The Julia-Wolff-Carathéodory theorem in polydisks

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Abstract

The classical Julia-Wolff-Carathéodory theorem gives a condition ensuring the existence of the non-tangential limit of both a bounded holomorphic function and its derivative at a given boundary point of the unit disk in the complex plane. This theorem has been generalized by Rudin to holomorphic maps between unit balls in $\mathbb{C}^n$, and by the author to holomorphic maps between strongly (pseudo)convex domains. Here we describe Julia-Wolff-Carathéodory theorems for holomorphic maps defined in a polydisk and with image either in the unit disk, or in another polydisk, or in a strongly convex domain. One of main tool for the proof is a general version of Lindelöf's principle valid for not necessarily bounded holomorphic functions.

0. Introduction

The classical Fatou theorem says that a bounded holomorphic function f defined on the unit disk $\Delta \subset \mathbb{C}$ admits non-tangential limit at almost every point of $\partial\Delta$, but it does not say anything about the behavior of $f(\zeta)$ as ζ approaches a specific point σ of the boundary. Of course, to be able to say something in this case one needs some hypotheses on f. For instance, one can assume that, in a very weak sense, $f(\zeta)$ goes to the boundary of the image of f as ζ goes to σ. This leads to the classical *Julia's lemma:*

Theorem 0.1: (Julia [Ju1]) *Let $f\colon \Delta \to \Delta$ be a bounded holomorphic function such that*

$$\liminf_{\zeta\to\sigma} \frac{1-|f(\zeta)|}{1-|\zeta|} = \alpha < +\infty \tag{0.1}$$

for some $\sigma \in \partial\Delta$. Then f has non-tangential limit $\tau \in \partial\Delta$ at σ, and furthermore

$$\frac{|\tau - f(\zeta)|^2}{1-|f(\zeta)|^2} \le \alpha \frac{|\sigma-\zeta|^2}{1-|\zeta|^2} \tag{0.2}$$

for all $\zeta \in \Delta$.

This statement has a very interesting geometrical interpretation. The *horocycle* $E(\sigma, R) \subset \Delta$ of *center* $\sigma \in \partial\Delta$ and *radius* $R > 0$ is the set

$$E(\sigma, R) = \left\{ \zeta \in \Delta \;\middle|\; \frac{|\sigma-\zeta|^2}{1-|\zeta|^2} < R \right\}.$$

Geometrically, $E(\sigma,R)$ is an euclidean disk of radius $R/(R+1)$ internally tangent to $\partial\Delta$ at σ. Therefore (0.2) becomes $f\big(E(\sigma,R)\big)\subseteq E(\tau,\alpha R)$ for all $R>0$, and the existence of the non-tangential limit more or less follows from (0.2) and from the fact that horocycles touch the boundary in exactly one point.

A horocycle can be thought of as the limit of Poincaré disks of fixed euclidean radius and centers going to the boundary; in a sense, thus, we can interpret horocycles as Poincaré disks centered at the boundary, and Julia's lemma as a Schwarz-Pick lemma at the boundary. This suggests that α might be considered as a sort of dilatation coefficient: f expands horocycles by a ratio of α. If σ were an internal point and $E(\sigma,R)$ an infinitesimal euclidean disk actually centered at σ, one would be tempted to say that α is the absolute value of the derivative of f at σ. This is exactly the content of the classical *Julia-Wolff-Carathéodory theorem:*

Theorem 0.2: *Let $f\colon\Delta\to\Delta$ be a bounded holomorphic function such that*

$$\liminf_{\zeta\to\sigma}\frac{1-|f(\zeta)|}{1-|\zeta|}=\alpha<+\infty$$

for some $\sigma\in\partial\Delta$, and let $\tau\in\partial\Delta$ be the non-tangential limit of f at σ. Then both the incremental ratio $\big(\tau-f(\zeta)\big)/(\sigma-\zeta)$ and the derivative $f'(\zeta)$ have non-tangential limit $\alpha\overline{\sigma}\tau$ at σ.

So condition (0.1) forces the existence of the non-tangential limit of both f and its derivative at σ. This theorem results from the work of several people: Julia [Ju2], Wolff [Wo], Carathéodory [C], Landau and Valiron [L-V], R. Nevanlinna [N] and others (see Burckel [B] and [A4] for proofs, history and applications).

Before describing what happens for functions of several complex variables, it is worthwhile to take a closer look to the notion of non-tangential limit in Δ. First of all, the non-tangential limit can be defined in two equivalent ways. We can say that a function $f\colon\Delta\to\mathbb{C}$ has *non-tangential limit* $L\in\mathbb{C}$ at $\sigma\in\partial\Delta$ if $f\big(\gamma(t)\big)\to L$ as $t\to1^-$ for every curve $\gamma\colon[0,1)\to\Delta$ such that $\gamma(t)\to\sigma$ non-tangentially as $t\to1^-$. Or, we can say that f has non-tangential limit $L\in\mathbb{C}$ if $f(\zeta)\to L$ as $\zeta\to\sigma$ staying inside any *Stolz region* $H(\sigma,M)$ of *vertex* σ and *amplitude* $M>1$, where

$$H(\sigma,M)=\left\{\zeta\in\Delta\;\middle|\;\frac{|\sigma-\zeta|}{1-|\zeta|}<M.\right\}.$$

Since Stolz regions are angle-shaped nearby the vertex σ, and the angle is going to π as $M\to+\infty$, it is clear that these two definitions are equivalent; as we shall see later on, this is not anymore the case in $\mathbb{C}^n$ with $n>1$.

The second observation is that to check the existence of a non-tangential limit it suffices to study the function along one single curve. This is the content of *Lindelöf's principle* (see Rudin [R] for a modern proof):

Theorem 0.3: (Lindelöf [Li]) *Let $f\colon\Delta\to\mathbb{C}$ be a bounded holomorphic function. Assume there are $\sigma\in\partial\Delta$ and a continuous curve $\gamma\colon[0,1)\to\Delta$ with $\gamma(t)\to\sigma$ as $t\to1^-$ so that*

$$\lim_{t\to1^-} f\bigl(\gamma(t)\bigr)=L\in\mathbb{C}.$$

Then f has non-tangential limit L at σ.

All these results have been generalized in several ways to functions and maps defined in domains of $\mathbb{C}^n$ (see the introductions to [A3, 5] for a brief historical summary); it turns out that the best way to express the generalizations (or, at least, the best way according to the point of view discussed in this paper) is in terms of geometric function theory.

The aim of geometric function theory is to describe how the geometrical shape of a domain influences the analytical behavior of holomorphic functions defined on it. A typical (and of the utmost importance) result of this kind is the Schwarz-Pick lemma: a holomorphic self-map of Δ is necessarily a contraction with respect to the Poincaré distance (which is a geometrical object depending only on the shape of Δ).

The Kobayashi distance of a domain $D\subset\subset\mathbb{C}^n$ is a direct generalization of the Poincaré distance of Δ (see, e.g., [K1, 2], [J-P] and [A4] for definition and properties); in particular, it is contracted by holomorphic maps. The main claim of this paper is that all the results discussed before (Julia's lemma, Julia-Wolff-Carathéodory theorem, Lindelöf's principle) are better understood if expressed using the Kobayashi distance (and its infinitesimal analogue, the Kobayashi metric). In previous papers (see [A3, 5]) we have already shown how to do so in strongly convex and strongly pseudoconvex domains; here we shall show how the same techniques can be applied to polydisks (and conceivably to other domains as well).

The first step consists in understanding the correct generalization to several variables of the notion of non-tangential limit. As first discovered by Korányi and Stein discussing Fatou's theorem in $\mathbb{C}^n$ ([Ko, K-S, S]), the natural approach regions for the study of boundary behavior of holomorphic functions are not cones, but regions approaching the boundary non-tangentially along the normal direction, and tangentially (at least parabolically) along the complex tangential directions. Korányi and Stein defined their approach regions in euclidean (local) terms; later, Krantz [Kr] suggested a definition using both the Kobayashi distance and the euclidean normal line. In [A3] we generalized Stolz regions as follows: if $D\subset\subset\mathbb{C}^n$ is a bounded domain and $z_0\in D$, then the (*small*) *Korányi region* $H_{z_0}(x,M)$ of *vertex* $x\in\partial D$, *pole* $z_0\in D$ and *amplitude* $M>1$ is given by

$$H_{z_0}(x,M)=\Bigl\{z\in D\Bigm|\limsup_{w\to x}[k_D(z,w)-k_D(z_0,w)]+k_D(z_0,z)<\log M\Bigr\},$$

where k_D is the Kobayashi distance of D. The rationale behind this definition is the following: the lim sup measure the "distance" (normalized as to be always finite, possibly negative) of z from the boundary point x; so $z\in H_{z_0}(x,M)$ if and only if the average between the "distance" of z from x and the distance of z from the pole z_0 is bounded by $\frac12\log M$. By the way, changing the pole amounts to shifting the amplitudes, and thus it is completely irrelevant.

It is easy to check that the Stolz region $H(\sigma, M) \subset \Delta$ is exactly the Korányi region of vertex σ, pole the origin and amplitude M. Therefore we shall say that a function $f: D \to \mathbb{C}$ has *K-limit* $L \in \mathbb{C}$ at $x \in \partial D$ if $f(z) \to L$ as $z \to x$ inside any Korányi region of vertex x.

Since if D is strongly pseudoconvex ([A5]) our Korányi regions are comparable (actually equal if $D = B^n$, the unit ball of $\mathbb{C}^n$) with Korányi-Stein's and Krantz's approach regions (with the latter even when D is a polydisk; see Remark 1.5), our K-limit coincides with their admissible limit, but our approach regions are completely described in terms of the Kobayashi distance only.

We remarked before that the non-tangential limit in Δ can be defined in two different ways. We just generalized the first way; Čirka [Č] discovered that the second one leads to a different notion of limit, which is the one needed to generalize Lindelöf's principle.

Let $D \subset\subset \mathbb{C}^n$ be a bounded domain, and $x \in \partial D$. A *x-curve* is a continuous curve $\sigma: [0,1) \to D$ such that $\sigma(t) \to x$ as $t \to 1^-$. Then Čirka showed that the correct version of Lindelöf's principle should be something like this: if $f: D \to \mathbb{C}$ is a bounded holomorphic function such that $f\bigl(\sigma^o(t)\bigr) \to L \in \mathbb{C}$ as $t \to 1^-$ for a given x-curve σ^o belonging to a specific class of x-curves, then $f\bigl(\sigma(t)\bigr) \to L$ as $t \to 1^-$ for all x-curves σ belonging to a (possibly sub)class of x-curves.

The main point here clearly is the identification of the correct class of curves. Several authors (see, e.g., [Č, C-K, D, D-Z, Kh]) have suggested several possibilities, more or less intrinsic. In [A3] we found a general technique (inspired by [C-K]) to generate such classes. Let $D \subset\subset \mathbb{C}^n$ be a bounded domain, and $x \in \partial D$. A *projection device* at x is given by the following data: a holomorphic immersion $\varphi_x: \Delta \to D$ extending continuously to the boundary so that $\varphi_x(1) = x$; a neighborhood U_0 of x in $\mathbb{C}^n$; and a device associating to every x-curve σ contained in U_0 a x-curve σ_x contained in $U_0 \cap \varphi_x(\Delta)$. For instance, if D is strongly convex containing the origin, we can take as φ_x a parametrization of the intersection of D with the complex line $\mathbb{C}x$ passing through 0 and x, and we can associate to every x-curve its orthogonal projection into $\mathbb{C}x$. In [A5] we described two other projection devices, another one is introduced in this paper (see Section 1), and many others are easily devised.

Given a projection device at $x \in \partial D$, we say that a x-curve σ is *restricted* if the curve $\tilde{\sigma}_x = \varphi_x^{-1} \circ \sigma_x$ goes to 1 non-tangentially in Δ, and that the x-curve σ is *special* if $k_D\bigl(\sigma(t), \sigma_x(t)\bigr) \to 0$ as $t \to 1^-$. We say that a function $f: D \to \mathbb{C}$ has *restricted K-limit* L at x (with respect to the given projection device) if $f\bigl(\sigma(t)\bigr) \to L$ as $t \to 1^-$ along any special restricted x-curve σ. Then

Theorem 0.4: ([A3]) *Let $D \subset\subset \mathbb{C}^n$ be a bounded domain equipped with a projection device at $x \in \partial D$. Let $f: D \to \mathbb{C}$ be a bounded holomorphic function, and assume there is a special x-curve σ^o such that*

$$\lim_{t \to 1^-} f\bigl(\sigma^o(t)\bigr) = L \in \mathbb{C}.$$

Then f has restricted K-limit L at x.

For instance, if we use the projection device defined before, Theorem 0.4 recovers Čirka's results for the case of strongly convex domains.

We should remark that whereas Theorem 0.4 holds in this very general setting, to generalize Theorem 0.2 we shall need a similar result for functions which are only bounded in Korányi regions. At present there is no general proof of such a statement, but only (usually hard; cf. Theorem 2.2) proofs working slightly differently in each case, depending on the actual geometry of the domain under consideration. It would be interesting to have a more general proof.

The next step is the generalization of Julia's lemma. Recalling the description of the horocycles as Poincaré disks at the boundary, it is natural to define in a domain $D \subset\subset \mathbb{C}^n$ the *small horosphere* $E_{z_0}(x,R)$ of *center* $x \in \partial D$, *pole* $z_0 \in D$ and *radius* $R > 0$ by

$$E_{z_0}(x,R) = \bigl\{ z \in D \bigm| \limsup_{w\to x} [k_D(z,w) - k_D(z_0,w)] < \tfrac{1}{2}\log R \bigr\};$$

the *big horosphere* $F_{z_0}(x,R)$ is analogously defined replacing the lim sup by a lim inf.

