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Finsler Metrics of Constant Curvature and the Characterization of Tube Domains

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1. Introduction

In this paper we would like to give an instance of how (real!) Finsler geometry may find applications in problems of complex analysis. In particular we shall show how the understanding of the classification of Finsler manifolds with vanishing horizontal flag curvature together with the theory of complex Monge-Ampère equation enables us to formulate characterizations, up to biholomorphic maps, of special classes of domains in $\mathbb{C}^n$.

Let us start with a simple example. Let $D \subset \mathbb{R}^n$ be a smooth strictly convex domain and suppose that $0 \in D$ is the baricenter of D. The Minkowski functional μ of D relative to the baricenter 0 is defined by $\mu(0) = 0$ and for $y \in \mathbb{R}^n \setminus \{0\}$ by:

$$\mu(y) = \inf\{1/t \mid t > 0 \quad \text{and} \quad ty \notin D\}.$$

Then $\mu \in C^0(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n \setminus \{0\})$ and $\rho = \mu^2$ is a strictly convex function such that $R(t, x) = \rho(tx)$ defines a smooth function on $\mathbb{R} \times \mathbb{R}^n \setminus \{0\}$. Let us identify

$$T\mathbb{R}^n \simeq \mathbb{R}^n \times \mathbb{R}^n \simeq \mathbb{R}^n \oplus \mathbb{R}^n \simeq \mathbb{R}^n + i\mathbb{R}^n \simeq \mathbb{C}^n.$$

The function μ defines a Minkowski metric $F\colon T\mathbb{R}^n \to \mathbb{R}_+$ in the obvious way: if $y \in T_x(\mathbb{R}^n)$ then $F(x; y) = F(x + iy) = \mu(y)$. Evidently F is a smooth strictly convex real Finsler metric whose indicatrix at every point is given by the domain D. Furthermore, if $G = F^2$ we recall that, because of the homogeneity property $G(x; ty) = \rho(ty) = t^2\rho(y)$ which holds for all $t \in \mathbb{R}$, one has for $y \neq 0$

$$G(y) = \frac{\partial^2 G}{\partial y^\alpha \partial y^\beta}(y)\, y^\alpha y^\beta,$$

where we are using the Einstein convention.

It is interesting to look at this construction from a different point of view. The "indicatrix bundle" for the metric F is the trivial bundle $\mathbb{R}^n \times D$ which, under the above indicated identification of the tangent bundle of $\mathbb{R}^n$ with $\mathbb{C}^n$, is exactly the tube domain $\mathbb{R}^n + iD$. If we denote by $\tau\colon \mathbb{C}^n \to \mathbb{R}_+$ the function defined by $\tau(z) = G(\mathrm{Im} z) = F^2(\mathrm{Im} z)$ then $\tau \in C^0(\mathbb{C}^n) \cap C^\infty(\mathbb{C}^n \setminus \{\mathrm{Im} z = 0\})$ and the

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function $(t,z)=(t,x+iy)\mapsto\tau(x+ity)$ is of class C^∞ on $\mathbb{R}\times(\mathbb{C}^n\setminus\{\mathrm{Im}z=0\})$. A simple calculation shows that in addition τ satisfies on $\mathbb{C}^n\setminus\{\mathrm{Im}z=0\}$ the following properties:

$$dd^c\tau>0, \tag{1.1}$$

$$(dd^c\sqrt{\tau})^n\equiv 0. \tag{1.2}$$

Because of (1.1), $dd^c\tau$ is the Kähler form of a metric h defined on $\mathbb{C}^n\setminus\{\mathrm{Im}z=0\}$ which, according to our identifications, is exactly the complement of the zero section of the tangent bundle of $\mathbb{R}^n$. Since an elementary computation yields

$$\frac{\partial^2\tau}{\partial z^\alpha\partial\bar z^\beta}(x+iy)=\frac{1}{4}\frac{\partial^2 G}{\partial y^\alpha\partial y^\beta}(y),$$

using the homogeneity property of G one sees that the real metric $g=\mathrm{Re}h$ associated to h induces on $\mathbb{R}^n$ exactly the Finsler metric F defined above (see [**AP**] for details on the relation between Finsler metrics on a manifold and Riemannian metrics on the complement of the zero section of its tangent bundle).

