V_{r+1}, χ_{r+1}) and $(\tilde{V}_{r+1}, \tilde{\chi}_{r+1})$ are centered in $[v]$, respectively in $\tilde{F}([v])$, and adapted to L. The proof then goes on as in the previous proposition. □

We are finally ready to prove the main result of this paper:

Theorem 2.4: (Diagonalization Theorem) *Let $F \in \mathrm{End}(\mathbb{C}^n, O)$ be such that dF_O is invertible and non-diagonalizable. Assume that dF_O is in Jordan canonical form, with $\rho \geq 1$ blocks of lenghts $\mu_1 \geq \cdots \geq \mu_\rho \geq 1$ associated respectively to the eigenvalues $\lambda_1, \ldots, \lambda_\rho \in \mathbb{C}$. Set $\mathcal{M} = \{\mu_1, \ldots, \mu_\rho\}$, and let $(M^0, \ldots, M^{\ell(\mathcal{M})})$ be the sequence of blow-ups associated to $\mathcal{M}$. Then for $1 \leq k \leq \ell(\mathcal{M})$ there exists a unique $\tilde{F}_k \in \mathrm{End}(M^k, E^k)$ such that $F \circ \pi_k = \pi_k \circ \tilde{F}_k$, and we have $\tilde{F}_k(\mathbf{e}_k) = \mathbf{e}_k$. Furthermore, $d(\tilde{F}_{\ell(\mathcal{M})})_{\mathbf{e}_{\ell(\mathcal{M})}}$ is diagonalizable, with eigenvalues $\tilde{\lambda}_1$, 1, $\lambda_2/\lambda_1, \ldots, \lambda_\rho/\lambda_1$ of multiplicity 1, $\mu_1 - 1$, $\mu_2, \ldots, \mu_\rho$ respectively, where $\tilde{\lambda}_1 = \lambda_1$ if $\mu_1 > \mu_2$, and $\tilde{\lambda}_1 = \lambda_1^2/\lambda_2$ if $\mu_1 = \mu_2$. More precisely, writing $\chi_{\ell(\mathcal{M})} \circ \tilde{F}_{\ell(\mathcal{M})} \circ \chi_{\ell(\mathcal{M})}^{-1} = (\tilde{f}_1, \ldots, \tilde{f}_n)$, and denoting by a_{11}^j the coefficient of $(z_1)^2$ in the power series expansion of f_j, if $\mu_1 > \mu_2$ we have*

$$\tilde{f}_j(w) = \begin{cases} w_1\big(\lambda_1 - a_{11}^{\mu_1} w_1 + 2w_2 + O(\|w\|^2)\big) & \text{if } j = 1, \\ w_j\big(1 - \frac{1}{\lambda_1} w_j + \frac{1}{\lambda_1} w_{j+1} + O(\|w\|^2)\big) & \text{if } 2 \leq j \leq \mu_1 - 1, \\ w_{\mu_1}\big(1 + \frac{a_{11}^{\mu_1}}{\lambda_1} w_1 - \frac{1}{\lambda_1} w_{\mu_1} + O(\|w\|^2)\big) & \text{if } j = \mu_1, \\ \frac{\lambda_l}{\lambda_1} w_j - \frac{\lambda_l}{\lambda_1^2} w_{j-\nu_l+1} w_j + \frac{1}{\lambda_1} w_{j-\nu_l+1} w_{j+1} + O(\|w\|^3) & \text{if } j \in \mathcal{P}'_{\mu_1,l} \setminus \{\nu_l + \mu_l\},\ 2 \leq l \leq \rho, \\ \frac{\lambda_l}{\lambda_1} w_j - \frac{\lambda_l}{\lambda_1^2} w_{\mu_l+1} w_j + O(\|w\|^3) & \text{if } j = \nu_l + \mu_l,\ \mu_l < \mu_1 - 1, \\ \frac{\lambda_l}{\lambda_1} w_j + \frac{a_{11}^j}{\lambda_1} w_1 w_{\mu_1} - \frac{\lambda_l}{\lambda_1^2} w_{\mu_1} w_j + O(\|w\|^3) & \text{if } j = \nu_l + \mu_l,\ \mu_l = \mu_1 - 1, \end{cases}$$

