Chapter 2.7
Angular derivatives

As we have already said several times, this book is mainly devoted to three theorems: the Wolff-Denjoy theorem, Shields' theorem, and the Julia-Wolff-Carathéodory theorem. In chapter 2.4 we followed the Wolff-Denjoy theorem up to its final version in convex domains; in chapter 2.5 we worked out the generalization of Shields' theorem. In this chapter we shall deal with the Julia-Wolff-Carathéodory theorem, picking up the last loose threads, and ending the book.

Our plan of attack is divided into three parts. First of all, we shall define K-regions in generic bounded domains. The trick is the same used for the horospheres in chapter 2.4: they are defined by means of certain limits expressed via the Kobayashi distance. Using K-regions we can define K-limits in strongly convex domains; in particular, we shall show that in Theorem 2.4.16, if D is strongly convex, we can infer the existence of the K-limit at the point $x \in \partial D$, a much stronger statement than the existence of the non-tangential limit.

The second part concerns restricted K-limits and the Lindelöf theorem. Using complex geodesics, we shall define special and restricted curves in a strongly convex domain (cf. section 2.2.3), and we shall prove a Lindelöf theorem for (not necessarily) bounded holomorphic functions in a strongly convex domain, having exactly the same statement as Theorem 2.2.25.

Finally, in the third part we shall deal with the main theorem. The idea is that the right statement must be the one of Theorem 2.2.29, expressed in another language: we must replace radial approach by approach along complex geodesics, vectors orthogonal to $x \in B^n$ by vectors tangent to ∂D, and orthogonal projections by holomorphic retractions associated to complex geodesics. In this way we can preserve the main features of Theorem 2.2.29 in the new setting, and we can naturally apply the Julia lemma described in chapter 2.4. However, the actual proof remains quite a difficult task, requiring a very accurate investigation of the boundary behavior of the objects involved. It will be needed, in a decisive way, Theorem 2.3.70; indeed, it will turn out that the different behavior of the Kobayashi metric along normal and complex tangential directions is the reason behind the different exponents appearing both in Theorem 2.2.29 and in the statement of the Julia-Wolff-Carathéodory theorem for strongly convex domains.

2.7.1 *K-regions*

In section 2.4.2 we defined the small horosphere $E^D_{z_0}(x, R)$ and the big horosphere $F^D_{z_0}(x, R)$ in a domain $D \subset\subset \mathbf{C}^n$ of center $x \in \partial D$, radius $R > 0$ and pole $z_0 \in D$. If the domain D is clearly indicated by the context, we shall often drop the superscript D by the notation; on the other hand, if the domain D is an euclidean ball and the pole z_0 is the center of the ball, we shall often drop the subscript z_0.

In this section we shall be mainly concerned with another related concept. The *small K-region* $H^D_{z_0}(x,M)$ and the *big K-region* $K^D_{z_0}(x,M)$ of *vertex* $x \in \partial D$, *amplitude* $M>0$ and *pole* $z_0 \in D$ are given by

$$H^D_{z_0}(x,R) = \bigl\{ z \in D \bigm| \limsup_{w\to x}\bigl[k_D(z,w) - k_D(z_0,w)\bigr] + k_D(z_0,z) < \log M \bigr\},$$
$$K^D_{z_0}(x,R) = \bigl\{ z \in D \bigm| \liminf_{w\to x}\bigl[k_D(z,w) - k_D(z_0,w)\bigr] + k_D(z_0,z) < \log M \bigr\}.$$

Clearly, we shall often write $H_{z_0}(x,M)$ and $K_{z_0}(x,M)$, as well as $H^B(x,M)$ and $K^B(x,M)$ in an euclidean ball B if the pole is the origin.

Since for any z, z_0, $w \in D$ we have

$$|k_D(z,w) - k_D(z_0,w)| \le k_D(z_0,z),$$

it follows that

$$\begin{aligned} 0 &\le \liminf_{w\to x}\bigl[k_D(z,w) - k_D(z_0,w)\bigr] + k_D(z_0,z) \\ &\le \limsup_{w\to x}\bigl[k_D(z,w) - k_D(z_0,w)\bigr] + k_D(z_0,z) \le 2\,k_D(z_0,z). \end{aligned}$$

The next lemma collects several elementary properties of K-regions:

Lemma 2.7.1: *Let D be a bounded domain of $\mathbf{C}^n$, $z_0 \in D$ and $x \in \partial D$. Then:*

(i) *for every $M>0$ we have $H_{z_0}(x,M) \subset K_{z_0}(x,M)$;*

(ii) *for every $0 < M_1 < M_2$ we have*

$$H_{z_0}(x,M_1) \subset H_{z_0}(x,M_2) \qquad \text{and} \qquad K_{z_0}(x,M_1) \subset K_{z_0}(x,M_2);$$

(iii) *$H_{z_0}(x,M) = K_{z_0}(x,M) = \emptyset$ if $M \le 1$;*

(iv) *$B_k\bigl(z_0, \frac{1}{2}\log M\bigr) \subset H_{z_0}(x,M)$ for all $M>1$;*

(v) *$\bigcup_{M>1} H_{z_0}(x,M) = \bigcup_{M>1} K_{z_0}(x,M) = D$;*

(vi) *for every $M>1$ we have $H_{z_0}(x,M) \subset E_{z_0}(x,M^2)$ and $K_{z_0}(x,M) \subset F_{z_0}(x,M^2)$;*

(vii) *for every $M>1$ and $R>0$ set $r = \frac{1}{2}\log(M^2/R)$; then*

$$H_{z_0}(x,M) \setminus B_k(z_0,r) \subset E_{z_0}(x,R) \qquad \text{and} \qquad K_{z_0}(x,M) \setminus B_k(z_0,r) \subset F_{z_0}(x,R).$$

Proof: Everything immediately follows from the definitions, **q.e.d.**

The dependence of K-regions on the pole is quite inessential:

Lemma 2.7.2: *Let D be a bounded domain in $\mathbf{C}^n$, $z_0 \in D$ and $x \in \partial D$. Then:*
(i) *if $\varphi\colon D \to D$ is an automorphism of D continuous up to the boundary then for all $M > 1$ we have*

$$\varphi\bigl(H_{z_0}(x,M)\bigr) = H_{\varphi(z_0)}\bigl(\varphi(x),M\bigr) \qquad \text{and} \qquad \varphi\bigl(K_{z_0}(x,M)\bigr) = K_{\varphi(z_0)}\bigl(\varphi(x),M\bigr);$$

(ii) *choose $z_1 \in D$ and set*

$$\log L = \limsup_{w\to x}\bigl[k_D(z_1,w) - k_D(z_0,w)\bigr] + k_D(z_0,z_1) \ge 0.$$

Then for any $M > 1$ we have $H_{z_1}(x,M) \subset H_{z_0}(x,LM)$ and $K_{z_1}(x,M) \subset K_{z_0}(x,LM)$.

Proof: (i) Obvious.

(ii) We have

$$k_D(z,w) - k_D(z_0,w) = \bigl[k_D(z,w) - k_D(z_1,w)\bigr] + \bigl[k_D(z_1,w) - k_D(z_0,w)\bigr];$$

hence

$$\limsup_{w\to x}\bigl[k_D(z,w) - k_D(z_0,w)\bigr] + k_D(z_0,z) \le \limsup_{w\to x}\bigl[k_D(z,w) - k_D(z_1,w)\bigr] + k_D(z_1,z) + \log L.$$

Analogously,

$$k_D(z,w) - k_D(z_1,w) = \bigl[k_D(z,w) - k_D(z_0,w)\bigr] + \bigl[k_D(z_0,w) - k_D(z_1,w)\bigr],$$

and hence

$$\liminf_{w\to x}\bigl[k_D(z,w) - k_D(z_1,w)\bigr] + k_D(z_1,z) \ge \liminf_{w\to x}\bigl[k_D(z,w) - k_D(z_0,w)\bigr] + k_D(z_0,z) - \log L,$$

q.e.d.

Now we wish to have a better idea of the shape of K-regions. We start by showing that in B^n they are exactly the Korányi regions introduced in section 2.2.3 (and indeed K stands for Korányi):

Proposition 2.7.3: *Let $x \in \partial B^n$. Then for every $M > 1$ we have*

$$H_0^{B^n}(x,M) = K_0^{B^n}(x,M) = \left\{ z \in B^n \;\middle|\; \frac{|1 - (z,x)|}{1 - \|z\|} < M \right\}.$$

Proof: In Proposition 2.2.20 we showed that for all $z \in B^n$

$$\lim_{w\to x}\bigl[k_{B^n}(z,w) - k_{B^n}(0,w)\bigr] = \tfrac{1}{2}\log\frac{|1-(z,x)|^2}{1-\|z\|^2};$$

since

$$k_{B^n}(0,z) = \tfrac{1}{2}\log\frac{1+\|z\|}{1-\|z\|},$$

the assertion follows, **q.e.d.**

Therefore we explicitely know the K-regions in B^n. Using Proposition 2.4.12, the interested reader can investigate the (quite complicated) shape of K-regions in Δ^n; however, here we shall be mainly concerned with K-regions in strongly (pseudo)convex domains, whose shape is suggested by the following result and by Proposition 2.7.6:

Proposition 2.7.4: *Let $D \subset\subset \mathbf{C}^n$ be a strongly pseudoconvex domain, and choose $z_0 \in D$ and $x \in \partial D$. Then there are an euclidean ball B contained in D, tangent to ∂D at x, and a constant $\varepsilon_0 > 0$ such that*

$$\forall M > 1 \qquad K^B(x, \varepsilon_0 M) \subset H^D_{z_0}(x, M). \tag{2.7.1}$$

Proof: Let $\varepsilon > 0$ be given by Theorem 2.3.56; then, recalling Theorems 2.3.51 and 2.3.52, for every $z \in B(x, \varepsilon)$ we have

$$\limsup_{w \to x} \big[k_D(z, w) - k_D(z_0, w)\big] + k_D(z_0, z) \le \tfrac{1}{2} \log\left(1 + \frac{\|z - x\|}{d(z, \partial D)}\right) + \tfrac{1}{2} \log \frac{\|z - x\|}{d(z, \partial D)} + C, \tag{2.7.2}$$

for a suitable constant $C \in \mathbf{R}$ depending only on x and z_0.

Let B be an euclidean ball tangent to ∂D at x and contained in $D \cap B(x, \varepsilon)$; then

$$\forall M > 1 \qquad K^B(x, M) = \left\{ z \in B \;\middle|\; \frac{\|z - x\|}{d(z, \partial B)} < M \right\}. \tag{2.7.3}$$

Fix $M > 1$, write $C = -\log \varepsilon_1$ and set $\varepsilon_0 = \varepsilon_1/2$. If $\varepsilon_1 M - 1 \le 1$, then $\varepsilon_0 M \le 1$, $K^B(x, \varepsilon_0 M) = \emptyset$ and (2.7.1) is trivially satisfied. So assume $\varepsilon_1 M - 1 > 1$; in particular,

$$\varepsilon_0 M < \varepsilon_1 M - 1. \tag{2.7.4}$$

Take $z \in K^B(x, \varepsilon_0 M)$. By (2.7.4), $z \in K^B(x, \varepsilon_1 M - 1)$; hence, since $d(z, \partial B) \le d(z, \partial D)$, (2.7.2) and (2.7.3) yield

$$\limsup_{w \to x} \big[k_D(z, w) - k_D(z_0, w)\big] + k_D(z_0, z) < \tfrac{1}{2} \log\left[M\left(M - \frac{1}{\varepsilon_1}\right)\right] < \log M,$$

and (2.7.1) follows, **q.e.d.**

An immediate consequence is

Corollary 2.7.5: *Let $D \subset\subset \mathbf{C}^n$ be a strongly pseudoconvex domain. Then for any $z_0 \in D$ and $x \in \partial D$ there exists $\varepsilon_0 > 0$ such that*

$$\forall M > 1/\varepsilon_0 \qquad \overline{H_{z_0}(x, M)} \cap \partial D = \overline{K_{z_0}(x, M)} \cap \partial D = \{x\}.$$