More examples in this vein may be constructed. It is known that (see [**GS**], [**L**], [**LS**], [**PW1**], [**S**]) given a complete Riemannian manifold M of nonnegative sectional curvature it is possible to define a complex structure on its tangent bundle TM so that M sits in TM as a totally real submanifold of top dimension. Moreover it is possible to define a smooth (in fact real analytic) function $\tau_0\colon TM\to[0,+\infty)$ such that $\tau_0^{-1}(0)=M$ (where we identify M with the zero section in TM) and satisfying (1.1) on TM and (1.2) on the complement of the zero section of TM. Deforming τ_0 along the leaves of the Monge-Ampère foliation as indicated in [**PW2**] it is possible to construct many continuous functions $\tau\colon TM\to[0,+\infty)$ with $\tau^{-1}(0)=M$ and such that on $\tilde M=TM\setminus M$, the complement of the zero section, τ is of class C^∞ and satisfies on (1.1) and (1.2). As in the example of tube domains it is possible to define a Riemannian metric on $\tilde M$ and hence a real Finsler metric on M. It is a natural question to ask if, imposing some natural condition on the function τ and on the Finsler metric induced on M, it is possible to obtain a characterization, up to biholomorphic maps, of tube domains as "indicatrix bundles".

To give our result we need to specify the kind of singularity the function τ has to possess. Let N be a complex manifold of (complex) dimension n. Let M be a totally real submanifold of N of (real) dimension n. Let f be a function on N with $f\in C^0(N)\cap C^\infty(N\setminus M)$. We say that f is *transversally regular along* M if for any point $p\in M$ and any coordinate system $(z_1,\dots,z_n)=(x_1+iy_1,\dots,x_n+iy_n)=(x+iy)$ on a neighborhood U of p so that $M\cap U=\{y=0\}$, the function

$$(t,z)=(t,x+iy)\mapsto f(x+ity)$$

is of class C^∞ on $\mathbb{R}\times(U\setminus\{y=0\})$.

Furthermore, we shall need the notion of horizontal flag curvature of a real Finsler metric. Given a Finsler manifold (M,F) a standard construction (see [**AP**]) yields a splitting $T\tilde M=\mathcal{H}\oplus\mathcal{V}$ of the tangent bundle to $\tilde M=TM\setminus M$ (the complement of the zero section) in the sum of the vertical bundle $\mathcal{V}$ and a horizontal bundle $\mathcal{H}$; furthermore associated to F there is a Riemannian metric $\langle\,,\rangle$ on $T\tilde M$, a metric connection ∇ (the so-called Cartan connection), and a natural section

$\chi\colon \tilde{M} \to \mathcal{H}$. If we denote by Ω the curvature operator associated to ∇, the *horizontal flag curvature* of the Finsler metric F is the symmetric bilinear form $R\colon \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ given by

$$R(H,K) = \langle \Omega(\chi, H)K, \chi \rangle.$$

This is the curvature term appearing in the second variation formula for Finsler metrics, and so it seems to be the correct generalization of the Riemannian curvature (see [**AP**] and [**BC**] for details).

With these notions we are ready to state our characterization:

THEOREM 1.1. *Let M be a simply connected differentiable manifold of (real) dimension n and suppose that on TM is defined a complex structure so that M sits in TM as a totally real submanifold of top dimension and there exists a function $\tau\colon TM \to [0, +\infty)$ such that*

(i) $\tau \in C^0(TM) \cap C^\infty(TM \setminus M)$ *and τ is transversally regular along M;*
(ii) $\tau_0^{-1}(0) = M$*;*
(iii) $dd^c\tau > 0$ *on* $\tilde{M} = TM \setminus M$*;*
(iv) $(dd^c\sqrt{\tau})^n \equiv 0$ *on* $\tilde{M}$*;*
(v) *the metric defined by $dd^c\tau$ on $\tilde{M}$ induces on M a real Finsler metric F of vanishing horizontal flag curvature.*

Then M is isometric to $\mathbb{R}^n$ equipped with a Minkowski metric F_0. Furthermore TM is biholomorphic to $\mathbb{C}^n$ and, if D is the indicatrix of the metric F_0, then the tubular neighborhood $\mathcal{T}(M) = \{z \in TM \mid \tau(z) < 1\}$ of M is biholomorphic to the tube domain $\mathbb{R}^n + iD$.

