HOMOGENEOUS VECTOR FIELDS AND MEROMORPHIC CONNECTIONS

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This note describes how to use geodesics of meromorphic connections to study real integral curves of homogeneous vector fields in $\mathbb{C}^2$.

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1. Introduction

In this short note we shall summarize a method recently introduced[1] to study the *real* dynamics of *complex* homogeneous vector fields. Besides its intrinsic interest, this is an useful problem to study because the discrete dynamics of the time 1-map is encoded in the real integral curves of the vector field, and time 1-maps of homogeneous vector fields are prototypical examples of holomorphic maps tangent to the identity at the origin (that is, of holomorphic self-maps $f\colon \mathbb{C}^n \to \mathbb{C}^n$ with $f(O) = O$ and $df_O = \mathrm{id}$). Indeed, Camacho[2,3] has proved that every (germ of a) holomorphic self-map tangent to the identity in $\mathbb{C}$ is locally topologically conjugated to the time-1 map of a homogeneous vector field, and it is natural to conjecture that such a statement should hold for generic holomorphic self-maps in several variables too; so understanding the real dynamics of complex homogeneous vector fields will go a long way toward the understanding of the dynamics of holomorphic self-maps tangent to the identity in a full neighborhood of the origin, one of the main open problems in contemporary local dynamics in several complex variables.

The main idea is that, roughly speaking, integral curves for homogeneous vector fields are geodesics for a meromorphic connection on a projective space. To understand the dynamics of geodesics of meromorphic connections is another very interesting problem, and it naturally splits in two parts: study of the global dynamics of geodesics (e.g., recurrence properties and Poincaré-Bendixson-like theorems), and study of the local dynamics nearby the poles of the connection (via normal forms and local conjugacies).

Due to space limitations, we shall describe our results in dimension 2 only; but part of the construction can be extended to any dimension, up to replace meromorphic connections by partial meromorphic connections. See Refs. 1,4 for details and proofs.

2. The construction

A *homogeneous vector field* of *degree* $\nu+1\geq 2$ in $\mathbb{C}^2$ is a vector field of the form

$$Q=Q^1\frac{\partial}{\partial z^1}+Q^2\frac{\partial}{\partial z^2}\;,$$

where Q^1, Q^2 are homogeneous polynomials of degree $\nu+1$ in two complex variables. A homogeneous vector field Q is *dicritical* if it is of the form

$$Q=P_\nu(z)\left(z^1\frac{\partial}{\partial z^1}+z^2\frac{\partial}{\partial z^2}\right)$$

where P_ν is a homogeneous polynomial of degree ν; *non-dicritical* otherwise.

Let $[\cdot]\colon\mathbb{C}^n\setminus\{O\}\to\mathbb{P}^1(\mathbb{C})$ be the canonical projection. A direction $[v]\in\mathbb{P}^1(\mathbb{C})$ is a *characteristic direction* for a homogeneous vector field Q if the complex line $L_{[v]}=\mathbb{C}v$ is Q-invariant (and then $L_{[v]}$ is a *characteristic line* of Q). We shall moreover say that a characteristic direction $[v]$ is *degenerate* if $Q|_{L_{[v]}}\equiv O$, and *non-degenerate* otherwise. It is easy to check that all directions are characteristic if Q is dicritical, and that a non-dicritical homogeneous vector field only has a finite number of characteristic directions. The dynamics on a characteristic line is one-dimensional, and very easy to study; so from now on we shall deal with non-dicritical homogeneous vector fields only, and we shall mainly be interested in the dynamics outside characteristic lines.

Let $\pi\colon M\to\mathbb{C}^2$ be the blow-up of the origin in $\mathbb{C}^2$, with exceptional divisor $E=\mathbb{P}^1(\mathbb{C})$. Let $p\colon N_E^{\otimes\nu}\to E$ be the ν-th tensor power of the normal bundle of E into M. There exists a natural ν-to-one holomorphic covering map $\chi_\nu\colon\mathbb{C}^2\setminus\{O\}\to N_E^{\otimes\nu}\setminus E$ generalizing the usual biholomorphism between $\mathbb{C}^2\setminus\{O\}$ and $N_E\setminus E=M\setminus E$: in the coordinates (ζ,v) induced by the canonical chart of M in $\pi^{-1}(z^1\neq 0)$ the map χ_ν is given by $\zeta(z)=z^2/z^1$ and $v(z)=(z^1)^\nu$, where ζ is the coordinate on $E=\mathbb{P}^1(\mathbb{C})$

and v is the coordinate on the fiber of $N_E^{\otimes\nu}$; in particular, $p\circ\chi_\nu(z)=[z]$ for all $z\in\mathbb{C}^2\setminus\{O\}$.

