Isometries of the Teichmüller metric

by Marco Abate[1]† and Giorgio Patrizio[2]†

[1] Dipartimento di Matematica, Università di Ancona, via Brecce Bianche, 60131 Ancona, Italy
[2] Dipartimento di Matematica "U. Dini", Università di Firenze, Viale Morgagni 67/A, 50137 Firenze, Italy

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0. Introduction

The aim of this paper is to apply some ideas coming from several complex variables and from complex differential geometry to questions about Teichmüller spaces.

Differential geometrical techniques have played a significant role in the study Teichmüller spaces. In fact, at least since the work of Kravets ([Kra]), it was understood that the Teichmüller metric — though not Riemannian — may be considered in the framework of differential geometry as a Finsler metric. The difficulty of this approach has advised to pursue other directions which led for instance to the definition of the Weil-Petersson metric which, being Kählerian, behaves much better and it is quite useful in a number of applications. On the other hand, the fundamental work of Royden ([R1]), who realized that the Teichmüller metric is exactly the Kobayashi metric of a Teichmüller space (see Gardiner [G] for the infinite dimensional case), shows that the Teichmüller metric not only is naturally defined, but it is also deeply related to the complex structure. For instance, as a consequence of this result, Royden was able to compute the automorphism group of finite dimensional Teichmüller spaces. Furthermore Royden (again [R1]) proved that, in the finite dimensional case, the Kobayashi-Teichmüller metric has a certain amount of smoothness which makes reasonable to address in terms of curvature questions such as existence of isometries. In this regard Royden conjectured that Teichmüller disks, which are isometries at one point from the unit disk into a Teichmüller space, are in fact global isometries with respect to the hyperbolic distance of the disk and the Teichmüller distance. In relation to this conjecture he studied the geometry of complex Finsler metrics ([R2]) and, under some further assumptions, was able to prove that Teichmüller disks are infinitesimal isometries at every point. The conjecture was later fully proved in [EKK] using different methods.

Motivated mainly by the goal of achieving a better understanding of the Kobayashi metric and of its applications in function theory, recently we made an effort to develop an efficient approach to complex Finsler geometry (see [AP]). Some progresses made in this direction suggested to return to Royden's original ideas and search for new possible applications in Teichmüller theory. First of all it is of interest to understand exactly the role of the curvature of Teichmüller metric. It is known that it is not true that Teichmüller spaces have nonpositive sectional curvature, and so they are not "hyperbolic" in the usual, "real", sense ([MW]). But, on the other hand, they have several properties

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in common with hyperbolic manifolds. It turns out that Teichmüller spaces have constant negative holomorphic curvature and therefore are definitely (Kobayashi) hyperbolic from the complex point of view. Here we provide a proof (both in the finite and in the infinite dimensional case) for this fact, that may already be known but does not seem to be stated in the literature. As a consequence, without unnecessary hypotheses, it is easy to show that Teichmüller disks are infinitesimal isometries. As in other situations ([AP]), for topological reasons, this is not enough to conclude that they are isometries for the distances and therefore it does not seem to be not possible to recover the full Royden conjecture via differential geometry only.

Another useful guideline is to remember that the Bers realization of Teichmüller spaces is a pseudoconvex domain in a complex (Banach) space and therefore it carries a nice complex analytic function theory. Inspired by similar results in several complex variables we tried to characterize Teichmüller spaces in terms of isometries of intrinsic metrics. Generalizing Royden's theorem ([R1]) on isometries, our main result in the finite dimensional case is the following:

Theorem 0.1: *A taut connected complex manifold N is biholomorphic to a finite dimensional Teichmüller space $T(\Gamma)$ if and only if there exists a holomorphic map $F\colon N \to T(\Gamma)$ which is an isometry for the Kobayashi metric at one point.*

The proof depends strongly on the uniqueness of extremal disks, and so cannot be applied to the infinite dimensional case, where, as additional difficulty, a satisfactory theory of holomorphic mappings is yet to be developed. Nevertheless, again in the vein of Royden's work, we have the following partial result.

Theorem 0.2: *Let $T(\Gamma_1)$ and $T(\Gamma_2)$ be infinite dimensional Teichmüller spaces. Then a holomorphic map $F\colon T(\Gamma_1) \to T(\Gamma_2)$ is biholomorphic if and only if it satisfies the following assumptions:*

(i) *F is a Fredholm map of index 0, i.e., $\dim \operatorname{Ker} dF_x = \dim \operatorname{Coker} dF_x < \infty$ for all $x \in M$;*
(ii) *F has discrete fibers and closed image;*
(iii) *F is an isometry for the Kobayashi-Teichmüller metric at one point.*

Most likely Theorem 0.2 may be improved, but to do so it is necessary a better understanding of both function theory and differential geometry of Teichmüller spaces in the infinite dimensional case. The latter could be quite illuminating although there is the additional difficulty that in this case the Teichmüller metric is not even C^1 ([Zh]).

We feel that it may be fruitful to further pursue the application in Teichmüller theory of ideas coming from several complex variables and complex differential geometry. For instance, the Bers realization of a finite dimensional Teichmüller space is a very interesting bounded pseudoconvex topologically trivial domain in $\mathbb{C}^n$ whose function theory deserves further study on his own.