Proof: Let $\varepsilon_0 > 0$ be given by Proposition 2.7.4; then $x \in \overline{H_{z_0}(x, M)}$ as soon as $\varepsilon_0 M > 1$. On the other hand, Lemma 2.7.1.(vi) yields $K_{z_0}(x, M) \subset F_{z_0}(x, M^2)$; hence the assertion follows from Theorem 2.4.14, **q.e.d.**

Later on we shall see that, for strongly convex domains, in the latter statement we can replace ε_0 by 1; see Proposition 2.7.8.(iii).

To control the shape of K-regions from the other side, we need a slight restriction: if D should be contained in an euclidean ball tangent to D at a point $x \in \partial D$, then D should be strongly convex near x. Therefore we have the following statement:

Proposition 2.7.6: *Let $D \subset\subset \mathbf{C}^n$ be a C^2 domain, and choose $z_0 \in D$ and $x \in \partial D$. Assume that D is strongly convex near x, and let B be an euclidean ball containing D and tangent to ∂D at x. Then there exists $\varepsilon_0 > 0$ such that*

$$\forall M > 1 \qquad\qquad H^D_{z_0}(x, M) \subset K^B(x, M/\varepsilon_0).$$

Proof: Let $\mathbf{n}_x$ denote the outer unit normal vector to ∂D (and ∂B) at x, and for every $\delta > 0$ set $z_\delta = x - \delta\mathbf{n}_x$. Choose $\sigma > 0$ so that z_σ is in D and lies between x and the center of B. If r is the radius of B, for $\delta < \sigma$ we have

$$k_B(z_\sigma, z_\delta) \geq \tfrac{1}{2} \log \frac{\sigma}{2 - \sigma/r} - \tfrac{1}{2} \log \delta \geq \tfrac{1}{2} \log \frac{\sigma}{2} - \tfrac{1}{2} \log \delta.$$

Now, Theorem 2.3.51 provides us with $c > 0$ independent of δ such that

$$k_D(z_\sigma, z_\delta) \leq \tfrac{1}{2} \log c - \tfrac{1}{2} \log d(z_\delta, \partial D) = \tfrac{1}{2} \log c - \tfrac{1}{2} \log \delta,$$

where the latter equality holds for δ small.

Therefore if $z \in D$ we have

$$\lim_{\delta \to 0} \big[k_B(z, z_\delta) - k_B(z_\sigma, z_\delta)\big] + k_B(z_\sigma, z) \\ \leq \limsup_{D \ni w \to x} \big[k_D(z, w) - k_D(z_\sigma, w)\big] + k_D(z_\sigma, z) + \tfrac{1}{2} \log \frac{\sigma}{2c}.$$

In other words we have shown that

$$\forall M > 1 \qquad\qquad H^D_{z_\sigma}(x, M) \subset K^B_{z_\sigma}(x, M/\varepsilon_1),$$

where $\varepsilon_1 = (2c/\sigma)^{1/2}$. Hence Lemma 2.7.2.(ii) implies that $\varepsilon_0 = L_1\varepsilon_1/L_2$ is as we need, where $\frac{1}{2} \log L_1$ is the Kobayashi distance in B between z_σ and the center of B, and

$$\log L_2 = \limsup_{D \ni w \to x} \big[k_D(z_\sigma, w) - k_D(z_0, w)\big] + k_D(z_0, z_\sigma),$$

q.e.d.

We end this section with a couple of remarks regarding K-regions in strongly convex domains. First of all, recalling Theorem 2.6.47 we immediately have

Proposition 2.7.7: *Let $D \subset\subset \mathbf{C}^n$ be a strongly convex C^3 domain. Then for any $z_0 \in D$, $x \in \partial D$ and $M > 1$ we have*

$$H_{z_0}(x, M) = K_{z_0}(x, M).$$

Proof: Indeed Theorem 2.6.47 implies the existence of the limit in the definition of K-regions, **q.e.d.**

Secondly, we can also correlate horospheres, K-regions and complex geodesics (cf. Theorem 2.6.45):

Proposition 2.7.8: *Let $D \subset\subset \mathbf{C}^n$ be a strongly convex C^3 domain; fix $z_0 \in D$, $x \in \partial D$ and let $\varphi_x \in \mathrm{Hol}(\Delta, D)$ be the unique complex geodesic such that $\varphi_x(0) = z_0$ and $\varphi_x(1) = x$. Then for all $R > 0$ and $M > 1$ we have:*

(i) $\varphi_x\big(E^\Delta(1, R)\big) = \varphi_x(\Delta) \cap E^D_{z_0}(x, R)$;
(ii) $\varphi_x\big(K^\Delta(1, M)\big) = \varphi_x(\Delta) \cap K^D_{z_0}(x, M)$;
(iii) $\varphi_x(0, 1) \subset K^D_{z_0}(x, M)$. *In particular,* $\overline{K^D_{z_0}(x, M)} \cap \partial D = \{x\}$ *for all* $M > 1$.