The proof of Theorem 1.1 has two main steps. The first, which will be performed in Section 2, is to characterize flat simply connected Finsler manifolds. The second is to make the necessary adjustment in the arguments given in [**PW1**] for the case of Riemannian manifolds to show that TM is biholomorphic to $\mathbb{C}^n$ and $\mathcal{T}(M)$ to a tube domain. We shall outline this part in Section 3. Section 2 uses heavily results of [**AP**], and Section 3 of [**PW1**]; hence we shall rely freely upon these two references for notations, notions and technicalities, giving here only the elements needed to reconstruct the proofs.

2. The Cartan-Ambrose-Hicks Theorem for real Finsler metrics

We need a classification of "Finsler space forms" at least in the flat case. To this end it is necessary to compare Finsler manifold with "the same" horizontal flag curvature as it is done in Riemannian geometry. Results of this kind are probably available in the literature (see [**R**] for instance). Nevertheless, using the machinery introduced in [**AP**] it is possible to give a proof which follows very closely the line of the corresponding Riemaniann result (see [**CE**]).

Let M and $\overline{M}$ be two complete real Finsler manifolds of dimension n, with Finsler metrics F and $\overline{F}$ respectively, and take $p \in M$ and $\overline{p} \in \overline{M}$. Assume there is a homogeneous isometry $I\colon T_pM \to T_{\overline{p}}\overline{M}$. For any geodesic σ issuing from p in M, denote by A_σ the parallel transport along σ, and by $\overline{\sigma} = \exp_{\overline{p}} \circ I \circ \exp_p^{-1} \circ \sigma$ the geodesic in $\overline{M}$ issuing from $\overline{p}$ in the direction $I\big(\dot{\sigma}(0)\big)$. In particular, if σ connects p to $q \in M$, and $\overline{\sigma}$ connects $\overline{p}$ to $\overline{q}$ in $\overline{M}$, we get a homogeneous isometry $I_\sigma\colon T_qM \to T_{\overline{q}}\overline{M}$ by setting $I_\sigma = A_{\overline{\sigma}} \circ I \circ A_{-\sigma}$. Furthermore, for any $v \in T_qM$ we get a map $I_\sigma^v\colon \mathcal{H}_v \to \mathcal{H}_{I_\sigma(v)}$ by setting

$$I_\sigma^v = \chi_{I_\sigma(v)} \circ I_\sigma \circ \chi_v^{-1}.$$

Denote by Ω the curvature operator of M, and by $\overline{\Omega}$ the curvature operator of $\overline{M}$. We shall say that I *preserves the horizontal flag curvature* if

$$\overline{\Omega}\big(\overline{\chi}, I_\sigma(v)^H\big)\overline{\chi} = I_\sigma^{\dot\sigma}\big(\Omega(\chi, v^H)\chi\big)$$

for all geodesics σ and all $v \in T_\sigma M$, where χ (respectively, $\overline{\chi}$) is the horizontal radial vector field of M (respectively, of $\overline{M}$; see [**AP**] for definitions).

Then we have the following:

Theorem 2.1 (Cartan-Ambrose-Hicks Theorem for Finsler metrics). *Let M and $\overline{M}$ be two complete real Finsler manifolds of dimension n, with M simply connected. Assume that for some $p \in M$ and $\overline{p} \in \overline{M}$ there is a homogeneous isometry $I\colon T_pM \to T_{\overline{p}}\overline{M}$ preserving the horizontal flag curvature. Then the map $\Phi\colon M \to \overline{M}$ given by $\sigma(t) \mapsto \Phi\big(\sigma(t)\big) = \overline{\sigma}(t)$ for any geodesic σ issuing from p is well defined. Furthermore, Φ is a local isometry and hence a covering map.*

Proof. It is very similar to the classical one. It suffices to remark that if J is a Jacobi field along σ in M, and $\overline{J} = I_\sigma \circ J$, then $\overline{J}$ is a Jacobi field along $\overline{\sigma}$ in $\overline{M}$. This follows from the fact that I preserves the horizontal flag curvature, and from the equality

$$I_\sigma^{\dot\sigma}\big(\chi(\dot\sigma)\big) = \overline{\chi}\big(\dot{\overline{\sigma}}\big).$$