In condition (0.1), both numerator and denominator can be interpreted as distances from the boundary. Since, when D is complete hyperbolic and $z_0 \in D$ is chosen once for all, $\exp\bigl(-2k_D(z_0,z)\bigr)$ is going to zero exactly when z is going to the boundary (and it behaves exactly as the euclidean distance from the boundary when D is strongly pseudoconvex; see [A1]), we can use it as a replacement for the euclidean distance from the boundary. This leads to the following version of Julia's lemma:

Theorem 0.5: ([A3]) *Let $D_1 \subset\subset \mathbb{C}^n$, $D_2 \subset\subset \mathbb{C}^m$ be two bounded domains, with D_1 complete hyperbolic. Fix $z_1 \in D_1$ and $z_2 \in D_2$. Let $f\colon D_1 \to D_2$ be a holomorphic map such that*

$$\liminf_{w\to x} \bigl[k_{D_1}(z_1,w) - k_{D_2}\bigl(z_2,f(w)\bigr)\bigr] = \tfrac{1}{2}\log\alpha < +\infty \tag{0.3}$$

for some $x \in \partial D$ and $\alpha > 0$. Then there exists a $y \in \partial D_2$ such that

$$\forall R > 0 \qquad\qquad f\bigl(E_{z_1}(x,R)\bigr) \subseteq F_{z_2}(y,\alpha R).$$

Furthermore, if $\overline{F_{z_2}(y,R)} \cap \partial D_2 = \{y\}$ for all $R > 0$ (for instance, if D_2 is strongly pseudoconvex) then f has K-limit y at x.

It should be remarked that in [A3] this theorem was stated only for self-maps of a complete hyperbolic domain, but the same proof goes through in the general case too.

Very often, (0.3) is equivalent to

$$\liminf_{w\to x} \frac{d\bigl(f(w),\partial D_2\bigr)}{d(w,\partial D_1)} < +\infty,$$

and so this result encompasses most of the other known generalizations of Julia's lemma (see, e.g., [M, H, W]).

We are left with the generalization of the Julia-Wolff-Carathéodory theorem. With respect to the one-dimensional case there is an obvious difference: instead of only one derivative we have to consider a whole (Jacobian) matrix of them, and there is no reason they should all behave in the same way. And indeed, as first showed by Rudin [R] in B^n, they do not. It turns out that the geometry of the domain (and, in particular, the boundary behavior of the Kobayashi metric) plays a main role in determining the correct statement of a Julia-Wolff-Carathéodory theorem.

Rudin proved the following Julia-Wolff-Carathéodory theorem for the unit ball of $\mathbb{C}^n$:

Theorem 0.6: (Rudin [R]) *Let $f\colon B^n\to B^m$ be a holomorphic map such that*

$$\tfrac{1}{2}\log\liminf_{w\to x}\frac{1-\|f(w)\|}{1-\|w\|}=\liminf_{w\to x}\bigl[k_{B^n}(0,w)-k_{B^m}\bigl(0,f(w)\bigr)\bigr]=\tfrac{1}{2}\log\alpha<+\infty,$$

for some $x\in B^n$, where $\|\cdot\|$ denote the euclidean norm. Let $y\in\partial B^m$ be the K-limit of f at x, as given by Theorem 0.5. Equip B^n with the projection device defined before in terms of the orthogonal projection, set $f_y(z)=\bigl(f(z),y\bigr)y$ (where $(\cdot\,,\cdot)$ is the canonical hermitian product) and denote by df_z the differential of f at z. Then:

(i) *The incremental ratio $\bigl[1-\bigl(f(z),y\bigr)\bigr]/[1-(z,x)]$ has restricted K-limit α at x;*
(ii) *The map $[f(z)-f_y(z)]/[1-(z,x)]^{1/2}$ has restricted K-limit 0 at x;*
(iii) *The function $\bigl(df_z(x),y\bigr)$ has restricted K-limit α at x;*
(iv) *The map $[1-(z,x)]^{1/2}d(f-f_y)_z(x)$ has restricted K-limit 0 at x;*
(v) *If $x^\perp$ is any vector orthogonal to x, the function $\bigl(df_z(x^\perp),y\bigr)/[1-(z,x)]^{1/2}$ has restricted K-limit 0 at x;*
(vi) *If $x^\perp$ is any vector orthogonal to x, the map $d(f-f_y)_z(x^\perp)$ is bounded in every Korányi region of vertex x.*

In [A3, 5] we generalized this theorem to strongly convex and strongly pseudoconvex domains, showing explicitly that the different behavior of the partial derivatives is due to the different behavior of the Kobayashi metric along normal directions or complex tangential directions.

Recently, Jafari [J] made a preliminary study of the case of the polydisk along the Šilov boundary; unfortunately, one of his statements is wrong (see Remarks 4.2 and 4.5). This paper is devoted to a full description of what happens in the polydisk. Our main result (Theorem 4.1) deals with bounded holomorphic functions:

Theorem 0.7: *Let $f\colon\Delta^n\to\Delta$ be a holomorphic function, and $x=(x_1,\ldots,x_n)\in\partial\Delta^n$. Assume there is $\alpha>0$ such that*

$$\tfrac{1}{2}\log\liminf_{w\to x}\frac{1-|f(w)|}{1-|||w|||}=\liminf_{w\to x}\bigl[k_{\Delta^n}(0,w)-k_{\Delta}\bigl(0,f(w)\bigr)\bigr]=\tfrac{1}{2}\log\alpha<+\infty,$$

where $|||\cdot|||$ denote the sup-norm. Let $\tau\in\partial\Delta$ be the K-limit of f at x, as given by Theorem 0.5. Equip Δ^n with the canonical projection device at x (to be described in Section 1). Then:

(i) *the incremental ratio $\bigl(\tau-f(z)\bigr)/\bigl(1-\tilde p_x(z)\bigr)$ has restricted K-limit $\alpha\tau$ at x, where the function $\tilde p_x\colon\Delta^n\to D$ is defined in (1.2);*
(ii) *If $|x_j|=1$, then the incremental ratio $\bigl(\tau-f(z)\bigr)/(x_j-z_j)$ has restricted K-limit $\alpha\tau\overline{x_j}$ at x;*
(iii) *the partial derivative $\partial f/\partial x=df_z(x)$ has restricted K-limit $\alpha\tau$ at x;*
(iv) *If $|x_j|<1$ then the partial derivative $\partial f/\partial z_j$ has restricted K-limit 0 at x;*
(v) *If $|x_j|=1$ then $\partial f/\partial z_j$ has restricted K-limit at x.*

This theorem is proved in Section 4. Section 1 is devoted to define Korányi regions and the canonical projection device in Δ^n. In Section 2 we prove the generalization of

Lindelöf's principle, and in Section 3 of Julia's lemma, along the lines described before. Finally, in Section 5 we shall discuss some more Julia-Wolff-Carathéodory theorems for maps from Δ^n into another polydisk, or into a strongly convex domains, showing how it is possible to apply in several different situations the machinery we developed here.

1. The canonical projection device and Korányi regions

As described in the introduction, to prove a Julia-Wolff-Carathéodory theorem two tools are needed: a Lindelöf principle, and a Julia's lemma. The aim of this section is to introduce concepts and definitions needed by both of them.

We shall denote by $\|\cdot\|$ the euclidean norm on $\mathbb{C}^n$, and by $(\cdot\,,\cdot)$ the canonical hermitian product

$$(z,w) = z_1\overline{w_1} + \cdots + z_n\overline{w_n}.$$

Let $|||\cdot|||$ denote the sup norm

$$|||z||| = \max_j\{|z_j|\};$$

the *unit polydisk* $\Delta^n \subset \mathbb{C}^n$ is the unit ball for this norm, that is $\Delta^n = \{z \in \mathbb{C}^n \mid |||z||| < 1\}$.

The Kobayashi distance k_{Δ^n} of Δ_n is given by (see, e.g., [J-P, Example 3.1.8])

$$k_{\Delta^n}(z,w) = \tfrac{1}{2}\log\frac{1 + |||\gamma_z(w)|||}{1 - |||\gamma_z(w)|||},$$

where $\gamma_z\colon \Delta^n \to \Delta^n$ is the automorphism of Δ^n given by

$$\gamma_z(w) = \left(\frac{w_1 - z_1}{1 - \overline{z_1}w_1}, \ldots, \frac{w_n - z_n}{1 - \overline{z_n}w_n}\right).$$

Since $t \mapsto \frac{1}{2}\log(1+t)/(1-t)$ is an increasing function, we get

$$k_{\Delta_n}(z,w) = \max_j\left\{\tfrac{1}{2}\log\frac{1 + \left|\frac{w_j - z_j}{1 - \overline{z_j}w_j}\right|}{1 - \left|\frac{w_j - z_j}{1 - \overline{z_j}w_j}\right|}\right\} = \max_j\{\omega(z_j, w_j)\}, \tag{1.1}$$

where ω is the Poincaré distance on the unit disk $\Delta \subset \mathbb{C}$.

A *complex geodesic* in a domain $D \subset \mathbb{C}^n$ is a a holomorphic map $\varphi\colon \Delta \to D$ which is an isometry with respect to the Poincaré distance on Δ and the Kobayashi distance on D. It is easy to prove (see, e.g., [A4, Proposition 2.6.10]) that $\varphi \in \mathrm{Hol}(\Delta, \Delta^n)$ is a complex geodesic iff at least one component of φ is an automorphism of Δ. In particular, given two points of Δ^n there is always a complex geodesic containing both of them in its image, and given a point of Δ^n and a tangent vector there is always a complex geodesic containing the point and tangent to the vector.

We shall need a particular class of complex geodesics in Δ^n. Given $x \in \partial\Delta^n$, the *complex geodesic associated* to x is the map $\varphi_x \in \mathrm{Hol}(\Delta, \Delta^n)$ given by

$$\varphi_x(\zeta) = \zeta x.$$

Since there is at least one component x_j of x with $|x_j| = 1$, every φ_x is a complex geodesic, and we have a canonical family of complex geodesics whose images fill up all of Δ^n and meet only at the origin.

To better describe the boundary behavior of functions and maps we shall need some more terminology. Let $x = (x_1, \dots, x_n) \in \partial\Delta^n$; the *Šilov degree* d_x of x is the number of components of x with absolute value 1:

$$d_x = \#\{j \mid |x_j| = 1\}.$$

In particular, $d_x = n$ iff x belongs to the *Šilov boundary* $(\partial\Delta)^n = \partial\Delta \times \cdots \times \partial\Delta \subset \partial\Delta^n$ of Δ^n. A *Šilov component* of x is a component x_j such that $|x_j| = 1$; an *internal component* of x is a component x_j such that $|x_j| < 1$. The *Šilov part* $\check{x} = (\check{x}_1, \dots, \check{x}_n) \in \partial\Delta^n$ of x is defined by

$$\check{x}_j = \begin{cases} x_j & \text{if } |x_j| = 1, \\ 0 & \text{if } |x_j| < 1; \end{cases}$$

notice that $(x, \check{x}) = d_x$. The *internal part* $\mathring{x} \in \Delta^n$ of x is defined by $\mathring{x} = x - \check{x}$, so that $x = \check{x} + \mathring{x}$. Finally, we shall say that a tangent vector $v = (v_1, \dots, v_n) \in \mathbb{C}^n$ has *no Šilov components with respect to* x if $v_j = 0$ whenever $|x_j| = 1$.

We have already associated to each $x \in \partial\Delta^n$ a complex geodesic φ_x. The *left-inverse function associated to* x is the holomorphic function $\tilde{p}_x \colon \Delta^n \to \Delta$ given by

$$\tilde{p}_x(z) = \frac{1}{d_x}(z, \check{x}); \tag{1.2}$$

the *holomorphic retraction* associated to x is the holomorphic map $p_x \colon \Delta^n \to \Delta^n$ given by

$$p_x(x) = \frac{1}{d_x}(z, \check{x})x = \varphi_x \circ \tilde{p}_x(z).$$

Notice that $\tilde{p}_x \circ \varphi_x = \mathrm{id}_\Delta$ (hence the name left-inverse), $p_x \circ p_x = p_x$ (hence the name retraction), and $p_x \circ \varphi_x = \varphi_x$, so that p_x is a retraction of Δ^n onto the image of φ_x.

A *x-curve* is a continuous curve $\sigma \colon [0, 1) \to \Delta^n$ such that $\sigma(t) \to x$ as $t \to 1^-$. We shall say that something happens *eventually* to a x-curve σ if it happens to $\sigma(t)$ for t close enough to 1. If $\sigma = (\sigma_1, \dots, \sigma_n)$, then an *internal component* (respectively, a *Šilov component*) of σ is a component σ_j such that x_j is an internal (respectively, Šilov) component of x.

We are now ready to define the projection device (see the introduction) we shall use. Let $x \in \partial\Delta^n$; the *canonical projection device* at x is given by the complex geodesic φ_x, the neighborhood $U_0 = \mathbb{C}^n$ and by associating to any x-curve σ the x-curve

$$\sigma_x = p_x \circ \sigma = \frac{1}{d_x}(\sigma, \check{x})x;$$

it is a x-curve whose image is contained in $\varphi_x(\Delta)$, and it does not depend on the internal components of σ. Furthermore,

$$|||\sigma_x||| = \frac{1}{d_x}|(\sigma, \check{x})|, \tag{1.3}$$

and

$$|||\sigma - \sigma_x||| = \max_j \left\{ \left| \sigma_j - \frac{(\sigma, \check{x})}{d_x} x_j \right| \right\}. \tag{1.4}$$

We shall also set $\tilde{\sigma}_x = \tilde{p}_x \circ \sigma$, which is a 1-curve in Δ such that $\sigma_x = \varphi_x \circ \tilde{\sigma}_x$. Notice that $|||\sigma_x||| = |\tilde{\sigma}_x|$.

We shall say that a x-curve σ is *special* if

$$\lim_{t \to 1^-} k_{\Delta^n}\bigl(\sigma(t), \sigma_x(t)\bigr) = 0.$$

This is equivalent to requiring that

$$\max_j \left\{ \left| \frac{\sigma_j - \frac{1}{d_x}(\sigma, \check{x}) x_j}{1 - \overline{\sigma_j} \frac{1}{d_x}(\sigma, \check{x}) x_j} \right| \right\} \to 0.$$

Let x_j be an internal component of x. Then

$$\overline{\sigma_j} \frac{1}{d_x}(\sigma, \check{x}) x_j \to |x_j|^2 < 1;$$

therefore σ is special iff

$$\max_{|x_j|=1} \left\{ \left| \frac{\sigma_j - \frac{1}{d_x}(\sigma, \check{x}) x_j}{1 - \overline{\sigma_j} \frac{1}{d_x}(\sigma, \check{x}) x_j} \right| \right\} \to 0. \tag{1.5}$$

In particular, being special imposes no restrictions on the internal components of a x-curve.