whereas if $\mu_1 = \mu_2$ we have

$$\tilde{f}_j(w) = \begin{cases} w_1\big(\frac{\lambda_1^2}{\lambda_2} - \frac{\lambda_1^2}{\lambda_2^2} a_{11}^{\nu_2+\mu_2} w_1 + \frac{2\lambda_1}{\lambda_2} w_2 + O(\|w\|^2)\big) & \text{if } j = 1, \\ w_j\big(1 - \frac{1}{\lambda_1} w_j + \frac{1}{\lambda_1} w_{j+1} + O(\|w\|^2)\big) & \text{if } 2 \leq j \leq \mu_1 - 1, \\ w_{\mu_1}\big(1 - \frac{1}{\lambda_1} w_{\mu_1} + O(\|w\|^2)\big) & \text{if } j = \mu_1, \\ \frac{\lambda_l}{\lambda_1} w_j - \frac{\lambda_l}{\lambda_1^2} w_{j-\nu_l+1} w_j + \frac{1}{\lambda_1} w_{j-\nu_l+1} w_{j+1} + O(\|w\|^3) & \text{if } j \in \mathcal{P}'_{\mu_1,l} \setminus \{\nu_l + \mu_l\},\ 2 \leq l \leq \rho, \\ w_{\mu_2+\nu_2}\big(\frac{\lambda_2}{\lambda_1} + \frac{a_{11}^{\nu_2+\mu_2}}{\lambda_1} w_1 + O(\|w\|^2)\big) & \text{if } j = \nu_2 + \mu_2, \\ \frac{\lambda_l}{\lambda_1} w_j + O(\|w\|^3) & \text{if } j = \nu_l + \mu_l,\ \mu_l < \mu_1, \\ \frac{\lambda_l}{\lambda_1} w_j + \frac{a_{11}^j}{\lambda_1} w_1 w_{\nu_2+\mu_2} + O(\|w\|^3) & \text{if } j = \nu_l + \mu_l,\ \mu_l = \mu_1,\ 3 \leq l \leq \rho. \end{cases}$$

Proof: Proposition 2.1 yields the existence of $\tilde{F}_1$; since $\tilde{F}_1|_{E^1}$ is induced by the differential of F at the origin, we see that $\mathbf{e}_1$ is a fixed point of $\tilde{F}_1$, and more generally that $\tilde{F}_1(Y^k) = Y^k$ for $k = 1, \ldots, \mu_1$.

By Proposition 2.3, $d(\tilde{F}_1)_{[v]}$ is invertible for all $[v] \in Y^{\mu_1}$. In particular, $\tilde{F}_1$ is non-degenerate along X^1, and so Proposition 2.1 yields $\tilde{F}_2$. Since $d\tilde{F}_1$ is invertible along Y^2, we can invoke Proposition 2.2 to prove that $d\tilde{F}_2$ is non-degenerate along X^2, and thus we get $\tilde{F}_3$. Furthermore, being $d\tilde{F}_2$ invertible along X^2, it is invertible along the proper transform of Y^3 too, because outside of $E^2 \subset X^2$ it is given by $d\tilde{F}_1$. Then we can again invoke Proposition 2.2 to prove that $\tilde{F}_3$ is non-degenerate along X^3, and Proposition 2.1 yields $\tilde{F}_4$. Repeating this procedure we clearly get $\tilde{F}_k$ for all k.

To show that $\mathbf{e}_k$ is a fixed point of $\tilde{F}_k$ it suffices to notice that for $k = 2, \ldots, \mu_1$ we have

$$\tilde{F}_1([\partial/\partial w_k]) = [\lambda_1(\partial/\partial w_k) + (\partial/\partial w_{k-1})],$$

and $[\partial/\partial w_{k-1}] \in Y^{k-1}$; analogously, if $\mu_2 = \mu_1$ then $\tilde{F}_1([\partial/\partial w_{\nu_2+\mu_2}]) = [\lambda_2(\partial/\partial w_{\nu_2+\mu_2}) + (\partial/\partial w_{\nu_2+\mu_2-1})]$ and $[\partial/\partial w_{\nu_2+\mu_2-1}] \in Y^{\mu_1}$.

We are left to prove that $d(\tilde{F}_{\ell(\mathcal{M})})_{\mathbf{e}_{\ell(\mathcal{M})}}$ is diagonalizable. From $F \circ \pi_{\ell(\mathcal{M})} = \pi_{\ell(\mathcal{M})} \circ \tilde{F}_{\ell(\mathcal{M})}$ we easily get

$$F \circ (\chi_0 \circ \pi_{\ell(\mathcal{M})} \circ \chi_{\ell(\mathcal{M})}^{-1}) = (\chi_0 \circ \pi_{\ell(\mathcal{M})} \circ \chi_{\ell(\mathcal{M})}^{-1}) \circ \tilde{F}\ . \tag{2.5}$$

Since we know that, writing $F = (f_1, \ldots, f_n)$,

$$f_j(z) = \begin{cases} \lambda_l z_j + z_{j+1} + \sum\limits_{h,k=1}^n a_{hk}^j z_h z_k + O(\|z\|^3) & \text{if } \nu_l + 1 \leq j < \nu_l + \mu_l, \\ \lambda_l z_j + \sum\limits_{h,k=1}^n a_{hk}^j z_h z_k + O(\|z\|^3) & \text{if } j = \nu_l + \mu_l, \end{cases}$$

for $1 \leq l \leq \rho$, it is not difficult to check, using (1.5) and (1.6), that the $\tilde{f}_j$'s have the claimed form, and we are done. $\square$

3. Parabolic curves

From now on we shall assume that $\mathrm{sp}(dF_O) = \{1\}$; in particular, the Diagonalization Theorem 2.4 yields a map tangent to the identity. This allows us to bring into play Hakim's theory, that we shall now briefly summarize.

Set $\Delta = \{\zeta \in \mathbb{C} \mid |\zeta - 1| < 1\}$. A *holomorphic curve at the origin* is a holomorphic injective map $\varphi\colon \Delta \to \mathbb{C}^n \setminus \{O\}$ such that φ extends continuosly to $0 \in \partial\Delta$ with $\varphi(0) = O$.