The paper is organized as follows. In Section 1 there is a quick outline of the necessary notions of complex Finsler geometry. Section 2 and 3 are devoted to the study of the Kobayashi-Teichmüller metric with regard to curvature and complex geodesics. Precise statement and proofs of Theorem 0.1 and 0.2 are in Section 4.

1. Complex Finsler metrics

We shall need some facts about complex Finsler geometry and intrinsic metrics on complex manifolds of finite and infinite dimension. Most of them are standard, but not easy to find in the literature for the infinite dimensional case; so, for reader's convenience and to set notations, in this section we give a short overview of the subject.

A *complex Finsler metric* F on a complex (Banach) manifold M is an upper semicontinuous function $F: T^{1,0}M \to \mathbb{R}^+$ satisfying

(i) $F(p; v) > 0$ for all $p \in M$ and $v \in T_p^{1,0}M$ with $v \neq 0$;
(ii) $F(p; \lambda v) = |\lambda| F(p; v)$ for all $p \in M$, $v \in T_p^{1,0}M$ and $\lambda \in \mathbb{C}$.

We shall systematically denote by $G: T^{1,0}(M) \to \mathbb{R}^+$ the function $G = F^2$. Using condition (ii) and the usual identification between real and holomorphic tangent bundles, the definition of length of a smooth curve in a Riemannian manifold makes sense in this context too; so we may again associate to F a topological distance on M, and we shall say that F is *complete* if this distance is. For the same reason, it makes sense to call (*real*) *geodesics* the extremals of the length functional. For an introduction to real and complex Finsler geometry (in the finite dimensional case) we refer to [AP].

An important role in the study of complex Finsler metric is played by the notion of holomorphic curvature. Let us start by considering the case of the unit disk Δ in the complex plane. A *pseudohermitian metric* μ_g of *scale* g on Δ is the upper semicontinuous pseudometric on the tangent bundle of Δ defined by

$$\mu_g = g\, d\zeta \otimes d\bar{\zeta}, \tag{1.1}$$

where $g: \Delta \to \mathbb{R}^+$ is a non-negative upper semicontinuous function such that $S_g = g^{-1}(0)$ is a discrete subset of Δ.

If g is a C^2 positive function (i.e., μ_g is a standard hermitian metric on Δ), then the Gaussian curvature of μ_g is defined by

$$K(\mu_g) = -\frac{1}{2g} \triangle \log g, \tag{1.2}$$

where $\triangle$ denotes the usual Laplacian

$$\triangle u = 4\, \frac{\partial^2 u}{\partial\zeta \partial\bar{\zeta}}. \tag{1.3}$$

In the general case (cf. [He]) we shall consider the (*lower*) *generalized Laplacian* of an upper semicontinuous function u defined by

$$\triangle u(\zeta) = 4 \liminf_{r \to 0} \frac{1}{r^2} \left\{ \frac{1}{2\pi} \int_0^{2\pi} u(\zeta + re^{i\theta})\, d\theta - u(\zeta) \right\}. \tag{1.4}$$

It is well known that for a function u of class C^2 in a neighborhood of the point ζ_0 the Laplacian (1.4) actually reduces to (1.3).

Let μ_g be a pseudohermitian metric on Δ. Then the *Gaussian curvature* $K(\mu_g)$ of μ_g is the function defined on $\Delta \setminus S_g$ by (1.2), using the generalized Laplacian (1.4). In particular, if μ_g is a standard hermitian metric then $K(\mu_g)$ reduced to the usual Gaussian curvature. It should be noted that this notion of holomorphic curvature is completely equivalent to the classical one based on the consideration of supporting metrics (see [R2], [He]).

Now consider a complex manifold M with a complex Finsler metric F, and take a point $p \in M$ and a non-zero tangent vector $v \in T_p^{1,0}M$. The *holomorphic curvature* $K_F(p;v)$ of F at $(p;v)$ is given by

$$K_F(p;v) = \sup\{K(\varphi^*G)(0)\},$$

where the supremum is taken with respect to the family of all holomorphic maps $\varphi\colon \Delta \to M$ with $\varphi(0) = p$ and $\varphi'(0) = \lambda v$ for some $\lambda \in \mathbb{C}^*$, and $K(\varphi^*G)$ is the Gaussian curvature discussed so far of the pseudohermitian metric φ^*G on Δ.

Clearly, the holomorphic curvature depends only on the complex line spanned by v in $T_p^{1,0}M$, and not on v itself. Furthermore, the holomorphic curvature defined in this way is invariant under holomorphic isometries, and when F is a honest smooth hermitian metric on M it coincides with the usual holomorphic sectional curvature of F at $(p;v)$ (see [Wu]).

We shall use the classical Ahlfors lemma which compares a generic pseudohermitian metric with an extremal one (usually the Poincaré metric), and the important result of Heins [He, Theorem 7.1; see also R3] which takes care of the case of equality of the metrics at one point. For $a > 0$, let $g_a\colon \Delta \to \mathbb{R}^+$ be defined by

$$g_a(\zeta) = \frac{1}{a(1-|\zeta|^2)^2};$$

then $\mu_a = g_a\, d\zeta \otimes d\bar\zeta$ is a hermitian metric of constant Gaussian curvature $K(\mu_a) = -4a$. Of course, μ_1 is the standard Poincaré metric on Δ.