The homogeneity of Q has a first important consequence: whereas the push-forward of a vector field in general is not a vector field, the push-forward $d\chi_\nu(Q)$ of Q by χ_ν is a global holomorphic vector field G defined on the total space of $N_E^{\otimes\nu}$, and vanishing only on the zero section and on the fibers over the degenerate characteristic directions.

The point is that G is, in a suitable sense, the geodesic field of a meromorphic connection. To explain why, we need two more objects. First of all, using Q it is possible to define a global morphism $X_Q\colon N_E^{\otimes\nu}\to TE$ vanishing only over the characteristic directions of Q (and hence it gives an isomorphism between $N_S^{\otimes\nu}$ and TS, where $S\subset E$ is the complement in E of the characteristic directions). If we denote by ∂_1 the local generator of N_E in the canonical chart of M in $\pi^{-1}(\{z^1\neq 0\})$, and by $\partial/\partial\zeta$ the local generator of TE, the local expression of X_Q is

$$X_Q(\partial_1^{\otimes\nu})=[Q^2(1,\zeta)-\zeta Q^1(1,\zeta)]\frac{\partial}{\partial\zeta}\ .$$

Notice that $[1:\zeta]$ is a characteristic direction of Q if and only if $Q^2(1,\zeta)-\zeta Q^1(1,\zeta)=0$.

Furthermore, we can also define a meromorphic connection ∇ on $N_E^{\otimes\nu}$. The global definition of ∇ is a bit involved;[1,4] in the usual coordinates is locally expressed by

$$\nabla_{\partial/\partial\zeta}\partial_1^{\otimes\nu}=-\frac{\nu Q^1(1,\zeta)}{Q^2(1,\zeta)-\zeta Q^1(1,\zeta)}\partial_1^{\otimes\nu}\ .$$

Notice that ∇ is actually holomorphic on S; its poles are contained in the characteristic directions.

Mixing X_Q and ∇ we can get a linear connection, that is a connection ∇^o defined on the tangent space of S; it suffices to set

$$\nabla^o_v w=\nabla_v X_Q^{-1}(w)$$

for any tangent vector fields v and w on S. It is clear that ∇^o is a holomorphic linear connection on S (and a meromorphic linear connection on E); we can then use it to define the notion of geodesic in this context. A smooth curve $\sigma\colon I\to S$, where $I\subseteq\mathbb{R}$ is an interval, is a *geodesic* for ∇^o if σ' is ∇^o-parallel, that is $\nabla^o_{\sigma'}\sigma'\equiv O$. In local coordinates, this equation is equivalent to the clearly geodesic-looking equation

$$\sigma''+(k\circ\sigma)(\sigma')^2=0\ ,$$

where k is the meromorphic function defined by $\nabla^o_{\partial/\partial\zeta}(\partial/\partial\zeta) = k(\partial/\partial\zeta)$. Again, the poles of ∇^o are contained in the set of characteristic directions of Q. Furthermore, to each pole p of ∇^o is associated a *residue* $\mathrm{Res}_p(\nabla^o) \in \mathbb{C}$, locally defined as the residue of the meromorphic function k just introduced (but again the definition is independent of the local coordinates). Similarly, one can define the residue $\mathrm{Res}_p(\nabla) \in \mathbb{C}$ of ∇ at a pole $p \in E$; the difference between the two residues is given by the order of vanishing of X_Q.