Now take $F \in \mathrm{End}(\mathbb{C}^n, O)$. We shall say that a holomorphic curve at the origin φ, or its image $D = \varphi(\Delta)$, is *F-invariant* if $F\bigl(\varphi(\Delta)\bigr) \subseteq \varphi(\Delta)$; that it is *stable* if it is F-invariant and $(F|_D)^k \to O$ uniformly on compact subsets of D. A *parabolic curve* is, by definition, a stable holomorphic curve at the origin. Finally, we shall say that φ is *tangent* to $v \in \mathbb{P}^{n-1}(\mathbb{C})$ if $[\varphi(\zeta)] \to v$ as $\zeta \to 0$.

Now let $P_2\colon \mathbb{C}^n \to \mathbb{C}^n$ be a $\mathbb{C}^n$-valued quadratic form. A *characteristic direction* of P_2 is a $v \in \mathbb{C}^n \setminus \{O\}$ such that $P_2(v) = \lambda v$. If $\lambda = 0$ then v is *degenerate;* otherwise it is a *non-degenerate* characteristic direction.

Then (the part we shall need of) Hakim's results can be summarized as follows:

Theorem 3.1: (Hakim [H2, 3]) *Let $F \in \mathrm{End}(\mathbb{C}^n, O)$ be such that $dF_O = \mathrm{id}$. Let $P_2\colon \mathbb{C}^n \to \mathbb{C}^n$ be the quadratic part of the homogeneous expansion of F. If $z^o \in \mathbb{C}^n$, set $z^k = F^k(z^o)$, and denote by $[z^k]$ its image in $\mathbb{P}^{n-1}(\mathbb{C})$ when $z^k \neq O$. Then:*

(i) *if $z^k \to O$ and $[z^k] \to [v]$ then v is a characteristic direction of P_2;*
(ii) *if v is a non-degenerate characteristic direction of P_2, then F admits a parabolic curve tangent to $[v]$;*
(iii) *if v is a non-degenerate characteristic direction of P_2 with $P_2(v) = \lambda v$ and $D \subset \mathbb{C}^n$ is the parabolic curve given by part (ii), then for every $z^o \in D$ and $1 \le j \le n$ we have*

$$z_j^k = -\frac{v_j}{\lambda k} + o\left(\frac{1}{k}\right) .$$

Putting together Theorems 2.4 and 3.1 we are able to prove the existence of a parabolic curve for generic non-diagonalizable maps $F \in \mathrm{End}(\mathbb{C}^n, O)$ such that $\mathrm{sp}(dF_O) = \{1\}$. In this context, "generic" means $a_{11}^{\mu_1} \neq 0$ and $\mu_2 < \mu_1$.

Corollary 3.2: *Let $F \in \mathrm{End}(\mathbb{C}^n, O)$ be such that dF_O is non-diagonalizable and $\mathrm{sp}(dF_O) = \{1\}$. Assume without loss of generality that dF_O is in Jordan canonical form, and let $\mathcal{M}$ be the ρ-partition of n induced by the block structure of dF_O. Assume moreover that $\ell(\mathcal{M}) = \mu_1$ and that $a_{11}^{\mu_1} \neq 0$, where we are using the notations introduced in the previous sections. Then F admits a parabolic curve φ tangent to e_1. Furthermore, if $z^o \in \varphi(\Delta)$ and $z^k = F^k(z^o)$, then*

$$z_j^k = \begin{cases} (-1)^{\mu_1+j-1} \frac{2\mu_1-1}{a_{11}^{\mu_1}} \binom{2\mu_1-2}{\mu_1-1} \frac{(\mu_1+j-2)!}{k^{\mu_1+j-1}} + o\left(\frac{1}{k^{\mu_1+j-1}}\right), & \text{if } 1 \le j \le \mu_1, \\ o\left(\frac{1}{k^{\mu_1+j-\nu_l}}\right), & \text{if } 1 \le j-\nu_l \le \mu_l < \mu_1 - 1, \\ (-1)^{\mu_1+j-\nu_l} \frac{a_{11}^{\mu_l+\nu_l}(2\mu_1-1)(\mu_l+j-\nu_l)}{a_{11}^{\mu_1}} \binom{2\mu_1-2}{\mu_1-1} \frac{(\mu_1+j-\nu_l-2)!}{k^{\mu_1+j-\nu_l}} + o\left(\frac{1}{k^{\mu_1+j-\nu_l}}\right), & \text{if } 1 \le j-\nu_l \le \mu_l = \mu_1 - 1. \end{cases} \tag{3.1}$$