There is a geometric interpretation of special curves: a x-curve is special if the Šilov components approach the image of φ_x faster than the rate of approach of the projection σ_x to the boundary of Δ^n. More precisely,

Proposition 1.1: *Let $x \in \partial \Delta^n$. Then a x-curve σ is special if and only if*

$$\lim_{t \to 1^-} \max_{|x_j|=1} \left\{ \frac{|\sigma_j(t) - (\sigma_x)_j(t)|}{1 - |||\sigma_x(t)|||} \right\} = 0. \tag{1.6}$$

Proof: Assume first that (1.6) holds. Then for $j = 1, \ldots, n$ we have

$$\left| 1 - \overline{\sigma_j} \frac{1}{d_x}(\sigma, \check{x}) x_j \right| \geq 1 - \frac{1}{d_x} |(\sigma, \check{x})|,$$

because $|\sigma_j| < 1$ and $|x_j| \leq 1$. Therefore

$$\max_{|x_j|=1} \left| \frac{\sigma_j - \frac{1}{d_x}(\sigma, \check{x}) x_j}{1 - \overline{\sigma_j} \frac{1}{d_x}(\sigma, \check{x}) x_j} \right| \leq \max_{|x_j|=1} \left\{ \frac{|\sigma_j - (\sigma_x)_j|}{1 - |||\sigma_x|||} \right\},$$

by (1.3) and (1.4), and so σ is special, by (1.5).

Conversely, assume (1.6) is not satisfied. Write $\sigma=\sigma_x+\alpha$, where $\alpha\colon[0,1)\to\mathbb{C}^n$ is such that $(\alpha,\check{x})\equiv 0$ and $\alpha(t)\to 0$ as $t\to 1^-$. Since (1.6) is not satisfied, there are a Šilov component σ_j, an $\varepsilon>0$ and a sequence $t_k\to 1^-$ such that

$$\frac{|\alpha_j(t_k)|}{1-|\tilde{\sigma}_x(t_k)|^2}\ge\frac{|\sigma_j(t_k)-(\sigma_x)_j(t_k)|}{2(1-|||\sigma_x(t_k)|||)}\ge\varepsilon.$$

Now,

$$\frac{\left|1-\overline{\sigma_j(t_k)}\frac{1}{d_x}(\sigma(t_k),\check{x})x_j\right|}{1-|\tilde{\sigma}_x(t_k)|^2}=\left|1-\frac{\overline{\alpha_j(t_k)}\tilde{\sigma}_x(t_k)x_j}{1-|\tilde{\sigma}_x(t_k)|^2}\right|\le 1+\frac{|\alpha_j(t_k)|}{1-|\tilde{\sigma}_x(t_k)|^2}.$$

Noticing that the function $t\mapsto t/(1+t)$ is strictly increasing we get

$$\left|\frac{\sigma_j(t_k)-\frac{1}{d_x}(\sigma(t_k),\check{x})x_j}{1-\overline{\sigma_j(t_k)}\frac{1}{d_x}(\sigma(t_k),\check{x})x_j}\right|\ge\frac{|\alpha_j(t_k)|/(1-|\tilde{\sigma}_x(t_k)|^2)}{1+|\alpha_j(t_k)|/(1-|\tilde{\sigma}_x(t_k)|^2)}\ge\frac{\varepsilon}{1+\varepsilon}>0,$$

and so σ is not special. □

Remark 1.1: Instead of special x-curves, Jafari [J] considers x-curves *tangent to the diagonal,* that is such that

$$\lim_{t\to 1^-}\frac{|||\sigma(t)-\sigma_x(t)|||}{1-|||\sigma_x(t)|||}=0.$$

The previous proposition shows that if x belongs to the Šilov boundary (the only case considered by Jafari) then a x-curve is special iff it is tangent to the diagonal. But if x does not belong to the Šilov boundary, this is not true anymore. For instance, take $x=(1,0)\in\partial\Delta^2$, and define $\sigma\colon[0,1)\to\Delta^2$ by

$$\sigma(t)=\left(\tfrac{1}{2}(1+t),\tfrac{1}{2}(1-t)\right).$$

Then it is easy to check that σ is special but

$$\frac{|||\sigma-\sigma_x|||}{1-|||\sigma_x|||}\equiv 1,$$

and so σ is not tangent to the diagonal in the sense of Jafari.

We shall need another class of x-curves. We say that a x-curve σ is *restricted* if $\tilde{\sigma}_x$ approaches 1 non-tangentially. Notice that, again, being restricted imposes no conditions on the internal components.

There is a more quantitative way of saying that a x-curve is restricted. The *Stolz region* $H(1,M)\subset\Delta$ of *vertex* $\tau\in\partial\Delta$ and *amplitude* $M>1$ is given by

$$H(\tau,M)=\left\{\zeta\in\Delta\;\middle|\;\frac{|\tau-\zeta|}{1-|\zeta|}<M\right\}.$$

Geometrically, $H(\tau,M)$ is a sort of angle with vertex at τ, symmetric with respect to the real axis, and amplitude going to π as $M\to+\infty$; therefore a 1-curve in Δ approaches 1 non-tangentially iff it eventually belongs to a Stolz region $H(1,M)$ for some $M>1$. Thus we shall say that a x-curve σ in Δ^n is *M-restricted* if $\tilde\sigma_x$ eventually belongs to $H(1,M)$.

Closely related to Stolz regions are the horocycles. The *horocycle* $E(\tau,R)\subset\Delta$ of *center* $\tau\in\partial\Delta$ and *radius* $R>0$ is the set

$$E(\tau,R)=\left\{\zeta\in\Delta\;\middle|\;\frac{|\tau-\zeta|^2}{1-|\zeta|^2}<R\right\}.$$

Geometrically, $E(\tau,R)$ is an euclidean disk of euclidean radius $R/(1+R)$ internally tangent to $\partial\Delta$ in τ.

In the following we shall need a version of Stolz regions and horocycles in Δ^n. Take $x\in\partial\Delta^n$, $R>0$ and $M>1$. Then the (*small*) *horosphere* $E(x,R)$ of *center* x and *radius* R and the (*small*) *Korányi region* $H(x,M)$ of *vertex* x and *amplitude* M are defined by (see [A1, 3]):

$$E(x,R)=\bigl\{z\in\Delta^n\bigm|\limsup_{w\to x}\,[k_{\Delta^n}(z,w)-k_{\Delta^n}(0,w)]<\tfrac12\log R\bigr\},$$
$$H(x,M)=\bigl\{z\in\Delta^n\bigm|\limsup_{w\to x}\,[k_{\Delta^n}(z,w)-k_{\Delta^n}(0,w)]+k_{\Delta^n}(0,z)<\log M\bigr\}.$$

It is easy to check that if $x=\tau\in\partial\Delta$ then we recover the horocycles and Stolz regions previously defined.

Remark 1.2: Replacing the lim sup by lim inf one gets the definitions of big horosphere $F(x,R)$ and big Korányi region $K(x,M)$, but we shall not need them in this paper.

To understand the shape of horospheres and Korányi regions in the polydisk we need to compute the lim sup used to define them:

Proposition 1.2: *Given $x\in\partial\Delta^n$ and $z\in\Delta^n$ one has*

$$\begin{aligned}\limsup_{w\to x}[k_{\Delta^n}(z,w)-k_{\Delta^n}(0,w)]&=\tfrac12\log\max_{|x_j|=1}\left\{\frac{|x_j-z_j|^2}{1-|z_j|^2}\right\}\\&=\lim_{s\to1^-}[k_{\Delta^n}\bigl(z,\varphi_x(s)\bigr)-\omega(0,s)].\end{aligned}\tag{1.7}$$

Proof: First of all,

$$k_{\Delta^n}(z,w)-k_{\Delta^n}(0,w)=\log\left(\frac{1+|||\gamma_z(w)|||}{1+|||w|||}\right)+\tfrac12\log\left(\frac{1-|||w|||^2}{1-|||\gamma_z(w)|||^2}\right).$$

Since $|||\gamma_z(x)|||=|||x|||=1$, we need to study the behavior of the second addend only. Now

$$1-|||w|||^2=\min_h\{1-|w_h|^2\};$$
$$1-|||\gamma_z(w)|||^2=\min_j\left\{\frac{1-|z_j|^2}{|1-\overline{z_j}w_j|^2}(1-|w_j|^2)\right\}.$$

Hence
$$\frac{1-\|\!|w|\!\|^2}{1-\|\!|\gamma_z(w)|\!\|^2}=\max_j\left\{\frac{|1-\overline{z_j}w_j|^2}{1-|z_j|^2}\ \min_h\left\{\frac{1-|w_h|^2}{1-|w_j|^2}\right\}\right\}.$$
If $|x_j|<1$ we have
$$\min_h\left\{\frac{1-|w_h|^2}{1-|w_j|^2}\right\}\to 0$$
as $w\to x$; so we ought to consider only j's such that x_j is a Šilov component of x.

We clearly have
$$\min_h\left\{\frac{1-|w_h|^2}{1-|w_j|^2}\right\}\le 1;$$
therefore
$$\limsup_{w\to x}\frac{1-\|\!|w|\!\|^2}{1-\|\!|\gamma_z(w)|\!\|^2}\le\max_{|x_j|=1}\left\{\frac{|1-\overline{z_j}w_j|^2}{1-|z_j|^2}\right\}. \tag{1.8}$$
To prove the opposite inequality, set $w^\nu=(1-1/\nu)^{1/2}x$; we have
$$1-|(w^\nu)_h|^2=(1-|x_h|^2)+|x_h|^2/\nu.$$
Hence if $|x_j|=1$ we get
$$\lim_{\nu\to\infty}\min_h\left\{\frac{1-|(w^\nu)_h|^2}{1-|(w^\nu)_j|^2}\right\}=1,$$
and we have proved the first equality in (1.7).

For the second equality, notice that
$$k_{\Delta^n}\big(z,\varphi_x(s)\big)-\omega(0,s)=\log\left(\frac{1+\|\!|\gamma_z\big(\varphi_x(s)\big)|\!\|}{1+s}\right)+\tfrac12\log\left(\frac{1-s^2}{1-\|\!|\gamma_z\big(\varphi_x(s)\big)|\!\|^2}\right).$$
The first addend on the right goes to 0 as $s\to 1^-$. Next we have
$$1-\|\!|\gamma_z\big(\varphi_x(s)\big)|\!\|^2=\min_j\left\{\frac{1-|z_j|^2}{|1-s\overline{z_j}x_j|^2}(1-s^2|x_j|^2)\right\};$$
therefore
$$\frac{1-s^2}{1-\|\!|\gamma_z\big(\varphi_x(s)\big)|\!\|^2}=\max_j\left\{\frac{|1-s\overline{z_j}x_j|^2}{1-|z_j|^2}\ \frac{1-s^2}{1-s^2|x_j|^2}\right\}.$$
Since $(1-s^2)/(1-s^2|x_j|^2)\equiv 1$ if $|x_j|=1$ and it goes to 0 if $|x_j|<1$, we are done. □

Therefore for every $x\in\partial\Delta^n$, $R>0$ and $M>1$ we have
$$E(x,R)=\left\{z\in\Delta^n\ \middle|\ \max_{|x_j|=1}\left\{\frac{|x_j-z_j|^2}{1-|z_j|^2}\right\}<R\right\},$$
and
$$H(x,M)=\left\{z\in\Delta^n\ \middle|\ \frac{1+\|\!|z|\!\|}{1-\|\!|z|\!\|}\max_{|x_j|=1}\left\{\frac{|x_j-z_j|^2}{1-|z_j|^2}\right\}<M^2\right\}. \tag{1.9}$$
The shape of $E(x,R)$ is easily described: it is a product $E_1\times\cdots\times E_n$, where $E_j=E(x_j,R)$ is a horocycle in Δ if $|x_j|=1$, and $E_j=\Delta$ otherwise.

The shape of $H(x,M)$ is more complicated, but anyway we can prove the following:

Proposition 1.3: *Let $x \in \partial\Delta^n$ and $M > 1$. Then*

$$\bigcup_{t\geq 0} B_{\Delta^n}\bigl(\varphi_x(t), \tfrac{1}{2}\log M\bigr) \subseteq H(x,M) \subseteq H_1 \times \cdots \times H_n, \tag{1.10}$$

where $B_{\Delta^n}(z,r)$ is the open ball of center z and radius r with respect to the Kobayashi distance of Δ^n, $H_j = H(x_j, M)$ is a Stolz region in Δ if $|x_j| = 1$ and $H_j = \Delta$ otherwise.

Proof: For every $t \geq 0$ we have

$$\lim_{s\to 1^-} [k_{\Delta^n}\bigl(z, \varphi_x(s)\bigr) - \omega(0,s)] + k_{\Delta^n}(0,z) \leq 2k_{\Delta^n}\bigl(z, \varphi_x(t)\bigr),$$

because $k_{\Delta^n}\bigl(\varphi_x(t), \varphi_x(s)\bigr) - \omega(0,s) + k_{\Delta^n}\bigl(0, \varphi_x(t)\bigr) = 0$ as soon as $s \geq t$. Proposition 1.2 then implies the first inclusion in (1.10).

For the second inclusion, take $z \in H(x,M)$ and suppose that $|x_h| = 1$. Then

$$\frac{|x_h - z_h|^2}{(1-|z_h|)^2} = \frac{1+|z_h|}{1-|z_h|}\,\frac{|x_h - z_h|^2}{1-|z_h|^2} \leq \frac{1+\|\!|z|\!\|}{1-\|\!|z|\!\|}\max_{|x_j|=1}\left\{\frac{|x_j - z_j|^2}{1-|z_j|^2}\right\} < M^2,$$

and so $z_j \in H(x_j, M)$. □

Remark 1.3: Given $x \in \partial\Delta^n$, let $z = t\check{x} + v$, where $t \in [0,1)$, the vector v has no Šilov components with respect to x and $\|\!|v|\!\| \leq t$. Then it is easy to check that $z \in H(x,M)$ for all $M > 1$; in particular, in the second inclusion of (1.10) we are forced to take the whole unit disk as factor for the internal components.

Remark 1.4: The second inclusion in (1.10) implies that if $z \to x$ inside some $H(x,M)$ and x_j is a Šilov component of x, then $z_j \to x_j$ non-tangentially; on the other hand, the first inclusion implies that $z_j \to x_j$ unrestricted if x_j is an internal component.