The relations between integral curves of Q, integral curves of G and geodesics of ∇^o is summarized by the following:

Proposition 2.1. *Let Q be a non-dicritical homogeneous vector field in $\mathbb{C}^2$, and let $\hat{S}_Q$ be the complement in $\mathbb{C}^2$ of the characteristic lines of Q. Then for a real curve $\gamma\colon I \to \hat{S}_Q$ the following are equivalent:*

(i) *γ is an integral curve in $\mathbb{C}^2$ of Q;*
(ii) *$\chi_\nu \circ \gamma$ is an integral curve in $N_S^{\otimes\nu}$ of the geodesic field G;*
(iii) *$[\gamma]$ is a geodesic in S for the induced connection ∇^o.*

The big advantage of this approach is that we can now bring into play the differential geometry machinery developed to study geodesics of connections. It is true that the connection ∇^o in general is *not* globally induced by a metric, and thus the theory of our geodesics is subtly different from the usual theory of metric geodesics. However, ∇^o is locally induced by a conformal family of *flat* metrics, and the flatness enables the use of global results like the Gauss-Bonnet theorem. Furthermore, we can use the residues of ∇^o to express the relations between the holomorphic structure and the behavior of geodesics. All of this yields a fairly complete description of the recurrence properties of the geodesics, that is a Poincaré-Bendixson theorem for meromorphic connections:

Theorem 2.1. *Let $\sigma\colon [0,\varepsilon_0) \to S$ be a maximal geodesic for a meromorphic connection ∇^o on $\mathbb{P}^1(\mathbb{C})$, where $S = \mathbb{P}^1(\mathbb{C}) \setminus \{p_0, \ldots, p_r\}$ and $p_0, \ldots, p_r$ are the poles of ∇^o. Then either*

(i) *$\sigma(t)$ tends to a pole of ∇^o as $t \to \varepsilon_0$; or*
(ii) *σ is closed, and then surrounds poles $p_1, \ldots, p_g$ with $\sum_{j=1}^{g} \mathrm{Re}\,\mathrm{Res}_{p_j}(\nabla^o) = -1$; or*
(iii) *the ω-limit set of σ in $\mathbb{P}^1(\mathbb{C})$ is given by the support of a closed geodesic surrounding poles $p_1, \ldots, p_g$ with $\sum_{j=1}^{g} \mathrm{Re}\,\mathrm{Res}_{p_j}(\nabla^o) = -1$; or*

(iv) *the ω-limit set of σ in $\mathbb{P}^1(\mathbb{C})$ is a simple cycle of saddle connections (see below) surrounding poles $p_1, \dots, p_g$ with $\sum\limits_{j=1}^{g} \operatorname{Re}\operatorname{Res}_{p_j}(\nabla^o) = -1$; or*

(v) *σ intersects itself infinitely many times, and in this case every simple loop of σ surrounds a set of poles whose sum of residues has real part belonging to $(-3/2, -1) \cup (-1, -1/2)$.*

In particular, a recurrent geodesic either intersects itself infinitely many times or is closed.

In this statement, a *saddle connection* is a geodesic connecting two (not necessarily distinct) poles of ∇^o; and a *simple cycle of saddle connections* is a Jordan curve composed of saddle connections. Notice furthermore that a closed geodesic is not necessarily periodic: it is if and only if the sum of the imaginary parts of the residues at the poles it surrounds is zero.

As a consequence, we get a Poincaré-Bendixson theorem for homogeneous vector fields:

Theorem 2.2. *Let Q be a homogeneous holomorphic vector field on $\mathbb{C}^2$ of degree $\nu + 1 \geq 2$, and let $\gamma\colon [0, \varepsilon_0) \to \mathbb{C}^2$ be a recurrent maximal integral curve of Q. Then γ is periodic or $[\gamma]\colon [0, \varepsilon_0) \to \mathbb{P}^1(\mathbb{C})$ intersect itself infinitely many times.*

Proposition 2.1 and Theorem 2.1 are very helpful in describing the global behavior of integral curves away from the characteristic lines; to complete the picture we need to know what happens nearby the characteristic lines. It turns out that the best way of solving this problem is by studying the integral curves of G nearby the fibers over the characteristic directions; the advantage here is that G extends holomorphically everywhere, and this makes the local study easier.