Proof: The idea is to apply Theorem 3.1 to the lifting $\tilde{F}_{\mu_1}$ of F, and then use π_{μ_1} to project the result down to F. Not all the characteristic directions of the quadratic part of $\tilde{F}_{\mu_1}$ at $\mathbf{e}_{\mu_1}$ are allowable, though. Since we are working in M^{μ_1}, characteristic directions tangent to $\pi_{\mu_1}^{-1}(X^0)$ should be excluded, because the $\tilde{F}_{\mu_1}$-parabolic curve provided by Theorem 3.1.(ii) could be contained in the singular divisor, and thus it would be killed by π_{μ_1}. Now, (1.3) says that $\pi_{\mu_1}^{-1}(X^0)$ is given by $\{w_1 = 0\} \cup \cdots \cup \{w_{\mu_1} = 0\}$; therefore we must look for characteristic directions v with $v_1, \ldots, v_{\mu_1} \neq 0$. Characteristic directions not tangent to the singular divisor $\pi_k^{-1}(X^0)$ will be called *allowable.*

The explicit form of $\tilde{F}_{\mu_1}$ given in Theorem 2.4 shows that an allowable characteristic direction v for $\tilde{F}_{\mu_1}$

at $\mathbf{e}_{\mu_1}$ must satisfy

$$\begin{cases} -a_{11}^{\mu_1} v_1 + 2v_2 = \lambda, & \text{for } j = 1, \\ -v_j + v_{j+1} = \lambda, & \text{for } 2 \le j \le \mu_1 - 1, \\ a_{11}^{\mu_1} v_1 - v_{\mu_1} = \lambda, & \text{for } j = \mu_1, \\ (-v_j + v_{j+1}) v_{j-\nu_l+1} = \lambda v_j, & \text{for } j \in \mathcal{P}'_{\mu_1,l} \setminus \{\nu_l + \mu_l\},\ 2 \le l \le \rho, \\ -v_{\mu_l+1} v_j = \lambda v_j, & \text{for } j = \nu_l + \mu_l,\ \mu_l < \mu_1 - 1, \\ a_{11}^{j} v_1 v_{\mu_1} - v_{\mu_1} v_j = \lambda v_j, & \text{for } j = \nu_l + \mu_l,\ \mu_l = \mu_1 - 1. \end{cases}$$

The unique non-degenerate (i.e., with $\lambda \ne 0$) solution of this system is

$$v_j = \begin{cases} \frac{1}{a_{11}^{\mu_1}} (2\mu_1 - 1)\lambda, & \text{for } j = 1, \\ (\mu_1 + j - 2)\lambda, & \text{for } 2 \le j \le \mu_1, \\ 0, & \text{for } j = \nu_l + h,\ 1 \le h \le \mu_l,\ \mu_l < \mu_1 - 1, \\ \frac{a_{11}^{\nu_l+\mu_l}}{a_{11}^{\mu_1}} (\mu_l + h)\lambda, & \text{for } j = \nu_l + h,\ 1 \le h \le \mu_l,\ \mu_l = \mu_1 - 1. \end{cases}$$

This is an allowable solution; therefore Theorem 3.1.(ii) yields a $\tilde{F}_{\mu_1}$-stable holomorphic curve $\tilde{\varphi}$ at the origin tangent to v. Since v is not tangent to $\pi_{\mu_1}^{-1}(X^0)$, which is invariant under $\tilde{F}_{\mu_1}$, the image of the curve is contained in $M^{\mu_1} \setminus \pi_{\mu_1}^{-1}(X^0)$, which is exactly the subset of M^{μ_1} where π_{μ_1} is a biholomorphism with $\mathbb{C}^n \setminus \{O\}$. Therefore the holomorphic curve $\varphi = \pi_{\mu_1} \circ \tilde{\varphi}$ is a parabolic curve at the origin for F in $\mathbb{C}^n$, and (3.1) follows from Theorem 3.1.(iii) and (1.6). □

Remark 3.1: Let $\chi \in \mathrm{Aut}(\mathbb{C}^n, O)$ be a (germ of) biholomorphism of $\mathbb{C}^n$ keeping the origin fixed and such that the differential of $\hat{F} = \chi^{-1} \circ F \circ \chi$ is still in Jordan form; then $\hat{a}_{11}^{\mu_1} = \alpha\, a_{11}^{\mu_1}$ for a suitable $\alpha \ne 0$, and thus F is generic iff $\hat{F}$ is.

Remark 3.2: If $\rho = 1$ and $a_{11}^{\mu_1} = 0$ but $a_{11}^{\mu_1 - 1} \ne 0$, it turns out that $d(\tilde{F}_{\mu_1 - 1})_{\mathbf{e}_{\mu_1 - 1}}$ is already diagonalizable, and an argument similar to the one used in the previous proof yields a parabolic curve for F in this case too. On the other hand, if $\rho \ge 2$ and $\mu_2 = \mu_1$ then $\tilde{F}_{\mu_1+1}$ has *no* allowable non-degenerate characteristic directions at $\mathbf{e}_{\mu_1+1}$.

Remark 3.3: We are finally able to explain why diagonalizing simply by blowing-up points does not work. Indeed, it turns out that in that case the lifted map would have no allowable characteristic directions; all the relevant dynamics would be inside the singular divisor, and so one would not easily detect the parabolic curve whose existence is proved in Corollary 3.2.

When $n = 2$ (and thus $\rho = 1$ and $\mu_1 = 2$), we are also able to study the non-generic case $a_{11}^2 = 0$, obtaining interesting results. For instance, we shall see that (for the first time, as far as I know) a coefficient of the cubic part of F enters directly into play even when the quadratic part of F is not zero.