Then Ahlfors' and Heins' results may be stated as follows:

Proposition 1.1: (Ahlfors-Heins' Lemma) *Let $\mu_g = g\, d\zeta \otimes d\bar\zeta$ be a pseudohermitian metric on Δ such that $K(\mu_g) \le -4a$ on $\Delta \setminus S_g$ for some $a > 0$. Then $g \le g_a$. Assume there is $\zeta_0 \in \Delta \setminus S_g$ such that $g(\zeta_0) = g_a(\zeta_0)$. Then $\mu_g \equiv \mu_a$.*

The notion of holomorphic curvature given for complex Finsler metrics gives immediately a version of Ahlfors' Lemma even for the infinite dimensional case:

Proposition 1.2: *Let F be a complex Finsler metric on a complex manifold M. Assume that the holomorphic curvature of F is bounded above by a negative constant $-4a$, for some $a > 0$. Then*

$$\varphi^*F \le \mu_a \tag{1.5}$$

for all holomorphic maps $\varphi\colon \Delta \to M$.

Proof: By definition, φ^*F is a pseudohermitian metric on Δ; by assumption (and by the invariance of the Gaussian curvature under automorphisms of Δ), $K(\varphi^*F) \le -4a$. Then the assertion follows from Proposition 1.1. $\square$

As a consequence, we obtain a generalization of a well known criterion of hyperbolicity for the infinite dimensional case as well:

Corollary 1.3: *Let M be a complex manifold admitting a (complete) complex Finsler metric F with holomorphic curvature bounded above by a negative constant. Then M is (complete) hyperbolic.*

Proof: Up to multiplying F by a suitable constant, we may assume $K_F \leq -4$. Let d denote the distance induced by F on M, and ω the Poincaré distance on Δ. Then Proposition 1.2 yields

$$d\big(\varphi(\zeta_1), \varphi(\zeta_2)\big) \leq \omega(\zeta_1, \zeta_2),$$

for all ζ_1, $\zeta_2 \in \Delta$ and holomorphic maps $\varphi\colon \Delta \to M$. But this immediately implies (cf. [K, Proposition IV.1.4]) that the Kobayashi distance k_M of M is bounded below by d, and the assertion follows. □

For finite dimensional manifolds, Wong [W] and Suzuki [Su] have shown that the holomorphic curvature of the Carathéodory metric is bounded *above* by -4 for Carathéodory-hyperbolic manifolds, whereas the holomorphic curvature of the Kobayashi metric is bounded *below* by -4 for Kobayashi-hyperbolic manifolds.

Another interesting immediate consequence of Proposition 1.1 is an interpretation (even in the infinite dimensional case) in terms of curvature of a well known property of the Carathéodory metric. Let F be a complex Finsler metric on a manifold M. We shall say that a holomorphic map $\varphi\colon \Delta \to M$ is *infinitesimally extremal* at $\zeta_0 \in \Delta$ if it is an isometry at ζ_0 between the Poincaré metric on Δ and F, that is if

$$\varphi^* F(\zeta_0; 1) = F\big(\varphi(\zeta_0); \varphi'(\zeta_0)\big) = \frac{1}{1 - |\zeta_0|^2}.$$

We shall say that φ is an *infinitesimal complex geodesic* if it is infinitesimally extremal at every point of Δ. Then:

Proposition 1.4: *Let F be a complex Finsler metric on a manifold M with holomorphic curvature bounded above by -4. Let $\varphi\colon \Delta \to M$ be a holomorphic map. Then the following statements are equivalent:*

(i) *φ is infinitesimally extremal at one point $\zeta_0 \in \Delta$;*
(ii) *φ is an infinitesimal complex geodesic.*

Proof: By definition and the invariance of holomorphic curvature under automorphisms of the unit disk, the Gaussian curvature of $\varphi^* F$ is bounded above by -4. The assertion then follows from Ahlfors-Heins' Lemma. □

We close this section with a remark about the behavior of the holomorphic curvature for sequences of metrics which we shall need later.

Proposition 1.5: *Let F_k be a sequence of complex Finsler metrics on a manifold M with holomorphic curvature bounded above by -4 monotonically converging pointwise to a complex Finsler metric F. Then the holomorphic curvature of F is still bounded above by -4.*

Proof: It follows from the definition of holomorphic curvature and the monotone convergence theorem (cf. [He, 10.(c)]). □

2. Intrinsic metrics on the space of Beltrami differentials

Let $\mathbb{H}^+$ be the upper half plane in $\mathbb{C}$, and let M denote the unit ball in $L^\infty(\mathbb{H}^+,\mathbb{C})$. Given $\mu \in M$, let w_μ be the unique quasiconformal homeomorphism of $\mathbb{H}^+$ fixing the points 0, 1, ∞ and satisfying the Beltrami equation $w_{\bar z} = \mu w_z$. Then (see, e.g., [EE]) the *Teichmüller metric* $\sigma\colon T^{1,0}M \cong M \times L^\infty(\mathbb{H}^+,\mathbb{C}) \to \mathbb{R}^+$ is the complex Finsler metric on M defined by

$$\sigma(\mu;\nu) = \left\| \frac{|\nu|}{1-|\mu|^2} \right\|_\infty , \tag{2.1}$$

where $\|\cdot\|_\infty$ denotes the L^∞ norm; note that $|\nu(z)|/(1-|\mu(z)|^2)$ is the Poincaré length of the tangent vector $\nu(z)$ at the point $\mu(z) \in \Delta$. The *Teichmüller distance* on M is just the integrated distance d_σ of the Finsler metric σ. It is known that M with the Teichmüller metric is a complete Finsler manifold, and it is easy to check that

$$d_\sigma(\mu_1,\mu_2) = \tanh^{-1} \left\| \frac{\mu_1-\mu_2}{1-\overline{\mu_1}\mu_2} \right\|_\infty . \tag{2.2}$$