Remark 1.5: In [Kr] Krantz suggested to define approach regions in general domains as unions like the one in the left-hand member of (1.10), replacing the geodesic line $\varphi_x(t)$ by the euclidean normal line at x. Proposition 1.3 shows that our Korányi regions are comparable with Krantz's approach regions.

Another consequence of Proposition 1.2 is the following:

Corollary 1.4: *Take $x \in \partial\Delta^n$, $R > 0$ and $M > 1$. Then*

$$\varphi_x\bigl(E(1,R)\bigr) = E(x,R) \cap \varphi_x(\Delta) = p_x\bigl(E(x,R)\bigr)$$

and

$$\varphi_x\bigl(H(1,M)\bigr) = H(x,M) \cap \varphi_x(\Delta) = p_x\bigl(H(x,M)\bigr).$$

Proof: Since φ_x is a complex geodesic, we have $k_{\Delta^n}\bigl(\varphi_x(\zeta), \varphi_x(s)\bigr) = \omega(\zeta, s)$ for every $\zeta \in \Delta$ and $s \in [0,1)$; the assertions then follows from Proposition 1.2. □

If $n = 1$ we have seen that there is a strong relationship between non-tangential curves and Stolz regions. A similar fact holds for $n > 1$ too:

Proposition 1.5: *Take $x \in \partial\Delta^n$, and let $\sigma\colon[0,1)\to\Delta^n$ be a x-curve. Fix $M>1$. Then:*
(i) *σ is M-restricted iff $\sigma_x(t)\in H(x,M)$ eventually;*
(ii) *if $\sigma(t)\in H(x,M)$ eventually then σ is M-restricted;*
(iii) *if σ is special and M-restricted then for any $M_1>M$ we have $\sigma(t)\in H(x,M_1)$ eventually.*

Proof: (i) By definition, σ is M-restricted iff $\tilde\sigma_x(t)\in H(1,M)$ eventually iff

$$\lim_{s\to1^-}[\omega(\tilde\sigma_x(t),s)-\omega(0,s)]+\omega\bigl(0,\tilde\sigma_x(t)\bigr)<\log M$$

eventually. Being φ_x a complex geodesic, and since $\sigma_x=\varphi_x\circ\tilde\sigma_x$, this happens iff eventually

$$\lim_{s\to1^-}[k_{\Delta^n}\bigl(\sigma_x(t),\varphi_x(s)\bigr)-\omega(0,s)]+k_{\Delta^n}\bigl(0,\sigma_x(t)\bigr)<\log M,$$

that is iff $\sigma_x(t)\in H(x,M)$ eventually, by Proposition 1.2.

(ii) For any $z\in\Delta^n$ we have

$$\begin{aligned}\lim_{s\to1^-}&[k_{\Delta^n}\bigl(p_x(z),\varphi_x(s)\bigr)-\omega(0,s)]+k_{\Delta^n}\bigl(0,p_x(z)\bigr)\\&\le\lim_{s\to1^-}[k_{\Delta^n}\bigl(z,\varphi_x(s)\bigr)-\omega(0,s)]+k_{\Delta^n}(0,z),\end{aligned}$$

because $p_x\circ\varphi_x=\varphi_x$, and the assertion follows, again by Proposition 1.2.

(iii) We have

$$\begin{aligned}k_{\Delta^n}&\bigl(\sigma(t),\varphi_x(s)\bigr)-\omega(0,s)+k_{\Delta^n}\bigl(0,\sigma(t)\bigr)\\&\le2\kappa_{\Delta^n}\bigl(\sigma(t),\sigma_x(t)\bigr)+k_{\Delta^n}\bigl(\sigma_x(t),\varphi_x(s)\bigr)-\omega(0,s)+\kappa_{\Delta^n}\bigl(0,\sigma_x(t)\bigr),\end{aligned}$$

and the assertion follows from (i) and Proposition 1.2. □

Remark 1.6: There are restricted x-curves that does not eventually belong to any Korányi region. For instance, take $x=(1,1)\in\partial\Delta^2$, and let $\sigma\colon[0,1)\to\Delta^2$ be given by

$$\sigma(t)=\bigl(t+i(1-t)^{1/2},t-i(1-t)^{1/2}\bigr).$$

Clearly $\tilde\sigma_x(t)=t$, and so σ is M-restricted for any $M>1$. On the other hand,

$$1-|||\sigma(t)|||=O(1-t)=1-|\sigma_j(t)|^2,\qquad|1-\sigma_j(t)|^2=O(1-t),$$

and so

$$\frac{1+|||\sigma(t)|||}{1-|||\sigma(t)|||}\max_j\left\{\frac{|1-\sigma_j(t)|^2}{1-|\sigma_j(t)|^2}\right\}=O\bigl((1-t)^{-1}\bigr)\to+\infty,$$

as claimed. Notice that

$$\frac{|||\sigma(t)-\sigma_x(t)|||}{1-|||\sigma_x(t)|||}=\frac{1}{(1-t)^{1/2}}\to+\infty,$$

and σ is not special.

We end this section with a result we shall use to build examples later on:

Lemma 1.6: *Let α_1, $\alpha_2 \in \mathrm{Hol}(\Delta^2,\mathbb{C})$ be so that* $\mathrm{Re}\,\alpha_j(z) > |\mathrm{Im}\,\alpha_j(z)|$ *for all* $z \in \Delta^2$ *and* $j = 1, 2$. *Then the function*

$$f(z) = \frac{\alpha_1(z) - \alpha_2(z)}{\alpha_1(z) + \alpha_2(z)}$$

is a holomorphic function from Δ^2 *into* Δ.

Proof: The hypothesis ensures that $\mathrm{Re}\bigl(\alpha_1(z)\overline{\alpha_2(z)}\bigr) > 0$ for all $z \in \Delta^2$. Therefore

$$\forall z \in \Delta^2 \qquad |\alpha_1(z) + \alpha_2(z)|^2 > |\alpha_1(z) - \alpha_2(z)|^2,$$

and we are done. □

2. The Lindelöf principle

In the previous section we have defined, via the canonical projection device, special and restricted x-curves. We say that a map $f\colon \Delta^n \to \mathbb{C}^m$ has *restricted K-limit* $L \in \mathbb{C}^m$ at $x \in \partial\Delta^n$ if $f\bigl(\sigma(t)\bigr) \to L$ as $t \to 1^-$ for any special restricted x-curve σ; we shall write

$$\tilde{K}\text{-}\lim_{z\to x} f(z) = L.$$

We say that f has *K-limit* $L \in \mathbb{C}^m$ at x if $f(z) \to L$ as $z \to x$ inside any Korányi region $H(x, M)$.

Remark 2.1: By Proposition 1.5.(iii), if f has K-limit L at x then it has restricted K-limit L at x too. The converse is false, even for bounded holomorphic functions: let $f\colon \Delta^2 \to \mathbb{C}$ be given by

$$f(z_1, z_2) = \frac{(1 - z_1)^{1/2} - (1 - z_2)^{1/2}}{(1 - z_1)^{1/2} + (1 - z_2)^{1/2}}.$$

By Lemma 1.6, the image of f is contained in Δ. Set $x = (1, 1) \in \partial\Delta^2$, and take a special restricted x-curve σ. Write $\sigma = \sigma_x + \alpha$; then

$$f(\sigma) = \frac{(1 - \tilde{\sigma}_x - \alpha_1)^{1/2} - (1 - \tilde{\sigma}_x - \alpha_2)^{1/2}}{(1 - \tilde{\sigma}_x - \alpha_1)^{1/2} + (1 - \tilde{\sigma}_x - \alpha_2)^{1/2}} = \frac{\left(1 - \frac{\alpha_1}{1-\tilde{\sigma}_x}\right)^{1/2} - \left(1 - \frac{\alpha_2}{1-\tilde{\sigma}_x}\right)^{1/2}}{\left(1 - \frac{\alpha_1}{1-\tilde{\sigma}_x}\right)^{1/2} + \left(1 - \frac{\alpha_2}{1-\tilde{\sigma}_x}\right)^{1/2}}.$$

Being σ special we have

$$\left|\frac{\alpha_j}{1 - \tilde{\sigma}_x}\right| \le \frac{|||\alpha|||}{1 - |\tilde{\sigma}_x|} = \frac{|||\sigma - \sigma_x|||}{1 - |||\sigma_x|||} \to 0,$$

and so f has restricted K-limit 0 at x.

On the other hand, for $\lambda \in (0, 1)$ let $\sigma^\lambda\colon [0, 1) \to \Delta^2$ be the x-curve

$$\sigma^\lambda(t) = \bigl(t, t + \lambda(1 - t)\bigr).$$

Then

$$\frac{1+|||\sigma^\lambda|||}{1-|||\sigma^\lambda|||}\max_j\left\{\frac{|1-\sigma_j^\lambda|^2}{1-|\sigma_j^\lambda|^2}\right\}=\frac{(1+\lambda)+(1-\lambda)t}{(1+t)(1-\lambda)}\le\frac{2}{1-\lambda},$$

that is $\sigma^\lambda(t)\in H\bigl(x,2/(1-\lambda)\bigr)$. But $f\bigl(\sigma^\lambda(t)\bigr)\equiv\bigl(1-(1-\lambda)^{1/2}\bigr)/\bigl(1+(1-\lambda)^{1/2}\bigr)$, and so f has no K-limit at x.

In the next section we shall use yet again another kind of limit, stronger than K-limit. Take $x\in\partial\Delta^n$; we shall say that a x-curve σ is *peculiar* if $\sigma(t)\in E(x,R)$ eventually for all $R>0$. Recalling the shape of horospheres, this means that

$$\lim_{t\to1^-}\max_{|x_j|=1}\left\{\frac{|x_j-\sigma_j(t)|^2}{1-|\sigma_j(t)|^2}\right\}=0;\tag{2.1}$$

as usual, being peculiar imposes no restrictions on the internal components.

We say that a function $f\colon\Delta^n\to\mathbb{C}^m$ admits *restricted E-limit* $L\in\mathbb{C}^m$ at $x\in\partial\Delta^n$ if $f\bigl(\sigma(t)\bigr)\to L$ as $t\to1^-$ for any peculiar x-curve σ.

Remark 2.2: One could obviously say that a function $f\colon\Delta^n\to\mathbb{C}^m$ has *E-limit* L at x if $f(z)\to L$ as $z\to x$ inside any horosphere $E(x,R)$, but we shall not use this definition.

Remark 2.3: It is easy to check that for every $M>1$ and $R>0$ one has

$$H(x,M)\setminus B_{\Delta^n}(0,r)\subset E(x,R),$$

where $r=\frac12\log(M^2/R)$; it follows that if f has restricted E-limit L at x then it has K-limit L there. To prove that the converse does not hold, not even for bounded holomorphic functions, we need a couple of preliminary observations.

First of all, take $z\in H\bigl((1,1),M\bigr)$. Then

$$\frac{1}{2M^2}\le\frac{1-|z_2|}{1-|z_1|}\le2M^2.\tag{2.2}$$

Indeed we have

$$\begin{aligned}M^2\ge\frac{1+|||z|||}{1-|||z|||}\max_j\left\{\frac{|1-z_j|^2}{1-|z_j|^2}\right\}&\ge\frac{1+|z_1|}{1-|z_1|}\,\frac{|1-z_2|^2}{1-|z_2|^2}\\&\ge\frac{1+|z_1|}{1-|z_1|}\,\frac{(1-|z_2|)^2}{1-|z_2|^2}\ge\frac12\,\frac{1-|z_2|}{1-|z_1|},\end{aligned}\tag{2.3}$$

and the right inequality in (2.2) follows; the other one is obtained reversing the roles of z_1 and z_2. As a consequence,

$$\frac{1}{2M^2}\le\left|\frac{1-z_2}{1-z_1}\right|\le2M^2.\tag{2.4}$$

Indeed (2.2) and (2.3) imply

$$\left|\frac{1-z_2}{1-z_1}\right|^2\le\frac{|1-z_2|^2}{(1-|z_1|)^2}\le M^2\frac{1-|z_2|^2}{1-|z_1|^2}\le4M^4$$

for all $z \in H\big((1,1),M\big)$.

Now fix $\alpha<1$ and define $f\in\mathrm{Hol}(\Delta^2,\mathbb{C})$ by

$$f(z_1,z_2)=\frac{(1-z_1)^{\alpha/2}-(1-z_2)^{1/2}}{(1-z_1)^{\alpha/2}+(1-z_2)^{1/2}};$$

Lemma 1.6 ensures us that $f(\Delta^2)\subset\Delta$. Since $\alpha<1$ we have

$$f(z_1,z_2)=\frac{1-\left(\frac{1-z_2}{1-z_1}\right)^{\alpha/2}(1-z_2)^{(1-\alpha)/2}}{1+\left(\frac{1-z_2}{1-z_1}\right)^{\alpha/2}(1-z_2)^{(1-\alpha)/2}},$$

and so, by (2.4), f has K-limit 1 at $(1,1)$. On the other hand, for $\lambda\in(0,1)$ let σ^λ be the $(1,1)$-curve given by

$$\sigma^\lambda(t)=\big(t,t-\lambda(1-t)^\alpha\big).$$

It is easy to check that σ^λ is peculiar, but

$$f\big(\sigma^\lambda(t)\big)=\frac{(1-t)^{\alpha/2}-[(1-t)+\lambda(1-t)^\alpha]^{1/2}}{(1-t)^{\alpha/2}+[(1-t)+\lambda(1-t)^\alpha]^{1/2}}=\frac{1-[\lambda+(1-t)^{1-\alpha}]^{1/2}}{1+[\lambda+(1-t)^{1-\alpha}]^{1/2}}\to\frac{1-\lambda^{1/2}}{1+\lambda^{1/2}},$$

and so f has no restricted E-limit at $(1,1)$.

Remark 2.4: The classical Lindelöf principle implies that examples like the previous one cannot exist in Δ. Indeed, if $f\in\mathrm{Hol}(\Delta,\Delta)$ has limit L along any given 1-curve, then L is the non-tangential limit of f at 1; therefore if f restricted to any other 1-curve admits a limit, that limit should be L.

Using these definitions it is very easy to prove a Lindelöf principle (Theorem 0.4):

Theorem 2.1: *Let $f\colon\Delta^n\to\mathbb{C}^m$ be a bounded holomorphic function. Given $x\in\partial\Delta^n$, assume there is a special x-curve σ^o such that*

$$\lim_{t\to1^-}f\big(\sigma^o(t)\big)=L\in\mathbb{C}^m.$$

Then f has restricted K-limit L at x.