Proof: Since $\varphi_x(t) \to x$ as $t \to 1$, Theorem 2.6.47 yields

$$\lim_{w\to x}\big[k_D(z, w) - k_D(z_0, w)\big] = \lim_{t\to 1}\big[k_D\big(z, \varphi_x(t)\big) - \omega(0, t)\big]. \tag{2.7.5}$$

Then for any $\zeta \in \Delta$ we have

$$\lim_{w\to x}\big[k_D\big(\varphi_x(\zeta), w\big) - k_D(z_0, w)\big] = \lim_{t\to 1}\big[\omega(\zeta, t) - \omega(0, t)\big] = \tfrac{1}{2}\log\frac{|1-\zeta|^2}{1-|\zeta|^2},$$

and

$$k_D\big(z_0, \varphi_x(\zeta)\big) = \omega(0, \zeta) = \tfrac{1}{2}\log\frac{1+|\zeta|}{1-|\zeta|},$$

and the assertions follow, **q.e.d.**

2.7.2 *The Lindelöf theorem*

In this section we shall prove two Lindelöf's theorems in strongly convex domains, generalizing both Theorem 2.2.25 and Proposition 2.2.26.

We start by recalling some facts from chapter 2.6. From now on, D will be a bounded strongly convex C^3 domain of $\mathbf{C}^n$, and z_0 a fixed point of D, the *pole*. For each $x \in \partial D$ let $\varphi_x \in \mathrm{Hol}(\Delta, D)$ be the unique complex geodesic such that $\varphi_x(0) = z_0$ and $\varphi_x(1) = x$ (cf. Theorem 2.6.45); we recall that $\varphi_x \in C^1(\overline{\Delta})$, by Theorem 2.6.43. Associated to φ_x there is the dual map $\varphi_x^* \in A^1_n(\Delta)$, normalized so that $\varphi_x^*(1) = \overline{\mathbf{n}_x}$, where $\mathbf{n}_x$ is, as usual, the outer unit normal vector to ∂D at x; note that this is *not* the normalization used in chapter 2.6. Finally, denote by $\tilde{p}_x$, respectively p_x, the left inverse of φ_x and the holomorphic retraction associated to φ_x.

A *x-curve* is a continuous curve $\sigma\colon[0,1)\to D$ such that $\sigma(t)\to x$ as $t\to 1$. If σ is a x-curve, we shall set $\sigma_x=p_x\circ\sigma$, and $\tilde\sigma_x=\tilde p_x\circ\sigma$; σ_x is a x-curve in $\varphi_x(\Delta)$, and $\tilde\sigma_x$ is a 1-curve in Δ.

To generalize Theorem 2.2.25, we must single out two particular classes of curves. A x-curve σ is *special* if

$$\lim_{t\to 1}k_D\bigl(\sigma(t),\sigma_x(t)\bigr)=0, \tag{2.7.6}$$

and it is *restricted* if $\sigma_x\to x$ non-tangentially; note that, since $\varphi_x(\overline{\Delta})$ is transversal to ∂D (by Lemma 2.6.33), σ is restricted iff $\tilde\sigma_x\to 1$ non-tangentially.

We shall say that a function $f\colon D\to\mathbf{C}$ has *restricted K-limit* $L\in\mathbf{C}$ at $x\in\partial D$ if $f\bigl(\sigma(t)\bigr)\to L$ as $t\to 1$ for any restricted special x-curve σ; we shall write

$$K'\text{-}\lim_{z\to x}f(z)=L.$$

We are now able to prove the announced generalization of the classical Lindelöf theorem:

Theorem 2.7.9: *Let $D\subset\subset\mathbf{C}^n$ be a strongly convex C^3 domain, and $x\in\partial D$. Let $f\colon D\to\mathbf{C}$ be a bounded holomorphic function, and assume there is a special x-curve σ^o such that*

$$\lim_{t\to 1}f\bigl(\sigma^o(t)\bigr)=L\in\mathbf{C} \tag{2.7.7}$$

exists. Then f has restricted K-limit L at x.

Proof: Clearly we can assume $f(D)\subset\subset\Delta$. Let σ be any special x-curve. Since

$$\omega\Bigl(f\bigl(\sigma(t)\bigr),f\bigl(\sigma_x(t)\bigr)\Bigr)\le k_D\bigl(\sigma(t),\sigma_x(t)\bigr),$$

it follows that the limit of $f\bigl(\sigma(t)\bigr)$ as $t\to 1$ exists iff the limit of $f\bigl(\sigma_x(t)\bigr)$ as $t\to 1$ exists, and the two are equal.

In particular, (2.7.7) implies that

$$\lim_{t\to 1}f\bigl(\sigma_x^o(t)\bigr)=L.$$

But then, if σ is a restricted x-curve Theorem 1.3.23 applied to $f\circ\varphi_x$ implies

$$\lim_{t\to 1}f\bigl(\sigma_x(t)\bigr)=L,$$

and so, by the previous observation, $f\bigl(\sigma(t)\bigr)\to L$ as $t\to 1$ for any restricted special x-curve σ, **q.e.d.**

Two observations are in order. First of all, the definition (2.7.6) of a special curve is global, depending on the globally defined Kobayashi distance of the domain, whereas the existence of a limit at a boundary point should be a purely local fact. The contrast is only apparent: indeed, the localization principle for the Kobayashi distance Theorem 2.3.65 shows that a x-curve σ is special iff

$$\lim_{t\to 1}k_{D\cap V}\bigl(\sigma(t),\sigma_x(t)\bigr)=0$$

for any (and hence all) neighbourhood V of x in $\mathbf{C}^n$.

