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FORMAL POINCARÉ-DULAC RENORMALIZATION FOR HOLOMORPHIC GERMS

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ABSTRACT. We shall describe an alternative approach to a general renormalization procedure for formal self-maps, originally suggested by Chen-Della Dora and Wang-Zheng-Peng, giving formal normal forms simpler than the classical Poincaré-Dulac normal form. As example of application we shall compute a complete list of normal forms for bi-dimensional superattracting germs with non-vanishing quadratic term; in most cases, our normal forms will be the simplest possible ones (in the sense of Wang-Zheng-Peng). We shall also discuss a few examples of renormalization of germs tangent to the identity, revealing interesting second-order resonance phenomena.

1. Introduction. In the study of a class of holomorphic dynamical systems, an important goal often is the classification under topological, holomorphic or formal conjugation. In particular, for each dynamical system in the class one would like to have a definite way of choosing a (hopefully simpler, possibly unique) representative in the same conjugacy class; a *normal form* of the original dynamical system.

The formal classification of one-dimensional germs is well-known (see, e.g., [2]): if

$$f(z) = \lambda z + a_\mu z^\mu + O_{\mu+1} \in \mathbb{C}[\![z]\!]$$

is a one-dimensional formal power series with complex cofficients and vanishing constant term, where $a_\mu \neq 0$ and $O_{\mu+1}$ is a remainder term of order at least $\mu + 1$, then f is formally conjugated to:

– $g(z) = \lambda z$ if $\lambda \neq 0$ and λ is not a root of unity;
– $g(z) = z^\mu$ if $\lambda = 0$; and to

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– $g(z) = \lambda z - z^{kq+1} + \alpha z^{2kq+1}$ if λ is a primitive q-th root of unity, for suitable $k \geq 1$ and $\alpha \in \mathbb{C}$ that are formal invariants (and where $q = 1$ and $k = \mu - 1$ when $\lambda = 1$).

In several variables, the most famous kind of normal form for local holomorphic dynamical systems (i.e., germs of holomorphic vector fields at a singular point, or germs of holomorphic self-maps with a fixed point) is the *Poincaré-Dulac normal form* with respect to formal conjugation. Let us recall very quickly its definition, at least in the setting we are interested here, that is of formal self-maps with a fixed point that we can assume to be the origin in $\mathbb{C}^n$.

Let $F \in \widehat{\mathcal{O}}^n$ be a formal transformation in n complex variables, where $\widehat{\mathcal{O}}^n$ denotes the space of n-tuples of power series in n variables with vanishing constant term, and let Λ denote the (not necessarily invertible) linear term of F; up to a linear change of variables, we can assume that Λ is in Jordan normal form. For simplicity, given a linear map $\Lambda \in M_{n,n}(\mathbb{C})$ we shall denote by $\widehat{\mathcal{O}}^n_\Lambda$ the set of formal transformations in $\widehat{\mathcal{O}}^n$ with Λ as linear part. If $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of Λ, we shall say that a multi-index $Q = (q_1, \ldots, q_n) \in \mathbb{N}^n$ with $q_1 + \cdots + q_n \geq 2$ is *Λ-resonant* if there is $j \in \{1, \ldots, n\}$ such that $\lambda_1^{q_1} \cdots \lambda_n^{q_n} = \lambda_j$. If this happens, we shall say that the monomial $z_1^{q_1} \cdots z_n^{q_n} e_j$ is *Λ-resonant,* where $\{e_1, \ldots, e_n\}$ is the canonical basis of $\mathbb{C}^n$. Then (see, e.g., [7, 33, 34, 35]) given $F \in \widehat{\mathcal{O}}^n_\Lambda$ it is possible to find a (not unique, in general) invertible formal transformation $\Phi \in \widehat{\mathcal{O}}^n_I$ with identity linear part such that $G = \Phi^{-1} \circ F \circ \Phi$ contains only Λ-resonant monomials.

The formal transformation G is a *Poincaré-Dulac normal form* of F; notice that, since $\Phi \in \widehat{\mathcal{O}}^n_I$, the linear part of G is still Λ. More generally, we shall say that a $G \in \widehat{\mathcal{O}}^n_\Lambda$ is *in Poincaré-Dulac normal form* if G contains only Λ-resonant monomials.

The importance of this result cannot be underestimated, and it has been applied uncountably many times; however it has some limitations. For instance, if $\Lambda = O$ or $\Lambda = I$ then *all* monomials are resonant; and thus in these cases any $F \in \widehat{\mathcal{O}}^n_\Lambda$ is in Poincaré-Dulac normal form, and a further simplification (a *renormalization*) is necessary. Actually, even when a Poincaré-Dulac normal form is different from the original germ, it is often possible to further simplify the germ by applying invertible transformations preserving the property of being in Poincaré-Dulac normal form.

The idea of renormalizing Poincaré-Dulac normal forms is by now well-established in the context of vector fields, where the renormalized normal forms are often called hypernormal forms, and can be obtained by using several different techniques; a (far from exhaustive) list of relevant papers one might consult is [5, 6, 8, 9, 10, 11, 14, 15, 16, 23, 24, 25, 26, 27, 28, 29, 31, 32, 36]; see also [30] for a fine introduction to the subject. On the other hand, with a few exceptions (see, for instance, [12, 18, 19]) this idea has been exploited in the context of self-maps only recently. One example is [3], where it is applied to a particular class of self-maps with identity linear part. Another, more recent, example can be found in [13], where it is applied to self-maps with invertible linear part whose resonances with respect to some of the eigenvalues are generated over $\mathbb{N}$ by one multi-index. More important for our aims are [37, 38], where the authors, inspired by [27, 18, 19, 39], construct an a priori infinite sequence of renormalizations giving simpler and simpler normal forms.

Let us roughly describe the main ideas underlying the theory of renormalization of formal transformations. For each $\nu \geq 2$ let $\mathcal{H}^\nu$ denote the space of n-tuples of homogeneous polynomials in n variables of degree ν. Then every $F \in \widehat{\mathcal{O}}^n_\Lambda$ admits a *homogeneous expansion*

$$F = \Lambda + \sum_{\nu\geq 2} F_\nu ,$$

where $F_\nu \in \mathcal{H}^\nu$ is the *ν-homogeneous term* of F. We shall also use the notation $\{G\}_\nu$ to indicate the ν-homogeneous term of a formal transformation G, and denote by $L_\Lambda\colon \widehat{\mathcal{O}}^n \to \widehat{\mathcal{O}}^n$ the map

$$L_\Lambda(H) = H \circ \Lambda - \Lambda H .$$

If $\Phi = I + \sum_{\nu\geq 2} H_\nu \in \widehat{\mathcal{O}}^n_I$ is the homogeneous expansion of an invertible formal transformation then it turns out that $L_\Lambda(\mathcal{H}^\nu) \subseteq \mathcal{H}^\nu$ and

$$\{\Phi^{-1} \circ F \circ \Phi\}_\nu = F_\nu - L_\Lambda(H_\nu) + R_\nu \tag{1.1}$$

for all $\nu \geq 2$, where R_ν is a remainder term depending only on F_ρ and H_σ with ρ, $\sigma < \nu$. This suggests to consider for each $\nu \geq 2$ splittings of the form

$$\mathcal{H}^\nu = \operatorname{Im} L^\nu_\Lambda \oplus \mathcal{N}^\nu \quad \text{and} \quad \mathcal{H}^\nu = \operatorname{Ker} L^\nu_\Lambda \oplus \mathcal{M}^\nu ,$$

where $L^\nu_\Lambda = L_\Lambda|_{\mathcal{H}^\nu}$, and $\mathcal{N}^\nu$ and $\mathcal{M}^\nu$ are suitable complementary subspaces. Then (1.1) implies that we can inductively choose $H_\nu \in \mathcal{M}^\nu$ so that $\{\Phi^{-1} \circ F \circ \Phi\}_\nu \in \mathcal{N}^\nu$ for all $\nu \geq 2$; we shall say that $G = \Phi^{-1} \circ F \circ \Phi$ is a *first order normal form* of F (with respect to the chosen complementary subspaces). Furthermore, it is not difficult to see that the quadratic (actually, the first non-linear non-vanishing) homogeneous term of G is a formal invariant, that is it is the same for all first order normal forms of F. Notice that when $\Lambda = O$ or $\Lambda = I$ we have $L_\Lambda \equiv O$, and thus in these cases every $F \in \widehat{\mathcal{O}}^n_\Lambda$ is a first order normal form.

When Λ is diagonal, $\operatorname{Ker} L_\Lambda$ is generated by the resonant monomials, and $\operatorname{Im} L_\Lambda$ is generated by the non-resonant monomials. Furthermore, for each $\nu \geq 2$ we have the splitting $\mathcal{H}^\nu = \operatorname{Im} L^\nu_\Lambda \oplus \operatorname{Ker} L^\nu_\Lambda$, and thus taking $\mathcal{N}^\nu = \operatorname{Ker} L^\nu_\Lambda$ and $\mathcal{M}^\nu = \operatorname{Im} L^\nu_\Lambda$ we have recovered the classical Poincaré-Dulac normal form (when Λ has a nilpotent part the situation is only slightly more complicated; see [30, Section 4.5] for details).

Summing up, a Poincaré-Dulac formal normal form is composed by homogeneous terms contained in a complementary space of the image of the operator L_Λ. Furthermore, the quadratic homogeneous term is uniquely determined, and we can still act on the normal form by transformations having all homogeneous terms in the kernel of L_Λ.

The k-th renormalization follows the same pattern. Assume that F is in $(k-1)$-th normal form. Then there is a suitable (not necessarily linear if $k \geq 3$) operator $\mathcal{L}^k$, depending on the first k homogeneous terms of F, so that we can bring F in a normal form G whose all homogeneous terms belong to a chosen complementary subspace[1] of the image of $\mathcal{L}^k$, and the first $k+1$ homogeneous terms of G are uniquely determined; we shall say that G is in *k-th order normal form* (with respect to the chosen subspaces).

A formal transformation G is in *infinite order normal form* if it is in k-th normal form for all k, with respect to some choice of complementary subspaces and using

[1]When $k \geq 3$ one has to choose a complementary subspace to a vector space of maximal dimension contained in the image of $\mathcal{L}^k$. Actually, [38] talks of "the" subspace of maximal dimension contained in $\mathcal{L}^k$, but a priori it might not be unique.

the operators $\mathcal{L}^k$ defined using the first k homogeneous terms of G. The main result of [38] then states that every element of $\widehat{\mathcal{O}}^n_\Lambda$ can be brought to a (possibly not unique) infinite order normal form by a sequence of formal conjugations tangent to the identity.

In the first section of this paper we shall describe an alternative approach, equivalent to the one proposed by Wang-Zheng-Peng but possibly simpler, to the determination of higher order normal forms, based on homogeneous polynomials and symmetric multilinear maps instead of on higher order derivatives. We shall concentrate in particular on second order normal forms because, as we shall see, in most cases the second order normal forms we shall obtain will automatically be infinite order normal forms.

To apply these procedures we need a rule for choosing complementary subspaces. It turns out that an efficient way of doing this is by taking orthogonal complements with respect to the Fischer Hermitian product, defined by (see [22])

$$\langle z_1^{p_1}\cdots z_n^{p_n} e_h, z_1^{q_1}\cdots z_n^{q_n} e_k\rangle = \begin{cases} 0 & \text{if } h\neq k \text{ or } p_j\neq q_j \text{ for some } j;\\ \dfrac{p_1!\cdots p_n!}{(p_1+\cdots+p_n)!} & \text{if } h=k \text{ and } p_j=q_j \text{ for all } j.\end{cases}$$

With this choice, as we shall see in Sections 2 and 3, the expression of the second order (and often infinite order) normal forms can be quite simple. For instance, in Section 2 we shall apply this procedure to the case of superattracting (i.e., with $\Lambda = O$) 2-dimensional formal transformations, case that has no analogue in the vector field setting, proving the following

Theorem 1.1. *Let $F\in\widehat{\mathcal{O}}^2_O$ be of the form $F(z,w)=F_2(z,w)+O_3$. Then:*

(i) *if $F_2(z,w)=(z^2,zw)$ or $F_2(z,w)=(-z^2,-z^2-zw)$ then F is formally conjugated to a unique infinite order normal form*

$$G(z,w)=F_2(z,w)+\big(\varphi(w)-z\psi'(w),2\psi(w)\big)\,,$$

where φ, $\psi\in\mathbb{C}[\![\zeta]\!]$ are power series of order at least 3;

(ii) *if $F_2(z,w)=(-zw,-z^2-w^2)$ then F is formally conjugated to a unique infinite order normal form*

$$G(z,w)=F_2(z,w)+\big(-2\varphi(z+w)+2\psi(w-z),\varphi(z+w)+\psi(w-z)\big)\,,$$

where φ, $\psi\in\mathbb{C}[\![\zeta]\!]$ are power series of order at least 3;

(iii) *if $F_2(z,w)=(zw,zw+w^2)$ then F is formally conjugated to a unique infinite order normal form*

$$G(z,w)=F_2(z,w)+\big(w\varphi'(z)+\psi(z),2\varphi(z)-w\varphi'(z)-\psi(z)\big)\,,$$

where φ, $\psi\in\mathbb{C}[\![\zeta]\!]$ are power series of order at least 3;

(iv) *if $F_2(z,w)=\big(-\rho z^2,(1-\rho)zw\big)$ with $\rho\neq 0$, 1 then F is formally conjugated to a unique infinite order normal form*

$$G(z,w)=F_2(z,w)+\big((\rho-1)z\varphi'(w)+\psi(w),-2\rho\varphi(z)\big)\,,$$

where φ, $\psi \in \mathbb{C}[\![\zeta]\!]$ are power series of order at least 3;

(v) *if $F_2(z,w) = \big(-z^2 + zw, w^2\big)$ then F is formally conjugated to a unique infinite order normal form*

$$G(z,w) = F_2(z,w) + \big(\varphi(\tfrac{z}{2} + w), -\tfrac{1}{4}\varphi(\tfrac{z}{2} + w) + \psi(z)\big) ,$$

where φ, $\psi \in \mathbb{C}[\![\zeta]\!]$ are power series of order at least 3;

(vi) *if $F_2(z,w) = \big(\rho z^2 + zw, (1+\rho)zw + w^2\big)$ with $\rho \neq 0$, -1 then F is formally conjugated to a unique infinite order normal form*

$$\begin{aligned} G(z,w) = F_2(z,w) + \Bigg(&\frac{1}{\rho}\left[\frac{1-\sqrt{-\rho}}{2m_\rho^2}\varphi(m_\rho z + w) + \frac{1+\sqrt{-\rho}}{2n_\rho^2}\varphi(n_\rho z + w)\right] \\ &+ \frac{1+\rho}{2\sqrt{-\rho}}\left(\frac{1}{m_\rho^2}\psi(m_\rho z + w) - \frac{1}{n_\rho^2}\psi(n_\rho z + w)\right), \\ &\frac{1-\sqrt{-\rho}}{2}\varphi(m_\rho z + w) + \frac{1+\sqrt{-\rho}}{2}\varphi(n_\rho z + w) \\ &+ \frac{\rho(1+\rho)}{2\sqrt{-\rho}}\big(\psi(m_\rho z + w) - \psi(n_\rho z + w)\big)\Bigg) \end{aligned}$$

where $\sqrt{-\rho}$ is any square root of $-\rho$,

$$m_\rho = \frac{\sqrt{-\rho} - \rho}{\rho(1+\rho)} , \quad n_\rho = -\frac{\sqrt{-\rho} + \rho}{\rho(1+\rho)} ,$$

and φ, $\psi \in \mathbb{C}[\![\zeta]\!]$ are power series of order at least 3;