Proof: (see [A3]). Clearly we can assume $m=1$ and $f(\Delta^n)\subset\subset\Delta$. Let σ be any special x-curve. We have

$$\omega\left(f\big(\sigma(t)\big),f\big(\sigma_x(t)\big)\right)\le k_{\Delta^n}\big(\sigma(t),\sigma_x(t)\big)\to0;$$

therefore the limit of $f\big(\sigma(t)\big)$ as $t\to1^-$ exists iff the limit of $f\big(\sigma_x(t)\big)$ as $t\to1^-$ does, and the two are equal. In particular, $f\big(\sigma_x^o(t)\big)\to L$ as $t\to1^-$. Hence, by the classical Lindelöf principle (see [R]), $f\big(\sigma_x(t)\big)\to L$ for any restricted x-curve σ and, by the previous observation, $f\big(\sigma(t)\big)\to L$ for any special restricted x-curve σ. □

Remark 2.5: The bounded holomorphic functions $f \in \mathrm{Hol}(\Delta^2, \Delta)$ described in Remarks 2.1 and 2.3 show that if $f\bigl(\sigma(t)\bigr) \to L$ as $t \to 1^-$ but σ is not special then it is not necessarily true that f has restricted K-limit L.

It turns out that to prove a Julia-Wolff-Carathéodory theorem a stronger result is needed. Given $x \in \partial\Delta^n$, if $f\colon \Delta^n \to \mathbb{C}^m$ is such that for every $M > 1$ there is a constant $c_M > 0$ such that $\|f(z)\| \le c_M$ for all $z \in H(x, M)$, we shall say that f is *K-bounded* at x. Then:

Theorem 2.2: *Given $x \in \partial\Delta^n$, let $f\colon \Delta^n \to \mathbb{C}^m$ be a holomorphic map K-bounded at x. Assume there is a restricted special x-curve σ^o such that*

$$\lim_{t \to 1^-} f\bigl(\sigma^o(t)\bigr) = L \in \mathbb{C}^m.$$

Then f has restricted K-limit L at x.

Proof: We can of course assume $m = 1$. First of all we claim that if σ is an M-restricted special x-curve, then for all $M_1 > M$ we have

$$\lim_{t \to 1^-} k_{H(x,M_1)}\bigl(\sigma(t), \sigma_x(t)\bigr) = 0. \tag{2.5}$$

For any $t \in [0, 1)$ let us consider the map $\psi_t\colon \mathbb{C} \to \mathbb{C}^n$ given by

$$\psi_t(\zeta) = \sigma_x(t) + \zeta\bigl(\sigma(t) - \sigma_x(t)\bigr).$$

Clearly, $\psi_t(0) = \sigma_x(t)$ and $\psi_t(1) = \sigma(t)$. Assume for the moment that we have proved that for every $R > 1$ there exists $t_0 = t_0(R) \in [0, 1)$ such that

$$\forall t_0 < t < 1 \qquad \psi_t(\Delta_R) \subset H(x, M_1), \tag{2.6}$$

where $\Delta_R = \{\zeta \in \mathbb{C} \mid |\zeta| < R\}$. Set

$$R(t) = \sup\{r > 0 \mid \psi_t(\Delta_r) \subset H(x, M_1)\}.$$

Inclusion (2.6) implies that $R(t) \to +\infty$ as $t \to 1^-$; since, by definition,

$$k_{H(x,M_1)}\bigl(\sigma(t), \sigma_x(t)\bigr) \le \inf\left\{ \frac{1}{R} \;\middle|\; \exists \varphi \in \mathrm{Hol}\bigl(\Delta_R, H(x, M_1)\bigr) : \varphi(0) = \sigma_x(t), \varphi(1) = \sigma(t) \right\},$$

equation (2.5) follows from (2.6).

Now we prove (2.6). If we write $\sigma = \tilde\sigma_x x + \alpha$, clearly we have $\psi_t(\zeta) = \tilde\sigma_x(t)x + \zeta\alpha(t)$, with $\alpha(t) \to 0$ as $t \to 1^-$. In particular, for every $R > 1$ there exists $t_1 = t_1(R) \in [0, 1)$ such that

$$\forall \zeta \in \Delta_R \quad \forall t_1 < t < 1 \quad \|\!|\psi_t(\zeta)|\!\| = \max_{|x_h|=1} \bigl\{ |\tilde\sigma_x(t)x_h + \zeta\alpha_h(t)| \bigr\}. \tag{2.7}$$

Assume, by contradiction, that (2.6) is false. Then there exist $M_1 > M$ and $R_0 > 1$ such that for any $t_0 \in [0,1)$ there are $t' = t'(t_0) \in (t_0,1)$ and $\zeta_0 = \zeta_0(t_0) \in \Delta_{R_0}$ such that $\psi_{t'}(\zeta_0) \notin H(x,M_1)$. Since $\sigma_x(t') \in H(x,M_1)$ eventually (Proposition 1.5), we can choose $t_1 = t_1(R_0) \in (0,1)$ such that $\sigma_x(t') \in H(x,M_1)$ for all $t_0 > t_1$; being $H(x,M_1)$ open we can also assume that $\psi_{t'}(\zeta_0) \in \partial H(x,M_1)$ but $\psi_{t'}(\zeta) \in H(x,M_1)$ for all $\zeta \in \Delta_{|\zeta_0|}$.

Recalling (1.9) and (2.7) we get

$$\max_{|x_h|=1}\left\{\frac{1+|\tilde\sigma_x x_h+\zeta_0\alpha_h|}{1-|\tilde\sigma_x x_h+\zeta_0\alpha_h|}\right\}\max_{|x_j|=1}\left\{\frac{|1-\tilde\sigma_x-\zeta_0\alpha_j\overline{x_j}|^2}{1-|\tilde\sigma_x x_j+\zeta_0\alpha_j|^2}\right\}=M_1^2,$$

where everything is evaluated at $t = t'$. So there are, possibly different, Šilov components x_{h_0} and x_{j_0} such that

$$M_1^2=\left(\frac{|1-\tilde\sigma_x-\zeta_0\alpha_{j_0}\overline{x_{j_0}}|}{1-|\tilde\sigma_x+\zeta_0\alpha_{j_0}\overline{x_{j_0}}|}\right)^2\left[\frac{1+|\tilde\sigma_x x_{h_0}+\zeta_0\alpha_{h_0}|}{1+|\tilde\sigma_x x_{j_0}+\zeta_0\alpha_{j_0}|}\Big/\frac{1-|\tilde\sigma_x x_{h_0}+\zeta_0\alpha_{h_0}|}{1-|\tilde\sigma_x x_{j_0}+\zeta_0\alpha_{j_0}|}\right]. \tag{2.8}$$

Now, if $\zeta \in \Delta_{R_0}$ we have

$$\begin{aligned}\frac{1+|\zeta|\dfrac{|\alpha_{j_0}|}{1-|\tilde\sigma_x|}}{1-|\zeta|\dfrac{|\alpha_{h_0}|}{1-|\tilde\sigma_x|}}=\frac{1-|\tilde\sigma_x|+|\zeta||\alpha_{j_0}|}{1-|\tilde\sigma_x|-|\zeta||\alpha_{h_0}|}&\ge\frac{1-|\tilde\sigma_x x_{j_0}+\zeta\alpha_{j_0}|}{1-|\tilde\sigma_x x_{h_0}+\zeta\alpha_{h_0}|}\\&\ge\frac{1-|\tilde\sigma_x|-|\zeta||\alpha_{j_0}|}{1-|\tilde\sigma_x|+|\zeta||\alpha_{h_0}|}=\frac{1-|\zeta|\dfrac{|\alpha_{j_0}|}{1-|\tilde\sigma_x|}}{1+|\zeta|\dfrac{|\alpha_{h_0}|}{1-|\tilde\sigma_x|}}.\end{aligned} \tag{2.9}$$

Being σ special, Proposition 1.1 yields

$$\max_{|x_j|=1}\frac{|\alpha_j|}{1-|\tilde\sigma_x|}\to 0; \tag{2.10}$$

therefore (2.9) implies that

$$\frac{1-|\tilde\sigma_x x_{j_0}+\zeta\alpha_{j_0}|}{1-|\tilde\sigma_x x_{h_0}+\zeta\alpha_{h_0}|}\to 1 \tag{2.11}$$

uniformly for $\zeta \in \Delta_{R_0}$. Now fix $\varepsilon > 0$ so that $M_1' = M_1/(1+\varepsilon) > M$; by (2.11) we can choose a sequence $t_0^k \to 1^-$ so that (2.8) holds for all t_0^k with the same j_0 and h_0, and moreover

$$\left[\frac{1+|\tilde\sigma_x x_{h_0}+\zeta\alpha_{h_0}|}{1+|\tilde\sigma_x x_{j_0}+\zeta\alpha_{j_0}|}\Big/\frac{1-|\tilde\sigma_x x_{h_0}+\zeta\alpha_{h_0}|}{1-|\tilde\sigma_x x_{j_0}+\zeta\alpha_{j_0}|}\right]\le(1+\varepsilon)^2$$

for all $\zeta \in \Delta_{R_0}$ and all t_0^k (where everything is evaluated at $t'(t_0^k)$, as usual). Recalling (2.8) we then get

$$\frac{|1-\tilde\sigma_x-\zeta_0\alpha_{j_0}\overline{x_{j_0}}|}{1-|\tilde\sigma_x+\zeta_0\alpha_{j_0}\overline{x_{j_0}}|}\ge\frac{M_1}{1+\varepsilon}=M_1'>M,$$

again for all t_0^k. Writing $v_k=\zeta_0(t_0^k)\alpha_{j_0}\bigl(t'(t_0^k)\bigr)\overline{x_{j_0}}$ we obtain

$$|1-\tilde\sigma_x|+|v_k|\ge|1-\tilde\sigma_x-v_k|\ge M_1'(1-|\tilde\sigma_x+v_k|)\ge M_1'(1-|\tilde\sigma_x|-|v_k|);$$

therefore, being σ M-restricted, for k large enough we have

$$M+\frac{|v_k|}{1-|\tilde\sigma_x|}\ge\frac{|1-\tilde\sigma_x|}{1-|\tilde\sigma_x|}+\frac{|v_k|}{1-|\tilde\sigma_x|}\ge M_1'\left(1-\frac{|v_k|}{1-|\tilde\sigma_x|}\right),$$

whence

$$\frac{|v_k|}{1-|\tilde\sigma_x|}\ge\frac{M_1'-M}{1+M_1'},$$

and so

$$R_0\ge\frac{M_1'-M}{1+M_1'}\,\frac{1-\bigl|\tilde\sigma_x\bigl(t'(t_0^k)\bigr)\bigr|}{\bigl|\alpha_{j_0}\bigl(t'(t_0^k)\bigr)\bigr|}.$$

Letting $k\to+\infty$, that is letting $t'(t_0^k)\to1^-$, we finally get a contradiction, because of (2.10).

Summing up, we have proved that (2.6) holds, and therefore (2.5) holds too; we can now finish the proof of the theorem. Let $M>1$ so that σ^o is M-restricted, and fix $M_1>M$; then (2.5) holds. On $H(x,M_1)$ the function f is bounded by c, say; therefore

$$k_{\Delta_c}\left(f\bigl(\sigma^o(t)\bigr),f\bigl(\sigma_x^o(t)\bigr)\right)\le k_{H(x,M_1)}\bigl(\sigma^o(t),\sigma_x^o(t)\bigr),$$

and so

$$\lim_{t\to1^-}f\bigl(\sigma_x^o(t)\bigr)=L. \tag{2.12}$$

Finally, let σ be any restricted special x-curve. The classical Lindelöf principle applied to $f\circ\varphi_x$ together with (2.12) implies

$$\lim_{t\to1^-}f\bigl(\sigma_x(t)\bigr)=L;$$

hence, arguing as before, we find that $f\bigl(\sigma(t)\bigr)\to L$ as $t\to1^-$, and we are done. □

3. Julia's lemma

Now we are ready to deal with Julia's lemma in the polydisk, at least for functions; the general case is discussed in the last section (but see also Theorem 0.5).

The standard Julia's lemma in the disk says that if $f\colon\Delta\to\Delta$ is a bounded holomorphic function such that the rate of approach of $f(\zeta)$ to $\partial\Delta$ is comparable to the rate of approach of ζ to $\sigma\in\partial\Delta$, then f sends horocycles centered at σ into horocycles centered at some $\tau\in\partial\Delta$ — and then f has non-tangential limit τ at σ.

If $f\in\mathrm{Hol}(\Delta^n,\Delta)$ and we want a version of Julia's lemma in the polydisk, the natural thing to do is to compare the rate of approach of $f(w)$ to $\partial\Delta^n$ with the rate of approach of w to $x\in\partial\Delta^n$, that is to study

$$\frac{1-|f(w)|}{1-\|w\|}$$

when $w\to x$. Now, it is easy to check that

$$\tfrac12\log\liminf_{w\to x}\frac{1-|f(w)|}{1-\|w\|}=\liminf_{w\to x}\bigl[k_{\Delta^n}(0,w)-\omega\bigl(0,f(w)\bigr)\bigr]; \tag{3.1}$$

since we have defined horospheres and the like in terms of the Kobayashi distance, the natural statement for a Julia lemma is:

Theorem 3.1: *Let $f\colon \Delta^n \to \Delta$ be a bounded holomorphic function, and let $x \in \partial\Delta^n$ be such that*

$$\liminf_{w\to x}\bigl[k_{\Delta^n}(0,w) - \omega\bigl(0,f(w)\bigr)\bigr] \le \tfrac{1}{2}\log\alpha < +\infty. \tag{3.2}$$

Then there exists $\tau \in \partial\Delta$ such that

$$\forall R > 0 \qquad f\bigl(E(x,R)\bigr) \subseteq E(\tau,\alpha R). \tag{3.3}$$

Furthermore, f admits restricted E-limit τ at x.