Then using the theory developed in [**AP**] it is possible to repeat almost word by word the proof of the classical Cartan-Ambrose-Hicks Theorem described in Theorem 1.36 of [**CE**] □

Corollary 2.2. *Let (M, F) be a complete simply connected real Finsler manifold of dimension n with vanishing horizontal flag curvature. Let $\hat F$ be the restriction of F to any tangent space T_pM. Then $\hat F$ defines a Minkowski metric on $\mathbb{R}^n \simeq TM$ so that $\exp_p\colon (\mathbb{R}^n, \hat F) \to (M, F)$ is an isometry.*

Proof. This follows from Theorem 2.1 and from the fact that the vanishing of the horizontal flag curvature implies $\Omega(\chi, H)\chi \equiv 0$, thanks to Proposition 1.4.5 of [**AP**]. □

3. The proof of Theorem 1.1

Corollary 2.2 proves the first part of the conclusion of Theorem 1.1 and shows in particular that TM is diffeomorphic to $T\mathbb{R}^n \simeq \mathbb{R}^n + i\mathbb{R}^n \simeq \mathbb{C}^n$. Following the ideas of [**PW1**] we shall now show that in fact they are biholomorphic. To this end we must define a suitable map from $\mathbb{R}^n + i\mathbb{R}^n \simeq \mathbb{C}^n$ to TM.

We must recall a few results on Monge-Ampère foliations. Since τ satisfies (1.1) and (1.2), according to [**PW1, Section 3**], $TM \setminus M$ is foliated by Riemann surfaces which are exactly the maximal complex submanifolds along which $\sqrt{\tau}$ is harmonic. Furthermore the leaves of this foliation are totally geodesic flat (with respect to the Kähler metric $dd^c\tau$) submanifolds of $TM \setminus M$, and the normalized gradient $\nabla\tau/\|\nabla\tau\|$ of τ is tangent to the geodesic flow normal to M. In fact as τ is transversally regular along M, if $p \in M \subset TM$ and v is tangent to M (considered as a totally real submanifold of TM), it is meaningful to consider geodesics on TM starting from p tangent to Jv, where J is the complex structure of TM.

If $\sigma\colon\mathbb{R}\to M$ is a (normalized) geodesic of M with respect to the Finsler metric induced on M by the metric $dd^c\tau$ on $TM\setminus M$, define a map $\Sigma_\sigma\colon\mathbb{R}^2\to TM$ by

$$\Sigma_\sigma(s,t)=\exp_{\sigma(t)}(-sJ\dot\sigma'(t)).$$

Then the proof of Theorem 5.1 of [**PW1**] shows that the image L_σ of Σ_σ is a complex submanifold of TM such that $\sqrt{\tau}$ restricted to $L_\sigma\setminus M=L_\sigma\cap\{\tau>0\}$ is harmonic. In other words for each normalized geodesic σ of M, the submanifold L_σ is an extended leaf of the Monge-Ampère foliation associated to $\sqrt{\tau}$. Let us note that for any normalized geodesic σ of M the restriction of the metric $dd^c\tau$ to $L_\sigma\cap\{\tau>0\}$ extends to a flat metric on L_σ which therefore is, being simply connected, biholomorphically isometric to $\mathbb{C}$ via the map $(t+is)\mapsto\Sigma_\sigma(s,t)$.

Let F_0 be the Minkowski metric on $\mathbb{R}^n$ given by Corollary 2.2, so that there exists an isometry $\Phi\colon(\mathbb{R}^n,F_0)\to(M,F)$ where F is the Finsler metric on M induced by τ. Let also τ_0 be the function on $\mathbb{C}^n$ obtained by taking the square of the Minkowski functional of the indicatrix of F_0. Then the same considerations developed above for the Monge-Ampère foliation on TM may be repeated for $T\mathbb{R}^n\simeq\mathbb{C}^n$ equipped with the function τ_0. Imitating the proof of Theorem 5.2 of [**PW1**], we define a map $\Psi\colon\mathbb{R}^n+i\mathbb{R}^n\simeq\mathbb{C}^n\to TM$ in the following way. Let σ_0 be any normalized geodesic for $(\mathbb{R}^n,F_0)$ — i.e., a straight line suitably parametrized — and let Σ_{σ_0} be the parametrization of the corresponding extended leaf. Let $\sigma=\Phi(\sigma_0)$ be the geodesic of (M,F) image of σ_0 via the isometry Φ, and Σ_σ the associated parametrization. Then we set