(vii) *if $F_2(z,w) = \big(\rho(-z^2 + zw), (1-\rho)(zw - w^2)\big)$ with $\rho \neq 0$, 1 then F is formally conjugated to a unique infinite order normal form*

$$\begin{aligned} G(z,w) =& F_2(z,w) \\ &+ \left(z\frac{\partial}{\partial z}\big[\varphi(z+w) + \psi(z+w)\big] - \varphi(z+w),\right. \\ &\left.\frac{\rho-1}{\rho}\left(z\frac{\partial}{\partial z}\big[\varphi(z+w) - \psi(z+w)\big] - 3\varphi(z+w) + 2\psi(z+w)\right)\right) \end{aligned}$$

where φ, $\psi \in \mathbb{C}[\![\zeta]\!]$ are power series of order at least 3;

(ix) *if $F_2(z,w) = (-z^2, -w^2)$ then F is formally conjugated to a unique infinite order normal form*

$$G(z,w) = F_2(z,w) + \big(\varphi(w), \psi(z)\big)$$

where φ, $\psi \in \mathbb{C}[\![\zeta]\!]$ are power series of order at least 3;

(x) *if $F_2(z,w) = (-\rho z^2, (1-\rho)zw - w^2)$ with $\rho \neq 0$, 1 then F is formally conjugated to a unique infinite order normal form*

$$G(z,w) = F_2(z,w) + \left(\varphi(w) + \frac{(1-\rho)^2}{4\rho}\psi\left(\frac{2}{1-\rho}z + w\right), \psi\left(\frac{2}{1-\rho}z + w\right)\right)$$

where φ, $\psi \in \mathbb{C}[\![\zeta]\!]$ are power series of order at least 3;

(xi) *if* $F_2(z,w) = (-\rho z^2 + (1-\tau)zw, (1-\rho)zw - \tau w^2)$ *with* ρ, $\tau \neq 0, 1$ *and* $\rho + \tau \neq 1$ *then* F *is formally conjugated to a unique infinite order normal form*

$$\begin{aligned} G(z,w) = F_2(z,w) + \Bigg(&\frac{\tau}{\rho}\Bigg[\frac{\sqrt{\rho+\tau-1}+\sqrt{\rho\tau}}{2m_{\rho,\tau}^2}\varphi(m_{\rho,\tau}z+w) \\ &\quad + \frac{\sqrt{\rho+\tau-1}-\sqrt{\rho\tau}}{2n_{\rho,\tau}^2}\varphi(n_{\rho,\tau}z+w) \\ &\quad + \frac{1}{m_{\rho,\tau}^2}\psi(m_{\rho,\tau}z+w) - \frac{1}{n_{\rho,\tau}^2}\psi(n_{\rho,\tau}z+w)\Bigg], \\ &\frac{\sqrt{\rho+\tau-1}+\sqrt{\rho\tau}}{2}\varphi(m_{\rho,\tau}z+w) \\ &+ \frac{\sqrt{\rho+\tau-1}-\sqrt{\rho\tau}}{2}\varphi(n_{\rho,\tau}z+w) \\ &+ \psi(m_{\rho,\tau}z+w) - \psi(n_{\rho,\tau}z+w)\Bigg) , \end{aligned}$$

where

$$m_{\rho,\tau} = \frac{\sqrt{\rho\tau}\sqrt{\rho+\tau-1}-\rho\tau}{\rho(\rho-1)} , \qquad n_{\rho,\tau} = -\frac{\sqrt{\rho\tau}\sqrt{\rho+\tau-1}+\rho\tau}{\rho(\rho-1)} .$$

and φ, $\psi \in \mathbb{C}[\![\zeta]\!]$ *are power series of order at least 3.*

Notice that the uniqueness of the infinite normal form implies that the power series φ and ψ appearing in this statement are formal invariants of the original map, in stark contrast with the one-dimensional case where the only formal invariant is the degree of the first non-linear non-vanishing term. Furthermore, the possibility of expressing the normal forms by using only two power series of one variables (and their derivatives) comes from the use of Fischer Hermitian product, which is particularly suited to this aim; other choices of complementary subspaces would lead to much more involved normal forms.

In [1] we showed that the quadratic terms considered in Theorem 1.1 form an almost complete list of all possible quadratic terms up to linear change of coordinates; the only missing possibilities are four degenerate cases where one of the coordinates is identically zero. In these remaining cases we shall anyway be able to compute a second order normal form:

Proposition 1.2.. *Let* $F \in \widehat{\mathcal{O}}_O^2$ *be of the form* $F(z,w) = F_2(z,w) + O_3$*. Then:*

(i) *if* $F_2(z,w) = (0, -z^2)$ *then* F *is formally conjugated to a unique second order normal form*

$$G(z,w) = F_2(z,w) + \big(\Phi(z,w), \psi(w)\big) ,$$

where $\psi \in \mathbb{C}[\![\zeta]\!]$ *and* $\Phi \in \mathbb{C}[\![z,w]\!]$ *are power series of order at least 3;*

(ii) *if* $F_2(z,w) = (0, zw)$ *then* F *is formally conjugated to a unique second order normal form*

$$G(z,w) = F_2(z,w) + \big(\Phi(z,w), 0\big) ,$$

where $\Phi \in \mathbb{C}[\![z,w]\!]$ *is a power series of order at least 3;*

(iii) *if* $F_2(z,w) = (-z^2, 0)$ *then* F *is formally conjugated to a unique second order normal form*

$$G(z,w) = F_2(z,w) + \big(\psi(w), \Phi(z,w)\big) ,$$

where $\psi \in \mathbb{C}[\![\zeta]\!]$ *and* $\Phi \in \mathbb{C}[\![z,w]\!]$ *are power series of order at least 3;*

(iv) *if* $F_2(z,w) = (z^2 - zw, 0)$ *then* F *is formally conjugated to a unique second order normal form*

$$G(z,w) = F_2(z,w) + \big(0, \Phi(z,w)\big) ,$$

where $\Phi \in \mathbb{C}[\![z,w]\!]$ *is a power series of order at least 3.*

Finally, in Section 3 we shall also discuss a few interesting examples with $\Lambda = I$, displaying in particular the appearance of non-trivial second-order resonance phenomena. For instance, we shall prove the following

Proposition 1.3.. *Let* $F \in \widehat{\mathcal{O}}_I^2$ *be of the form* $F(z,w) = (z,w) + F_2(z,w) + O_3$, *with*

$$F_2(z,w) = \big(-\rho z^2, (1-\rho)zw\big)$$

and $\rho \neq 0$. *Put*

$$\mathcal{E} = \big([0,1] \cap \mathbb{Q}\big) \cup \left\{ -\frac{1}{n} \;\middle|\; n \in \mathbb{N}^* \right\}$$

and

$$\mathcal{F} = \big([0,1] \cap \mathbb{Q}\big) \cup \left\{ 1 + \frac{1}{n}, 1 + \frac{2}{n} \;\middle|\; n \in \mathbb{N}^* \right\} .$$

Then:

(i) *if* $\rho \notin \mathcal{E} \cup \mathcal{F}$ *then* F *is formally conjugated to a unique second order normal form*

$$\begin{aligned} G(z,w) = {} & (z,w) + F_2(z,w) \\ & + \big(az^3 + \varphi(w) + (1-\rho)z\psi'(w), (1-\rho)w\psi'(w) + (3\rho - 1)\psi(z)\big), \end{aligned}$$

where φ, $\psi \in \mathbb{C}[\![\zeta]\!]$ *are power series of order at least 3, and* $a \in \mathbb{C}$*;*

(ii) *if* $\rho = 1 + \frac{1}{n} \in \mathcal{F} \setminus \mathcal{E}$ *then* F *is formally conjugated to a unique second order normal form*

$$\begin{aligned} G(z,w) = {} & (z,w) + F_2(z,w) \\ & + \left(a_0 z^3 + a_1 z^2 w^{n+1} + \varphi(w) - \frac{1}{n} z\psi'(w), \right. \\ & \left. - \frac{1}{n} w\psi'(w) + \left(2 + \frac{3}{n}\right)\psi(w) \right) , \end{aligned}$$

where φ, $\psi \in \mathbb{C}[\![\zeta]\!]$ *are power series of order at least 3, and* a_0, $a_1 \in \mathbb{C}$*;*

(iii) *if* $\rho = 1 + \frac{2}{m} \in \mathcal{F} \setminus \mathcal{E}$ *with* m *odd then* F *is formally conjugated to a unique second order normal form*

$$\begin{aligned} G(z,w) = {} & (z,w) + F_2(z,w) \\ & + \left(a_0 z^3 + \varphi(w) - \frac{2}{m} z\big(w\psi'(w) + \psi(w)\big), \right. \\ & \left. -\frac{2}{m} w^2\psi'(w) + \left(2 + \frac{4}{m}\right) w\psi(w) \right) , \end{aligned}$$

where φ, $\psi \in \mathbb{C}[\![\zeta]\!]$ are power series of order at least respectively 3 and 2, and $a_0 \in \mathbb{C}$;

(iv) *if $\rho = -\frac{1}{n} \in \mathcal{E} \setminus \mathcal{F}$ then F is formally conjugated to a unique second order normal form*

$$\begin{aligned} G(z,w) = (z,w) &+ F_2(z,w) \\ &+ \left(a_0 z^3 + \varphi(w) + \left(1 + \frac{1}{n}\right) z\big(w\psi'(w) + \psi(w)\big), \right. \\ &\qquad \left. a_1 z^{n+2} + \psi(z) + \left(1 + \frac{1}{n}\right) w^2\psi'(w) - \frac{2}{n} w\psi(w) \right) , \end{aligned}$$

where φ, $\psi \in \mathbb{C}[\![\zeta]\!]$ are power series of order at least respectively 3 and 2, and a_0 $a_1 \in \mathbb{C}$;

(v) *if $\rho = 1 \in \mathcal{E} \cap \mathcal{F}$ then F is formally conjugated to a unique second order normal form*

$$G(z,w) = (z,w) + F_2(z,w) + \big(\varphi_1(w) + z^3\psi(w), \varphi_2(w) + z\varphi_3(w)\big) \ ,$$

where φ_1, $\varphi_2 \in \mathbb{C}[\![\zeta]\!]$ are power series of order at least 3, $\varphi_2 \in \mathbb{C}[\![\zeta]\!]$ is a power series of order at least 2, and $\varphi_3 \in \mathbb{C}[\![\zeta]\!]$ is a power series;

(vi) *if $\rho = a/b \in (0,1) \cap \mathbb{Q} \subset \mathcal{E} \setminus \mathcal{F}$ then F is formally conjugated to a unique second order normal form*

$$\begin{aligned} G(z,w) = (z,w) &+ F_2(z,w) \\ &+ \left(\varphi(w) + z^3\varphi_0(z^{b-a}w^a) + (b-a)\frac{\partial}{\partial w}\big(z^2 w\chi(z^{b-a}w^a)\big) \right. \\ &\qquad + \left(1 - \frac{a}{b}\right) z\big(w\psi'(w) + \psi(w)\big), \\ &\qquad \left. a\frac{\partial}{\partial z}\big(z^2 w\chi(z^{b-a}w^a)\big) + \left(1 - \frac{a}{b}\right) w^2\psi'(w) + 2\frac{a}{b} w\psi(w) \right) , \end{aligned}$$

where φ, $\psi \in \mathbb{C}[\![\zeta]\!]$ are power series of order at least 3, and φ_0, $\chi \in \mathbb{C}[\![\zeta]\!]$ are power series of order at least 1.

In future papers we plan to study the dynamics of the normal forms we obtained, and to discuss the convergence of the normalizing transformations.

2. Renormalization. In this section we shall present the renormalization procedure for formal transformations, concentrating on the parts that will be useful for our aims. One of the main differences between our approach and the one followed by Wang, Zheng and Peng is that we shall systematically use the relations between homogeneous polynomials and symmetric multilinear maps instead of relying on higher order derivatives as in [38].

Let us start collecting a few results on homogeneous polynomials and maps we shall need later.

Definition 2.1. We shall denote by $\mathcal{H}^d$ the space of *homogenous maps of degree d*, i.e., of n-tuples of homogeneous polynomials of degree $d \geq 1$ in the variables $(z_1, \dots, z_n)$. It is well known (see, e.g., [17, pp. 79–88]) that to each $P \in \mathcal{H}^d$ is associated a *unique* symmetric multilinear map $\tilde{P} \colon (\mathbb{C}^n)^d \to \mathbb{C}^n$ such that

$$P(z) = \tilde{P}(z, \dots, z)$$

for all $z \in \mathbb{C}^n$. We also set $\mathcal{H} = \prod\limits_{d \geq 2} \mathcal{H}^d$.

Roughly speaking, the symmetric multilinear map associated to a homogeneous map H encodes the derivatives of H. For instance, it is easy to check that for each $H \in \mathcal{H}^d$ we have

$$(\operatorname{Jac} H)(z) \cdot v = d\,\tilde{H}(v, z, \ldots, z) \tag{2.1}$$

for all z, $v \in \mathbb{C}^n$.

Later on we shall need to compute the multilinear map associated to a homogeneous map obtained as a composition. The formula we are interested in is contained in the next lemma.

Lemma 2.2. *Assume that $P \in \mathcal{H}^d$ is of the form*

$$P(z) = \tilde{K}\big(H_{d_1}(z), \ldots, H_{d_r}(z)\big) ,$$

where $\tilde{K}$ is r-multilinear, $d_1 + \cdots + d_r = d$, and $H_{d_j} \in \mathcal{H}^{d_j}$ for $j = 1, \ldots, r$. Then

$$\tilde{P}(v, w, \ldots, w) = \frac{1}{d} \sum_{j=1}^{r} d_j \tilde{K}\big(H_{d_1}(w), \ldots, \tilde{H}_{d_j}(v, w, \ldots, w), \ldots, H_{d_r}(w)\big)$$

for all v, $w \in \mathbb{C}^n$.

Proof. Write $z = w + \varepsilon v$. Then

$$\begin{aligned}
P(w) &+ d\varepsilon \tilde{P}(v, w, \ldots, w) + O(\varepsilon^2) \\
&= P(w + \varepsilon v) \\
&= \tilde{K}\big(\tilde{H}_{d_1}(w + \varepsilon v, \ldots, w + \varepsilon v), \ldots, \tilde{H}_{d_r}(w + \varepsilon v, \ldots, w + \varepsilon v)\big) \\
&= \tilde{K}\big(H_{d_1}(w), \ldots, H_{d_r}(w)\big) \\
&\quad + \varepsilon \sum_{j=1}^{r} d_j \tilde{K}\big(H_{d_1}(w), \ldots, \tilde{H}_{d_j}(v, w, \ldots, w), \ldots, H_{d_r}(w)\big) + O(\varepsilon^2) ,
\end{aligned}$$

and we are done. □

Definition 2.3. Given a linear map $\Lambda \in M_{n,n}(\mathbb{C})$, we define a linear operator $L_\Lambda \colon \mathcal{H} \to \mathcal{H}$ by setting

$$L_\Lambda(H) = H \circ \Lambda - \Lambda H .$$

We shall say that a homogeneous map $H \in \mathcal{H}^d$ is Λ-*resonant* if $L_\Lambda(H) = O$, and we shall denote by $\mathcal{H}^d_\Lambda = \operatorname{Ker} L_\Lambda \cap \mathcal{H}^d$ the subspace of Λ-resonant homogeneous maps of degree d. Finally, we set $\mathcal{H}_\Lambda = \prod\limits_{d \geq 2} \mathcal{H}^d_\Lambda$.