The group G of automorphisms of $\mathbb{H}^+$ acts naturally on M as a group of linear isometries via the action

$$\forall (A,\mu) \in G \times M \qquad (A,\mu) \mapsto \mu^A = \frac{(\mu \circ A)\bar{A}'}{A'}. \tag{2.3}$$

Now let Γ be a Fuchsian group, i.e., a subgroup of the automorphism group of $\mathbb{H}^+$ acting properly discontinuously on $\mathbb{H}^+$. Then

$$L^\infty(\Gamma) = \bigl\{ \mu \in L^\infty(\mathbb{H}^+,\mathbb{C}) \bigm| \mu = \mu^A \ \forall A \in G \bigr\} \tag{2.4}$$

is a closed subspace of $L^\infty(\mathbb{H}^+,\mathbb{C})$, and hence the space of *Beltrami differentials relative to* Γ defined by

$$M(\Gamma) = M \cap L^\infty(\Gamma) \tag{2.5}$$

is the unit ball in the Banach space $L^\infty(\Gamma)$. Evidently M is the space of Beltrami differentials relative to the trivial group. The Teichmüller metric and distance on $M(\Gamma)$ — which we denote again by σ and d_σ respectively — are obtained by restriction, and again $M(\Gamma)$ is a complete Finsler manifold.

It turns out that the Teichmüller metric and distance on $M(\Gamma)$ agree with the Kobayashi and the Carathéodory metric and distance. This fact, partly observed in [EKK, Proposition 1], is a direct consequence of more general results due to Harris [H] and Vesentini [V]. In fact it is also a simple corollary of a theorem of Dineen-Timoney-Vigué ([DTV]) which says that the Kobayashi metric (distance) agrees with the Carathéodory metric (distance) on any convex set in a complex Banach space. Here we state the result precisely and give a simple direct proof.

Theorem 2.1: *Let Γ be a Fuchsian group. Then the Teichmüller, Carathéodory and Kobayashi metrics (distances) of $M(\Gamma)$ coincide.*

Proof: First of all we claim that that it is enough to prove the result at the origin (i.e., to show that Teichmüller, Kobayashi and Carathéodory metrics agree at the origin and that Teichmüller, Kobayashi and Carathéodory distances from the the origin agree). To this end observe that for any $0 \neq \mu \in M(\Gamma)$ there exists a holomorphic automorphism F_μ of $M(\Gamma)$ defined by

$$F_\mu(\lambda) = \frac{\mu - \lambda}{1 - \bar{\mu}\lambda}.$$

Clearly $F_\mu(0) = \mu$; furthermore it is easy to check that F_μ is an isometry for both Teichmüller metric and distance, and it is an isometry for Kobayashi and Carathéodory metrics and distances being a biholomorphic map, so that our claim follows.

Now, it is well known (see for instance [A]) that the Kobayashi and Carathéodory metric of a unit ball in a complex Banach space X at the origin agree with the norm $\|\cdot\|$ of X — and thus with the Teichmüller metric at the origin if $X = M(\Gamma)$, by (2.1). Analogously, it is also known (see again [A]) that the Kobayashi and Carathéodory distance from the origin of the unit ball in X are given by $\tanh^{-1}\|\cdot\|$, and thus they agree with the Teichmüller distance from the origin if $X = M(\Gamma)$, by (2.2). $\square$

Let F be a complex Finsler metric on a manifold M. We shall say that a holomorphic map $\varphi\colon \Delta \to M$ is *extremal* at $\zeta_0 \in \Delta$ if

$$\forall \zeta \in \Delta \qquad\qquad d_F\big(\varphi(\zeta_0), \varphi(\zeta)\big) = \omega(\zeta_0, \zeta),$$

where d_F is the distance induced by F and ω is the Poincaré distance. We shall say that φ is a *complex geodesic* if it is extremal at all points of Δ, that is if it is a global isometry between the Poincaré distance and d_F.