$$\Psi\big(\Sigma_{\sigma_0}(s,t)\big)=\Sigma_\sigma(s,t).$$

The map Ψ is well defined, bijective, holomorphic when restricted to any complexification in $\mathbb{C}^n$ of straight line of $\mathbb{R}^n$, and such that the map

$$(x,r,y)\mapsto\Psi(x+iry)$$

is smooth for all $r\in\mathbb{R}$, $x\in\mathbb{R}^n$, $y\in\mathbb{R}^n\setminus\{0\}$. In fact, under the correct identifications of the corresponding complex structures, the map Ψ is just the differential of the isometry Φ, and thus it is smooth. But then it follows from a Hartogs theorem due to Sibony and Wong (see [**SW**]) that Ψ is holomorphic and hence biholomorphic.

As τ_0 is the distance squared from $\mathbb{R}^n$ in $\mathbb{C}^n$ and τ the distance squared from M in TM, it follows also that if D is the indicatrix of F_0 then $\mathbb{R}^n+iD$ is mapped biholomorphically by Ψ onto the "indicatrix bundle" $\mathcal{T}M$ — and thus the proof is complete.

References

[AP] M. Abate, G. Patrizio, *Finsler Metrics — A Global Approach*, Lecture Notes in Mathematics 1591, Springer-Verlag, Berlin, 1994.

[BC] D. Bao, S.S. Chern, *On a notable connection in Finsler geometry*, Houston J. Math. **19** (1993), 138-180.

[CE] J. Cheeger, D.G. Ebin, *Comparison Theorems in Riemannian Geometry*, North-Holland, Amsterdam, 1975.

[GS] V. Guillelmin, M. Stenzel, *Grauert tubes and the homogeneous Monge-Ampère equation*, J. Diff. Geom. **34** (1991), 561–570.

[L] L. Lempert, *Complex structures on the tangent bundle of Riemannian manifolds*, Complex Analysis and Geometry, Plenum Press, New York, 1993, pp. 235–251.

[LS] L. Lempert, R. Szöke, *Global solutions of the homogeneous complex Monge-Ampère equation and complex structure on the tangent bundle of Riemannian manifolds*, Math. Ann. **290** (1991), 689–712.

[PW1] G. Patrizio, P.M. Wong, *Stein manifolds with compact symmetric centers*, Math. Ann. **289** (1991), 355–382.

[PW2] G. Patrizio, P.M. Wong, *Monge-Ampère functions with large center*, Proc. Sympos. Appl. Math., vol. 52, part 2, Amer. Math. Soc., Providence, RI, 1991, pp. 435–447.

[R] H. Rund, *The Differential Geometry of Finsler Spaces*, Springer, Berlin, 1959.

[SW] N. Sibony, P.M. Wong, *Some results on global analytic sets*, Séminaire Lelong-Skoda, Années 1978/79, Lecture Notes in Mathematics 822, Springer-Verlag, Berlin, 1980, pp. 222–237.

[S] R. Szöke, *Complex structures on the tangent bundle of Riemannian manifolds*, Math. Ann. **291** (1991), 409–428.

Open problems

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1. Classify simply connected Kähler Finsler manifolds with constant negative holomorphic curvature.

2. Under which hypotheses does weakly Kähler imply Kähler?

3. Is it true that the Kobayashi metric of strongly convex bounded domains in $\mathbb{C}^n$ is Kähler? (It is always weakly Kähler).

4. Describe the (necessarily affine) isometric biholomorphisms of complex Minkowski spaces (i.e., of $\mathbb{C}^n$ endowed with a complex Minkowski metric).

5. Classify under biholomorphic equivalence tubular neighborhoods of the zero section in the tangent bundle of complete Riemannian manifolds with nonnegative sectional curvature.

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