So, assume $n = 2$ and $a_{11}^2 = 0$, and write

$$\begin{aligned} f_1(z) &= z_1 + z_2 + a_{11}^1 (z_1)^2 + 2a_{12}^1 z_1 z_2 + a_{22}^1 (z_2)^2 + \cdots, \\ f_2(z) &= z_2 + 2a_{12}^2 z_1 z_2 + a_{22}^2 (z_2)^2 + a_{111}^2 (z_1)^3 + \cdots. \end{aligned}$$

We shall describe our results in terms of the following quantities:

$$\varepsilon = a_{11}^1 + a_{12}^2, \qquad \text{and} \qquad \eta = (a_{11}^1 - a_{12}^2)^2 + 2a_{111}^2;$$

they are projective invariants of F under change of coordinates. More precisely, let again $\chi \in \mathrm{Aut}(\mathbb{C}^n, O)$ be a (germ of) biholomorphism of $\mathbb{C}^n$ keeping the origin fixed and such that the differential of $\hat{F} = \chi^{-1} \circ F \circ \chi$ is still in Jordan form; then $\hat{a}_{11}^2 = 0$, $\hat{\varepsilon} = \alpha\varepsilon$ and $\hat{\eta} = \alpha^2 \eta$ for a suitable $\alpha \ne 0$.

Then:

Corollary 3.3: *Let $F \in \mathrm{End}(\mathbb{C}^2, O)$ be such that dF_O is non-diagonalizable and $\mathrm{sp}(dF_O) = \{1\}$. Assume that dF_O is in Jordan canonical form, and that F is non-generic, that is $a^2_{11} = 0$. Assume moreover that $(\varepsilon, \eta) \neq (0, 0)$, where ε and η are the invariants just defined. Then:*

(i) *if $\eta \neq 0$, ε^2, then F admits two distinct parabolic curves at the origin;*
(ii) *if $\eta = \varepsilon^2 \neq 0$, or $\eta = 0 \neq \varepsilon^2$, then F admits one parabolic curve at the origin.*

In both cases, the parabolic curves are tangent to e_1. Furthermore, if z^o belongs to the image of one of the curves and $z^k = F^k(z^o)$, then $z^k_1 = c_1/k + o(1/k)$ and $z^k_2 = c_2/k^2 + o(1/k^2)$ for suitable $c_1 \neq 0$ and $c_2 \in \mathbb{C}$.

Proof: The point is that one blow-up is enough to diagonalize such a map; in fact, in this case the local expansion of $\tilde{F}_1$ nearby $\mathbf{e}_1$ is given by

$$\tilde{f}_j(w) = \begin{cases} w_1 + a^1_{11}(w_1)^2 + w_1 w_2 + O(\|w\|^3), & \text{if } j = 1, \\ w_2 + a^2_{111}(w_1)^2 + (2a^2_{12} - a^1_{11}) w_1 w_2 - (w_2)^2 + O(\|w\|^3), & \text{if } j = 2. \end{cases}$$

A direction $[v] \in \mathbb{P}^1(\mathbb{C})$ is allowable iff $v_1 \neq 0$; therefore we can assume $v_1 = 1$, and finding the allowable characteristic directions boils down to solving a quadratic equation whose discriminant is η. The allowable characteristic directions then are multiple of

$$v_\pm = \left(1, \frac{a^2_{12} - a^1_{11} \pm \sqrt{\eta}}{2}\right),$$

and $v_\pm$ is degenerate iff $\varepsilon \pm \sqrt{\eta} = 0$. Theorem 3.1 thus yields the assertion, exactly as in the previous corollary. □

Remark 3.4: If $\varepsilon = \eta = 0$ several things might happen; we can even have more than two stable holomorphic curves at the origin. See [A] and [CD] for examples.

Remark 3.5: A $\mathbb{C}^n$-valued quadratic form P_2 on $\mathbb{C}^n$ induces on the projective space a holomorphic map $\hat{P}_2 \colon \mathbb{P}^{n-1}(\mathbb{C}) \setminus Z \to \mathbb{P}^{n-1}(\mathbb{C})$, where Z is the image in $\mathbb{P}^{n-1}(\mathbb{C})$ of the cone $P_2^{-1}(O) \setminus \{O\} \subset \mathbb{C}^n$. If $v \in \mathbb{C}^n$ is a non-degenerate characteristic direction for P_2, then its image $[v] \in \mathbb{P}^{n-1}(\mathbb{C})$ is a fixed point of $\hat{P}_2$. In particular, we may then consider the linear map

$$A_{[v]} = d(\hat{P}_2)_{[v]} - \mathrm{id} \colon T_{[v]}\bigl(\mathbb{P}^{n-1}(\mathbb{C})\bigr) \to T_{[v]}\bigl(\mathbb{P}^{n-1}(\mathbb{C})\bigr)\,.$$