Second, our Theorem 2.7.9 recovers Čirka's Theorem 2.2.25. Indeed, it suffices to show that a x-curve in B^n is special in this new sense iff it satisfies (2.2.33):

Proposition 2.7.10: *Fix a point $x \in \partial B^n$. Then a x-curve σ is special iff*

$$\lim_{t \to 1} \frac{\|\sigma(t) - \sigma_x(t)\|^2}{1 - \|\sigma_x(t)\|^2} = 0.$$

Proof: The complex geodesic $\varphi_x \colon \overline{\Delta} \to \overline{D}$ is given by $\varphi_x(\zeta) = \zeta x$, and the associated holomorphic retraction is $p_x(z) = (z, x)x$. In particular, $\sigma_x = (\sigma, x)x$ and

$$(\sigma, \sigma_x) = |(\sigma, x)|^2 = \|\sigma_x\|^2.$$

Now, for any $z \in B^n$ let $\gamma_z \colon B^n \to B^n$ be an automorphism of B^n such that $\gamma_z(z) = 0$. Then

$$k_{B^n}\big(\sigma(t), \sigma_x(t)\big) = \tfrac{1}{2} \log \frac{1 + \big\|\gamma_{\sigma(t)}\big(\sigma_x(t)\big)\big\|}{1 - \big\|\gamma_{\sigma(t)}\big(\sigma_x(t)\big)\big\|} \to 0$$

as $t \to 1$ iff $\big\|\gamma_{\sigma(t)}\big(\sigma_x(t)\big)\big\| \to 0$ as $t \to 1$. Since

$$1 - \big\|\gamma_{\sigma(t)}\big(\sigma_x(t)\big)\big\|^2 = \frac{\big(1 - \|\sigma(t)\|^2\big)\big(1 - \|\sigma_x(t)\|^2\big)}{\big|1 - \big(\sigma(t), \sigma_x(t)\big)\big|^2} = \frac{1 - \|\sigma(t)\|^2}{1 - \big|\big(\sigma(t), x\big)\big|^2}, \tag{2.7.8}$$

it follows that

$$\big\|\gamma_{\sigma(t)}\big(\sigma_x(t)\big)\big\|^2 = \frac{\big\|\sigma(t)\big\|^2 - \big|\big(\sigma(t), x\big)\big|^2}{1 - \big|\big(\sigma(t), x\big)\big|^2} = \frac{\|\sigma(t) - \sigma_x(t)\|^2}{1 - \|\sigma_x(t)\|^2},$$

and we are done, **q.e.d.**

In other words, in B^n a x-curve σ is special iff $\|\sigma - \sigma_x\|^2$ goes to 0 faster than the distance of σ_x from the boundary. This is what happens in general:

Proposition 2.7.11: *Let $D \subset\subset \mathbf{C}^n$ be a strongly convex C^3 domain, and fix $z_0 \in D$ and $x \in \partial D$. Let σ be a restricted x-curve. If σ is special, then*

$$\lim_{t \to 1} \frac{\|\sigma(t) - \sigma_x(t)\|^2}{d\big(\sigma_x(t), \partial D\big)} = 0. \tag{2.7.9}$$

Conversely, if (2.7.9) *holds and there is an euclidean ball $B \subset D$ tangent to ∂D at x such that $\sigma(t) \in B$ eventually, then σ is special.*

Proof: Assume σ special. Up to a translation and a rescaling, we can assume $D \subset B^n$ and that D and B^n are tangent to each other at x. Since σ is restricted, σ_x is non-tangential; so we can replace $d\big(\sigma_x(t), \partial D\big)$ by $d\big(\sigma_x(t), \partial B^n\big)$ in the assertion, and we should show that

$$\lim_{t \to 1} \frac{\|\sigma(t) - \sigma_x(t)\|^2}{1 - \|\sigma_x(t)\|^2} = 0, \tag{2.7.10}$$

knowing that $k_D\big(\sigma(t),\sigma_x(t)\big)\to 0$ as $t\to 1$. Since $k_{B^n}\le k_D$, this implies, as in (2.7.8),

$$1-\frac{\big(1-\|\sigma(t)\|^2\big)\big(1-\|\sigma_x(t)\|^2\big)}{\big|1-\big(\sigma(t),\sigma_x(t)\big)\big|^2}\to 0.$$

Therefore it suffices to show that there exists $\varepsilon>0$ such that

$$\left[1-\frac{\big(1-\|\sigma(t)\|^2\big)\big(1-\|\sigma_x(t)\|^2\big)}{\big|1-\big(\sigma(t),\sigma_x(t)\big)\big|^2}\right]\cdot\frac{1-\|\sigma_x(t)\|^2}{\|\sigma(t)-\sigma_x(t)\|^2}\ge\varepsilon \tag{2.7.11}$$

for all t close enough to 1 and so that $\sigma(t)\neq\sigma_x(t)$.

Analogously, if (2.7.9) holds and there is an euclidean ball $B\subset D$ tangent to ∂D at x such that $\sigma(t)\in B$ eventually (and we can assume $B=B^n$), then (2.7.10) still holds. Furthermore, if there is $M<+\infty$ such that

$$\left[1-\frac{\big(1-\|\sigma(t)\|^2\big)\big(1-\|\sigma_x(t)\|^2\big)}{\big|1-\big(\sigma(t),\sigma_x(t)\big)\big|^2}\right]\cdot\frac{1-\|\sigma_x(t)\|^2}{\|\sigma(t)-\sigma_x(t)\|^2}\le M \tag{2.7.12}$$

it follows that

$$\lim_{t\to 1}k_B\big(\sigma(t),\sigma_x(t)\big)=0;$$

but $k_B\ge k_D$, and the converse assertion follows too.

So it remains to prove (2.7.11) and (2.7.12), assuming (as we may) $\|x\|=1$ and $\mathbf{n}_x=x$.