An immediate consequence of Theorem 2.1 is the following:

Corollary 2.2: *Let Γ be a Fuchsian group, and $\varphi\colon \Delta \to M(\Gamma)$ a holomorphic map. Then the following statements are equivalent:*

(i) *φ is infinitesimally extremal at one point with respect to the Teichmüller metric of $M(\Gamma)$;*
(ii) *φ is an infinitesimal complex geodesic with respect to the Teichmüller metric of $M(\Gamma)$;*
(iii) *φ is extremal at one point with respect to the Teichmüller metric of $M(\Gamma)$;*
(iv) *φ is a complex geodesic with respect to the Teichmüller metric of $M(\Gamma)$.*

Proof: Since the four properties are equivalent for the Carathéodory metric ([V]), the claim is a consequence of Theorem 2.1. $\square$

3. Complex geodesics on Teichmüller spaces

Let us start by recalling a few facts about Teichmüller spaces, following [EE] and [G]. Let $\mathbb{H}^-$ denote the lower half plane in $\mathbb{C}$ and consider the Banach space $B = \mathrm{Hol}(\mathbb{H}^-, \mathbb{C})$ with the norm

$$||\phi||_B = \sup\big\{|z - \bar{z}|^2 |\phi(z)| \;\big|\; z \in \mathbb{H}^-\big\}. \tag{3.1}$$

Then $G = \mathrm{Aut}(\mathbb{H}^+) = \mathrm{Aut}(\mathbb{H}^-)$ acts on B as a group of linear isometries via the action $G \times B \to B$ defined by

$$(A, \phi) \mapsto \phi^A = (\phi \circ A)(A')^2. \tag{3.2}$$

If $\Gamma \subset G$ is a Fuchsian group, we denote by $B(\Gamma)$ the subspace of Γ-invariant functions, i.e., the subspace

$$B(\Gamma) = \{\phi \in B \mid \phi = \phi^A\}. \tag{3.3}$$

If $\mu \in M$ then there exists a unique homeomorphism w^μ of the Riemann sphere in itself which leaves 0, 1, ∞ fixed and such that w^μ is holomorphic on $\mathbb{H}^-$ and $w^\mu \circ (w_\mu)^{-1}$ is holomorphic on $\mathbb{H}^+$. Thanks to Nehari's theorem a map $\Phi: M \to B$ is well defined by $\Phi(\mu) = [w^\mu]$, where $[\,\cdot\,]$ denotes the Schwarzian derivative. The image $T = \Phi(M)$ of Φ is called the *universal Teichmüller space;* if Γ is a Fuchsian group then

$$T(\Gamma) = \Phi\bigl(M(\Gamma)\bigr) \subset B(\Gamma)$$

is called the *Teichmüller space* of Γ. It is known that this presentation of Teichmüller spaces is equivalent to the presentation as moduli spaces of Riemann surfaces. Furthermore Bers has proved that the map Φ is continuous and holomorphic and that the holomorphic and topological structures of $T(\Gamma)$ are just the quotient structure induced by $\Phi: M(\Gamma) \to T(\Gamma)$. We can then define the *Teichmüller metric* $\tau_\Gamma: T(\Gamma) \times B(\Gamma) \to \mathbb{R}$ on $T(\Gamma)$ using the quotient map Φ as follows:

$$\tau_\Gamma(t; \psi) = \inf\bigl\{\sigma(\mu; \nu) \bigm| \mu \in M(\Gamma),\, \nu \in L^\infty(\Gamma) \text{ with } t = \Phi(\mu),\, d\Phi_\mu(\nu) = \psi\bigr\}, \tag{3.4}$$

where σ is the Teichmüller metric on $M(\Gamma)$. Notice that (3.4) is well posed since σ is invariant under right translations (see [EE] for details). In an analogous way one defines the *Teichmüller distance* d_{τ_Γ}:

$$d_{\tau_\Gamma}(s, t) = \inf\bigl\{d_\sigma(\alpha, \beta) \bigm| \alpha, \beta \in M(\Gamma) \text{ with } s = \Phi(\alpha),\, t = \Phi(\beta)\bigr\}. \tag{3.5}$$

The Teichmüller distance is always complete. Furthermore, O'Byrne ([O]) proved that in fact d_{τ_Γ} is exactly the integrated distance of τ_Γ, as it is desirable. This is also a consequence of the famous result of Royden ([R1]) which states that the Teichmüller metric coincides with the Kobayashi metric of $T(\Gamma)$. Actually Royden proved the equality in the case of finite dimensional Teichmüller spaces only, but later Gardiner (see [G]), by means of an approximation argument, proved that the equality also holds in the infinite dimensional case. Gardiner's argument has a consequence that we shall need later on:

Proposition 3.1: *Let Γ be any Fuchsian group. Then the holomorphic curvature of the Kobayashi-Teichmüller metric of $T(\Gamma)$ is identically equal to -4. As a consequence, $T(\Gamma)$ is Kobayashi complete hyperbolic.*

Proof: If $T(\Gamma)$ is finite dimensional, then it is known that the holomorphic curvature of finite dimensional Teichmüller spaces is bounded above by -4 ([G, Lemma 7.8]). Hence, by Corollary 1.3, $T(\Gamma)$ is complete hyperbolic so that the holomorphic curvature must also be bounded below by -4 ([W], [Su]) and the claim follows. Let us assume that $T(\Gamma)$ is not finite dimensional. Then, by Gardiner's approximation procedure (see [G]), there exist a sequence $\{T_j\}$ of finite dimensional Teichmüller spaces and a sequence of holomorphic maps $\pi_j: T(\Gamma) \to T_j$ such that the pull-back metrics $\pi_j^* \tau_{\Gamma_j}$ monotonically converge to τ_Γ. Since, as already remarked, the holomorphic curvature of finite dimensional Teichmüller spaces is bounded above by -4, Proposition 1.5 implies that the holomorphic curvature of τ_Γ is bounded above by -4. In particular, by Corollary 1.3, $T(\Gamma)$ is complete hyperbolic in this case too.