It turns out that this is the same matrix introduced by Hakim [H2, 3]. She proved that, under the hypotheses of Theorem 3.1, if $A_{[v]}$ has $d \geq 0$ eigenvalues with positive real part then the map actually admits a parabolic holomorphic $(d+1)$-manifold at the origin. In the case $n = 2$, $a^2_{11} = 0$ and $(\varepsilon, \eta) \neq (0, 0)$, we have

$$A_{[v_\pm]} = \mp 2 \frac{\sqrt{\eta}}{\varepsilon \pm \sqrt{\eta}}.$$

In particular, $A_{[v]} = -1$ when $\eta = \varepsilon^2 \neq 0$ (where, choosing ε as principal determination of $\sqrt{\eta}$, the non-degenerate characteristic direction is v_+), $A_{[v]} = 0$ when $\eta = 0 \neq \varepsilon$, and $\mathrm{Re}\, A_{[v_\pm]} > 0$ iff

$$\mathrm{Re}\left(\frac{\varepsilon}{\pm\sqrt{\eta}}\right) < -1,$$

when $\eta \neq 0$, ε^2. In particular, if $|\,\mathrm{Re}(\varepsilon/\sqrt{\eta})| > 1$ then the map F admits a parabolic basin of attraction for the origin.

Remark 3.6: It is not difficult to compute the matrix $A_{[v]}$ for the allowable characteristic direction described in the proof of Corollary 3.2; it is not so easy to compute the sign of the real part of the eigenvalues, though. For $n \leq 20$ we checked that the matrix $A_{[v]}$ has no eigenvalue with positive real part, and we suspect that this is true for all n.

Remark 3.7: Hakim [H3] proved that when $\tilde{F} \in \mathrm{End}(\mathbb{C}^n, O)$ is a global automorphism of $\mathbb{C}^n$ with $d\tilde{F}_O = \mathrm{id}$, and v is a non-degenerate characteristic direction, then the set Ω_v of orbits $z^k \to O$ such that $[z^k] \to [v]$ is an $\tilde{F}$-stable biholomorphic image of $\mathbb{C}^{d+1}$, where $d \geq 0$ is the number of eigenvalues of $A_{[v]}$ with positive real part (assuming, for simplicity, that $A_{[v]}$ has no purely imaginary eigenvalues). This is still true in our situation. Indeed, if our map F is a global automorphism of $\mathbb{C}^n$, then its lifting $\tilde{F}$ is a global automorphism of $M^{\mu_1} \setminus \pi_{\mu_1}^{-1}(X^0)$, which is biholomorphic to $\mathbb{C}^n \setminus \{O\}$. Furthermore, if v is an allowable characteristic direction, then Ω_v cannot intersect the singular divisor, because the latter is $\tilde{F}$-invariant whereas v is not tangent to it. This means that we can apply Hakim's result to $\tilde{F}$, and projecting down via π_{μ_1} we get an F-stable $(d+1)$-manifold biholomorphic to $\mathbb{C}^{d+1}$. In particular, then, Remark 3.5 yields yet another instance of the Fatou-Bieberbach phenomenon in $\mathbb{C}^2$.

4. Regular orbits

In the previous section we have shown that allowable (i.e., not tangent to the singular divisor) characteristic directions of the lifting of a map F give rise to parabolic curves. A priori, other characteristic directions might also give rise to parabolic curves, or possibly to F-orbits converging to the origin. The aim of this section is to show that this cannot happen, at least in the case $\rho = 1$, when dF_O is the Jordan $n \times n$ block J_n associated to the eigenvalue 1.

To state more precisely our result, we need some definitions. Let $\{z^k\} \subset \mathbb{C}^n \setminus \{O\}$ be a sequence converging to the origin. We shall say that $\{z^k\}$ is *0-regular* if $\{[z^k]\}$ converges to some $[v] \in \mathbb{P}^{n-1}(\mathbb{C})$; this is equivalent to saying that $\pi_1^{-1}(z^k)$ converges to some $[v] \in E^1$. We shall say that $\{z^k\}$ is *1-regular* if either $[v] \neq \mathbf{e}_1$ (and we shall specify this case saying that it is 1-regular *of first kind*) or $[v] = \mathbf{e}_1$ and $\{\chi_1 \circ \pi_1^{-1}(z^k)\}$ is 0-regular (and then $\{z^k\}$ is 1-regular *of second kind*). Now we proceed by induction. Let $\{z^k\}$ be $(r-1)$-regular. If it is $(r-1)$-regular of first kind, we shall also say that it is *r-regular (of first kind).* If it is $(r-1)$-regular of second kind, then $\pi_r^{-1}(z^k)$ converges to some $[v] \in E^r$. We shall say that $\{z^k\}$ is *r-regular* if either $[v] \neq \mathbf{e}_r$ (and we shall again say r-regular *of first kind*) or $[v] = \mathbf{e}_r$ and $\{\chi_r \circ \pi_r^{-1}(z^k)\}$ is 0-regular (and then $\{z^k\}$ is r-regular *of second kind*). We stress that we impose no conditions if $[v] \neq \mathbf{e}_r$; so for most sequences r-regularity is equivalent to 0-regularity.