For the sake of simplicity, we shall drop t in the following computations. We have

$$\begin{aligned}&\left[1-\frac{\big(1-\|\sigma\|^2\big)\big(1-\|\sigma_x\|^2\big)}{\big|1-\big(\sigma,\sigma_x\big)\big|^2}\right]\cdot\frac{1-\|\sigma_x\|^2}{\|\sigma-\sigma_x\|^2}\\ &\qquad=\left[\frac{1-\|\sigma_x\|^2}{|1-(\sigma,\sigma_x)|}\right]^2\frac{1}{\|\sigma-\sigma_x\|^2}\left[\frac{|1-(\sigma,\sigma_x)|^2}{1-\|\sigma_x\|^2}-(1-\|\sigma\|^2)\right].\end{aligned}$$

Now

$$\frac{|1-(\sigma,\sigma_x)|^2}{(1-\|\sigma_x\|^2)^2}=\frac{|1-\|\sigma_x\|^2+(\sigma_x-\sigma,\sigma_x)|^2}{(1-\|\sigma_x\|^2)^2}=\left|1+\frac{(\sigma_x-\sigma,\sigma_x)}{1-\|\sigma_x\|^2}\right|^2.$$

By definition,

$$|(\sigma_x-\sigma,\sigma_x)|=\big|\big\langle\sigma_x-\sigma,\overline{\sigma_x}-\varphi_x^*(\tilde\sigma_x)\big\rangle\big|\le\|\sigma-\sigma_x\|\cdot\big\|\sigma_x-\overline{\varphi_x^*(\tilde\sigma_x)}\big\|. \tag{2.7.13}$$

Now let $\psi\in C^1(\overline{\Delta})$ be given by $\psi=\overline{\varphi_x^*}-\varphi$. Clearly, $\psi(1)=0$; hence

$$\begin{aligned}\big\|\sigma_x-\overline{\varphi_x^*(\tilde\sigma_x)}\big\|=\|\psi(1)-\psi(\tilde\sigma_x)\|&\le c_1|1-\tilde\sigma_x|=c_1|\tilde p_x(x)-\tilde p_x(\sigma_x)|\\ &\le c_2\|x-\sigma_x\|\le c_3(1-\|\sigma_x\|^2),\end{aligned} \tag{2.7.14}$$

for suitable c_1, c_2, $c_3 > 0$, because ψ and $\tilde{p}_x$ are C^1 functions, and σ_x goes to x non-tangentially. Therefore

$$\frac{|1-(\sigma,\sigma_x)|^2}{(1-\|\sigma_x\|^2)^2} = |1+o(1)|^2. \tag{2.7.15}$$

Write $\sigma = \zeta\sigma_x + v$, where v is orthogonal to σ_x. Since $\sigma - \sigma_x \to 0$, it is clear that $\zeta \to 1$ and $v \to 0$. Then

$$\begin{aligned}\frac{|1-(\sigma,\sigma_x)|^2}{1-\|\sigma_x\|^2} - (1-\|\sigma\|^2) &= \frac{\|\sigma-\sigma_x\|^2 + |(\sigma,\sigma_x)|^2 - \|\sigma\|^2\|\sigma_x\|^2}{1-\|\sigma_x\|^2} \\ &= \frac{|1-\zeta|^2\|\sigma_x\|^2}{1-\|\sigma_x\|^2} + \|v\|^2\end{aligned}$$

and

$$\|\sigma-\sigma_x\|^2 = |1-\zeta|^2\|\sigma_x\|^2 + \|v\|^2.$$

By definition we have $\big\langle \sigma-\sigma_x, \varphi_x^*(\tilde{\sigma}_x)\big\rangle \equiv 0$, that is

$$(1-\zeta)\big\langle \sigma_x, \varphi_x^*(\tilde{\sigma}_x)\big\rangle = \big\langle v, \varphi_x^*(\tilde{\sigma}_x)\big\rangle.$$

Since $\varphi_x^* \neq 0$, $(v,\sigma_x) \equiv 0$, $\sigma_x \to x$ and $\varphi_x^*(\tilde{\sigma}_x) \to \overline{x}$ as $t \to 1$, it follows that

$$|1-\zeta| = o(1)\|v\|;$$

in particular, there are c_4, $c_5 > 0$ such that

$$c_4\|v\| \le \|\sigma-\sigma_x\| \le c_5\|v\|.$$

Furthermore, recalling (2.7.13) and (2.7.14) it follows that

$$|1-\zeta| = \frac{|(\sigma_x-\sigma,\sigma_x)|}{\|\sigma_x\|^2} \le c_6\|\sigma-\sigma_x\|(1-\|\sigma_x\|^2) \le c_5c_6\|v\|(1-\|\sigma_x\|^2),$$

for a suitable $c_6 > 0$.

Putting all together we get

$$\frac{1}{c_5^2} \le \frac{1}{\|\sigma-\sigma_x\|^2}\left[\frac{|1-(\sigma,\sigma_x)|^2}{1-\|\sigma_x\|^2} - (1-\|\sigma\|^2)\right] \le \frac{1}{c_4^2}\big[1+o(1)\big]; \tag{2.7.16}$$

so (2.7.11) and (2.7.12) follow from (2.7.15) and (2.7.16), and we are done, **q.e.d.**

Since we introduced a restricted K-limit, there should exist a K-limit. The definition is very natural: we say that a function $f: D \to \mathbf{C}$ admits *K-limit* $L \in \mathbf{C}$ at $x \in \partial D$ if $f(z) \to L$ as $z \to x$ within $K_{z_0}(x,M)$ for all $M > 1$; here z_0 is any point of D. Clearly, by Lemma 2.7.2.(ii), this definition does not depend on z_0.

To study the relations among K-limits, restricted K-limits and non-tangential limits we need a definition and one lemma.

Let $D \subset\subset \mathbf{C}^n$ be a strongly convex C^3 domain, and fix $z_0 \in D$. If σ is a x-curve, σ_x is non-tangential iff

$$\lim_{t\to 1}\big[k_D\big(\sigma_x(s),\varphi_x(t)\big) - \omega(0,t)\big] + k_D\big(z_0,\sigma_x(s)\big) < \log M$$

for some $M > 1$ and every $s \in [0,1)$, by (2.7.5); we shall say that σ is *M-restricted*.