Proof: First of all choose a sequence $\{w_\nu\} \subset \Delta^n$ converging to x such that

$$\lim_{\nu\to\infty}\bigl[k_{\Delta^n}(0,w_\nu) - \omega\bigl(0,f(w_\nu)\bigr)\bigr] = \liminf_{w\to x}\bigl[k_{\Delta^n}(0,w) - \omega\bigl(0,f(w)\bigr)\bigr].$$

Up to a subsequence, we can assume that $f(w_\nu) \to \tau \in \overline{\Delta}$. Since Δ^n is complete hyperbolic, we have $k_{\Delta^n}(0,w_\nu) \to +\infty$; therefore $\omega\bigl(0,f(w_\nu)\bigr) \to +\infty$ as well, and $\tau \in \partial\Delta$.

Now take $z \in E(x,R)$; then

$$\begin{aligned}
\lim_{\zeta\to\tau}[\omega(f(z),\zeta) - \omega(0,\zeta)] &= \lim_{\nu\to\infty}\bigl[\omega\bigl(f(z),f(w_\nu)\bigr) - \omega\bigl(0,f(w_\nu)\bigr)\bigr]\\
&\le \liminf_{\nu\to\infty}\bigl[k_{\Delta^n}(z,w_\nu) - \omega\bigl(0,f(w_\nu)\bigr)\bigr]\\
&\le \liminf_{\nu\to\infty}[k_{\Delta^n}(z,w_\nu) - k_{\Delta^n}(0,w_\nu)] + \lim_{\nu\to\infty}\bigl[k_{\Delta^n}(0,w_\nu) - \omega\bigl(0,f(w_\nu)\bigr)\bigr]\\
&\le \limsup_{w\to x}[k_{\Delta^n}(z,w) - k_{\Delta^n}(0,w)] + \tfrac{1}{2}\log\alpha\\
&< \tfrac{1}{2}\log(\alpha R),
\end{aligned}$$

that is $f(z) \in E(\tau,\alpha R)$.

Finally, let σ be a peculiar x-curve. Then (3.3) implies that $f\bigl(\sigma(t)\bigr) \in E(\tau,R)$ eventually for all $R > 0$, and this may happen iff $f\bigl(\sigma(t)\bigr) \to \tau$ as $t \to 1^-$, and we are done. □

If the lim inf in (3.1) is equal to $\frac{1}{2}\log\alpha$, we shall say that f is *α-Julia* at x.

It turns out that to compute (3.1) it suffices to check what happens along the image of φ_x:

Lemma 3.2: *Let $f \in \mathrm{Hol}(\Delta^n,\Delta)$, and take $x \in \partial\Delta^n$. Then*

$$\liminf_{w\to x}\frac{1 - |f(w)|}{1 - \|\!|w|\!\|} = \liminf_{t\to 1^-}\frac{1 - \bigl|f\bigl(\varphi_x(t)\bigr)\bigr|}{1 - t}. \tag{3.4}$$

Proof: Let us call α the left-hand side of (3.4), and β the right-hand side. Since $\varphi_x(t) \to x$ as $t \to 1^-$, clearly $\alpha \le \beta$; in particular, if $\alpha = +\infty$ we are done. So assume $\alpha < +\infty$; we must show that $\beta \le \alpha$.

Since α is finite, we can apply Theorem 3.1 (and its proof). So there is $\tau \in \partial\Delta$ such that

$$\forall z \in \Delta^n \qquad \frac{|\tau - f(z)|^2}{1 - |f(z)|^2} \le \alpha \max_{|x_j|=1}\left\{\frac{|x_j - z_j|^2}{1 - |z_j|^2}\right\}.$$

Now, if $\zeta\in\Delta$ then for $z=\varphi_x(\zeta)=\zeta x$ one has

$$\max_{|x_j|=1}\left\{\frac{|x_j-z_j|^2}{1-|z_j|^2}\right\}=\frac{|1-\zeta|^2}{1-|\zeta|^2};$$

therefore

$$\sup_{\zeta\in\Delta}\left\{\frac{\bigl|\tau-f\bigl(\varphi_x(\zeta)\bigr)\bigr|^2}{1-\bigl|f\bigl(\varphi_x(\zeta)\bigr)\bigr|^2}\Big/\frac{|1-\zeta|^2}{1-|\zeta|^2}\right\}\le\alpha. \tag{3.5}$$

Set $t_k=(k-1)/(k+1)$ for every $k\in\mathbb{N}$. Clearly $t_k\in\Delta$ and $t_k\to 1^-$ as $k\to+\infty$; moreover, $|1-t_k|^2/(1-|t_k|^2)=1/k$. It follows that $f\bigl(\varphi_x(t_k)\bigr)\in\overline{E(\tau,\alpha/k)}$, by (3.5). Now, $E(\tau,\alpha/k)$ is an euclidean disk of radius $\alpha/(k+\alpha)$; therefore

$$1-\bigl|f\bigl(\varphi_x(t_k)\bigr)\bigr|\le\bigl|\tau-f\bigl(\varphi_x(t_k)\bigr)\bigr|\le\frac{2\alpha}{k+\alpha}.$$

Since $1-|t_k|=2/(k+1)$ it follows that

$$\beta\le\limsup_{k\to\infty}\frac{1-\bigl|f(\varphi_x(t_k))\bigr|}{1-|t_k|}\le\lim_{k\to\infty}\alpha\,\frac{k+1}{k+\alpha}=\alpha,$$

and we are done. $\square$

In particular, thus, to check whether a given bounded holomorphic function f is α-Julia at $x\in\partial\Delta^n$ it suffices to study the function $t\mapsto f(tx)$.

Remark 3.1: The $\liminf$ (3.4) is always positive. Indeed,

$$\omega\bigl(0,f(w)\bigr)\le\omega\bigl(0,f(0)\bigr)+\omega\bigl(f(0),f(w)\bigr)\le\omega\bigl(0,f(0)\bigr)+k_{\Delta^n}(0,w);$$

therefore

$$k_{\Delta^n}(0,w)-\omega\bigl(0,f(w)\bigr)\ge-\omega\bigl(0,f(0)\bigr)>-\infty$$

and

$$\frac{1-|f(w)|}{1-\|w\|}\ge\frac{1-|f(0)|}{2(1+|f(0)|)}>0$$

for all $w\in\Delta^n$.

4. The Julia-Wolff-Carathéodory theorem

We are finally ready to state and prove the Julia-Wolff-Carathéodory Theorem 0.7 for bounded holomorphic functions in the polydisk. If $v\in\mathbb{C}^n$ and $f\colon\Delta^n\to\mathbb{C}$, we set

$$\frac{\partial f}{\partial v}(z)=df_z(v)=\sum_j v^j\,\frac{\partial f}{\partial z^j}(z).$$

Theorem 4.1: *Let $f\in\mathrm{Hol}(\Delta^n,\Delta)$ be a bounded holomorphic function, and $x\in\partial\Delta^n$. Assume there is $\alpha>0$ such that*

$$\liminf_{w\to x}\frac{1-|f(w)|}{1-\|w\|}=\alpha<+\infty.$$

Let $\tau \in \partial\Delta$ be the restricted E-limit of f at x, as given by Theorem 3.1. Then:

(i)
$$\tilde{K}\text{-}\lim_{z\to x} \frac{\tau - f(z)}{1 - \tilde{p}_x(z)} = \alpha\tau;$$

(ii) *If x_j is a Šilov component of x, then*

$$\tilde{K}\text{-}\lim_{z\to x} \frac{\tau - f(z)}{x_j - z_j} = \alpha\tau\overline{x_j};$$

(iii)
$$\tilde{K}\text{-}\lim_{z\to x} \frac{\partial f}{\partial x}(z) = \tilde{K}\text{-}\lim_{z\to x} \frac{\partial f}{\partial \check{x}}(z) = \alpha\tau;$$

(iv) *If x_j is an internal component of x, then*

$$\tilde{K}\text{-}\lim_{z\to x} \frac{\partial f}{\partial z_j}(z) = 0;$$

(v) *If x_j is a Šilov component of x, then $\partial f/\partial z_j$ has restricted K-limit at x.*

Remark 4.1: If x_j is an internal component of x, that is $|x_j| < 1$, then the incremental ratio $\bigl(\tau - f(z)\bigr)/(x_j - z_j)$ is *not* well-defined, because there are $z \in \Delta^n$ with $z_j = x_j$.

Remark 4.2: If x_j is a Šilov component of x, then $\partial f/\partial z_j$ might have a restricted K-limit at x different from the restricted K-limit of the corresponding incremental ratio. For instance, choose $0 < \beta < \alpha < 1$ and let $f \in \mathrm{Hol}(\Delta^2, \Delta)$ be given by

$$f(z_1, z_2) = 1 + \tfrac{1}{2}(\alpha + \beta)(z_1 - 1) + \tfrac{1}{2}(\alpha - \beta)(z_2 - 1).$$

Then it is easy to check that f is α-Julia at $x = (1, 1)$ but

$$\frac{\partial f}{\partial z_1} \equiv \frac{\alpha + \beta}{2}$$

is different from α. Notice that, on the other hand, $\partial f/\partial x \equiv \alpha$, as it should be.

Remark 4.3: In general, as we shall see in Propositions 4.8, 4.9 and Remark 4.6, the restricted K-limit of $\partial f/\partial z_j$ at x is of the form $\beta_j\tau\overline{x_j}$, where all β_j's are non-negative, $\beta_j = 0$ if x_j is an internal component of x, and $\beta_1 + \cdots + \beta_n = \alpha$.

The proof of Theorem 4.1 will fill the rest of this section. The idea is to show that the given functions are K-bounded, have limit along a special restricted curve — usually $t \mapsto \varphi_x(t)$ — and then apply Theorem 2.2. We begin with:

Lemma 4.2: *Let $f \in \mathrm{Hol}(\Delta^n, \Delta)$ be α-Julia at $x \in \partial\Delta^n$, and let $\tau \in \partial\Delta$ be its restricted E-limit at x. Fix $M > 1$. Then for all $z \in H(x, M)$ we have*

$$\left|\frac{\tau - f(z)}{1 - \tilde{p}_x(z)}\right| \le 2\alpha M^2 \qquad \text{and} \qquad \left|\frac{\tau - f(z)}{x_j - z_j}\right| \le 2\alpha M^2,$$

where x_j is any Šilov component of x.

Proof: Take $z \in H(x, M)$ and set

$$\tfrac{1}{2}\log R = \log M - k_{\Delta^n}(0, z). \tag{4.1}$$

Clearly, $z \in E(x, R)$; therefore $f(z) \in E(\tau, \alpha R)$ and thus

$$\lim_{s\to 1^-} \big[\omega\big(f(z), s\tau\big) - \omega(0, s\tau)\big] - \omega\big(0, f(z)\big) < \log(\alpha R)$$

(notice that $-\omega\big(0, f(z)\big) \le \omega(f(z), s\tau) - \omega(0, s\tau)$ for all $s < 1$). So

$$\log(\alpha R) > \tfrac{1}{2}\log\frac{|\tau - f(z)|^2}{1 - |f(z)|^2} - \tfrac{1}{2}\log\frac{1 + |f(z)|}{1 - |f(z)|} = \log\frac{|\tau - f(z)|}{1 + |f(z)|},$$

and (4.1) yields

$$\log\frac{|\tau - f(z)|}{1 + |f(z)|} < \log\alpha + \log M^2 - \log\frac{1 + |||z|||}{1 - |||z|||} \le \log(\alpha M^2) - \log\frac{1 + |\tilde{p}_x(z)|}{1 - |\tilde{p}_x(z)|},$$

because $|\tilde{p}_x(z)| \le |||z|||$. Hence

$$\log\left|\frac{\tau - f(z)}{1 - \tilde{p}_x(z)}\right| \le \log\frac{|\tau - f(z)|}{1 - |\tilde{p}_x(z)|} < \log\left(\alpha M^2 \frac{1 + |f(z)|}{1 + |\tilde{p}_x(z)|}\right) \le \log(2\alpha M^2).$$

Analogously, if x_j is a Šilov component of x we get

$$\log\left|\frac{\tau - f(z)}{x_j - z_j}\right| \le \log\frac{|\tau - f(z)|}{1 - |||z|||} < \log\left(\alpha M^2 \frac{1 + |f(z)|}{1 + |||z|||}\right) \le \log(2\alpha M^2).$$

□

Lemma 4.3: *Let $f \in \mathrm{Hol}(\Delta^n, \Delta)$ be α-Julia at $x \in \partial\Delta^n$, and let $\tau \in \partial\Delta$ be its restricted E-limit at x. Then*

$$\lim_{t\to 1^-} \frac{\tau - f\big(\varphi_x(t)\big)}{1 - t} = \lim_{t\to 1^-} (f \circ \varphi_x)'(t) = \alpha\tau.$$

Proof: Indeed, Lemma 3.2 shows that $f \circ \varphi_x$ is α-Julia at $1 \in \partial\Delta$, and the assertion follows from the classical Julia-Wolff-Carathéodory Theorem 0.2. □

And so:

Corollary 4.4: *Let $f \in \mathrm{Hol}(\Delta^n, \Delta)$ be α-Julia at $x \in \partial\Delta^n$, and let $\tau \in \partial\Delta$ be its restricted E-limit at x. Then*

$$\tilde{K}\text{-}\lim_{z \to x} \frac{\tau - f(z)}{1 - \tilde{p}_x(z)} = \alpha\tau \qquad \textit{and} \qquad \tilde{K}\text{-}\lim_{z \to x} \frac{\tau - f(z)}{x_j - z_j} = \alpha\tau\overline{x_j},$$

where x_j is any Šilov component of x.