When Λ is diagonal, then the Λ-resonant monomials are exactly the resonant monomials appearing in the classical Poincaré-Dulac theory.

Definition 2.4. If $Q = (q_1, \ldots, q_n) \in \mathbb{N}^n$ is a multi-index and $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, we shall put $z^Q = z_1^{q_1} \cdots z_n^{q_n}$. Given $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n) \in M_{n,n}(\mathbb{C})$, we shall say that $Q \in \mathbb{N}^n$ with $q_1 + \cdots + q_n \geq 2$ is Λ-*resonant on the j-th coordinate* if

$\lambda_1^{q_1}\cdots\lambda_n^{q_n}=\lambda_j$. If Q is Λ-resonant on the j-th coordinate, we shall also say that the monomial $z^Q e_j$ is Λ-*resonant*, where $\{e_1,\dots,e_n\}$ is the canonical basis of $\mathbb{C}^n$.

Remark 1. If $\Lambda=\mathrm{diag}(\lambda_1,\dots,\lambda_n)\in M_{n,n}(\mathbb{C})$ is diagonal, and $z^Q e_j\in\mathcal{H}^d$ is a homogeneous monomial (with $q_1+\cdots+q_n=d$), then (identifying the matrix Λ with the vector, still denoted by Λ, of its diagonal entries) we have

$$L_\Lambda(z^Q e_j)=(\Lambda^Q-\lambda_j)z^Q e_j\ .$$

Therefore $z^Q e_j$ is Λ-resonant if and only if Q is Λ-resonant in the j-th coordinate. In particular, a basis of $\mathcal{H}^d_\Lambda$ is given by the Λ-resonant monomials, and we have

$$\mathcal{H}^d=\mathcal{H}^d_\Lambda\oplus\mathrm{Im}\,L_\Lambda|_{\mathcal{H}^d}$$

for all $d\geq 2$.

It is possible to detect the Λ-resonance by using the associated multilinear map:

Lemma 2.5. *If $\Lambda\in M_{n,n}(\mathbb{C})$ and $H\in\mathcal{H}^d$ then H is Λ-resonant if and only if*

$$\tilde H(\Lambda v_1,\dots,\Lambda v_d)=\Lambda\tilde H(v_1,\dots,v_d) \tag{2.2}$$

for all $v_1,\dots,v_d\in\mathbb{C}^n$. In particular, if $H\in\mathcal{H}^d_\Lambda$ then

$$\big((\mathrm{Jac}\,H)\circ\Lambda\big)\cdot\Lambda=\Lambda\cdot(\mathrm{Jac}\,H)\ . \tag{2.3}$$

Proof. One direction is trivial. Conversely, assume $H\in\mathcal{H}^d_\Lambda$. By definition, H is Λ-resonant if and only if $\tilde H(\Lambda w,\dots,\Lambda w)=\Lambda\tilde H(w,\dots,w)$ for all $w\in\mathbb{C}^n$. Put $w=z+\varepsilon v_1$; then

$$\begin{aligned}\tilde H(\Lambda z,\dots,\Lambda z)&+\varepsilon d\,\tilde H(\Lambda v_1,\Lambda z,\dots,\Lambda z)+O(\varepsilon^2)\\&=\tilde H\big(\Lambda(z+\varepsilon v_1),\dots,\Lambda(z+\varepsilon v_1)\big)\\&=\Lambda\tilde H(z+\varepsilon v_1,\dots,z+\varepsilon v_1)\\&=\Lambda\tilde H(z,\dots,z)+\varepsilon d\,\Lambda\tilde H(v_1,z,\dots,z)+O(\varepsilon^2)\ ,\end{aligned}$$

and thus

$$\tilde H(\Lambda v_1,\Lambda z,\dots,\Lambda z)=\Lambda\tilde H(v_1,z,\dots,z)\ ; \tag{2.4}$$

in particular (2.3) is a consequence of (2.1).

Now put $z=z_1+\varepsilon v_2$ in (2.4). We get

$$\begin{aligned}\tilde H(\Lambda v_1,\Lambda z_1,\dots,\Lambda z_1)&+\varepsilon(d-1)\tilde H(\Lambda v_1,\Lambda v_2,\Lambda z_1,\dots,\Lambda z_1)+O(\varepsilon^2)\\&=\tilde H\big(\Lambda v_1,\Lambda(z_1+\varepsilon v_2),\dots,\Lambda(z_1+\varepsilon v_2)\big)\\&=\Lambda\tilde H(v_1,z_1+\varepsilon v_2,\dots,z_1+\varepsilon v_2)\\&=\Lambda\tilde H(v_1,z_1,\dots,z_1)\\&\quad+\varepsilon(d-1)\Lambda\tilde H(v_1,v_2,z,\dots,z)+O(\varepsilon^2)\ ,\end{aligned}$$

and hence

$$\tilde{H}(\Lambda v_1, \Lambda v_2, \Lambda z_1, \dots, \Lambda z_1) = \Lambda \tilde{H}(v_1, v_2, z_1, \dots, z_1)$$

for all v_1, v_2, $z_1 \in \mathbb{C}^n$. Proceeding in this way we get (2.2). □

We now introduce the operator needed for the second order normalization.

Definition 2.6. Given $P \in \mathcal{H}^\mu$ and $\Lambda \in M_{n,n}(\mathbb{C})$, let $L_{P,\Lambda}\colon \mathcal{H}^d \to \mathcal{H}^{d+\mu-1}$ be given by

$$L_{P,\Lambda}(H)(z) = d\,\tilde{H}\big(P(z), \Lambda z, \dots, \Lambda z\big) - \mu \tilde{P}\big(H(z), z, \dots, z\big)\ .$$

Remark 2. Equation (2.1) implies that

$$d\,\tilde{H}\big(P(z), \Lambda z, \dots, \Lambda z\big) = (\operatorname{Jac} H)(\Lambda z) \cdot P(z)\ .$$

Therefore

$$L_{P,\Lambda}(H) = \big((\operatorname{Jac} H) \circ \Lambda\big) \cdot P - (\operatorname{Jac} P) \cdot H\ ;$$

In the notations of [38] we have $L_{P,\Lambda}(H) = [H, P]$, and $L_{P,\Lambda}|_{\mathcal{H}^d_\Lambda} = \mathcal{T}_d[P]$ when $P \in \mathcal{H}^\mu_\Lambda$.

Using multilinear maps it is easy to prove the following useful fact:

Lemma 2.7. *Take $\Lambda \in M_{n,n}(\mathbb{C})$ and $P \in \mathcal{H}^\mu_\Lambda$. Then $L_{P,\Lambda}(\mathcal{H}^d_\Lambda) \subseteq \mathcal{H}^{d+\mu-1}_\Lambda$ for all $d \geq 2$.*

Proof. Using Lemma 2.5 and the definition of $L_{P,\Lambda}$, if $H \in \mathcal{H}^d_\Lambda$ we get

$$\begin{aligned} L_{P,\Lambda}(H)(\Lambda z) &= d\,\tilde{H}\big(P(\Lambda z), \Lambda^2 z, \dots, \Lambda^2 z\big) - \mu \tilde{P}\big(H(\Lambda z), \Lambda z, \dots, \Lambda z\big) \\ &= d\,\tilde{H}\big(\Lambda P(z), \Lambda^2 z, \dots, \Lambda^2 z\big) - \mu \tilde{P}\big(\Lambda H(z), \Lambda z, \dots, \Lambda z\big) \\ &= d\,\Lambda \tilde{H}\big(P(z), \Lambda z, \dots, \Lambda z\big) - \mu \Lambda \tilde{P}\big(H(z), z, \dots, z\big) \\ &= \Lambda L_{P,\Lambda}(H)(z)\ . \end{aligned}$$

□

To state and prove the main technical result of this section we fix a few more notations.

Definition 2.8. We shall denote by $\widehat{\mathcal{O}}^n = \prod_{d \geq 1} \mathcal{H}^d$ the space of n-tuples of formal power series with vanishing constant term. Furthermore, given $\Lambda \in M_{n,n}(\mathbb{C})$ we shall denote by $\widehat{\mathcal{O}}^n_\Lambda$ the subset of $F \in \widehat{\mathcal{O}}^n$ with $dF_O = \Lambda$. Every $F \in \widehat{\mathcal{O}}^n$ can be written in a unique way as a formal sum

$$F = \sum_{d \geq 1} F_d \tag{2.5}$$

with $F_d \in \mathcal{H}^d$; formula (2.5) is the *homogeneous expansion* of F, and F_d is the *d-homogeneous term* of F. We shall often write $\{F\}_d$ for F_d. In particular, if $F \in \widehat{\mathcal{O}}^n_\Lambda$ then $\{F\}_1 = \Lambda$.

The homogeneous terms behave in a predictable way with respect to composition and inverse: indeed it is easy to see that if $F = \sum\limits_{d\geq 1} F_d$ and $G = \sum\limits_{d\geq 1} G_d$ are two elements of $\widehat{\mathcal{O}}^n$ then

$$\{F \circ G\}_d = \sum_{\substack{1\leq r\leq d\\ d_1+\cdots+d_r=d}} \tilde{F}_r(G_{d_1}, \dots , G_{d_r}) \tag{2.6}$$

for all $d \geq 1$; and that if $\Phi = I + \sum\limits_{d\geq 2} H_d$ belongs to $\widehat{\mathcal{O}}^n_I$ then the homogeneous expansion of the inverse transformation $\Phi^{-1} = I + \sum\limits_{d\geq 2} K_d$ is given by

$$K_d = -H_d - \sum_{\substack{2\leq r\leq d-1\\ d_1+\cdots+d_r=d}} \tilde{K}_r(H_{d_1}, \dots , H_{d_r}) \tag{2.7}$$

for all $d \geq 2$. In particular we have

Lemma 2.9. *Take $\Phi = I + \sum\limits_{d\geq 2} H_d \in \widehat{\mathcal{O}}^n_I$, and let $\Phi^{-1} = I + \sum\limits_{d\geq 2} K_d$ be the homogeneous expansion of the inverse. Then if $H_2, \dots , H_d$ are Λ-resonant for some $d \geq 2$ and $\Lambda \in M_{n,n}(\mathbb{C})$ then K_d also is Λ-resonant.*

Proof. We argue by induction. Assume that $H_2, \dots , H_d$ are Λ-resonant. If $d = 2$ then $K_2 = -H_2$ and thus K_2 is clearly Λ-resonant. Assume the assertion true for $d - 1$; in particular, $K_2, \dots , K_{d-1}$ are Λ-resonant. Then

$$\begin{aligned} K_d \circ \Lambda &= -H_d \circ \Lambda - \sum_{\substack{2\leq r\leq d-1\\ d_1+\cdots+d_r=d}} \tilde{K}_r(H_{d_1} \circ \Lambda, \dots , H_{d_r} \circ \Lambda) \\ &= \Lambda H_d - \sum_{\substack{2\leq r\leq d-1\\ d_1+\cdots+d_r=d}} \tilde{K}_r(\Lambda H_{d_1}, \dots , \Lambda H_{d_r}) = \Lambda K_d \end{aligned}$$

because $K_2, \dots , K_{d-1}$ are Λ-resonant (and we are using Lemma 2.5). □

Definition 2.10. Given $\Lambda \in M_{n,n}(\mathbb{C})$, we shall say that $F \in \widehat{\mathcal{O}}^n$ is *Λ-resonant* if $F \circ \Lambda = \Lambda F$. Clearly, F is Λ-resonant if and only if $\{F\}_d \in \mathcal{H}^d_\Lambda$ for all $d \in \mathbb{N}$.

The main technical result of this section is the following:

Theorem 2.11. *Given $F \in \widehat{\mathcal{O}}^n_\Lambda$, let $F = \Lambda + \sum\limits_{d\geq\mu} F_d$ be its homogeneous expansion, with $F_\mu \neq O$. Then for every $\Phi \in \widehat{\mathcal{O}}^n_I$ with homogeneous expansion $\Phi = I + \sum\limits_{d\geq 2} H_d$ and every $\nu \geq 2$ we have*

$$\{\Phi^{-1} \circ F \circ \Phi\}_\nu = F_\nu - L_\Lambda(H_\nu) - L_{F_\mu,\Lambda}(H_{\nu-\mu+1}) + Q_\nu + R_\nu\ , \tag{2.8}$$

where Q_ν depends only on Λ and on H_γ with $\gamma < \nu$, while R_ν depends only on F_ρ with $\rho < \nu$ and on H_γ with $\gamma < \nu-\mu+1$, and we put $L_{F_\mu,\Lambda}(H_1) = O$. Furthermore, we have:

(i) *if $H_2, \dots , H_{\nu-1} \in \mathcal{H}_\Lambda$ then $Q_\nu = O$; in particular, if Φ is Λ-resonant then $L_\Lambda(H_\nu) = Q_\nu = O$ for all $\nu \geq 2$;*

(ii) *if Φ is Λ-resonant then* $\{\Phi^{-1}\circ F\circ\Phi\}_\nu = O$ *for* $2\le\nu<\mu$, $\{\Phi^{-1}\circ F\circ\Phi\}_\mu = F_\mu$, *and*

$$\{\Phi^{-1}\circ F\circ\Phi\}_{\mu+1} = F_{\mu+1} - L_{F_\mu,\Lambda}(H_2)\ ;$$

(iii) *if* $F=\Lambda$ *then* $R_\nu = O$ *for all* $\nu\ge 2$*;*

(iv) *if* $F_2,\dots,F_{\nu-1}$ *and* $H_2,\dots,H_{\nu-\mu}$ *are Λ-resonant then* R_ν *is Λ-resonant.*

Proof. Using twice (2.6) we get

$$\begin{aligned}
&\{\Phi^{-1}\circ F\circ\Phi\}_\nu\\
&\quad=\sum_{\substack{1\le s\le\nu\\ \nu_1+\cdots+\nu_s=\nu}}\tilde K_s(\{F\circ\Phi\}_{\nu_1},\dots,\{F\circ\Phi\}_{\nu_s})\\
&\quad=\sum_{\substack{1\le s\le\nu\\ \nu_1+\cdots+\nu_s=\nu}}\ \sum_{\substack{1\le j\le s\\ 1\le r_j\le\nu_j\\ d_{j1}+\cdots+d_{jr_j}=\nu_j}}\tilde K_s\bigl(\tilde F_{r_1}(H_{d_{11}},\dots,H_{d_{1r_1}}),\dots,\tilde F_{r_s}(H_{d_{s1}},\dots,H_{d_{sr_s}})\bigr)\\
&\quad=T_\nu+S_1(\nu)+\sum_{s\ge2}S_s(\nu)\ ,
\end{aligned}$$

where $\Phi^{-1}=I+\sum_{d\ge2}K_d$ is the homogeneous expansion of Φ^{-1}, and:

$$T_\nu=\sum_{\substack{1\le s\le\nu\\ \nu_1+\cdots+\nu_s=\nu}}\tilde K_s(\Lambda H_{\nu_1},\dots,\Lambda H_{\nu_s})$$

is obtained considering only the terms with $r_1=\cdots=r_s=1$;

$$S_1(\nu)=\sum_{\substack{\mu\le r\le\nu\\ d_1+\cdots+d_r=\nu}}\tilde F_r(H_{d_1},\dots,H_{d_r})$$

contains the terms with $s=1$ and $r_1>1$; and

$$\begin{aligned}
&S_s(\nu)\\
&=\sum_{\nu_1+\cdots+\nu_s=\nu}\ \sum_{\substack{1\le j\le s\\ 1\le r_j\le\nu_j\\ \max\{r_1,\dots,r_s\}\ge\mu}}\ \sum_{\substack{1\le j\le s\\ d_{j1}+\cdots+d_{jr_j}=\nu_j}}\tilde K_s\bigl(\tilde F_{r_1}(H_{d_{11}},\dots,H_{d_{1r_1}}),\dots,\tilde F_{r_s}(H_{d_{s1}},\dots,H_{d_{sr_s}})\bigr)
\end{aligned}$$

contains the terms with fixed $s\ge2$ and at least one r_j greater than 1 (and thus greater than or equal to μ, because $F_2=\ldots=F_{\mu-1}=O$ by assumption).