Lemma 2.7.12: *Let $D \subset\subset \mathbf{C}^n$ be a strongly convex C^3 domain; fix $z_0 \in D$ and $x \in \partial D$. Let σ be a x-curve. Then:*

(i) *if $\sigma(t) \in K_{z_0}(x, M)$ for t close to 1, then σ is M-restricted;*
(ii) *if σ is special and M-restricted, then for any $M_1 > M$ we have $\sigma(t) \in K_{z_0}(x, M_1)$ for all t sufficiently close to 1;*
(iii) *every non-tangential x-curve is special and restricted.*

Proof: (i) For any $z \in D$ we have

$$\lim_{t\to 1}\big[k_D\big(p_x(z), \varphi_x(t)\big) - \omega(0,t)\big] \le \lim_{t\to 1}\big[k_D\big(z, \varphi_x(t)\big) - \omega(0,t)\big]$$

and $k_D\big(z_0, p_x(z)\big) \le k_D(z_0, z)$ because $p_x \circ \varphi_x = \varphi_x$, and (i) follows.

(ii) We have

$$\begin{aligned} k_D\big(\sigma(s), \varphi_x(t)\big) &- \omega(0,t) + k_D\big(z_0, \sigma(s)\big) \\ &\le 2\,k_D\big(\sigma(s), \sigma_x(s)\big) + k_D\big(\sigma_x(s), \varphi_x(t)\big) - \omega(0,t) + k_D\big(z_0, \sigma_x(t)\big); \end{aligned}$$

hence

$$\lim_{t\to 1}\big[k_D\big(\sigma(s), \varphi_x(t)\big) - \omega(0,t)\big] + k_D\big(z_0, \sigma(s)\big) < \log M + 2\,k_D\big(\sigma(s), \sigma_x(s)\big),$$

and (ii) follows.

(iii) Let σ be a non-tangential x-curve, $B \subset D$ the euclidean ball tangent to ∂D at x given by Proposition 2.7.4, and $\varepsilon_0 > 0$ the corresponding constant; up to a translation and a rescaling, we can assume $B = B^n$. Since σ is non-tangential, there is $M > 1$ such that $\sigma(t) \in K^B(x, \varepsilon_0 M)$ eventually; by Proposition 2.7.4 and part (i), σ is M-restricted. In particular, being σ_x non-tangential, there is $c_1 > 0$ such that

$$\forall t \in [0,1) \qquad \|\sigma_x(t) - x\| \le c_1\, d\big(\sigma_x(t), \partial D\big). \tag{2.7.17}$$

Next, σ satisfies (2.2.36), for it is non-tangential; hence there is $c_2 > 0$ such that

$$\forall t \in [0,1) \qquad \|\sigma(t) - x\| \le c_2 \big|1 - \big(\sigma(t), x\big)\big|. \tag{2.7.18}$$

Now, by definition $\langle \sigma, \varphi_x^* \circ \tilde\sigma_x\rangle \equiv \langle \sigma_x, \varphi_x^* \circ \tilde\sigma_x\rangle$; then

$$1 - (\sigma, x) = \langle \sigma - x, \varphi_x^* \circ \tilde\sigma_x - \varphi_x^*(1)\rangle + \langle x - \sigma_x, \varphi_x^* \circ \tilde\sigma_x\rangle,$$

and so

$$\begin{aligned} |1 - (\sigma, x)| \le c_3\big[\|\varphi_x^* \circ \tilde\sigma_x - \varphi_x^*(1)\| + \|x - \sigma_x\|\big] &\le c_4\big[|1 - \tilde\sigma_x| + \|x - \sigma_x\|\big] \\ &\le c_5 \|x - \sigma_x\|, \end{aligned} \tag{2.7.19}$$

for suitable c_3, c_4, $c_5 > 0$, because φ_x^* and $\tilde p_x$ are C^1 maps.

Putting together (2.7.17), (2.7.18) and (2.7.19) we find

$$\|\sigma - \sigma_x\| \le \|\sigma - x\| + \|x - \sigma_x\| \le c_6\, d(\sigma_x, \partial D),$$

for a suitable $c_6 > 0$; hence

$$\lim_{t\to 1} \frac{\|\sigma(t) - \sigma_x(t)\|^2}{d\big(\sigma_x(t), \partial D\big)} = 0,$$

and, by Proposition 2.7.11, σ is special, **q.e.d.**

So Lemma 2.7.12 shows that K-limit implies restricted K-limit, and that restricted K-limit implies non-tangential limit, exactly as in the ball.

We end this section proving a Lindelöf theorem for not necessarily bounded holomorphic functions, generalizing Proposition 2.2.26. We shall say that a function $f\colon D\to\mathbf{C}$ is *K-bounded* at $x\in\partial D$ if f is bounded in every K-region $K_{z_0}(x,M)$, where z_0 is any point of D; it is clear that this definition does not depend on z_0. Then

Theorem 2.7.13: *Let $D\subset\subset\mathbf{C}^n$ be a strongly convex C^3 domain, and choose $x\in\partial D$. Let $f\colon D\to\mathbf{C}$ be a holomorphic function K-bounded at x, and assume there is a restricted special x-curve σ^o such that*

$$\lim_{t\to1} f\bigl(\sigma^o(t)\bigr)=L\in\mathbf{C}.$$

Then f has restricted K-limit L at x.

Proof: First of all, we claim that if σ is a M-restricted special x-curve, then there exists $M_0>M$ such that

$$\lim_{t\to1} k_{K_{z_0}(x,M_0)}\bigl(\sigma(t),\sigma_x(t)\bigr)=0. \tag{2.7.20}$$

Let $B\subset D$ be the euclidean ball tangent to ∂D at x given by Proposition 2.7.4, and $\varepsilon_0>0$ the corresponding constant. Choose $M_1>M$ so that $M_1\varepsilon_0>M$; in particular,

$$\emptyset\ne K^B(x,\varepsilon_0M_1)\subset K^D_{z_0}(x,M_1).$$

Finally, since σ_x is non-tangential and σ is M-restricted, $\sigma_x(t)\in K^B(x,\varepsilon_0M_1)$ for t close enough to 1. Choose $M_0>M_1$.

