Preface

COMPLEX FINSLER GEOMETRY

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The correct notion of complex Finsler metrics is probably due to Rizza ([RI]) who also derived some direct consequences. Almost immediately afterwards, Rund ([RU]) developed the subject by defining a connection and presenting the equation for geodesics in close analogy with the real case. Since then there has been a great deal of contributions mostly dealing with the extension of the typical real Finsler geometry arguments and topics to the complex setting. In this direction we like to remember the work of Fukui about the use of the Cartan connection in complex geometry (see [FU] and the literature quoted there).

For their technical nature, this type of contribution was directed mainly at the experts of the field rather than to draw the attention for possible applications in other areas. Furthermore, the lack of consideration of explicit examples made the choice of the "right" notions in the complex setting difficult and sometimes rather artificial. Just to mention one instance, there are many possible definitions of Kählerianity in Finsler geometry which reduce in the hermitian case to the usual definition of Kähler metrics. The strongest ones give great advantages in the arguments but, unfortunately, the lack of examples raises the doubt that perhaps metrics satisfying such strong conditions occur very infrequently (see [AP2] for a discussion of this issue).

In recent years complex Finsler geometry has attracted a renewed interest as examples of such metrics appear in a very natural way in the geometric theory of several complex variables, thereby suggesting new directions for the investigation of the differential geometry of complex Finsler metrics. In fact, the attempt to find metrics which contain information about the complex structure of non-compact complex manifolds of dimension two or higher has led to at least two important developments. The first one is the consideration of the Bergmann metric, a truly hermitian Kähler metric when it is defined. The second one is the definition of the Carathéodory metric ([CAR]) and of the Kobayashi metric ([KO1], [RO1]), which are (in general not smooth) complex Finsler metrics.

Interesting properties of the Kobayashi metric are described in [PA3], [JP] and [KO3]. Its importance for complex manifolds is discussed, for example, by Lang ([L]).

The Carathéodory and Kobayashi metrics share in higher dimension the properties of the hyperbolic metric for Riemann surfaces, even though they are merely defined as complex Finsler metrics. One should expect that they are hermitian essentially only for the unit ball (and its quotients); this has been shown to be the case, albeit under additional hypotheses ([ST]). Unfortunately, in general, even for hyperbolic complex manifolds, these two metrics do not have enough regularity to allow much hope for a differential geometrical study.

Nevertheless there are some important exceptions.

The first, which motivated the work of Royden ([RO2]), is the Teichmüller metric. As a consequence again of results of Royden ([RO1]), the Teichmüller metric is the Kobayashi

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metric of a Teichmüller space and, in many cases, it has decent regularity (at least class $C^{1,1}$).

The second example is the case of smoothly bounded strictly convex domains in $\mathbb{C}^n$ for which, thanks to the fundamental work of Lempert ([LE1], [LE2]), it is known that the Kobayashi and Carathéodory metrics agree, are smooth (in the Finslerian sense, that is, the square of the fundamental function is smooth away from the zero section of the tangent bundle), and they even have strictly convex indicatrix at every point. It is also known ([PA2]) that the Kobayashi metric of a smooth strictly pseudoconvex complete circular domain is smooth in a neighborhood of its center of symmetry.

A different pole of interest is provided by the study of vector bundles. Thanks to a nice observation of Kobayashi's ([KO2], see also [KO5]), the negativity of a vector bundle is equivalent to the existence of a negatively curved Finsler metric on it.

The work of Kobayashi ([KO2]) not only proposes a new line of applications which still raises a good number of open interesting questions (see [KO5]), it also suggests the modern approach to the theory of connections and curvature. The Finsler geometry of a vector bundle over a complex manifold is studied by considering the hermitian geometry of a naturally related bundle. In this way, for the essential role played by the complex structure, there is no ambiguity in the choice of connection. In fact, the "right" connection, already introduced by Rund ([RU]) based on local considerations, is the Chern connection of the associated hermitian structure. Furthermore, standard hermitian technology suggests the notion of curvature and in particular that of holomorphic curvature ([KN], [WO1], [RO3]).

Kobayashi also defined ([KO5]) the notion of Einstein-Finsler structures (for background, one can see his monograph [KO4]). However, as he himself pointed out, in the above definition his mean curvature as it stands has no intrinsic meaning; so one must rephrase his definition in terms of some curvature tensor. This has recently been done by Aikou ([A2]), who also showed that if a holomorphic vector bundle over a Kähler manifold admits an Einstein-Finsler structure, then it is semi-stable.

The study of other special Finsler structures on complex manifolds is an interesting project which should be carried out. Works contributing to this area include, for example, [I], [PW], and [WO2].

Motivated by the problem of showing that Teichmüller disks are totally geodesic inside Teichmüller spaces, Royden ([RO2]) investigates (among other things) the notion of holomorphic curvature. He states that, as for hermitian manifolds ([WU]), holomorphic curvature can be realized variationally by considering the pulled back metrics via holomorphic disks parallel to the given direction (see [AP1] and [AP2] for complete proofs). This geometric interpretation of curvature is very useful and it may be used under much weaker regularity assumptions (see also [WO1]). Such flexibility may be important for applications to the study of intrinsic metrics of complex manifolds. Also, to [RO3] can be traced the weaker notions of Kählerianity, which we feel are more appropriate.

In [A1], Aikou considered the classification of complex manifolds modelled on a complex Minkowski space whose sectional curvature is constant and non-positive. But the criterion obtained is not a sufficient one. Looking ahead, one can nonetheless consider the question for the case of positive sectional curvature, as well as the general case of complex Finsler manifolds.

Using Cartan's moving frame technique, Faran ([FA]) studied the problem of local equivalence of complex Finsler metrics, giving the complex analogue of the corresponding real results obtained by Chern [CH]. In this setting, Faran recovered exactly the connection introduced by Rund [RU] as a fundamental tool and he was able to give a characterization of the Kobayashi metric in case the latter is smooth (see also [PA1] and [AP1]).

The work of Pang ([PA1]) and Abate-Patrizio (see in particular [AP1], [AP2]) originated from the attempt to understand the geometry of the Kobayashi metric of strictly convex domains from the viewpoint of differential geometry. In particular, it was important to discover necessary and sufficient conditions for the existence of totally geodesic holomorphic embeddings of Riemann surfaces of constant curvature into complex Finsler manifolds. This program leads naturally to the consideration of Kähler Finsler manifolds of constant holomorphic curvature. A step towards their classification is made in [AP3]. The complete classification is still an open (and challenging) problem. Furthermore, [AP2] contains an introduction to the modern theory of differential geometry of complex Finsler metrics.

In a different direction, Bland and Kalka (see [BK] for instance) are also motivated by the search of complex Finsler metrics which reflect as much as possible the function theory of the underlying manifold. In analogy with the hermitian case, they try to find "optimal" complex Finsler metrics by carrying out suitable variations of curvature functions. Along this line they have obtained some interesting results which invite further investigation. Perhaps their approach might even lead to new examples of complex Finsler metrics.

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