Let us first study T_ν. The summand corresponding to $s=1$ is ΛH_ν; the summand corresponding to $s=\nu$ is $K_\nu\circ\Lambda$; therefore

$$T_\nu=\Lambda H_\nu+K_\nu\circ\Lambda+\sum_{\substack{2\le s\le\nu-1\\ \nu_1+\cdots+\nu_s=\nu}}\tilde K_s(\Lambda H_{\nu_1},\dots,\Lambda H_{\nu_s})=-L_\Lambda(H_\nu)+Q_\nu\ ,$$

where, using (2.7) to express K_ν,

$$Q_\nu=\sum_{\substack{2\le s\le\nu-1\\ \nu_1+\cdots+\nu_s=\nu}}\Bigl[\tilde K_s(\Lambda H_{\nu_1},\dots,\Lambda H_{\nu_s})-\tilde K_s(H_{\nu_1}\circ\Lambda,\dots H_{\nu_s}\circ\Lambda)\Bigr]$$

depends only on Λ and H_γ with $\gamma < \nu$ because $2 \le s \le \nu - 1$ in the sum. In particular, if $H_1, \dots, H_{\nu-1} \in \mathcal{H}_\Lambda$ then $Q_\nu = O$, and (i) is proved.

Now let us study $S_1(\nu)$. First of all, we clearly have $S_1(\nu) = O$ for $2 \le \nu < \mu$, and $S_1(\mu) = F_\mu$. When $\nu > \mu$ we can write

$$\begin{aligned} S_1(\nu) &= F_\nu + \sum_{\substack{\mu \le r \le \nu-1 \\ d_1+\cdots+d_r=\nu}} \tilde{F}_r(H_{d_1}, \dots, H_{d_r}) \\ &= F_\nu + \mu \tilde{F}_\mu(H_{\nu-\mu+1}, I, \dots, I) \\ &\quad + \sum_{\substack{d_1+\cdots+d_\mu=\nu \\ 1<\max\{d_j\}<\nu-\mu+1}} \tilde{F}_\mu(H_{d_1}, \dots, H_{d_\mu}) + \sum_{\substack{\mu+1 \le r \le \nu-1 \\ d_1+\cdots+d_r=\nu}} \tilde{F}_r(H_{d_1}, \dots, H_{d_r}) \,. \end{aligned}$$

in particular, $S_1(\mu+1) = F_{\mu+1} + \mu \tilde{F}_\mu(H_2, I, \dots, I)$. Notice that the two remaining sums depend only on F_ρ with $\rho < \nu$ and on H_γ with $\gamma < \nu - \mu + 1$ (in the first sum is clear; for the second one, if $d_j \ge \nu - \mu + 1$ for some j we then would have $d_1 + \cdots + d_r \ge \nu - \mu + 1 + r - 1 \ge \nu + 1$, impossible). Summing up we have

$$S_1(\nu) = \begin{cases} O & \text{for } 2 \le \nu < \mu, \\ F_\mu & \text{for } \nu = \mu, \\ F_{\mu+1} + \mu \tilde{F}_\mu(H_2, I, \dots, I) & \text{for } \nu = \mu + 1, \\ F_\nu + \mu \tilde{F}_\mu(H_{\nu-\mu+1}, I, \dots, I) + R^1_\nu & \text{for } \nu > \mu + 1, \end{cases}$$

where

$$R^1_\nu = \sum_{\substack{d_1+\cdots+d_\mu=\nu \\ 1<\max\{d_j\}<\nu-\mu+1}} \tilde{F}_\mu(H_{d_1}, \dots, H_{d_\mu}) + \sum_{\substack{\mu+1 \le r \le \nu-1 \\ d_1+\cdots+d_r=\nu}} \tilde{F}_r(H_{d_1}, \dots, H_{d_r})$$

depends only on F_ρ with $\rho < \nu$ and on H_γ with $\gamma < \nu - \mu + 1$.

Let us now discuss $S_s(\nu)$ for $s \ge 2$. First of all, the condition $\max\{r_1, \dots, r_s\} \ge \mu$ implies

$$\mu + s - 1 \le r_1 + \cdots + r_s \le \nu_1 + \cdots + \nu_s = \nu \,,$$

that is $s \le \nu - \mu + 1$. In particular, $S_s(\nu) = O$ if $\nu \le \mu$ or if $s > \nu - \mu + 1$. Moreover, if we had $d_{ij} \ge \nu - \mu + 1$ for some $1 \le i \le s$ and $1 \le j \le r_s$ we would get

$$\nu = d_{11} + \cdots + d_{sr_s} \ge \nu - \mu + 1 + r_1 + \cdots + r_s - 1 \ge \nu - \mu + 1 + \mu + s - 1 - 1 = \nu + s - 1 > \nu \,,$$

impossible. This means that $S_s(\nu)$ depends only on F_ρ with $\rho < \nu$ for all s, on H_γ with $\gamma < \nu - \mu + 1$ when $s < \nu - \mu + 1$, and that $S_{\nu-\mu+1}(\nu)$ depends on $H_{\nu-\mu+1}$ just because it contains $\tilde{K}_{\nu-\mu+1}$. Furthermore, the conditions $\max\{r_1, \dots, r_{\nu-\mu+1}\} \ge \mu$ and $\nu_1 + \ldots + \nu_{\nu-\mu+1} = \nu$ imply that

$$\begin{aligned} S_{\nu-\mu+1}(\nu) &= (\nu - \mu + 1) \tilde{K}_{\nu-\mu+1}(F_\mu, \Lambda, \dots, \Lambda) \\ &= -(\nu - \mu + 1) \tilde{H}_{\nu-\mu+1}(F_\mu, \Lambda, \dots, \Lambda) + R^2_\nu \,, \end{aligned}$$

where (using (2.7) and Lemma 2.2)

$$R^2_\nu = \sum_{\substack{2 \le r \le \nu-\mu \\ d_1+\cdots+d_r=\nu-\mu+1}} \sum_{j=1}^{r} d_j \tilde{K}_r\big(H_{d_1} \circ \Lambda, \dots, \tilde{H}_{d_j}(F_\mu, \Lambda, \dots, \Lambda), \dots, H_{d_r} \circ \Lambda\big)$$

depends only on Λ, F_μ and H_γ with $\gamma < \nu - \mu + 1$.

Putting everything together, we have

$$\begin{aligned}\{\Phi^{-1}\circ F\circ\Phi\}_\nu &= T_\nu + S_1(\nu) + \sum_{s=2}^{\nu-\mu+1} S_s(\nu)\\ &= F_\nu - L_\Lambda(H_\nu) + Q_\nu + \begin{cases} O & \text{if } 2\le\nu\le\mu,\\ -L_{F_\mu,\Lambda}(H_2) & \text{if } \nu=\mu+1,\\ -L_{F_\mu,\Lambda}(H_{\nu-\mu+1}) + R_\nu & \text{if } \nu>\mu+1,\end{cases}\end{aligned}$$

where

$$R_\nu = R_\nu^1 + R_\nu^2 + \sum_{s=2}^{\nu-\mu} S_s(\nu)$$

depends only on F_ρ with $\rho < \mu$ and on H_γ with $\gamma < \nu - \mu + 1$. In particular, if $F = \Lambda$ then we have $S_s(\nu) = O$ for all $s \ge 1$ and hence $R_\nu = O$ for all $\nu \ge 2$.

In this way we have proved (2.8) and parts (i), (ii) and (iii). Concerning (iv), it suffices to notice that if $F_2, \dots, F_{\nu-1}$ and $H_2, \dots, H_{\nu-\mu+1}$ are Λ-resonant, then also R_ν^1, $S_2(\nu), \dots, S_{\nu-\mu}(\nu)$ and R_ν^2 (by Lemmas 2.5 and 2.9) are Λ-resonant. □

Remark 3. In [38, Theorem 2.4] the remainder term R_ν is expressed by using combinations of higher order derivatives instead of combinations of multilinear maps.

We can now introduce the second order normal forms, using the Fischer Hermitian product to provide suitable complementary spaces.

Definition 2.12. The *Fischer Hermitian product* on $\mathcal{H}$ is defined by

$$\langle z_1^{p_1}\cdots z_n^{p_n} e_h, z_1^{q_1}\cdots z_n^{q_n} e_k\rangle = \begin{cases} 0 & \text{if } h\ne k \text{ or } p_j\ne q_j \text{ for some } j;\\ \dfrac{p_1!\cdots p_n!}{(p_1+\cdots+p_n)!} & \text{if } h=k \text{ and } p_j=q_j \text{ for all } j.\end{cases}$$

Definition 2.13. Given $\Lambda \in M_{n,n}(\mathbb{C})$, we shall say that $G \in \widehat{\mathcal{O}}^n_\Lambda$ is *in second order normal form* if $G = \Lambda$ or ($G \ne \Lambda$ and) the homogeneous expansion $G = \Lambda + \sum_{d\ge\mu} G_d$ of G satisfies the following conditions:

(a) $G_\mu \ne O$;
(b) $G_d \in \mathcal{H}^d \cap (\operatorname{Im} L_{G_\mu,\Lambda})^\perp$ for all $d > \mu$ (where we are using the Fischer Hermitian product).

Given $F \in \widehat{\mathcal{O}}^n_\Lambda$, we shall say that $G \in \widehat{\mathcal{O}}^n_\Lambda$ is *a second order normal form* of F if G is in second order normal form and $G = \Phi^{-1}\circ F\circ\Phi$ for some $\Phi \in \widehat{\mathcal{O}}^n_I$.

We can now prove the existence of second order normal forms:

Theorem 2.14. *Let $\Lambda \in M_{n,n}(\mathbb{C})$ be given. Then each $F \in \widehat{\mathcal{O}}^n_\Lambda$ admits a second order normal form. More precisely, if $F = \Lambda + \sum_{d\ge\mu} F_d$ is in Poincaré-Dulac normal form (and $F \not\equiv \Lambda$) then there exists a unique Λ-resonant $\Phi = I + \sum_{d\ge 2} H_d \in \widehat{\mathcal{O}}^n_I$ such that $H_d \in (\operatorname{Ker} L_{F_\mu,\Lambda})^\perp$ for all $d \ge 2$ and $G = \Phi^{-1}\circ F\circ\Phi$ is in second order normal form. Furthermore, if Λ is diagonal we also have $G_d \in \mathcal{H}^d_\Lambda$ for all $d \ge \mu$.*

Proof. By the classical theory we can assume that F is in Poincaré-Dulac normal form. If $F \equiv \Lambda$ we are done; assume then that $F \not\equiv \Lambda$.

First of all, by Theorem 2.11 if Φ is Λ-resonant we have $\{\Phi^{-1} \circ F \circ \Phi\}_d = F_d$ for all $d \le \mu$. Now consider the splittings

$$\mathcal{H}^d = \operatorname{Im} L_{F_\mu,\Lambda}|_{\mathcal{H}_\Lambda^{d-\mu+1}} \oplus (\operatorname{Im} L_{F_\mu,\Lambda}|_{\mathcal{H}_\Lambda^{d-\mu+1}})^\perp$$

and

$$\mathcal{H}_\Lambda^{d-\mu+1} = \operatorname{Ker} L_{F_\mu,\Lambda}|_{\mathcal{H}_\Lambda^{d-\mu+1}} \oplus (\operatorname{Ker} L_{F_\mu,\Lambda}|_{\mathcal{H}_\Lambda^{d-\mu+1}})^\perp .$$

If $d = \mu + 1$ we can find a unique $G_{\mu+1} \in (\operatorname{Im} L_{F_\mu,\Lambda})^\perp \cap \mathcal{H}^{\mu+1}$ and a unique $H_2 \in (\operatorname{Ker} L_{F_\mu,\Lambda})^\perp \cap \mathcal{H}_\Lambda^2$ such that $F_{\mu+1} = G_{\mu+1} + L_{F_\mu,\Lambda}(H_2)$. Then Theorem 2.11 yields

$$\{\Phi^{-1} \circ F \circ \Phi\}_{\mu+1} = F_{\mu+1} - L_{F_\mu,\Lambda}(\{\Phi\}_2) = G_{\mu+1} + L_{F_\mu,\Lambda}(H_2) - L_{F_\mu,\Lambda}(\{\Phi\}_2) ;$$

so to get $\{\Phi^{-1} \circ F \circ \Phi\}_{\mu+1} \in (\operatorname{Im} L_{F_\mu,\Lambda})^\perp \cap \mathcal{H}^{\mu+1}$ with $\{\Phi\}_2 \in (\operatorname{Ker} L_{F_\mu,\Lambda})^\perp \cap \mathcal{H}_\Lambda^2$ we must necessarily take $\{\Phi\}_2 = H_2$.

Assume, by induction, that we have uniquely determined

$$H_2, \dots , H_{d-\mu} \in (\operatorname{Im} L_{F_\mu,\Lambda})^\perp \cap \mathcal{H}_\Lambda ,$$

and thus $R_d \in \mathcal{H}^d$ in (2.8). Hence there is a unique $G_d \in (\operatorname{Im} L_{F_\mu,\Lambda})^\perp \cap \mathcal{H}^d$ and a unique $H_{d-\mu+1} \in (\operatorname{Ker} L_{F_\mu,\Lambda})^\perp \cap \mathcal{H}_\Lambda^{d-\mu+1}$ such that $F_d + R_d = G_d + L_{F_\mu,\Lambda}(H_{d-\mu+1})$. Thus to get $\{\Phi^{-1} \circ F \circ \Phi\}_d \in (\operatorname{Im} L_{F_\mu,\Lambda})^\perp \cap \mathcal{H}^d$ with $\{\Phi\}_{d-\mu+1} \in (\operatorname{Ker} L_{F_\mu,\Lambda})^\perp \cap \mathcal{H}_\Lambda^{d-\mu+1}$ the only possible choice is $\{\Phi\}_{d-\mu+1} = H_{d-\mu+1}$, and thus $\{\Phi^{-1} \circ F \circ \Phi\}_d = G_d$.

Finally, if Λ is diagonal then $F_d \in \mathcal{H}_\Lambda^d$ for all $d \ge \mu$. Furthermore, Lemma 2.7 yields $\operatorname{Im} L_{F_\mu,\Lambda}|_{\mathcal{H}_\Lambda^{d-\mu+1}} \subseteq \mathcal{H}_\Lambda^d$ for all $d \ge \mu$; recalling Theorem 2.11.(vi) we then see can we can always find $G_d \in \mathcal{H}_\Lambda^d$, and we are done. □

The definition and construction of k-th order normal forms is similar; the idea is to extract from the remainder term R_ν the pieces depending on H_γ with γ varying in a suitable range, and use them to build operators generalizing L_Λ and $L_{P,\Lambda}$. Since we shall not need it here we refer to [38] for details; for our needs it suffices to recall that given $F = \Lambda + \sum_{d\ge 2} F_d \in \widehat{\mathcal{O}}_\Lambda^n$ it is possible to introduce a sequence of (not necessarily linear) operators $\mathcal{L}^{(d)}[\Lambda, F_2, \dots , F_d]\colon \operatorname{Ker} \mathcal{L}^{(d-1)} \times \mathcal{H}^{d+1} \to \mathcal{H}^{d+1}$ for $d \ge 1$, with $\mathcal{L}^{(1)}[\Lambda](H_2) = L_\Lambda(H_2)$ and $\mathcal{L}^{(2)}[\Lambda, F_2](H_2, H_3) = L_\Lambda(H_3) + L_{F_2,\Lambda}(H_2)$.