To prove that the holomorphic curvature is bounded below by -4, fix $[\mu] \in T(\Gamma)$ and $\psi \in B(\Gamma) \cong T^{1,0}_{[\mu]} T(\Gamma)$. By [EE, Theorem 3.(c)], up to replacing Γ by an isomorphic Fuchsian group we can assume that $[\mu] = \Phi(0)$. Choose $\nu \in L^\infty(\Gamma)$ such that $\psi = d\Phi_0(\nu)$ and $\tau_\Gamma([\mu]; \psi) = \sigma(0; \nu)$; in other words, ν is infinitesimally extremal. Define $\tilde{\varphi}: \Delta \to M(\Gamma)$ by $\tilde{\varphi}(\zeta) = \zeta\nu / \|\nu\|_\infty$, and set $\varphi = \Phi \circ \tilde{\varphi}$. Then $\varphi(0) = [\mu]$ and $\tau_\Gamma\big(\varphi(0); \varphi'(0)\big) = 1$. By Proposition 1.4, then, $\varphi^* \tau_\Gamma$ is the Poincaré metric of Δ, which has Gaussian curvature identically -4. Thus the definition implies that the holomorphic curvature of τ_Γ at $([\mu]; \psi)$ is at least -4, and we are done. □

It is known that Teichmüller spaces have not nonpositive real sectional curvature, and that they are not hyperbolic in any reasonable real sense (see [MW] for instance). Nevertheless, as it has been underlined by many, Teichmüller spaces behave very much in a hyperbolic manner. We feel that the reason is purely a complex geometrical one, as illustrated by Proposition 3.1.

We are now able to complete Royden's program at the infinitesimal level:

Corollary 3.2: *Let Γ be a Fuchsian group, and $\varphi: \Delta \to T(\Gamma)$ a holomorphic map. Then the following statements are equivalent:*

(i) *φ is infinitesimally extremal at one point with respect to the Teichmüller metric of $T(\Gamma)$;*
(ii) *φ is an infinitesimal complex geodesic with respect to the Teichmüller metric of $T(\Gamma)$.*

Proof: It follows from Propositions 1.4 and 3.1. □

So it is clear that the hard part in Royden's program lies in passing from the metric to the distance; this has been accomplished in [EKK] where it is proved the analogous of Theorem 2.2 for $T(\Gamma)$ by means of the following lifting lemma: for any holomorphic map $\varphi: \Delta \to T(\Gamma)$ there exists a holomorphic map $\tilde{\varphi}: \Delta \to M(\Gamma)$ such that $\varphi = \Phi \circ \tilde{\varphi}$.

We end this section by discussing existence and uniqueness of (infinitesimal) complex geodesics:

Proposition 3.3: *Let Γ be a Fuchsian group. Then:*

(i) *for any point $[\mu] \in T(\Gamma)$ and tangent vector $\psi \in B(\Gamma)$ there exists an infinitesimal complex geodesic $\varphi: \Delta \to T(\Gamma)$ such that $\varphi(0) = [\mu]$ and $\varphi'(0)$ is a non-zero multiple of ψ. Furthermore, if $T(\Gamma)$ is finite dimensional then φ is uniquely determined.*
(ii) *for any couple of distinct points $[\mu_1]$, $[\mu_2] \in T(\Gamma)$ there exists a complex geodesic $\varphi: \Delta \to T(\Gamma)$ such that $\varphi(0) = [\mu_1]$ and $\varphi(r) = [\mu_2]$ for some $r > 0$. Furthermore, if $T(\Gamma)$ is finite dimensional then φ is uniquely determined.*

Proof: Using right translations, up to replacing Γ by an isomorphic Fuchsian group we can assume (see [EE, Theorem 3.(c)]) that $[\mu] = \Phi(0)$. Choose again $\nu \in L^\infty(\Gamma)$ such that $\psi = d\Phi_0(\nu)$ and $\tau_\Gamma([\mu];\psi) = \sigma(0;\nu)$, and define $\varphi\colon \Delta \to T(\Gamma)$ as in the proof of Proposition 3.1. By construction, φ is infinitesimally extremal at the origin, and thus the existence part of (i) is done, by Corollary 3.2. A similar argument yields the existence part of (ii), using [EKK, Theorem 5] instead of Corollary 3.2. The uniqueness is well known. □

We close this section underlining that on the Bers realization of a Teichmüller space (which is not starlike with respect to any of its points) the Kobayashi-Teichmüller metric has a geometry (e.g., good regularity, constant negative curvature, existence and uniqueness of complex geodesics, existence of pluricomplex Green functions, ...) which resembles in a striking way the geometry of invariant metrics on (strictly) convex domains, where the Kobayashi and Carathéodory metrics agree. It is also known ([Kr]) that Kobayashi and Carathéodory metrics agree on many directions, and we recall that Theorem 2.1 holds. Even though the Carathéodory metric need not to be preserved under projections, in light of all this it is very surprising that it seems that in general the Kobayashi-Teichmüller metric does not agree with Carathéodory metric (see [Kru] and references therein). This aspect should be better understood and deserves further investigation.

4. Isometries and biholomorphic maps into Teichmüller spaces

In this paragraph we would like to show how typical arguments involving intrinsic metrics may be useful in Teichmüller theory. Following ideas of [P], [Vi1, 2] and in particular [Gr] we can show the following result (that is, Theorem 0.1) in the vein of Royden's characterization of automorphisms of Teichmüller spaces as isometries of the Teichmüller metric.