Proof: The first limit follows immediately from Lemmas 4.2, 4.3 and Theorem 2.2. For the second limit it suffices to remark that

$$\frac{\tau - f(z)}{x_j - z_j} = \frac{\tau - f(z)}{1 - \tilde{p}_x(z)} \, \frac{1 - \tilde{p}_x(z)}{x_j - z_j}$$

and

$$\frac{1 - \tilde{p}_x\big(\varphi_x(t)\big)}{x_j - \big(\varphi_x(t)\big)_j} = \overline{x_j},$$

and then apply again Lemmas 4.2, 4.3 and Theorem 2.2. □

We have proved parts (i) and (ii) of Theorem 4.1; furthermore, by Lemma 4.3 we know that $\partial f/\partial x$ has limit $\alpha\tau$ along the x-curve $t \mapsto \varphi_x(t)$. So we must now deal with the K-boundedness of the partial derivatives. To do so we need two further lemmas:

Lemma 4.5: *Take $M_1 > M > 1$ and set $r = (M_1 - M)/(M_1 + M) < 1$. Fix $x \in \partial\Delta^n$ and let $\psi \in \mathrm{Hol}(\Delta, \Delta^n)$ be a complex geodesic such that $z_0 = \psi(0) \in H(x, M)$. Then $\psi(\Delta_r) \subset H(x, M_1)$.*

Proof: Let $\delta = \frac{1}{2}\log(M_1/M) > 0$; then $\zeta \in \Delta_r$ iff $\omega(0, \zeta) < \delta$. Then

$$\begin{aligned}
\lim_{s \to 1^-} &\big[k_{\Delta^n}\big(\psi(\zeta), \varphi_x(s)\big) - \omega(0, t)\big] + k_{\Delta^n}\big(0, \psi(\zeta)\big) \\
&\le 2k_{\Delta^n}\big(z_0, \psi(\zeta)\big) + \lim_{s \to 1^-}\big[k_{\Delta^n}\big(z_0, \varphi_x(s)\big) - \omega(0, t)\big] + k_{\Delta^n}(0, z_0) \\
&< 2\omega(0, \zeta) + \log M < \log M_1
\end{aligned}$$

for all $\zeta \in \Delta_r$. □

We shall denote by $\kappa_{\Delta^n} \colon \Delta^n \times \mathbb{C}^n \to \mathbb{R}^+$ the Kobayashi metric of Δ^n; it is well-known (see, e.g., [J-P, Example 3.5.6]) that

$$\kappa_{\Delta^n}(z; v) = \max_j \left\{ \frac{|v_j|}{1 - |z_j|^2} \right\}.$$

Lemma 4.6: *Take $x \in \partial\Delta^n$ and $M > 1$. Then for any $z \in H(x, M)$ and $v \in \mathbb{C}^n$ we have*

$$|1 - \tilde{p}_x(z)|\kappa_{\Delta^n}(z; v) \le 2M^3 |||v|||. \tag{4.2}$$

Proof: First of all,

$$(1-|||z|||)\kappa_{\Delta^n}(z;v)=\max_j\left\{\frac{|v_j|(1-|||z|||)}{1-|z_j|^2}\right\}\le |||v|||;$$

moreover if $z\in H(x,M)$ we have

$$\frac{|1-\tilde p_x(z)|}{1-|||z|||}\le M\,\frac{1-|||p_x(z)|||}{1-|||z|||},$$

by Corollary 1.4. Now, being $z\in H(x,M)$, for any $\varepsilon>0$ there is $t<1$ such that

$$\begin{aligned}\varepsilon+\log M&>k_{\Delta^n}\bigl(z,\varphi_x(t)\bigr)-\omega(0,t)+k_{\Delta^n}(0,z)\\&\ge k_{\Delta^n}\bigl(p_x(z),\varphi_x(t)\bigr)-k_{\Delta^n}\bigl(\varphi_x(t),0\bigr)+k_{\Delta^n}(0,z)\ge k_{\Delta^n}(0,z)-k_{\Delta^n}\bigl(0,p_x(z)\bigr);\end{aligned}$$

therefore

$$\log M\ge k_{\Delta^n}(0,z)-k_{\Delta^n}\bigl(0,p_x(z)\bigr)\ge \tfrac12\log\frac{1-|||p_x(z)|||}{2(1-|||z|||)},$$

and (4.2) follows. □

Then:

Proposition 4.7: *Let $f\in\mathrm{Hol}(\Delta^n,\Delta)$ be α-Julia at $x\in\partial\Delta^n$, and let $\tau\in\partial\Delta$ be its restricted E-limit at x. Then for every $v\in\mathbb{C}^n$ the partial derivative $\partial f/\partial v$ is K-bounded at x.*

Proof: Fix $v\in\mathbb{C}^n$, with $v\ne0$, and for every $z\in\Delta^n$ let $\psi_z\in\mathrm{Hol}(\Delta,\Delta^n)$ be a complex geodesic with $\psi_z(0)=z$ and $\psi_z'(0)=v/\kappa_{\Delta^n}(z;v)$. Clearly, we can choose ψ_z of the form $\gamma\circ\varphi_y$ for suitable $y\in\partial\Delta^n$ and γ automorphism of Δ^n.

Choose $r\in(0,1)$. Cauchy's formula yields

$$\begin{aligned}\frac{\partial f}{\partial v}(z)&=\kappa_{\Delta^n}(z;v)(f\circ\psi_z)'(0)=\frac{\kappa_{\Delta^n}(z;v)}{2\pi i}\int_{|\zeta|=r}\frac{f\bigl(\psi_z(\zeta)\bigr)}{\zeta^2}\,d\zeta\\&=\frac{1}{2\pi}\int_0^{2\pi}\frac{f\bigl(\psi_z(re^{1\theta})\bigr)-\tau}{\tilde p_x\bigl(\psi_z(re^{i\theta})\bigr)-1}\cdot\frac{\tilde p_x\bigl(\psi_z(re^{i\theta})\bigr)-1}{\tilde p_x(z)-1}\cdot\frac{\bigl(\tilde p_x(z)-1\bigr)\kappa_{\Delta^n}(z;v)}{re^{i\theta}}\,d\theta,\end{aligned}\tag{4.3}$$

where we used the fact that $\int_0^{2\pi}e^{-i\theta}\,d\theta=0$.

Fix $M>1$ and take $z\in H(x,M)$. Choose $M_1>M$ so that $(M_1-M)/(M_1+M)>r$. By Lemma 4.5 we have $\psi_z(\overline{\Delta_r})\subset H(x,M_1)$; so Lemma 4.2 yields

$$\left|\frac{f\bigl(\psi_z(re^{i\theta})\bigr)-\tau}{\tilde p_x\bigl(\psi_z(re^{i\theta})\bigr)-1}\right|\le 2\alpha M_1^2,$$

and the first factor in (4.3) is bounded. For the second factor we first remark that

$$\left|\frac{\tilde p_x\bigl(\psi_z(re^{i\theta})\bigr)-1}{\tilde p_x(z)-1}\right|\le\frac{\bigl|1-\tilde p_x\bigl(\psi_z(re^{i\theta})\bigr)\bigr|}{1-\bigl|\tilde p_x\bigl(\psi_z(re^{i\theta})\bigr)\bigr|}\cdot\frac{1-\bigl|\tilde p_x\bigl(\psi_z(re^{i\theta})\bigr)\bigr|}{1-|\tilde p_x(z)|}\le M_1\frac{1-|||p_x\bigl(\psi_z(re^{i\theta})\bigr)|||}{1-|||p_x(z)|||},$$

by Corollary 1.4 and Lemma 4.5. Now

$$\tfrac{1}{2}\log\frac{1-|||p_x\big(\psi_z(re^{i\theta})\big)|||}{2(1-|||p_x(z)|||)} \le k_{\Delta^n}\big(0,p_x(z)\big)-k_{\Delta^n}\Big(0,p_x\big(\psi_z(re^{i\theta})\big)\Big)$$
$$\le k_{\Delta^n}\Big(p_x(z),p_x\big(\psi_z(re^{i\theta})\big)\Big) \le k_{\Delta^n}\big(z,\psi_z(re^{i\theta})\big)=\tfrac{1}{2}\log\frac{1+r}{1-r},$$

and so the second factor in (4.3) is bounded too. By Lemma 4.6, the third factor is bounded by $2M^3|||v|||/r$, and we are done. □

Remark 4.4: Putting all together we have actually proved that

$$\left|\frac{\partial f}{\partial v}(z)\right| \le C\alpha M^6 |||v|||$$

for all $v\in\mathbb{C}^n$ and $z\in H(x,M)$, where C is a universal constant, obtained choosing the best $r\in(0,1)$.

Proposition 4.7 together with Lemma 4.3 yield

$$\tilde{K}\text{-}\lim_{z\to x}\frac{\partial f}{\partial x}(z)=\alpha\tau;$$

therefore to end the proof of Theorem 4.1 it suffices to show that parts (iv) and (v) hold. This is accomplished in the following propositions.

Proposition 4.8: *Let $f\in\mathrm{Hol}(\Delta^n,\Delta)$ be α-Julia at $x\in\partial\Delta^n$, and let $\tau\in\partial\Delta$ be its restricted E-limit at x. Take $v\in\mathbb{C}^n$ with no Šilov components with respect to x. Then*

$$\tilde{K}\text{-}\lim_{z\to x}\frac{\partial f}{\partial v}(z)=0.$$

Proof: For $t\in(0,1)$ let $\psi_t\in\mathrm{Hol}(\mathbb{C},\mathbb{C}^n)$ be given by

$$\psi_t(\zeta)=tx+\zeta v.$$

We have

$$|||\psi_t(\zeta)|||=\max_j\{|tx_j+\zeta v_j|\}\le\max\{t,t|||\mathring{x}|||+|\zeta|\,|||v|||\};$$

therefore if $r_t=(1-t)/|||v|||$ we have $\psi_t(\overline{\Delta_r})\subset\Delta^n$.

For any $\theta\in\mathbb{R}$ put $\sigma^\theta(t)=\psi_t(r_te^{i\theta})$. Clearly, σ^θ is a x-curve. Furthermore, $\sigma^\theta_x(t)=tx$, because v has no Šilov components with respect to x; so σ^θ is restricted and (cf. Proposition 1.1) special.

Now, $\psi_t(0)=tx=\varphi_x(t)$ and $\psi_t'(0)=v$ for any $t\in(0,1)$. Hence

$$\frac{\partial f}{\partial v}(tx)=(f\circ\psi_t)'(0)=\frac{1}{2\pi i}\int_{|\zeta|=r_t}\frac{f\circ\psi_t(\zeta)}{\zeta^2}\,d\zeta=\frac{1}{2\pi}\int_0^{2\pi}\frac{f\big(\psi_t(r_te^{i\theta})\big)-\tau}{r_te^{i\theta}}\,d\theta$$
$$=\frac{1}{2\pi}\int_0^{2\pi}\frac{f\big(\sigma^\theta(t)\big)-\tau}{\tilde{p}_x\big(\sigma^\theta(t)\big)-1}\cdot\frac{\tilde{p}_x\big(\sigma^\theta(t)\big)-1}{r_te^{i\theta}}\,d\theta.$$

Since σ^θ is a special restricted x-curve, we know that the first factor in the integrand converges boundedly to $\alpha\tau$ as $t \to 1^-$. For the second factor,

$$\frac{\tilde{p}_x\big(\sigma^\theta(t)\big) - 1}{r_t e^{i\theta}} = -\frac{|||v|||}{e^{i\theta}};$$

therefore

$$\lim_{t\to 1^-} \frac{\partial f}{\partial v}(tx) = -\frac{\alpha\tau |||v|||}{2\pi} \int_0^{2\pi} \frac{d\theta}{e^{i\theta}} = 0,$$

and the assertion follows from Proposition 4.7 and Theorem 2.2. □

Remark 4.5: The curve $\sigma^\theta(t) = tx + r_t e^{i\theta} v$, where $r_t = (1-t)/|||v|||$, is special iff $v = \lambda\check{x} + \mathring{v}$, where $\lambda \in \mathbb{C}$ and $\mathring{v}$ has no Šilov components with respect to x, whereas in the proof of [J, Theorem 5.(d)] it is mistakenly assumed to be special (that is, tangent to the diagonal) for all $v \in \mathbb{C}^n$.

In particular, then, the case of Šilov components requires a completely different proof:

Proposition 4.9: *Let $f \in \mathrm{Hol}(\Delta^n, \Delta)$ be α-Julia at $x \in \partial\Delta^n$, and let $\tau \in \partial\Delta$ be its restricted E-limit at x. Take a Šilov component x_j of x. Then $\partial f/\partial z_j$ has restricted K-limit $\beta\tau\overline{x_j}$ at x, where $\beta \geq 0$.*

Proof: Let M be the set of all $k = (k_1, \ldots, k_n) \in \mathbb{N}^n$ with $k_1, \ldots, k_n$ relatively prime and $|k| = k_1 + \cdots + k_n > 0$. Since the function $(\tau + f)/(\tau - f)$ has positive real part, the generalized Herglotz representation formula proved in [K-K] yields

$$\frac{\tau + f(z)}{\tau - f(z)} = \sum_{k\in M} \left[\int_{(\partial\Delta)^n} \frac{w^k + z^k}{w^k - z^k}\, d\mu_k(w) + C_k \right], \tag{4.4}$$

for suitable $C_k \in \mathbb{C}$ and positive Borel measures μ_k on $(\partial\Delta)^n$, where $z^k = z_1^{k_1}\cdots z_n^{k_n}$ and $w^k = w_1^{k_1}\cdots w_n^{k_n}$; the sum is absolutely converging.

Let $X_k = \{w \in (\partial\Delta)^n \mid w^k = x^k\}$, and set $\beta_k = \mu_k(X_k) \geq 0$ and $\mu_k^o = \mu_k - \mu_k|_{X_k}$, where $\mu_k|_{X_k}$ is the restriction of μ_k to X_k (i.e., $\mu_k|_{X_k}(E) = \mu_k(E \cap X_k)$ for every Borel subset E). Notice that $X_k = \varnothing$ (and so $\beta_k = 0$) as soon as $k_j > 0$ for some internal component x_j of x.