Definition 2.15. We shall say that $G = \Lambda + \sum\limits_{d\ge 2} G_d \in \widehat{\mathcal{O}}_\Lambda^n$ is in *infinite order normal form* if $G_d \in W_d^\perp$ for all $d \ge 2$, where W_d is a vector subspace of maximal dimension contained in the image of $\mathcal{L}^{(d-1)}[\Lambda, G_2, \dots , G_{d-1}]$. We shall also say that G is an *infinite order normal form of* $F \in \widehat{\mathcal{O}}_\Lambda^n$ if it is in infinite order normal form and it is formally conjugated to F.

We end this section quoting a result from [38] giving a condition ensuring that a second order normal form is actually an infinite order normal form:

Proposition 2.16. ([38, Theorem 4.9]) *Let $\Lambda \in M_{n,n}(\mathbb{C})$ be diagonal, and $F = \Lambda + \sum\limits_{d\ge 2} F_d \in \widehat{\mathcal{O}}_\Lambda^n$ with $F_2 \ne O$ and Λ-resonant. Assume that* $\operatorname{Ker} L_{F_2,\Lambda}|_{\mathcal{H}_\Lambda^d} = \{O\}$

for all $d \geq 2$. Then the second order normal form of F is the unique infinite order normal form of F.

3. Superattracting germs. In this section we shall completely describe the second order normal forms obtained when $n = \mu = 2$ and $\Lambda = O$, that is for 2-dimensional superattracting germs with non-vanishing quadratic term. Except in four degenerate instances, the second order normal forms will be infinite order normal forms, and will be expressed just in terms of two power series of *one* variable, thus providing a drastic simplification of the germs.

In [1] we showed that, up to a linear change of variable, we can assume that the quadratic term F_2 is of one (and only one) of the following forms:

(∞) $F_2(z,w) = (z^2, zw)$;
(1_{00}) $F_2(z,w) = (0, -z^2)$;
(1_{10}) $F_2(z,w) = \big(-z^2, -(z^2 + zw)\big)$;
(1_{11}) $F_2(z,w) = \big(-zw, -(z^2 + w^2)\big)$;
(2_{001}) $F_2(z,w) = (0, zw)$;
(2_{011}) $F_2(z,w) = (zw, zw + w^2)$;
($2_{10\rho}$) $F_2(z,w) = \big(-\rho z^2, (1-\rho)zw\big)$, with $\rho \neq 0$;
($2_{11\rho}$) $F_2(z,w) = \big(\rho z^2 + zw, (1+\rho)zw + w^2\big)$, with $\rho \neq 0$;
(3_{100}) $F_2(z,w) = (z^2 - zw, 0)$;
($3_{\rho 10}$) $F_2(z,w) = \big(\rho(-z^2 + zw), (1-\rho)(zw - w^2)\big)$, with $\rho \neq 0, 1$;
($3_{\rho\tau 1}$) $F_2(z,w) = \big(-\rho z^2 + (1-\tau)zw, (1-\rho)zw - \tau w^2\big)$, with $\rho, \tau \neq 0$ and $\rho + \tau \neq 1$

(where the labels refer to the number of characteristic directions and to their indices; see also [4]).

We shall use the standard basis $\{u_{d,j}, v_{d,j}\}_{j=0,\dots,d}$ of $\mathcal{H}^d$, where

$$u_{d,j} = (z^j w^{d-j}, 0) \qquad \text{and} \qquad v_{d,j} = (0, z^j w^{d-j}) \ ,$$

and we endow $\mathcal{H}^d$ with the Fischer Hermitian product, so that $\{u_{d,j}, v_{d,j}\}_{j=0,\dots,d}$ is an orthogonal basis and

$$\|u_{d,j}\|^2 = \|v_{d,j}\|^2 = \binom{d}{j}^{-1} .$$

When $\Lambda = O$, we have $\mathcal{H}_\Lambda = \mathcal{H}$, and the operator $L = L_{F_2,\Lambda}\colon \mathcal{H}^d \to \mathcal{H}^{d+1}$ is given by

$$L(H) = -\operatorname{Jac}(F_2) \cdot H \ .$$

To apply Proposition 2.16, we need to know when $\operatorname{Ker} L|_{\mathcal{H}^d} = \{O\}$. Since

$$\begin{aligned}\dim \operatorname{Ker} L|_{\mathcal{H}^d} + \dim \operatorname{Im} L|_{\mathcal{H}^d} &= \dim \mathcal{H}^d = \dim \mathcal{H}^{d+1} - 2 \\ &= \dim \operatorname{Im} L|_{\mathcal{H}^d} + \dim(\operatorname{Im} L|_{\mathcal{H}^d})^\perp - 2 \ ,\end{aligned}$$

we find that

$$\operatorname{Ker} L|_{\mathcal{H}^d} = \{O\} \quad \text{if and only if} \quad \dim(\operatorname{Im} L|_{\mathcal{H}^d})^\perp = 2 \ . \tag{3.1}$$

We shall now study separately each case.

• *Case* (∞).
In this case we have

$$L(u_{d,j}) = -2u_{d+1,j+1} - v_{d+1,j} \qquad \text{and} \qquad L(v_{d,j}) = -v_{d+1,j+1}$$

for all $d \geq 2$ and $j = 0, \dots, d$. Therefore

$$\operatorname{Im} L|_{\mathcal{H}^d} = \operatorname{Span}\left(u_{d+1,2}, \dots, u_{d+1,d+1}, 2u_{d+1,1} + v_{d+1,0}, v_{d+1,1}, \dots, v_{d+1,d+1}\right) ,$$

and thus

$$(\operatorname{Im} L|_{\mathcal{H}^d})^{\perp} = \operatorname{Span}\left(u_{d+1,0}, (d+1)u_{d+1,1} - 2v_{d+1,0}\right) .$$

In particular, thanks to (3.1) and Proposition 2.16, a second order normal form is automatically an infinite order normal form.

It then follows that every formal power series of the form

$$F(z,w) = (z^2 + O_3, zw + O_3)$$

(where O_3 denotes a remainder term of order at least 3) has a unique infinite order normal form

$$G(z,w) = \big(z^2 + \varphi(w) + z\psi'(w), zw - 2\psi(w)\big)$$

where φ, $\psi \in \mathbb{C}[\![\zeta]\!]$ are power series of order at least 3. Notice that (here and in later formulas) the appearance of the derivative (which simplifies the expression of the normal form) is due to the fact we are using the Fischer Hermitian product; using another Hermitian product might lead to more complicated normal forms.

• *Case* (1_{00}).
In this case we have

$$L(u_{d,j}) = 2v_{d+1,j+1} \qquad \text{and} \qquad L(v_{d,j}) = 0$$

for all $d \geq 2$ and $j = 0, \dots, d$. Therefore

$$\operatorname{Im} L|_{\mathcal{H}^d} = \operatorname{Span}\left(v_{d+1,1}, \dots, v_{d+1,d+1}\right) ,$$

and thus

$$(\operatorname{Im} L|_{\mathcal{H}^d})^{\perp} = \operatorname{Span}\left(u_{d+1,0}, \dots, u_{d+1,d+1}, v_{d+1,0}\right) .$$

This is a degenerate case, where we cannot use Proposition 2.16. Anyway, Theorem 2.14 still apply, and it follows that every formal power series of the form

$$F(z,w) = (O_3, -z^2 + O_3)$$

has a second order normal form

$$G(z,w) = \big(\Phi(z,w), -z^2 + \psi(w)\big)$$

where $\psi \in \mathbb{C}[\![\zeta]\!]$ and $\Phi \in \mathbb{C}[\![z,w]\!]$ are power series of order at least 3.

• *Case* (1_{10}).
In this case we have

$$L(u_{d,j}) = 2u_{d+1,j+1} + 2v_{d+1,j+1} + v_{d+1,j} \qquad \text{and} \qquad L(v_{d,j}) = v_{d+1,j+1}$$

for all $d \geq 2$ and $j = 0, \ldots, d$. Therefore

$$\operatorname{Im} L|_{\mathcal{H}^d} = \operatorname{Span}\left(2u_{d+1,1} + v_{d+1,0}, u_{d+1,2}, \ldots, u_{d+1,d+1}, v_{d+1,1}, \ldots, v_{d+1,d+1}\right),$$

and thus

$$(\operatorname{Im} L|_{\mathcal{H}^d})^\perp = \operatorname{Span}\left(u_{d+1,0}, (d+1)u_{d+1,1} - 2v_{d+1,0}\right).$$

It then follows that every formal power series of the form

$$F(z,w) = (-z^2 + O_3, -z^2 - zw + O_3)$$

has a unique infinite order normal form

$$G(z,w) = \big(-z^2 + \varphi(w) + z\psi'(w), -z^2 - zw - 2\psi(w)\big)$$

where φ, $\psi \in \mathbb{C}[\![\zeta]\!]$ are power series of order at least 3.

• *Case* (1_{11}).
In this case we have

$$L(u_{d,j}) = u_{d+1,j} + 2v_{d+1,j+1} \qquad \text{and} \qquad L(v_{d,j}) = u_{d+1,j+1} + 2v_{d+1,j}$$

for all $d \geq 2$ and $j = 0, \ldots, d$. It follows that

$$\begin{aligned} \operatorname{Im} L|_{\mathcal{H}^d} = \operatorname{Span}(&u_{d+1,0} - u_{d+1,2}, \ldots, u_{d+1,d-1} - u_{d+1,d+1}, \\ &v_{d+1,2} - v_{d+1,0}, \ldots, v_{d+1,d+1} - v_{d+1,d-1}, \\ &u_{d+1,0} + 2v_{d+1,1}, u_{d+1,1} + 2v_{d+1,0}), \end{aligned}$$

and a few computations yield

$$\begin{aligned} (\operatorname{Im} L|_{\mathcal{H}^d})^\perp &= \operatorname{Span}\left(\sum_{j=0}^{d+1} \binom{d+1}{j} (v_{d+1,j} - 2u_{d+1,j}), \right. \\ &\qquad\qquad \left. \sum_{j=0}^{d+1} (-1)^j \binom{d+1}{j} (v_{d+1,j} + 2u_{d+1,j}) \right) \\ &= \operatorname{Span}\left(\big(-2(z+w)^{d+1}, (z+w)^{d+1}\big), \big(2(w-z)^{d+1}, (w-z)^{d+1}\big)\right). \end{aligned}$$

It then follows that every formal germ of the form

$$F(z,w) = (-zw + O_3, -z^2 - w^2 + O_3)$$

has a unique infinite order normal form

$$G(z,w) = \big(-zw - 2\varphi(z+w) + 2\psi(w-z), -z^2 - w^2 + \varphi(z+w) + \psi(w-z)\big)$$

where φ, $\psi \in \mathbb{C}[\![\zeta]\!]$ are arbitrary power series of order at least 3. Again, the fact that the normal form is expressed in terms of power series evaluated in $z + w$ and $z - w$ is due to the fact we are using the Fischer Hermitian product.

• *Case* (2_{001}).
In this case we have

$$L(u_{d,j}) = -v_{d+1,j} \qquad \text{and} \qquad L(v_{d,j}) = -v_{d+1,j+1}$$

for all $d \geq 2$ and $j = 0, \dots, d$. It follows that

$$\operatorname{Im} L|_{\mathcal{H}^d} = \operatorname{Span}\left(v_{d+1,0}, \dots, v_{d+1,d+1}\right)$$

and hence

$$(\operatorname{Im} L|_{\mathcal{H}^d})^\perp = \operatorname{Span}\left(u_{d+1,0}, \dots, u_{d+1,d+1}\right) .$$

We are in a degenerate case; hence every formal germ of the form

$$F(z,w) = (O_3, zw + O_3)$$

has a second order normal form

$$G(z,w) = \bigl(\Phi(z,w), zw\bigr)$$

where $\Phi \in \mathbb{C}[\![z,w]\!]$ is a power series of order at least three.

• *Case* (2_{011}).
In this case we have

$$L(u_{d,j}) = -u_{d+1,j} - v_{d+1,j} \qquad \text{and} \qquad L(v_{d,j}) = -u_{d+1,j+1} - 2v_{d+1,j} - v_{d+1,j+1}$$

for all $d \geq 2$ and $j = 0, \dots, d$. It follows that

$$\begin{aligned}\operatorname{Im} L|_{\mathcal{H}^d} = \operatorname{Span}(&u_{d+1,0}, \dots, u_{d+1,d-1}, v_{d+1,0}, \dots, v_{d+1,d-1},\\ &u_{d+1,d} + v_{d+1,d}, u_{d+1,d+1} + v_{d+1,d+1} + 2v_{d+1,d}) ,\end{aligned}$$

and hence

$$\begin{aligned}&(\operatorname{Im} L|_{\mathcal{H}^d})^\perp\\ &\quad = \operatorname{Span}\left((d+1)u_{d+1,d} - (d+1)v_{d+1,d} + 2v_{d+1,d+1}, u_{d+1,d+1} - v_{d+1,d+1}\right) .\end{aligned}$$

It then follows that every formal germ of the form

$$F(z,w) = (zw + O_3, zw + w^2 + O_3)$$

has a unique infinite order normal form

$$G(z,w) = \bigl(zw + w\varphi'(z) + \psi(z), zw + w^2 + 2\varphi(z) - w\varphi'(z) - \psi(z)\bigr) ,$$

where φ, $\psi \in \mathbb{C}[\![\zeta]\!]$ are power series of order at least 3.

• *Case* $(2_{10\rho})$.
In this case we have

$$L(u_{d,j}) = 2\rho u_{d+1,j+1} + (\rho-1)v_{d+1,j} \qquad \text{and} \qquad L(v_{d,j}) = (\rho-1)v_{d+1,j+1}$$

for all $d \geq 2$ and $j = 0, \dots, d$. We clearly have two subcases to consider.

If $\rho = 1$ then
$$\operatorname{Im} L|_{\mathcal{H}^d} = \operatorname{Span}\left(u_{d+1,1}, \dots, u_{d+1,d+1}\right) ,$$
and hence
$$(\operatorname{Im} L|_{\mathcal{H}^d})^\perp = \operatorname{Span}\left(u_{d+1,0}, v_{d+1,0}, \dots, v_{d+1,d+1}\right) .$$
We are in the third degenerate case; hence every formal germ of the form

$$F(z,w) = (-z^2 + O_3, O_3)$$

has a second order normal form

$$G(z,w) = \big(-z^2 + \psi(w), \Phi(z,w)\big) ,$$

where $\psi \in \mathbb{C}[\![\zeta]\!]$ and $\Phi \in \mathbb{C}[\![z,w]\!]$ are power series of order at least 3.

If instead $\rho \neq 1$ (recalling that $\rho \neq 0$ too) then

$$\begin{aligned}&\operatorname{Im} L|_{\mathcal{H}^d}\\ &\quad = \operatorname{Span}\left(2\rho u_{d+1,1} + (\rho-1)v_{d+1,0}, u_{d+1,2}, \dots, u_{d+1,d+1}, v_{d+1,1}, \dots, v_{d+1,d+1}\right) ,\end{aligned}$$

and hence

$$(\operatorname{Im} L|_{\mathcal{H}^d})^\perp = \operatorname{Span}\left(u_{d+1,0}, (\rho-1)(d+1)u_{d+1,1} - 2\rho v_{d+1,0}\right) .$$

It then follows that every formal germ of the form

$$F(z,w) = \big(-\rho z^2 + O_3, (1-\rho)zw + O_3\big)$$

with $\rho \neq 0$, 1 has a unique infinite order normal form

$$G(z,w) = \big(-\rho z^2 + (\rho-1)z\varphi'(w) + \psi(w), (1-\rho)zw - 2\rho\varphi(z)\big) ,$$

where φ, $\psi \in \mathbb{C}[\![\zeta]\!]$ are power series of order at least 3.