Theorem 4.1: *Let Γ be a Fuchsian group so that $T(\Gamma)$ is finite dimensional, and let N be a taut connected complex manifold. Then a holomorphic map $F\colon N \to T(\Gamma)$ is biholomorphic if and only if it is an isometry for the Kobayashi metric at one point.*

Proof: If $F\colon N \to T(\Gamma)$ is biholomorphic then it is an isometry for the Kobayashi metric, and hence the claim in one direction is trivial.

Conversely, suppose that $F\colon N \to T(\Gamma)$ is a holomorphic map which is isometric at the point $p \in N$, i.e., such that

$$\kappa_N(p;v) = \kappa_{T(\Gamma)}\big(F(p);dF_p(v)\big) = \tau_\Gamma\big(F(p);dF_p(v)\big)$$

for every $v \in T^{1,0}_p N$, where κ_N is the Kobayashi metric of N and $\kappa_{T(\Gamma)}$ is the Kobayashi metric of $T(\Gamma)$.

Let $J(p)$ be the set of holomorphic maps $\varphi\colon \Delta \to N$ with $\varphi(0) = p$ and infinitesimally extremal at the origin with respect to κ_N; being N taut, for every $v \in T^{1,0}_p N$ there is at least one $\varphi \in J(p)$ such that $\varphi'(0)$ is a positive multiple of v. Take $\varphi \in J(p)$. The fact that F is an isometry at p implies that $F \circ \varphi$ is still infinitesimally extremal at the origin with respect to $\kappa_{T(\Gamma)}$; hence, by [EKK, Theorem 5], $F \circ \varphi$ is a complex geodesic in $T(\Gamma)$. But then recalling the decreasing property of the Kobayashi distance we get

$$\omega(\zeta_1,\zeta_2) \ge k_N\big(\varphi(\zeta_1),\varphi(\zeta_2)\big) \ge k_{T(\Gamma)}\big((F\circ\varphi)(\zeta_1),(F\circ\varphi)(\zeta_2)\big) = \omega(\zeta_1,\zeta_2), \tag{4.1}$$

for every $\zeta_1,\zeta_2 \in \Delta$, where ω is the Poincaré distance of Δ and $k_{T(\Gamma)}$ is the Teichmüller (Kobayashi) distance of $T(\Gamma)$; therefore every $\varphi \in J(p)$ is a complex geodesic with respect to the Kobayashi distance of N.

Now, as remarked in [Gr, Proposition 2], it is easy to show that the set

$$N_0 = \bigcup \{ \phi(\Delta) \mid \phi \in J(p) \}$$

is closed. On the other hand observe that by Proposition 3.3 the images of the complex geodesic of $T(\Gamma)$ through $[\mu] = F(p)$ fill all $T(\Gamma)$, and the images of two complex geodesics meeting at $[\mu]$ either coincide or meet only at $[\mu]$. We noticed that F sends complex geodesics through p in complex geodesics through $F(p)$; since complex geodesics are proper maps biholomorphic onto their image ([V]), it follows that $F|_{N_0} \colon N_0 \to T(\Gamma)$ is bijective. If we can show that N_0 is also open it will follow that $N_0 = N$ and that F is biholomorphic. For this aim it suffices to show that $f = F|_{N_0}$ is a homeomorphism of N_0, with the induced topology, onto $T(\Gamma)$. To this end we need only to check the continuity of f^{-1}. Let $\{[\nu_j]\}$ be a sequence of points in $T(\Gamma)$ with

$$\lim_{j\to\infty} [\nu_j] = [\nu_0].$$

For all j let $p_j = f^{-1}([\nu_j]) \in N_0$; we must show that $p_j \to p_0$. Take $\varphi_j \in J(p)$ such that $\varphi_j(r_j) = p_j$ for some $0 \le r_j < 1$. Now (4.1) implies that

$$k_{T(\Gamma)}([\mu], [\nu_j]) = \omega(0, r_j) = k_N(p, p_j). \tag{4.2}$$

Since the topology induced by the Teichmüller distance on $T(\Gamma)$ is equivalent to the manifold topology, the sequence $\{r_j\}$ is bounded away from 1.

To prove that $p_j \to p_0$ it suffices to show that every subsequence of $\{p_j\}$ admits a subsequence converging to p_0. So fix a subsequence of $\{p_j\}$, which we shall still denote by $\{p_j\}$. From what we have seen we get a subsequence $\{r_{j_k}\}$ converging to some $r_0 < 1$. On the other hand, because of the tautness of N, there exists a subsequence of $\{\varphi_{j_k}\}$, which we denote again by $\{\varphi_{j_k}\}$, converging uniformly on compact subsets of Δ to a holomorphic map $\varphi_0 \colon \Delta \to N$ such that $\varphi_0(0) = p$. Then $\{p_{j_k}\} = \{\varphi_{j_k}(r_{j_k})\}$ converges to some point $\varphi_0(r_0) \in N$. But, by construction φ_0 belongs to $J(p)$; therefore

$$f\big(\varphi_0(r_0)\big) = \lim_{k\to\infty} f\big(\varphi_{j_k}(r_{j_k})\big) = \lim_{k\to\infty} f(p_{j_k}) = \lim_{k\to\infty} [\nu_{j_k}] = [\nu_0].$$

It follows that $\varphi_0(r_0) = f^{-1}([\nu_0]) = p_0$, and so $p_{j_k} \to p_0$, as needed. □

So Royden's theorem about biholomorphic isometries of finite dimensional Teichmüller spaces is just a particular case of this result, because we have already remarked that every finite dimensional Teichmüller space is taut (see [A] for properties and examples of taut manifolds).