Using these notations (4.4) becomes

$$\frac{\tau + f(z)}{\tau - f(z)} = \sum_{k\in M} \left[\beta_k \frac{x^k + z^k}{x^k - z^k} + \int_{(\partial\Delta)^n} \frac{w^k + z^k}{w^k - z^k}\, d\mu_k^o(w) + C_k \right]. \tag{4.5}$$

In particular, if $z = tx = \varphi_x(t)$ we get

$$\frac{\tau + f(tx)}{\tau - f(tx)} = \sum_{k\in M} \left[\beta_k \frac{1 + t^{|k|}}{1 - t^{|k|}} + \int_{(\partial\Delta)^n} \frac{w^k + t^{|k|}x^k}{w^k - t^{|k|}x^k}\, d\mu_k^o(w) + C_k \right]. \tag{4.6}$$

Let us multiply both sides by $(1-t)$, and then take the limit as $t \to 1^-$. The left-hand side, by Corollary 4.4, tends to $2/\alpha$ (notice that $\alpha \neq 0$, by Remark 3.1). For the right-hand side, first of all we have

$$\frac{1+t^{|k|}}{1-t^{|k|}}(1-t) = \frac{1+t^{|k|}}{1+\cdots+t^{|k|-1}} \to \frac{2}{|k|}.$$

Next, if $|x^k|<1$ it is clear that

$$(1-t)\int\limits_{(\partial\Delta)^n} \frac{w^k+t^{|k|}x^k}{w^k-t^{|k|}x^k}\,d\mu_k^o(w) \to 0. \tag{4.7}$$

Otherwise, since $\mu_k^o(X_k)=0$, for every $\varepsilon>0$ there exists an open neighborhood A_ε of X_k in $(\partial\Delta)^n$ such that $\mu_k^o(A_\varepsilon)<\varepsilon$. Then

$$\begin{aligned}
(1-t)&\Bigg|\int\limits_{(\partial\Delta)^n} \frac{w^k+t^{|k|}x^k}{w^k-t^{|k|}x^k}\,d\mu_k^o(w)\Bigg| \\
&\le (1-t)\Bigg|\int\limits_{A_\varepsilon} \frac{w^k+t^{|k|}x^k}{w^k-t^{|k|}x^k}\,d\mu_k^o(w) + \int\limits_{(\partial\Delta)^n\setminus A_\varepsilon} \frac{w^k+t^{|k|}x^k}{w^k-t^{|k|}x^k}\,d\mu_k^o(w)\Bigg| \\
&\le 2\,\frac{1-t}{1-t^{|k|}}\varepsilon + (1-t)\Bigg|\int\limits_{(\partial\Delta)^n\setminus A_\varepsilon} \frac{w^k+t^{|k|}x^k}{w^k-t^{|k|}x^k}\,d\mu_k^o(w)\Bigg| \to \frac{2}{|k|}\varepsilon.
\end{aligned}$$

Since this happens for all $\varepsilon>0$, it follows that (4.7) holds in this case too. Summing up, we have found

$$\frac{1}{\alpha} = \sum_{k\in M}\frac{\beta_k}{|k|}; \tag{4.8}$$

in particular, the series on the right-hand side is converging.

Differentiating (4.5) with respect to z_j we get

$$\frac{\partial f}{\partial z_j}(z) = \overline{\tau}\big(\tau - f(z)\big)^2 \sum_{k\in M} k_j\frac{z^k}{z_j}\left[\beta_k\frac{x^k}{(x^k-z^k)^2} + \int\limits_{(\partial\Delta)^n}\frac{w^k}{(w^k-z^k)^2}\,d\mu_k^o(w)\right]. \tag{4.9}$$

Since we already know that $\partial f/\partial z_j$ is K-bounded, it suffices to show that $\partial f/\partial z_j$ has limit along the x-curve $t\mapsto tx$, where we have

$$\begin{aligned}
\frac{\partial f}{\partial z_j}(tx) = \overline{\tau}\left(\frac{\tau-f(tx)}{1-t}\right)^2 \sum_{k\in M} k_j t^{|k|-1}\overline{x_j}\Bigg[&\beta_k\left(\frac{1-t}{1-t^{|k|}}\right)^2 \\
&+ x^k(1-t)^2\int\limits_{(\partial\Delta)^n}\frac{w^k}{(w^k-t^{|k|}x^k)^2}\,d\mu_k^o(w)\Bigg].
\end{aligned}$$

The same argument used before shows that

$$(1-t)^2 \int\limits_{(\partial\Delta)^n} \frac{w^k}{(w^k - t^{|k|}x^k)^2}\, d\mu_k^o(w) \to 0$$

as $t \to 1^-$. Therefore

$$\lim_{t\to 1^-} \frac{\partial f}{\partial z_j}(tx) = \alpha^2 \tau \overline{x_j} \sum_{k\in M} \beta_k \frac{k_j}{|k|^2}, \tag{4.10}$$

where the series is converging because $\beta_k k_j/|k|^2 \le \beta_k/|k|$, and we are done. □

Remark 4.6: If x_j is an internal component of x, then the sum in (4.10) vanishes. Indeed, we have already remarked that $\beta_k = 0$ if $k_j > 0$, and so in this case $\beta_k k_j = 0$ always. Furthermore, (4.10) and (4.8) yield

$$\lim_{t\to 1^-} \frac{\partial f}{\partial x}(tx) = \sum_{j=1}^n x_j \lim_{t\to 1^-} \frac{\partial f}{\partial z_j}(tx) = \alpha^2\tau \sum_{j=1}^n |x_j|^2 \sum_{k\in M} \beta_k \frac{k_j}{|k|^2} = \alpha^2\tau \sum_{k\in M} \frac{\beta_k}{|k|} = \alpha\tau,$$

(where we used again the fact that $\beta_k k_j = 0$ always if $|x_j| < 1$), as it should be according to Theorem 4.1.(iii) and Remark 4.3.

Remark 4.7: If $d_x = 1$, then $\partial f/\partial v$ has restricted K-limit at x for all $v \in \mathbb{C}^n$. Indeed, in this case all $v \in \mathbb{C}^n$ are of the form $\lambda\check{x} + \mathring{v}$, where $\mathring{v}$ has no Šilov components with respect to x, and the assertion follows from Theorem 4.1.(iii) and (iv).

5. The multidimensional case

As discussed in the introduction, given the correct setup and enough geometrical information, it is possible to obtain Julia-Wolff-Carathéodory-like theorems for holomorphic maps between any kind of domains. In [A3, 5] we discussed the situation for maps between strongly convex and strongly pseudoconvex domains; in the previous section we studied the situation for functions from a polydisk into the unit disk in $\mathbb{C}$. This section is devoted to describe what happens for maps from a polydisk into another polydisk, or for maps from a polydisk into a strongly (pseudo)convex domain.

Let us start with a $f \in \mathrm{Hol}(\Delta^n, \Delta^m)$. The Julia condition

$$\liminf_{w\to x} \left[k_{\Delta^n}(0,w) - k_{\Delta^n}\big(0, f(w)\big)\right] = \tfrac{1}{2}\log\alpha < +\infty \tag{5.1}$$

translates in

$$\min_j \liminf_{w\to x} \frac{1-|f_j(w)|}{1-\|\!|w|\!\|} < +\infty. \tag{5.2}$$

In other words, (5.1) is equivalent to assuming that at least one component of f is α-Julia for some α, without saying anything else on the other components. Therefore if we write $f = \check{f} + \mathring{f}$, where $\check{f}$ contains the components of f satisfying a Julia condition (possibly with different α's), and $\mathring{f}$ the other components, we recover for $\check{f}$ results exactly like the one described in Theorems 3.1 and 4.1, whereas we cannot say anything about $\mathring{f}$. For instance, if $g \in \mathrm{Hol}(\Delta, \Delta)$ is given by

$$g(\zeta) = \exp\left(-\frac{\pi}{2} - i\log(1-\zeta)\right),$$

then $g(t)$ has no limit as $t \to 1^-$, and the map

$$f(z_1, z_2) = \bigl(z_1, \tfrac{1}{2}g(z_2)\bigr)$$

satisfies (5.2) with $x = (1,1)$ and $\alpha = 1$, but $f_2(t,t)$ has no limit as $t \to 1^-$. In particular, it is easy to check that

$$f\Bigl(E\bigl((1,1),1\bigr)\Bigr) \not\subseteq E\bigl((1,1),1\bigr),$$

and so in a general Julia lemma (like Theorem 0.5) one is forced to consider both small and big horospheres.

Remark 5.1: Even if it is less natural from the point of view of geometric function theory, one might of course consider maps $f \in \mathrm{Hol}(\Delta^n, \Delta^m)$ satisfying

$$\max_j \liminf_{w \to x} \frac{1 - |f_j(w)|}{1 - \|\!|w|\!\|} < +\infty$$

instead of (5.2). Then all components of f are α-Julia (for possibly different α's), and we recover Theorems 3.1 and 4.1 for all components of f.

The situation is more interesting if we consider maps $f \in \mathrm{Hol}(\Delta^n, D)$, where $D \subset\subset \mathbb{C}^m$ is a strongly convex domain. To state the result in this case we need some preparation. Fix once for all a point $z_0 \in D$; then for every $y \in \partial D$ there exists (see [L, A2]) a unique complex geodesic $\psi_y \in \mathrm{Hol}(\Delta, D) \cap C^1(\overline{\Delta}, \overline{D})$ such that $\psi_y(0) = z_0$ and $\psi_y(1) = y$. Associated to ψ_y there is a holomorphic retraction $q_y \colon \overline{D} \to \psi_y(\overline{\Delta})$ such that $q_y \circ q_y = q_y$; setting $\tilde{q}_y = \psi_y^{-1} \circ q_y \colon \overline{D} \to \overline{\Delta}$ we clearly have $\tilde{q}_y \circ \psi_y = \mathrm{id}_\Delta$. In particular, $\tilde{q}_y(y) = 1$. Given $f \in \mathrm{Hol}(\Delta^m, D)$ and $y \in \partial D$ we can associate to f the function $\tilde{f}_y = \tilde{q}_y \circ f$; roughly speaking, $\tilde{f}_y$ is the component of f in the direction of y, and $f - f_y$ (where $f_y = q_y \circ f$) contains the components of f “normal” to y.

Then the Julia-Wolff-Carathéodory theorem for this case is:

Theorem 5.1: *Let $D \subset\subset \mathbb{C}^m$ be a bounded strongly convex C^3 domain, and fix $z_0 \in D$. Let $x \in \partial D$ and $f \in \mathrm{Hol}(\Delta^n, D)$ be such that*

$$\liminf_{w \to x} \bigl[k_{\Delta^n}(0, w) - k_D\bigl(z_0, f(w)\bigr)\bigr] = \tfrac{1}{2}\log\alpha < +\infty, \tag{5.3}$$

where k_D is the Kobayashi distance of D. Then f has restricted E-limit $y \in \partial D$ at x, and the following maps are K-bounded at x:

(i) $[1 - \tilde{f}_y(z)]/[1 - \tilde{p}_x(z)]$;
(ii) $[f(z) - f_y(z)]/[1 - \tilde{p}_x(z)]^{1/2}$;
(iii) $[1 - \tilde{f}_y(z)]/[x_j - z_j]$, *where x_j is a Šilov component of x;*

(iv) $[f(z) - f_y(z)]/[x_j - z_j]^{1/2}$, *where x_j is a Šilov component of x;*
(v) $\partial \tilde{f}_y/\partial v$, *for all $v \in \mathbb{C}^n$;*
(vi) $[1 - \tilde{p}_x(z)]^{1/2} d(f - f_y)_z(v)$, *again for all $v \in \mathbb{C}^n$.*

Furthermore, function (i) *and function* (v) *for $v = x$, $\check{x}$ have restricted K-limit α at x; maps* (ii), (iv) *and function* (v) *for v without Šilov components with respect to x have restricted K-limit* 0 *at x; function* (iii) *has restricted K-limit $\alpha\overline{x_j}$ at x; function* (v) *for $v = x_j$, where x_j is a Šilov component of x, has restricted K-limit at x; and map* (vi) *for $v = \lambda\check{x} + \mathring{v}$ (where $\lambda \in \mathbb{C}$ and $\mathring{v}$ has no Šilov components with respect to x) has restricted K-limit* 0 *at x.*

Proof: The arguments needed are a mixture of the ones used to prove Theorem 4.1 and the ones used to prove the Julia-Wolff-Carathéodory theorem for strongly convex domains (see [A3]); so we shall just sketch the necessary modifications.

The existence of the restricted E-limit $y \in \partial D$ at x is proved exactly as in Theorem 3.1 (cf. [A3, Proposition 1.19]), and it follows from the fact that horospheres in strongly convex domains touch the boundary in exactly one point, as in the disk (see [A1]).

Next, (5.3) implies

$$\begin{aligned}\liminf_{t\to 1^-} \tfrac{1}{2}\log\frac{1 - |\tilde{f}_y(tx)|}{1-t} &= \liminf_{t\to 1^-}\big[k_{\Delta^n}(0, tx) - k_D\big(z_0, f_y(tx)\big)\big] \\ &\geq \liminf_{t\to 1^-}\big[k_{\Delta^n}(0, tx) - k_D\big(z_0, f(tx)\big)\big] \geq \tfrac{1}{2}\log\alpha.\end{aligned}$$

On the other hand, since

$$\tilde{f}_y \circ \varphi_x\big(E(1,R)\big) = \tilde{p}_y\Big(f \circ \varphi_x\big(E(1,R)\big)\Big) \subseteq \tilde{p}_y\big(E(y, \alpha R)\big) = E(1, \alpha R),$$

it follows that $\tilde{f}_y \circ \varphi_x$ is α-Julia; therefore the classical Julia-Wolff-Carathéodory theorem implies

$$\limsup_{t\to 1^-} \tfrac{1}{2}\log\frac{1 - |\tilde{f}_y(tx)|}{1-t} \leq \limsup_{t\to 1^-} \tfrac{1}{2}\log\frac{|1 - \tilde{f}_y(tx)|}{1-t} \leq \tfrac{1}{2}\log\alpha,$$

and $\tilde{f}_y \in \mathrm{Hol}(\Delta^n, \Delta)$ is α-Julia at x, by Lemma 3.2. Hence we can apply Theorem 4.1 to $\tilde{f}_y$, and all the assertions concerning functions (i), (iii) and (v) are proved.

So we are left with $f - f_y$. The proof of [A3, Proposition 3.7] applies word by word, replacing [A3, Proposition 3.4] by Lemma 4.2, and so maps (ii) and (iv) are K-bounded at x. Repeating the proof of [A3, Proposition 3.9] we get the restricted K-limit for map (ii), and arguing as in the proof of Corollary 4.4 we deal with map (iv) too.

Since, by Lemma 4.6, $\big(1 - \tilde{p}_x(z)\big)\kappa_{\Delta^n}(z; v)$ is K-bounded for all $v \in \mathbb{C}^n$, the arguments in the proof of [A3, Proposition 3.14] and Proposition 4.7 show that the map (vi) is K-bounded for all $v \in \mathbb{C}^n$. Finally, the proof of Proposition 4.8, suitably adapted as in [A3, Proposition 3.18] and recalling Remark 4.5, shows that the map (vi) has restricted K-limit 0 at x for the indicated v's, as claimed. □

We end this paper by remarking that, using the techniques described in [A5], one can localize the statement of Theorem 5.1 near $y \in \partial D$, obtaining a similar result for maps with values in a strongly pseudoconvex domain. We leave the details to the interested reader.

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