• *Case* $(2_{11\rho})$.
In this case we have

$$\begin{cases} L(u_{d,j}) = -2\rho u_{d+1,j+1} - u_{d+1,j} - (1+\rho)v_{d+1,j} \\ L(v_{d,j}) = -u_{d+1,j+1} - 2v_{d+1,j} - (1+\rho)v_{d+1,j+1} \end{cases} \tag{3.2}$$

for all $d \geq 2$ and $j = 0, \dots, d$. We clearly have two subcases to consider.

If $\rho = -1$ then

$$\begin{aligned}\operatorname{Im} L|_{\mathcal{H}^d} = \operatorname{Span}\big(&u_{d+1,0} - 2u_{d+1,1}, \dots, u_{d+1,d} - 2u_{d+1,d+1},\\ &u_{d+1,1} + 2v_{d+1,0}, \dots, u_{d+1,d} + 2v_{d+1,d}\big) ,\end{aligned}$$

and hence

$$\begin{aligned}(\operatorname{Im} L|_{\mathcal{H}^d})^\perp &= \operatorname{Span}\left(\sum_{j=0}^{d+1}\binom{d+1}{j}\frac{1}{2^j}(u_{d+1,j}-\frac{1}{4}v_{d+1,j}),v_{d+1,d+1}\right)\\ &= \operatorname{Span}\left(\left(\left(\tfrac{z}{2}+w\right)^{d+1},-\frac{1}{4}\left(\tfrac{z}{2}+w\right)^{d+1}\right),(0,z^{d+1})\right)\ .\end{aligned}$$

It then follows that every formal germ of the form

$$F(z,w)=(-z^2+zw+O_3,w^2+O_3)$$

has a unique infinite order normal form

$$G(z,w)=\left(-z^2+zw+\varphi(\tfrac{z}{2}+w),w^2-\frac{1}{4}\varphi(\tfrac{z}{2}+w)+\psi(z)\right)\ ,$$

where φ, $\psi\in\mathbb{C}[\![\zeta]\!]$ are power series of order at least 3.

If instead $\rho\neq-1$ (recalling that $\rho\neq 0$ too) then a basis of $\operatorname{Im} L|_{\mathcal{H}^d}$ is given by the vectors listed in (3.2), and a computation shows that $(\operatorname{Im} L|_{\mathcal{H}^d})^\perp$ is given by homogeneous maps of the form

$$\sum_{j=0}^{d+1}(a_j u_{d+1,j}+b_j v_{d+1,j})$$

where the coefficients a_j, b_j satisfy the following relations:

$$\begin{cases} c_jb_j=-\dfrac{2}{1+\rho}c_{j-1}b_{j-1}-\dfrac{1}{\rho(1+\rho)}c_{j-2}b_{j-2} & \text{for } j=2,\dots,d+1,\\ c_ja_j=\dfrac{1}{\rho}c_{j-2}b_{j-2} & \text{for } j=2,\dots,d+1,\\ a_0=(3\rho-1)b_0+2\dfrac{\rho(1+\rho)}{d+1}b_1\ , & \\ a_1=-2(d+1)b_0-(1+\rho)b_1\ , & \end{cases}$$

where $c_j^{-1}=\binom{d+1}{j}$ and b_0, $b_1\in\mathbb{C}$ are arbitrary. Solving these recurrence equations one gets

$$b_j=\frac{1}{2\sqrt{-\rho}}\binom{d+1}{j}\left[\frac{\rho(1+\rho)}{d+1}(m_\rho^j-n_\rho^j)b_1+\big(\rho(m_\rho^j-n_\rho^j)+\sqrt{-\rho}(m_\rho^j+n_\rho^j)\big)b_0\right]\ ,$$

where $\sqrt{-\rho}$ is any square root of $-\rho$, and

$$m_\rho=\frac{\sqrt{-\rho}-\rho}{\rho(1+\rho)}\ ,\qquad n_\rho=-\frac{\sqrt{-\rho}+\rho}{\rho(1+\rho)}\ .$$

It follows that the unique infinite order normal form of a formal germ of the form

$$F(z,w)=\big(\rho z^2+zw+O_3,(1+\rho)zw+w^2+O_3\big)$$

with $\rho \neq 0,\ -1$ is

$$\begin{aligned} G(z,w) = \Bigg(&\rho z^2 + zw + \frac{1}{\rho}\left[\frac{1-\sqrt{-\rho}}{2m_\rho^2}\varphi(m_\rho z + w) + \frac{1+\sqrt{-\rho}}{2n_\rho^2}\varphi(n_\rho z + w)\right] \\ &\quad + \frac{1+\rho}{2\sqrt{-\rho}}\left(\frac{1}{m_\rho^2}\psi(m_\rho z + w) - \frac{1}{n_\rho^2}\psi(n_\rho z + w)\right), \\ &(1+\rho)zw + w^2 + \frac{1-\sqrt{-\rho}}{2}\varphi(m_\rho z + w) + \frac{1+\sqrt{-\rho}}{2}\varphi(n_\rho z + w) \\ &\quad + \frac{\rho(1+\rho)}{2\sqrt{-\rho}}\big(\psi(m_\rho z + w) - \psi(n_\rho z + w)\big)\Bigg) \end{aligned}$$

where $\varphi,\ \psi \in \mathbb{C}[\![\zeta]\!]$ are power series of order at least 3.

• *Case* (3_{100}).
In this case we have

$$L(u_{d,j}) = u_{d+1,j} - 2u_{d+1,j+1} \qquad \text{and} \qquad L(v_{d,j}) = u_{d+1,j+1}$$

for all $d \geq 2$ and $j = 0, \ldots, d$. It follows that

$$\operatorname{Im} L|_{\mathcal{H}^d} = \operatorname{Span}(u_{d+1,0}, \ldots, u_{d+1,d+1})$$

and hence

$$(\operatorname{Im} L|_{\mathcal{H}^d})^\perp = \operatorname{Span}(v_{d+1,0}, \ldots, v_{d+1,d+1}) \ .$$

We are in the last degenerate case; hence every formal germ of the form

$$F(z,w) = (z^2 - zw + O_3, O_3)$$

has a second order normal form

$$G(z,w) = \big(z^2 - zw, \Phi(z,w)\big) \ ,$$

where $\Phi \in \mathbb{C}[\![z,w]\!]$ is a power series of order at least 3.

• *Case* $(3_{\rho 10})$.
In this case we have

$$\begin{cases} L(u_{d,j}) = \rho(2u_{d+1,j+1} - u_{d+1,j}) + (\rho - 1)v_{d+1,j} \\ L(v_{d,j}) = -\rho u_{d+1,j+1} + (\rho - 1)(v_{d+1,j+1} - 2v_{d+1,j}) \end{cases} \tag{3.3}$$

for all $d \geq 2$ and $j = 0, \ldots, d$. Then a basis of $\operatorname{Im} L|_{\mathcal{H}^d}$ is given by the homogeneous maps listed in (3.3), and a computation shows that $(\operatorname{Im} L|_{\mathcal{H}^d})^\perp$ is given by homogeneous maps of the form

$$\sum_{j=0}^{d+1}(a_j u_{d+1,j} + b_j v_{d+1,j})$$

where the coefficients a_j, b_j satisfy the following relations:

$$\begin{cases} c_{j+1}a_{j+1} = \dfrac{\rho - 1}{\rho}(c_{j+1}b_{j+1} - 2c_j b_j) & \text{for } j = 0, \ldots, d, \\ c_{j+1}b_{j+1} = 2c_j b_j - c_{j-1}b_{j-1} & \text{for } j = 1, \ldots, d, \\ c_0 a_0 = 2c_1 a_1 + \dfrac{\rho - 1}{\rho}c_0 b_0 \ , \end{cases}$$

where $c_j^{-1} = \binom{d+1}{j}$ and b_0, $b_1 \in \mathbb{C}$ are arbitrary. Solving these recurrence equations we find

$$\begin{cases} b_j = \binom{d+1}{j}\left[\frac{j}{d+1}b_1 - (j-1)b_0\right] & \text{for } j = 0, \dots, d+1, \\ a_j = \frac{\rho-1}{\rho}\binom{d+1}{j}\left[\frac{2-j}{d+1}b_1 + (j-3)b_0\right] & \text{for } j = 0, \dots, d+1, \end{cases}$$

where b_0, $b_1 \in \mathbb{C}$ are arbitrary. So every formal germ of the form

$$F(z,w) = \big(\rho(-z^2 + zw) + O_3, (1-\rho)(zw - w^2) + O_3\big)$$

with $\rho \neq 0$, 1 has a unique infinite order normal form

$$\begin{aligned} G(z,w) = \bigg(&\rho(-z^2 + zw) + z\frac{\partial}{\partial z}\big[\varphi(z+w) + \psi(z+w)\big] - \varphi(z+w), \\ &(1-\rho)(zw - w^2) \\ &\quad + \frac{\rho-1}{\rho}\left(z\frac{\partial}{\partial z}\big[\varphi(z+w) - \psi(z+w)\big] - 3\varphi(z+w) + 2\psi(z+w)\right)\bigg) \end{aligned}$$

where φ, $\psi \in \mathbb{C}[\![\zeta]\!]$ are power series of order at least 3.

• *Case* $(3_{\rho\tau 1})$.
In this case we have

$$L(u_{d,j}) = (\tau - 1)u_{d+1,j} + 2\rho u_{d+1,j+1} + (\rho - 1)v_{d+1,j}$$

and

$$L(v_{d,j}) = (\tau - 1)u_{d+1,j+1} + 2\tau v_{d+1,j} + (\rho - 1)v_{d+1,j+1}$$

for all $d \geq 2$ and $j = 0, \dots, d$. As before, we have a few subcases to consider.

Assume first $\rho = \tau = 1$. Then

$$\operatorname{Im} L|_{\mathcal{H}^d} = \operatorname{Span}\left(u_{d+1,1}, \dots, u_{d+1,d+1}, v_{d+1,0}, \dots, v_{d+1,d}\right)\ ;$$

hence

$$(\operatorname{Im} L|_{\mathcal{H}^d})^{\perp} = \operatorname{Span}\left(u_{d+1,0}, v_{d+1,d+1}\right)\ ,$$

It then follows that every formal germ of the form

$$F(z,w) = (-z^2 + O_3, -w^2 + O_3)$$

has a unique infinite order normal form

$$G(z,w) = \big(-z^2 + \varphi(w), -w^2 + \psi(z)\big)\ ,$$

where φ, $\psi \in \mathbb{C}[\![\zeta]\!]$ are power series of order at least 3.

Assume now $\rho \neq 1$. Then a computation shows that $(\operatorname{Im} L|_{\mathcal{H}^d})^{\perp}$ is given by homogeneous maps of the form

$$\sum_{j=0}^{d+1}(a_j u_{d+1,j} + b_j v_{d+1,j})$$

where the coefficients a_j, b_j satisfy the following relations:

$$\begin{cases} c_{j+1}a_{j+1} = \dfrac{\tau}{\rho}c_{j-1}b_{j-1} & \text{for } j=1,\dots,d, \\ c_{j+1}b_{j+1} = -\dfrac{2\tau}{\rho-1}c_jb_j - \dfrac{\tau(\tau-1)}{\rho(\rho-1)}c_{j-1}b_{j-1} & \text{for } j=1,\dots,d, \\ (\tau-1)c_1a_1+(\rho-1)c_1b_1+2\tau c_0b_0=0\ , \\ (\tau-1)c_0a_0+(\rho-1)c_0b_0+2\rho c_1a_1=0\ , \end{cases} \tag{3.4}$$

where $c_j^{-1}=\binom{d+1}{j}$ and b_0, $b_1\in\mathbb{C}$ are arbitrary.

When $\tau=1$ conditions (3.4) reduce to

$$\begin{cases} c_{j+1}a_{j+1} = \dfrac{1}{\rho}c_{j-1}b_{j-1} & \text{for } j=1,\dots,d, \\ c_{j+1}b_{j+1} = -\dfrac{2}{\rho-1}c_jb_j & \text{for } j=1,\dots,d, \\ (\rho-1)c_1b_1+2c_0b_0=0\ , \\ (\rho-1)c_0b_0+2\rho c_1a_1=0\ , \end{cases}$$

whose solution is

$$\begin{cases} a_j = \dbinom{d+1}{j}\dfrac{1}{\rho}\left(\dfrac{2}{1-\rho}\right)^{j-2}b_0 & \text{for } j=1,\dots,d+1, \\ b_j = \dbinom{d+1}{j}\left(\dfrac{2}{1-\rho}\right)^{j}b_0 & \text{for } j=0,\dots,d+1, \end{cases}$$

where a_0, $b_0\in\mathbb{C}$ are arbitrary. Therefore

$$\begin{aligned}(\operatorname{Im} L|_{\mathcal{H}^d})^\perp \\ &= \operatorname{Span}\left((w^{d+1},0),\left(\frac{(1-\rho)^2}{4\rho}\left(\frac{2}{1-\rho}z+w\right)^{d+1},\left(\frac{2}{1-\rho}z+w\right)^{d+1}\right)\right)\ ,\end{aligned}$$

and thus every formal germ of the form

$$F(z,w)=\bigl(-\rho z^2+O_3,(1-\rho)zw-w^2+O_3\bigr)$$

with $\rho\neq1$ has a unique infinite order normal form

$$\begin{aligned}G(z,w)=\biggl(&-\rho z^2+\varphi(w)+\frac{(1-\rho)^2}{4\rho}\psi\left(\frac{2}{1-\rho}z+w\right), \\ &(1-\rho)zw-w^2+\psi\left(\frac{2}{1-\rho}z+w\right)\biggr)\ ,\end{aligned}$$

where φ, $\psi\in\mathbb{C}[\![\zeta]\!]$ are arbitrary power series of order at least 3.

The case $\rho=1$ and $\tau\neq1$ is treated in the same way; we get that every formal germ of the form

$$F(z,w)=\bigl(-z^2+(1-\tau)zw+O_3,-\tau w^2+O_3\bigr)$$

with $\tau\ne1$ has a unique infinite order normal form

$$G(z,w)=\Biggl(-z^2+(1-\tau)zw+\psi\left(\frac{1-\tau}{2}z+w\right),\\ -\tau w^2+\varphi(z)+\frac{(1-\tau)^2}{4\tau}\psi\left(\frac{1-\tau}{2}z+w\right)\Biggr)\;,$$

where φ, $\psi\in\mathbb{C}[\![\zeta]\!]$ are power series of order at least 3.