Despite its apparent complexity, the condition of r-regularity is fairly natural; it is just a way to say that the different components of the sequence go to zero at comparable rates. For instance, if for $j = 1, \ldots, n$ there are $a_j \in \mathbb{C}^*$ and $\delta_j > 0$ such that

$$z_j^k = \frac{a_j}{k^{\delta_j}} + o\left(\frac{1}{k^{\delta_j}}\right) ,$$

then $\{z^k\}$ is r-regular for every r; and it is easy to provide examples of much more general r-regular sequences.

Now let $F \in \mathrm{End}(\mathbb{C}^n, O)$ be such that $dF_O = J_n$. Assume that F is generic, that is $a_{11}^n \neq 0$, and let $\tilde{F}$ be its lifting. We shall say that an F-orbit is *regular* if it converges to the origin and it is n-regular. A quick look to (1.5) and (1.6) shows that orbits obtained pushing down 0-regular orbits of $\tilde{F}$ tangent to allowable characteristic directions are regular; such orbits are called *standard,* and are the ones described in Corollary 3.2. Using this terminology, our aim is to prove that every regular orbit is standard. To do so, we need a lemma:

Lemma 4.1: *Let $\{w^k\} \subset \mathbb{C}^*$ be a sequence converging to 0. Assume there is another sequence $\{u^k\} \subset \mathbb{C}$ such that $u^k/w^k \to c \in \mathbb{C}$ and*

$$w^{k+1} = w^k(1 + u^k) + o\big((w^k)^2\big) .$$

Then $1/(kw^k) \to -c$. In particular, if $c \neq 0$ we have

$$w^k = -\frac{1}{ck} + o\left(\frac{1}{k}\right) .$$

Proof: Set $\varepsilon^k = w^{k+1} - w^k - u^k w^k$, so that $\varepsilon^k/(w^k)^2 \to 0$. We then have

$$\frac{1}{w^{h+1}} = \frac{1}{w^h} - \frac{u^h}{w^h} + \frac{(u^h)^2/w^h + (u^h - 1)\varepsilon^h/(w^h)^2}{1 + u^h + \varepsilon^h/w^h} .$$

Summing this equality for $h = 0, \ldots, k-1$ and dividing by k we find

$$\frac{1}{kw^k} = \frac{1}{kw^0} - \frac{1}{k}\sum_{h=0}^{k-1} \frac{u^h}{w^h} + \frac{1}{k}\sum_{h=0}^{k-1} \frac{(u^h)^2/w^h + (u^h-1)\varepsilon^h/(w^h)^2}{1+u^h+\varepsilon^h/w^h} ,$$

and the assertion follows from the convergence of the averages of a converging sequence. □

Then:

Theorem 4.2: *Let $F \in \mathrm{End}(\mathbb{C}^n, O)$ be such that $dF_O = J_n$. Assume that F is generic. Then every regular orbit of F is standard.*

Proof: Up to a linear change of coordinates we can assume $a^n_{11} = 1$. Let $\{z^k = F^k(z^o)\}$ be a regular orbit; we first of all want to prove, by induction, that $\pi_r^{-1}(z^k) \to \mathbf{e}_r$ for $r = 1, \ldots, n$.

First of all, 0-regularity yields $[z^k] \to [v] \in \mathbb{P}^{n-1}(\mathbb{C})$. But then v must be an eigenvector of dF_O; therefore $[v] = \mathbf{e}_1$, and thus $\pi_1^{-1}(z^k) \to \mathbf{e}_1$. Exactly the same argument shows that $\pi_2^{-1}(z^k) \to \mathbf{e}_2$.

Now assume that $\pi_r^{-1}(z^k) \to \mathbf{e}_r$ for some $2 \le r \le n-1$, and put $w^k = \chi_r \circ \pi_r^{-1}(z^k)$. The 0-regularity of $\{w^k\}$ implies that $[w^k] \to [v] \in \mathbb{P}^{n-1}(\mathbb{C})$; again, v must be (canonically identified to) an eigenvector of $d(\tilde{F}_r)_{\mathbf{e}_r}$. Now, a computation using (1.5) and (1.6) shows that for $1 < r < n$ we have

$$w^1_j = \begin{cases} w_1\bigl(1 - a^r_{11}w_1 + 2w_2 - w_{r+1} + O(\|w|^2)\bigr) & \text{if } j = 1,\\ w_j\bigl(1 - w_j + w_{j+1} + O(\|w|^2)\bigr) & \text{if } 1 < j < r,\\ w_r\bigl(1 + a^r_{11}w_1 - w_r + w_{r+1} + O(\|w|^2)\bigr) & \text{if } j = r,\\ a^j_{11}w_1 + w_j + w_{j+1} + 2a^j_{12}w_1w_2 - (a^r_{11}w_1 + w_{r+1})(a^j_{11}w_1 + w_j + w_{j+1}) + O(\|w|^3) & \text{if } r < j < n,\\ a^n_{11}w_1 + w_n + 2a^n_{12}w_1w_2 - (a^r_{11}w_1 + w_{r+1})(a^n_{11}w_1 + w_n) + O(\|w|^3) & \text{if } j = n. \end{cases} \tag{4.1}$$

In particolar, $d(\tilde{F}_r)_{\mathbf{e}_r}$ is represented by the matrix

$$\left|\begin{array}{cc|c} \multicolumn{2}{c|}{I_r} & O \\ \hline \begin{matrix} a^{r+1}_{11} \\ \vdots \\ a^n_{11} \end{matrix} & O & J_{n-r} \end{array}\right| .$$

Therefore $v = (0, v_2, \ldots, v_{r+1}, 0, \ldots, 0)$; to prove that $\pi_{r+1}^{-1}(z^k) \to \mathbf{e}_{r+1}$ it suffices to show that $v_{r+1} \neq 0$.