For infinite dimensional Teichmüller spaces the previous proof fails because the uniqueness of complex geodesics through pair of points does not hold. A different approach suggests a way to circumvent this problem: if N is a Teichmüller space, then it is already known that $N_0 = N$. Furthermore exactly as before one shows that F preserves the distance from the base point p, that is (4.2). If N is finite dimensional, it is then easy to prove that F is proper (essentially because closed Kobayashi balls are compact, which follows from Teichmüller spaces being complete hyperbolic). Standard theorems about proper holomorphic maps between equidimensional complex manifolds then imply that F is surjective, and that the cardinality of almost every fiber is constant (and finite), and the exceptional fibers have fewer elements. But (4.2) implies that this cardinality should be one, and so F is a biholomorphism.

In the infinite dimensional case this argument breaks down because (4.2) does not imply anymore that F is proper; moreover, one should be careful in talking about "equidimensionality". Nevertheless, a result due to Shoiykhet ([S]) suggests that further investigations in this direction may be fruitful. We need some preliminaries. Let $F\colon M \to N$ be a holomorphic map between complex Banach manifolds. We say that $x \in M$ is a *regular point* if the differential of F at x is surjective; otherwise we say that x is a *singular point*. We denote the set of singular points by S_F. Also we say that F is a *Fredholm map of index 0* if for all $x \in M$

$$\dim \operatorname{Ker} dF_x = \dim \operatorname{Coker} dF_x < \infty; \tag{4.3}$$

notice that this equality is a way of saying that M and N are equidimensional. Shoiykhet's result is summarized in the following

Proposition 4.2: *Let $F\colon M \to N$ be a holomorphic map between complex Banach manifolds such that F is a Fredholm map of index 0 with discrete fibers. Then*

(i) *the singular set S_F is an analytic set and in fact it is the zero set of some holomorphic function on M;*
(ii) *the image $F(M)$ of F is an open subset of N;*
(iii) *there is an integer $m \geq 1$, the multiplicity of F, such that for every $y \in F(M) \setminus F(S_F)$ the fiber $F^{-1}(y)$ has exactly m elements and for any $y \in F(S_F)$ the fiber $F^{-1}(y)$ has strictly less than m elements.*

Proof: Because of [S, Theorem 1], given any $x \in M$ there exists a neighborhood U_x such that the restriction $F|_{U_x}$ satisfies the claim of the proposition. It is straightforward to globalize the result. □

A proper map between complex manifolds has discrete fibers and closed image. So the following result (Theorem 0.2) is a generalization of Royden's theorem to the infinite dimensional case:

Theorem 4.3: *Let $T(\Gamma_1)$ and $T(\Gamma_2)$ be Teichmüller spaces (possibly infinite dimensional) relative to Fuchsian groups Γ_1 and Γ_2. Then a holomorphic map $F\colon T(\Gamma_1) \to T(\Gamma_2)$ is biholomorphic if and only if it is a Fredholm map of index 0 with discrete fibers and closed image which is an isometry for the Kobayashi-Teichmüller metric at one point.*

Proof: Clearly if F is biholomorphic then it is a bijective isometry at every point. Furthermore in the finite dimensional case the result is a corollary of Theorem 4.1. Let us then assume that F is a Fredholm map of index 0 with discrete fibers and closed image which is an isometry at some point $[\mu_0] \in T(\Gamma_1)$. In particular, by Proposition 4.2.(ii), F is surjective.

Now notice that for any point $[\mu] \in T(\Gamma_1)$ there is a complex geodesic $\varphi\colon \Delta \to T(\Gamma_1)$ such that $\phi(0) = [\mu_0]$ and $\phi(r) = [\mu]$ for some $0 \leq r < 1$. Then, as F is an infinitesimal isometry at $[\mu_0]$, the composition $F \circ \varphi$ is infinitesimally extremal at the origin, and hence a complex geodesic through $[\nu_0] = F([\mu_0])$. This implies that

$$k_{T(\Gamma_2)}\big([\nu_0], F([\mu])\big) = k_{T(\Gamma_1)}([\mu_0], [\mu])$$

for all $[\mu] \in T(\Gamma_1)$; in particular, F has multiplicity 1 in an open neighborhood of $[\mu_0]$. This is enough to conclude that F is biholomorphic. In fact, using (i) and (iii) of Proposition 4.2, it follows that F has multiplicity $m = 1$ on all $T(\Gamma_1)$ and $S_F = \emptyset$, so that the inverse of F, by the implicit function theorem, is holomorphic. □

It would be interesting to know whether the hypotheses (closed image, discrete fibers, and so on) in this theorem hold (as they do in the finite dimensional case) for any holomorphic map between infinite dimensional Teichmüller spaces which is an isometry at one point.

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