Finally assume ρ, $\tau\ne1$ (and $\rho+\tau\ne1$). Solving the recurrence equations (3.4) we find

$$\begin{aligned}b_j=&\frac{1}{2\sqrt{\rho\tau(\rho+\tau-1)}}\binom{d+1}{j}\\ &\times\Biggl[\frac{\rho(\rho-1)}{d+1}(m_{\rho,\tau}^j-n_{\rho,\tau}^j)b_1\\ &\quad+\bigl(\rho\tau(m_{\rho,\tau}^j-n_{\rho,\tau}^j)+\sqrt{\rho\tau(\rho+\tau-1)}(m_{\rho,\tau}^j+n_{\rho,\tau}^j)\bigr)b_0\Biggr]\;,\end{aligned}$$

for $j=0,\dots,d+1$, where $\sqrt{\rho\tau(\rho+\tau-1)}$ is any square root of $\rho\tau(\rho+\tau-1)$, and

$$m_{\rho,\tau}=\frac{\sqrt{\rho\tau(\rho+\tau-1)}-\rho\tau}{\rho(\rho-1)}\;,\qquad n_{\rho,\tau}=-\frac{\sqrt{\rho\tau(\rho+\tau-1)}+\rho\tau}{\rho(\rho-1)}\;.$$

Moreover, from (3.4) we also get

$$\begin{aligned}a_j=&\frac{\tau}{2\rho\sqrt{\rho\tau(\rho+\tau-1)}}\binom{d+1}{j}\\ &\times\Biggl[\frac{\rho(\rho-1)}{d+1}(m_{\rho,\tau}^{j-2}-n_{\rho,\tau}^{j-2})b_1\\ &\quad+\bigl(\rho\tau(m_{\rho,\tau}^{j-2}-n_{\rho,\tau}^{j-2})+\sqrt{\rho\tau(\rho+\tau-1)}(m_{\rho,\tau}^{j-2}+n_{\rho,\tau}^{j-2})\bigr)b_0\Biggr]\;,\end{aligned}$$

again for $j=0,\dots,d+1$. It follows that the unique infinite order normal form of a formal germ of the form

$$F(z,w)=\bigl(-\rho z^2+(1-\tau)zw+O_3,(1-\rho)zw-\tau w^2+O_3\bigr)$$

with ρ, $\tau\ne0$, 1 and $\rho+\tau\ne1$, is

$$\begin{aligned}G(z,w)=\Biggl(&-\rho z^2+(1-\tau)zw+\frac{\tau}{\rho}\Biggl[\frac{\sqrt{\rho+\tau-1}+\sqrt{\rho\tau}}{2m_{\rho,\tau}^2}\varphi(m_{\rho,\tau}z+w)\\ &\qquad+\frac{\sqrt{\rho+\tau-1}-\sqrt{\rho\tau}}{2n_{\rho,\tau}^2}\varphi(n_{\rho,\tau}z+w)\\ &\qquad+\frac{1}{m_{\rho,\tau}^2}\psi(m_{\rho,\tau}z+w)-\frac{1}{n_{\rho,\tau}^2}\psi(n_{\rho,\tau}z+w)\Biggr]\;,\\ &(1-\rho)zw-\tau w^2+\frac{\sqrt{\rho+\tau-1}+\sqrt{\rho\tau}}{2}\varphi(m_{\rho,\tau}z+w)\\ &\qquad+\frac{\sqrt{\rho+\tau-1}-\sqrt{\rho\tau}}{2}\varphi(n_{\rho,\tau}z+w)\\ &\qquad+\psi(m_{\rho,\tau}z+w)-\psi(n_{\rho,\tau}z+w)\Biggr)\;,\end{aligned}$$

where the square roots of $\rho\tau$ and of $\rho+\tau-1$ are chosen so that their product is equal to the previously chosen square root of $\rho\tau(\rho+\tau-1)$, and φ, $\psi\in\mathbb{C}[\![\zeta]\!]$ are power series of order at least 3 — and we have finished proving Theorem 1.1 and Proposition 1.2.

4. Germs tangent to the identity. In this section we shall assume $n=\mu=2$ and $\Lambda=I$, that is we shall be interested in 2-dimensional germs tangent to the identity of order 2. We shall keep the notations introduced in the previous section. It should be recalled that in his monumental work [20] (see [21] for a survey) Écalle studied the formal classification of germs tangent to the identity in dimension n, giving a complete set of formal invariants for germs satisfying a generic condition: the existence of at least one non-degenerate characteristic direction (an eigenradius, in Écalle's terminology). A *characteristic direction* of a germ tangent to the identity F is a non-zero direction v such that $F_\mu(v)=\lambda v$ for some $\lambda\in\mathbb{C}$, where F_μ is the first (nonlinear) non-vanishing term in the homogeneous expansion of F. The characteristic direction v is *non-degenerate* if $\lambda\neq 0$.

For this reason, we decided to discuss here the cases *without* non-degenerate characteristic directions, that is the cases (1_{00}), (1_{10}) and (2_{001}), that cannot be dealt with Écalle's methods. Furthermore, we shall also study the somewhat special case (∞), where all directions are characteristic; and we shall examine in detail case $(2_{10\rho})$, where interesting second-order resonance phenomena appear.

When $\Lambda=I$ the operator $L=L_{F_2,\Lambda}$ is given by

$$L(H)=\mathrm{Jac}(H)\cdot F_2-\mathrm{Jac}(F_2)\cdot H\ .$$

In particular, $L(F_2)=O$ always; therefore we cannot apply Proposition 2.16 (nor other similar conditions stated in [38]), and we shall compute the second order normal form only.

• *Case* (∞).
In this case we have

$$L(u_{d,j})=(d-2)u_{d+1,j+1}-v_{d+1,j}\qquad\text{and}\qquad L(v_{d,j})=(d-1)v_{d+1,j+1}$$

for all $d\geq 2$ and $j=0,\dots,d$. Therefore

$$\mathrm{Im}\,L|_{\mathcal{H}^d}=\begin{cases}\mathrm{Span}\big(u_{d+1,2},\dots,u_{d+1,d+1},\\ \qquad\qquad(d-2)u_{d+1,1}-v_{d+1,0},v_{d+1,1},\dots,v_{d+1,d+1}\big) & \text{for } d>2,\\ \mathrm{Span}\,(v_{3,0},\dots,v_{3,3}) & \text{for } d=2.\end{cases}$$

Thus

$$(\mathrm{Im}\,L|_{\mathcal{H}^d})^\perp=\begin{cases}\mathrm{Span}\,(u_{d+1,0},(d+1)u_{d+1,1}+(d-2)v_{d+1,0}) & \text{for } d>2,\\ \mathrm{Span}\,(u_{3,0},\dots,u_{3,3}) & \text{for } d=2.\end{cases}$$

It then follows that every formal power series of the form

$$F(z,w)=(z+z^2+O_3,w+zw+O_3)$$

has as second order normal form

$$\begin{aligned}G(z,w)=\big(&z+z^2+a_0z^3+a_1z^2w+a_2zw^2+\varphi(w)+z\psi'(w),\\ &zw+w\psi'(w)-3\psi(w)\big)\end{aligned}$$

where $\varphi\in\mathbb{C}[\![\zeta]\!]$ is a power series of order at least 3, $\psi\in\mathbb{C}[\![\zeta]\!]$ is a power series of order at least 4 and a_0, a_1, $a_2\in\mathbb{C}$.

• *Case* (1_{00}).
In this case we have

$$L(u_{d,j}) = (j-d)u_{d+1,j+2} + 2v_{d+1,j+1} \qquad \text{and} \qquad L(v_{d,j}) = (j-d)v_{d+1,j+2}$$

for all $d \geq 2$ and $j = 0, \dots, d$. Therefore

$$\operatorname{Im} L|_{\mathcal{H}^d} = \operatorname{Span}\left(2v_{d+1,1} - du_{d+1,2}, u_{d+1,3}, \dots, u_{d+1,d+1}, v_{d+1,2}, \dots, v_{d+1,d+1}\right) ,$$

and thus

$$(\operatorname{Im} L|_{\mathcal{H}^d})^{\perp} = \operatorname{Span}\left(u_{d+1,0}, u_{d+1,1}, v_{d+1,0}, u_{d+1,2} + v_{d+1,1}\right) .$$

It then follows that every formal power series of the form

$$F(z,w) = (z + O_3, w - z^2 + O_3)$$

has as second order normal form

$$G(z,w) = \big(z + w\varphi_1(w) + z\varphi_2(w) + z^2\psi(w), w - z^2 + w\varphi_3(w) + zw\psi(w)\big) ,$$

where φ_1, φ_2, $\varphi_3 \in \mathbb{C}[\![\zeta]\!]$ are power series of order at least 2, and $\psi \in \mathbb{C}[\![\zeta]\!]$ is a power series of order at least 1.

• *Case* (1_{10}).
In this case we have

$$L(u_{d,j}) = (2-d)u_{d+1,j+1} - (d-j)u_{d+1,j+2} + 2v_{d+1,j+1} + v_{d+1,j}$$

and

$$L(v_{d,j}) = (1-d)v_{d+1,j+1} - (d-j)v_{d+1,j+2}$$

for all $d \geq 2$ and $j = 0, \dots, d$. Therefore

$$\operatorname{Im} L|_{\mathcal{H}^d} = \begin{cases} \operatorname{Span}\big((2-d)u_{d+1,1} + v_{d+1,0}, u_{d+1,2}, \dots, u_{d+1,d+1}, & \\ \qquad\qquad v_{d+1,1}, \dots, v_{d+1,d+1}\big) & \text{for } d > 2, \\ \operatorname{Span}\left(v_{3,0} - 2u_{3,2}, u_{3,3}, v_{3,1}, v_{3,2}, v_{3,3}\right) & \text{for } d = 2, \end{cases}$$

and thus

$$(\operatorname{Im} L|_{\mathcal{H}^d})^{\perp} = \begin{cases} \operatorname{Span}\left(u_{d+1,0}, (d+1)u_{d+1,1} + (d-2)v_{d+1,0}\right) & \text{for } d > 2, \\ \operatorname{Span}\left(u_{3,0}, u_{3,1}, 3u_{3,2} + 2v_{3,0}\right) & \text{for } d = 2. \end{cases}$$

It then follows that every formal power series of the form

$$F(z,w) = (z - z^2 + O_3, w - z^2 - zw + O_3)$$

has as second order normal form

$$\begin{aligned} G(z,w) = \big(&z - z^2 + \varphi(w) + a_1 zw^2 + 3a_2 z^2 w + z\psi'(w), \\ &w - z^2 - zw + 2a_2 w^3 + w\psi'(w) - 3\psi(w)\big) , \end{aligned}$$

where $\varphi \in \mathbb{C}[\![\zeta]\!]$ is a power series of order at least 3, $\psi \in \mathbb{C}[\![\zeta]\!]$ is a power series of order at least 4, and a_1, $a_2 \in \mathbb{C}$.

• *Case* (2_{001}).
In this case we have

$$L(u_{d,j}) = (d-j)u_{d+1,j+1} - v_{d+1,j} \qquad \text{and} \qquad L(v_{d,j}) = (d-j-1)v_{d+1,j+1}$$

for all $d \geq 2$ and $j = 0, \dots, d$. It follows that

$$\operatorname{Im} L|_{\mathcal{H}^d} = \operatorname{Span}\left(du_{d+1,1} - v_{d+1,0}, u_{d+1,2}, \dots, u_{d+1,d}, v_{d+1,1}, \dots, v_{d+1,d+1}\right)$$

and hence

$$(\operatorname{Im} L|_{\mathcal{H}^d})^{\perp} = \operatorname{Span}\left(u_{d+1,0}, u_{d+1,d+1}, (d+1)u_{d+1,1} + dv_{d+1,0}\right) .$$

It then follows that every formal germ of the form

$$F(z,w) = (z + O_3, w + zw + O_3)$$

has as second order normal form

$$G(z,w) = \big(z + \varphi_1(z) + \varphi_2(w) + z\psi'(w), zw + w\psi'(w) - \psi(w)\big)$$

where φ_1, φ_2, $\psi \in \mathbb{C}[\![\zeta]\!]$ are power series of order at least 3.

• *Case* $(2_{10\rho})$.
In this case we have

$$\begin{cases} L(u_{d,j}) = (d-j-d\rho+2\rho)u_{d+1,j+1} + (\rho-1)v_{d+1,j} , \\ L(v_{d,j}) = (d-j-d\rho+\rho-1)v_{d+1,j+1} , \end{cases} \tag{4.1}$$

for all $d \geq 2$ and $j = 0, \dots, d$. Here we shall see the resonance phenomena we mentioned at the beginning of this section: for some values of ρ the dimension of the kernel of $L|_{\mathcal{H}^d}$ increases, and in some cases we shall end up with a normal form depending on power series evaluated in monomials of the form $z^{b-a}w^a$.

Let us put

$$E_d = \left\{ \frac{d-j-1}{d-1} \;\middle|\; j = 0, \dots, d \right\} \setminus \{0\} \quad \text{and} \quad F_d = \left\{ \frac{d-j}{d-2} \;\middle|\; j = 0, \dots, d-1 \right\}$$

(we are excluding 0 because $\rho \neq 0$ by assumption), where E_d is defined for all $d \geq 2$ whereas F_d is defined for all $d \geq 3$, and set

$$\mathcal{E} = \bigcup_{d\geq 2} E_d = \big((0,1] \cap \mathbb{Q}\big) \cup \left\{ -\frac{1}{n} \;\middle|\; n \in \mathbb{N}^* \right\}$$

and

$$\mathcal{F} = \bigcup_{d\geq 3} F_d = \big((0,1] \cap \mathbb{Q}\big) \cup \left\{ 1 + \frac{1}{n}, 1 + \frac{2}{n} \;\middle|\; n \in \mathbb{N}^* \right\} .$$

So $\mathcal{E}$ is the set of $\rho \in \mathbb{C}^*$ such that $L(v_{d,j}) = 0$ for some $d \geq 2$ and $0 \leq j \leq d$, while $\mathcal{F}$ is the set of $\rho \in \mathbb{C}^*$ such that $L(u_{d,j}) = (\rho - 1)v_{d+1,j}$ for some $d \geq 3$ and $0 \leq j \leq d-1$.

Let us first discuss the non-resonant case, when $\rho \notin \mathcal{E} \cup \mathcal{F}$. Then none of the coefficients in (4.1) vanishes, and thus

$$\operatorname{Im} L|_{\mathcal{H}^2} = \operatorname{Span}\left(2u_{3,1} + (\rho - 1)v_{3,0}, u_{3,2}, v_{3,1}, v_{3,2}, v_{3,3}\right)$$

and

$$\operatorname{Im} L|_{\mathcal{H}^d} = \operatorname{Span}\big((d - d\rho + 2\rho)u_{d+1,1} + (\rho - 1)v_{d+1,0}, \\ u_{d+1,2}, \dots, u_{d+1,d+1}, v_{d+1,1}, \dots, v_{d+1,d+1}\big) ,$$

for $d \geq 3$, and hence

$$(\operatorname{Im} L|_{\mathcal{H}^d})^\perp = \begin{cases} \operatorname{Span}\left(u_{d+1,0}, (1-\rho)(d+1)u_{d+1,1} + (d(1-\rho) + 2\rho)v_{d+1,0}\right) & \\ & \text{for } d \geq 3, \\ \operatorname{Span}\left(u_{3,0}, u_{3,3}, 3(1-\rho)u_{3,1} + 2v_{3,0}\right) & \text{for } d = 2. \end{cases}$$

It then follows that every formal germ of the form

$$F(z, w) = (z - \rho z^2 + O_3, w + (1 - \rho)zw + O_3)$$

with $\rho \notin \mathcal{E} \cup \mathcal{F}$ (and $\rho \neq 0$) has as second order normal form

$$G(z, w) = \big(z - \rho z^2 + az^3 + \varphi(w) + (1 - \rho)z\psi'(w), \\ w + (1 - \rho)zw + (1 - \rho)w\psi'(w) + (3\rho - 1)\psi(z)\big) ,$$

where φ, $\psi \in \mathbb{C}[\![\zeta]\!]$ are power series of order at least 3, and $a \in \mathbb{C}$.