The characteristic directions can be subdivided in three classes: the *apparent singularities*, which are the characteristic directions which are not poles of ∇, the *Fuchsian singularities*, which are poles of ∇ of order 1, and the *irregular singularities,* which are poles of ∇ of order greater than 1. Fuchsian singularities are generic; and non-degenerate characteristic directions are Fuchsian singularities. We have a complete formal description of all kinds of singularities, and a complete holomorphic description of Fuchsian and apparent singularities. For instance, the holomorphic classification of Fuchsian singularities, revealing in particular the existence of resonance phenomena, is the following

Theorem 2.3. *Let $z_0 \in \mathbb{P}^1(\mathbb{C})$ be a Fuchsian pole of ∇, that is assume* *that in local coordinates* (U_α, z_α) *centered at* z_0 *we can write*

$$G = z_\alpha^\mu (a_0 + a_1 z_\alpha + \cdots)\partial_\alpha - z_\alpha^{\mu-1}(b_0 + b_1 z_\alpha + \cdots)\frac{\partial}{\partial v_\alpha} ,$$

with $\mu \geq 1$ *and* a_0, $b_0 \neq 0$. *Put* $\rho = b_0/a_0$. *Then* μ *and* ρ *are (formal and) holomorphic invariants, and we can find a chart* (U, z) *centered in* p_0 *in which* G *is given by*

$$z^{\mu-1}\left(zv\partial - \rho v^2 \frac{\partial}{\partial v}\right)$$

if $\mu - 1 - \rho \notin \mathbb{N}^*$, *or by*

$$z^{\mu-1}\left(zv\partial - \rho(1 + az^n)v^2 \frac{\partial}{\partial v}\right)$$

for a suitable $a \in \mathbb{C}$ *(another formal and holomorphic invariant) if* $n = \mu - 1 - \rho \in \mathbb{N}^*$.

Putting together all previous results (and several similar results proved in Ref. 1) one gets a fairly complete description of the dynamics of a large class of homogeneous vector fields. An example of statement we are able to prove is the following:

Theorem 2.4. *Let* Q *be a non-dicritical homogeneous vector field on* $\mathbb{C}^2$ *of degree* $\nu+1 \geq 2$. *Assume that all characteristic directions of* Q *are Fuchsian singularities of order* 1 *(this is the generic case). Assume moreover that for no set of characteristic directions the real part of the sum of the residues of* ∇^o *is equal to* -1. *Let* $\gamma\colon [0, \varepsilon_0) \to \mathbb{C}^2$ *be a maximal integral curve of* Q. *Then:*

(a) *If* $\gamma(0)$ *belongs to a characteristic line* L *then the image of* γ *is contained in* L. *Moreover, either* $\gamma(t) \to O$ *(and this happens for a Zariski open dense set of initial conditions), or* $\|\gamma(t)\| \to +\infty$.

(b) *If* $\gamma(0)$ *does not belong to a characteristic line, then either*

  (i) γ *converges to the origin tangentially to a characteristic direction whose residue with respect to* ∇ *has negative real part; or*

  (ii) $\|\gamma(t)\| \to +\infty$ *tangentially to a characteristic direction whose residue with respect to* ∇ *has positive real part; or*

  (iii) $[\gamma]\colon [0, \varepsilon_0) \to \mathbb{P}^1(\mathbb{C})$ *intersects itself infinitely many times.*

  Furthermore, if (iii) never occurs then (i) holds for a Zariski open dense set of initial conditions.

In this theorem, the assumption on the sum of the residues of ∇^o is used just to exclude closed geodesics or simple cycles of saddle connections with the aim of simplifying the statement, but we have a fairly good understanding of the dynamics in those cases too. For instance, we have examples of homogeneous vector fields with periodic integral curves of arbitrarily high period accumulating the origin — and thus of holomorphic self-maps tangent to the identity with periodic orbits of arbitrarily high period accumulating the origin, an unexpected phenomenon the cannot happen in one variable. Furthermore, since the only constraint on the residues of ∇^o is that their sum must be -2, using Theorem 2.1.(v) it is easy to construct a large class of homogeneous vector fields with only Fuchsian singularities of order 1 where the case (b.iii) in Theorem 2.4 cannot occur; and so for this large class of homogeneous vector fields we have a complete description of the dynamics. To have a complete description of the dynamics of all homogeneos vector fields in $\mathbb{C}^2$ it remains to understand better what happens for irregular singularities and when there are geodesics intersecting themselves infinitely often; and we plan to attack these problems in future papers.

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