Assume, by contradiction, $v_{r+1} = 0$, and let $j_0 = \max\{2 \le j \le r \mid v_j \neq 0\}$. We know that $w^k_j/w^k_{j_0} \to v_j/v_{j_0}$ for all j; in particular, $w^k_j = O(w^k_{j_0})$ if $v_j \neq 0$, and $w^k_j = o(w^k_{j_0})$ if $v_j = 0$. Then (4.1) yields

$$w^{k+1}_{j_0} = w^k_{j_0}(1 - w^k_{j_0}) + o\bigl((w^k_{j_0})^2\bigr) ;$$

hence using Lemma 4.1 we find

$$w^k_{j_0} = \frac{1}{k} + o\left(\frac{1}{k}\right) ,$$

and so

$$w^k_j = \frac{v_j/v_{j_0}}{k} + o\left(\frac{1}{k}\right) \tag{4.2}$$

for all $j = 1, \ldots, n$.

We now claim that $v_j/v_{j_0} = j_0 - j + 1$ for all $j = 2, \ldots, j_0$. We argue by induction on $j_0 - j$. Take $j < j_0$ and assume that $v_{j+1}/v_{j_0} = j_0 - j$. Noticing that $w^k_j \neq 0$ for all k and $1 \le j \le r$ (because $\pi_r^{-1}(z^k)$ does not belong to the singular divisor), we can write

$$\frac{w^{k+1}_j}{w^k_j} = 1 - w^k_j + w^k_{j+1} + O\bigl((w^k_{j+1})^2\bigr) .$$

If $v_j = 0$ we would get

$$\frac{w_j^{k+1}}{w_j^k} = 1 + \frac{j_0 - j}{k} + o\left(\frac{1}{k}\right) ,$$

which is impossible because the infinite product $\prod_k (w_j^{k+1}/w_j^k)$ is converging to zero. Therefore $v_j \neq 0$; but then applying Lemma 4.1 to

$$w_j^{k+1} = w_j^k(1 - w_j^k + w_{j+1}^k) + o\big((w_j^k)^2\big)$$

and recalling (4.2) we get $v_j/v_{j_0} = j_0 - j + 1$, as claimed.

In particular we then have $v_2/v_{j_0} = j_0 - 1$, and so

$$\frac{w_1^{k+1}}{w_1^k} = 1 + \frac{2(j_0 - 1)}{k} + o\left(\frac{1}{k}\right) ,$$

which is impossible. The contradiction arises because we assumed $v_{r+1} = 0$; therefore we must have $v_{r+1} \neq 0$, as claimed.

Summing up, we have in particular proved that $\pi_n^{-1}(z^k) \to \mathbf{e}_n$; set $w^k = \chi_n \circ \pi_n^{-1}(z^k)$. Notice that, by construction, $w_j^k \neq 0$ for all k and j. By 0-regularity, $[w^k] \to [v] \in \mathbb{P}^{n-1}(\mathbb{C})$; Theorem 3.1 then says that v must be a characteristic direction of $\tilde{F}_n$ at $\mathbf{e}_n$, that is a solution of

$$\begin{cases} -v_1^2 + 2v_1v_2 = \lambda v_1 , & \\ -v_j^2 + v_jv_{j+1} = \lambda v_j & \text{for } j = 2, \ldots, n-1, \\ v_1v_n - v_n^2 = \lambda v_n . & \end{cases}$$

To end the proof we must show that v is allowable, that is that $v_j \neq 0$ for $j = 1, \ldots, n$.

Assume, by contradiction, that there is a j_0 such that $v_{j_0} \neq 0$ but $v_{j_0+1} = 0$ (where here by v_{n+1} we mean v_1). Then it is easy to prove that $v_j/v_{j_0} \in \mathbb{N}$ for all $j = 1, \ldots, n$; in particular, v_j/v_{j_0} is always non-negative. Now we have

$$w_{j_0}^{k+1} = w_{j_0}^k(1 - w_{j_0}^k) + o\big((w_{j_0}^k)^2\big) ;$$

therefore Lemma 4.1 yields $w_{j_0}^k = 1/k + o(1/k)$. Recalling (4.2) we then get $w_j^k = c_j/k + o(1/k)$ with $c_j \geq 0$ for all $j = 1, \ldots, n$. But then arguing exactly as in the first part of the proof we show that $v_{j_0-1}, \ldots, v_1 \neq 0$; and then we get $v_n \neq 0$, and going up we finally arrive to prove $v_{j_0+1} \neq 0$, contradiction. $\square$

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