Assume now $\rho \in \mathcal{F} \setminus \mathcal{E}$. Then $L(v_{d,j}) \neq O$ always, and thus $v_{d+1,j} \in \operatorname{Im} L|_{\mathcal{H}^d}$ for all $d \geq 2$ and all $j = 1, \dots, d + 1$. Since $\rho > 1$, if $d > 2$ it also follows that $u_{d+1,j+1} \in \operatorname{Im} L|_{\mathcal{H}^d}$ for $j = 1, \dots, d$.

Now, if $\rho = 1 + (1/n)$ then

$$\frac{d}{d-2} = \rho \quad \Longleftrightarrow \quad d = 2(n + 1) ,$$

and

$$\frac{d-1}{d-2} = \rho \quad \Longleftrightarrow \quad d = n + 2 .$$

Taking care of the case $d = 2$ separately, we then have

$$\operatorname{Im} L|_{\mathcal{H}^d} \\ = \begin{cases} \operatorname{Span}\big((d - d\rho + 2\rho)u_{d+1,1} + (\rho - 1)v_{d+1,0}, u_{d+1,2}, \dots, u_{d+1,d+1}, \\ \qquad v_{d+1,1}, \dots, v_{d+1,d+1}\big) & \text{for } d \geq 3,\ d \neq n + 2,\ 2(n + 1), \\ \operatorname{Span}\big(u_{d+1,1} + (\rho - 1)v_{d+1,0}, u_{d+1,3}, \dots, u_{d+1,d+1}, \\ \qquad v_{d+1,1}, \dots, v_{d+1,d+1}\big) & \text{for } d = n + 2, \\ \operatorname{Span}\big(u_{d+1,2}, \dots, u_{d+1,d+1}, \\ \qquad v_{d+1,0}, \dots, v_{d+1,d+1}\big) & \text{for } d = 2(n + 1), \\ \operatorname{Span}\left(2u_{3,1} + (\rho - 1)v_{3,0}, u_{3,2}, v_{3,1}, v_{3,2}, v_{3,3}\right) & \text{for } d = 2, \end{cases}$$

and hence

$$(\operatorname{Im} L|_{\mathcal{H}^d})^\perp = \begin{cases} \operatorname{Span}\left(u_{d+1,0}, (1-\rho)(d+1)u_{d+1,1} + (d(1-\rho)+2\rho)v_{d+1,0}\right) & \\ \qquad\qquad\qquad\qquad \text{for } d \geq 3,\ d \neq n+2,\ 2(n+1), & \\ \operatorname{Span}\left(u_{d+1,0}, u_{d+1,2}, (1-\rho)(d+1)u_{d+1,1} + v_{d+1,0}\right) & \\ \qquad\qquad\qquad\qquad \text{for } d = n+2, & \\ \operatorname{Span}\left(u_{d+1,0}, u_{d+1,1}\right) \qquad \text{for } d = 2(n+1), & \\ \operatorname{Span}\left(u_{3,0}, u_{3,3}, 3(1-\rho)u_{3,1} + 2v_{3,0}\right) \qquad \text{for } d = 2. & \end{cases}$$

It then follows that every formal germ of the form

$$F(z,w) = \left(z - \left(1 + \frac{1}{n}\right)z^2 + O_3, w - \frac{1}{n}zw + O_3\right)$$

with $n \in \mathbb{N}^*$ has as second order normal form

$$\begin{aligned} G(z,w) = \Bigg(z - \left(1+\frac{1}{n}\right)z^2 + \varphi(w) + (1-\rho)z\psi'(w) + a_0z^3 + a_1z^2w^{n+1}, \\ w - \frac{1}{n}zw + (1-\rho)w\psi'(w) + (3\rho-1)\psi(w)\Bigg)\ , \end{aligned}$$

where φ, $\psi \in \mathbb{C}[\![\zeta]\!]$ are power series of order at least 3, and a_0, $a_1 \in \mathbb{C}$.

If instead $\rho = 1 + (2/m)$ with m odd (if m is even we are again in the previous case) then

$$\frac{d}{d-2} = \rho \quad \Longleftrightarrow \quad d = m+2\ ,$$

whereas $\frac{d-1}{d-2} \neq \rho$ always. Hence

$$\operatorname{Im} L|_{\mathcal{H}^d} = \begin{cases} \operatorname{Span}\big((d - d\rho + 2\rho)u_{d+1,1} + (\rho-1)v_{d+1,0}, u_{d+1,2}, \dots, u_{d+1,d+1}, & \\ \qquad v_{d+1,1}, \dots, v_{d+1,d+1}\big) & \text{for } d \geq 3,\ d \neq m+2, \\ \operatorname{Span}\left(u_{d+1,2}, \dots, u_{d+1,d+1}, v_{d+1,0}, \dots, v_{d+1,d+1}\right) & \text{for } d = m+2, \\ \operatorname{Span}\left(2u_{3,1} + (\rho-1)v_{3,0}, u_{3,2}, v_{3,1}, v_{3,2}, v_{3,3}\right) & \text{for } d = 2, \end{cases}$$

and thus

$$(\operatorname{Im} L|_{\mathcal{H}^d})^\perp = \begin{cases} \operatorname{Span}\left(u_{d+1,0}, (1-\rho)(d+1)u_{d+1,1} + (d - d\rho + 2\rho)v_{d+1,0}\right) & \\ & \text{for } d \geq 3,\ d \neq m+2, \\ \operatorname{Span}\left(u_{d+1,0}, u_{d+1,1}\right) & \text{for } d = m+2, \\ \operatorname{Span}\left(u_{3,0}, u_{3,3}, 3(1-\rho)u_{3,1} + 2v_{3,0}\right) & \text{for } d = 2. \end{cases}$$

It then follows that every formal germ of the form

$$F(z,w) = \left(z - \left(1 + \frac{2}{m}\right)z^2 + O_3, w - \frac{2}{m}zw + O_3\right)$$

with $m \in \mathbb{N}^*$ odd has as second order normal form

$$G(z,w) = \left(z - \left(1 + \frac{2}{m}\right) z^2 + \varphi(w) + a_0 z^3 + (1-\rho) z (w\psi'(w) + \psi(w)), \right.$$
$$\left. w - \frac{2}{m} zw + (1-\rho) w^2 \psi'(w) + 2\rho w \psi(w) \right) ,$$

where $\varphi \in \mathbb{C}[\![\zeta]\!]$ is a power series of order at least 3, $\psi \in \mathbb{C}[\![\zeta]\!]$ is a power series of order at least 2, and $a_0 \in \mathbb{C}$.

Now let us consider the case $\rho = -1/n \in \mathcal{E} \setminus \mathcal{F}$. In this case the coefficients in the expression of $L(u_{d,j})$ are always different from zero (with the exception of $d = j = 2$), whereas

$$d - j - d\rho + \rho - 1 = 0 \quad \Longleftrightarrow \quad j = d = n + 1 .$$

It follows that

$$\operatorname{Im} L|_{\mathcal{H}^d}$$
$$= \begin{cases} \operatorname{Span}\big((d - d\rho + 2\rho) u_{d+1,1} + (\rho - 1) v_{d+1,0}, u_{d+1,2}, \ldots, u_{d+1,d+1}, \\ \qquad v_{d+1,1}, \ldots, v_{d+1,d+1}\big) & \text{for } d \geq 3,\ d \neq n+1, \\ \operatorname{Span}\big((d - d\rho + 2\rho) u_{d+1,1} + (\rho - 1) v_{d+1,0}, u_{d+1,2}, \ldots, u_{d+1,d+1}, \\ \qquad v_{d+1,1}, \ldots, v_{d+1,d}\big) & \text{for } d = n+1, \\ \operatorname{Span}\left(2u_{3,1} + (\rho-1) v_{3,0}, u_{3,2}, v_{3,1}, v_{3,2}, v_{3,3}\right) & \text{for } d = 2, \end{cases}$$

and thus

$$(\operatorname{Im} L|_{\mathcal{H}^d})^\perp$$
$$= \begin{cases} \operatorname{Span}\left(u_{d+1,0}, (1-\rho)(d+1) u_{d+1,1} + (d - d\rho + 2\rho) v_{d+1,0}\right) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } d \geq 3,\ d \neq n+1, \\ \operatorname{Span}\left(u_{d+1,0}, v_{d+1,d+1}, (1-\rho)(d+1) u_{d+1,1} + (d - d\rho + 2\rho) v_{d+1,0}\right) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } d = n+1, \\ \operatorname{Span}\left(u_{3,0}, u_{3,3}, 3(1-\rho) u_{3,1} + 2 v_{3,0}\right) \qquad\qquad \text{for } d = 2. \end{cases}$$

It then follows that every formal germ of the form

$$F(z,w) = \left(z + \frac{1}{n} z^2 + O_3, w + \left(1 + \frac{1}{n}\right) zw + O_3 \right)$$

with $n \in \mathbb{N}^*$ has as second order normal form

$$G(z,w) = \left(z + \frac{1}{n} z^2 + \varphi(w) + a_0 z^3 + (1-\rho) z (w\psi'(w) + \psi(w)), \right.$$
$$\left. w + \left(1 + \frac{1}{n}\right) zw + \psi(z) + a_1 z^{n+2} + (1-\rho) w^2 \psi'(w) + 2\rho w \psi(w) \right) ,$$

where $\varphi \in \mathbb{C}[\![\zeta]\!]$ is a power series of order at least 3, $\psi \in \mathbb{C}[\![\zeta]\!]$ is a power series of order at least 2, and a_0, $a_1 \in \mathbb{C}$.

Let us now discuss the extreme case $\rho = 1$. It is clear that

$$\operatorname{Im} L|_{\mathcal{H}^d} = \operatorname{Span}\left(u_{d+1,1}, u_{d+1,2}, u_{d+1,4}, \dots, u_{d+1,d+1}, v_{d+1,2}, \dots, v_{d+1,d+1}\right) ,$$

and hence

$$(\operatorname{Im} L|_{\mathcal{H}^d})^\perp = \operatorname{Span}\left(u_{d+1,0}, u_{d+1,3}, v_{d+1,0}, v_{d+1,1}\right) ,$$

It then follows that every formal germ of the form

$$F(z,w) = \bigl(z - z^2 + O_3, w + O_3\bigr)$$

has as second order normal form

$$G(z,w) = \bigl(z - z^2 + \varphi_1(w) + z^3\psi(w), w + \varphi_2(w) + z\varphi_3(w)\bigr) ,$$

where $\varphi_1, \varphi_2 \in \mathbb{C}[\![\zeta]\!]$ are power series of order at least 3, $\varphi_3 \in \mathbb{C}[\![\zeta]\!]$ is a power series of order at least 2, and $\psi \in \mathbb{C}[\![\zeta]\!]$ is a power series.

We are left with the case $\rho \in (0,1) \cap \mathbb{Q}$. Write $\rho = a/b$ with a, $b \in \mathbb{N}$ coprime and $0 < a < b$. Now

$$d - j - 1 - \frac{a}{b}(d-1) = 0 \quad \Longleftrightarrow \quad j = \frac{(d-1)(b-a)}{b} ;$$

since a and b are coprime, this happens if and only if $d = b\ell + 1$ and $j = (b-a)\ell$ for some $\ell \geq 1$. Analogously,

$$d - j - \frac{a}{b}(d-2) = 0 \quad \Longleftrightarrow \quad j = d - \frac{a(d-2)}{b} ;$$

again, being a and b coprime, this happens if and only if $d = b\ell+2$ and $j = (b-a)\ell+2$ for some $\ell \geq 0$. It follows that

$$\operatorname{Im} L|_{\mathcal{H}^d} = \begin{cases} \operatorname{Span}\bigl((d - d\rho + 2\rho)u_{d+1,1} + (\rho-1)v_{d+1,0}, u_{d+1,2}, \dots, u_{d+1,d+1}, \\ \qquad v_{d+1,1}, \dots, v_{d+1,d+1}\bigr) & \text{for } d \geq 3,\ d \not\equiv 1,\ 2 \pmod b \\ \operatorname{Span}\bigl((d - d\rho + 2\rho)u_{d+1,1} + (\rho-1)v_{d+1,0}, \\ \qquad u_{d+1,2}, \dots, \widehat{u_{d+1,(b-a)\ell+2}}, \dots, u_{d+1,d+1}, \\ \qquad v_{d+1,1}, \dots, \widehat{v_{d+1,(b-a)\ell+1}}, \dots, v_{d+1,d+1}, \\ \qquad \frac{a}{b}u_{d+1,(b-a)\ell+2} - \left(\frac{a}{b} - 1\right) v_{d+1,(b-a)\ell+1}\bigr) & \text{for } d = b\ell+1, \\ \operatorname{Span}\bigl((d - d\rho + 2\rho)u_{d+1,1} + (\rho-1)v_{d+1,0}, \\ \qquad u_{d+1,2}, \dots, \widehat{u_{d+1,(b-a)\ell+3}}, \dots, u_{d+1,d+1}t \\ \qquad v_{d+1,1}, \dots, v_{d+1,d+1}\bigr) & \text{for } d = b\ell+2, \\ \operatorname{Span}\left(2u_{3,1} + (\rho-1)v_{3,0}, u_{3,2}, v_{3,1}, v_{3,2}, v_{3,3}\right) & \text{for } d = 2, \end{cases}$$

(where the hat indicates that that term is missing from the list), and thus

$$(\operatorname{Im} L|_{\mathcal{H}^d})^\perp = \begin{cases} \operatorname{Span}\bigl(u_{d+1,0}, \\ \qquad (1-\rho)u_{d+1,1} + (d - d\rho + 2\rho)v_{d+1,0}\bigr) & \text{for } d \geq 3,\ d \not\equiv 1,\ 2 \pmod b, \\ \operatorname{Span}\bigl((b-a)(a\ell+1)u_{d+1,(b-a)\ell+2} + a\bigl((b-a)\ell+2\bigr)v_{d+1,(b-a)\ell+1}, u_{d+1,0}, \\ \qquad (1-\rho)u_{d+1,1} + (d - d\rho + 2\rho)v_{d+1,0}\bigr) & \text{for } d = b\ell+1, \\ \operatorname{Span}\bigl(u_{d+1,0}, u_{d+1,(b-a)\ell+3}, \\ \qquad (1-\rho)u_{d+1,1} + (d - d\rho + 2\rho)v_{d+1,0}\bigr) & \text{for } d = b\ell+2, \\ \operatorname{Span}\left(u_{3,0}, u_{3,3}, 3(1-\rho)u_{3,1} + 2v_{3,0}\right) & \text{for } d = 2. \end{cases}$$

It then follows that every formal germ of the form

$$F(z,w) = \left(z - \frac{a}{b}z^2 + O_3, w + \left(1 - \frac{a}{b}\right) zw + O_3\right)$$

with $a/b \in (0,1) \cap \mathbb{Q}$ and a, b coprime, has as second order normal form

$$\begin{aligned} &G(z,w) \\ &\quad = \Bigg(z - \frac{a}{b}z^2 + \varphi(w) + z^3\varphi_0(z^{b-a}w^a) + (b-a)\frac{\partial}{\partial w}\big(z^2 w\chi(z^{b-a}w^a)\big) \\ &\qquad\qquad + \left(1 - \frac{a}{b}\right) z(w\psi'(w) + \psi(w)), \\ &\qquad w + \left(1 - \frac{a}{b}\right) zw + a\frac{\partial}{\partial z}\big(z^2 w\chi(z^{b-a}w^a)\big) + \left(1 - \frac{a}{b}\right) w^2\psi'(w) + 2\frac{a}{b}w\psi(w)\Bigg), \end{aligned}$$

where φ, $\psi \in \mathbb{C}[\![\zeta]\!]$ are power series of order at least 3, and φ_0, $\chi \in \mathbb{C}[\![\zeta]\!]$ are power series of order at least 1 — and we have proved Proposition 1.3.

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