Consider now $p_x^{-1}\bigl(\sigma_x(t)\bigr)$. By definition, $p_x^{-1}\bigl(\sigma_x(t)\bigr)$ is the intersection of a complex affine hyperplane with D; moreover, this hyperplane tends to the complex tangent plane $T^{\mathbf{C}}_x(\partial D)$ to ∂D at x as t goes to 1. Therefore the function

$$\delta(t)=\inf\bigl\{\|v\|\bigm| \sigma_x(t)+v\in p_x^{-1}\bigl(\sigma_x(t)\bigr)\cap\partial K^B(x,\varepsilon_0M_0)\bigr\}$$

tends to 0 at the same rate as

$$\delta'(t)=\inf\bigl\{\|v\|\bigm| v\in T^{\mathbf{C}}_x(\partial D),\ \sigma_x(t)+v\in\partial K^B(x,\varepsilon_0M_0)\bigr\}.$$

On the other hand, since σ_x is non-tangential, $\delta'(t)$ tends to 0 at the same rate as the function $\delta_{\varepsilon_0M_0}$ defined in Lemma 2.2.23 calculated in the orthogonal projection of $\sigma_x(t)$ into the affine line $x+\mathbf{C}\mathbf{n}_x$ — it is easy to see that this projection still belongs to $K^B(x,\varepsilon_0M_1)$ for $\sigma_x(t)$ does. In conclusion

$$\delta(t)=O\bigl((1-\|\sigma_x(t)\|)^{1/2}\bigr)=O\bigl(d(\sigma_x(t),\partial D)^{1/2}\bigr), \tag{2.7.21}$$

again because σ_x is non-tangential. We point out that if $v\in\mathbf{C}^n$ is such that $\|v\|<\delta(t)$ and $\sigma_x(t)+v\in p_x^{-1}\bigl(\sigma_x(t)\bigr)$, then $\sigma_x(t)+v\in K^B(x,\varepsilon_0M_0)$.

Now consider the map $\psi_t\colon\mathbf{C}\to\mathbf{C}^n$ defined by

$$\psi_t(\zeta)=\sigma_x(t)+\zeta\bigl(\sigma(t)-\sigma_x(t)\bigr).$$

Clearly, $\psi_t(0) = \sigma_x(t)$ and $\psi_t(1) = \sigma(t)$; moreover, $\psi_t(\zeta)$ belongs to the affine hyperplane defining $p_x^{-1}\bigl(\sigma_x(t)\bigr)$ for all $\zeta \in \mathbf{C}$. Let

$$R(t) = \sup\{r > 0 \mid \psi_t(\Delta_r) \subset K_{z_0}(x, M_0)\};$$

Proposition 2.7.11 and (2.7.21) imply

$$R(t) \geq \frac{\delta(t)}{\|\sigma(t) - \sigma_x(t)\|} \longrightarrow +\infty$$

as $t \to 1$, and so (2.7.20) is proved.

In particular, (2.7.20) holds for σ^o, with a specific $M_0 > 1$. On $K_{z_0}(x, M_0)$, f is bounded by $R > 0$, say; hence

$$k_{\Delta_R}\bigl(f\bigl(\sigma^o(t)\bigr), f\bigl(\sigma^o_x(t)\bigr)\bigr) \leq k_{K_{z_0}(x,M_0)}\bigl(\sigma^o(t), \sigma^o_x(t)\bigr),$$

and so

$$\lim_{t \to 1} f\bigl(\sigma^o_x(t)\bigr) = L. \tag{2.7.22}$$

Finally, let σ be any restricted special x-curve. The classical Lindelöf Theorem 1.3.23 together with (2.7.22) implies

$$\lim_{t \to 1} f\bigl(\sigma_x(t)\bigr) = L;$$

hence, arguing as before, we find that $f\bigl(\sigma(t)\bigr) \to L$ as $t \to 1$, and we are done, **q.e.d.**

This theorem will be very handy in the next section, where we shall apply it exactly as in section 2.2.4 we applied Proposition 2.2.26 to prove Theorem 2.2.29.

2.7.3 *The Julia-Wolff-Carathéodory theorem in strongly convex domains*

We shall finally deal with the main concern of this chapter, the generalization to strongly convex domains of the Julia-Wolff-Carathéodory theorem.

Let us fix some notations. D is still a bounded strongly convex C^3 domain of $\mathbf{C}^n$, and z_0 a fixed point of D. We denote by $K_{z_0}\colon \overline{D} \to [0,1]$ the function $K_{z_0}(z) = \tanh\bigl(k_D(z_0, z)\bigr)$, and for any $x \in \partial D$ we write φ_x, φ_x^*, $\tilde{p}_x$ and p_x for (respectively) the unique complex geodesic such that $\varphi_x(0) = z_0$ and $\varphi_x(1) = x$, its dual map, its left inverse and the holomorphic retraction associated to φ_x. Moreover, we define the *geodesic normal vector* ν_x by $\nu_x = \varphi_x'(1)$. It is clear that ν_x is transversal to ∂D at x, but in general it is different from the real normal vector $\mathbf{n}_x$. We shall use τ_x to denote a generic complex tangent vector to ∂D at x. Finally, if $f\colon D \to D$ is a holomorphic map, we set $f_x = p_x \circ f\colon D \to D$ and $\tilde{f}_x = \tilde{p}_x \circ f\colon D \to \Delta$. Using these notations, we can state our main theorem:

Theorem 2.7.14: *Let $D \subset\subset \mathbf{C}^n$ be a strongly convex C^3 domain, and fix $z_0 \in D$. Let $f: D \to D$ be a holomorphic map such that for some $x \in \partial D$ we have*

$$\liminf_{w \to x} \bigl[k_D(z_0, w) - k_D\bigl(z_0, f(w)\bigr)\bigr] = \tfrac{1}{2}\log\alpha < +\infty \qquad (2.7.23)$$

for a suitable $\alpha > 0$. Then f has K-limit $y \in \partial D$ at x, and the following maps are K-bounded at x:

(i) $\bigl[1 - \tilde{f}_y(z)\bigr]/\bigl[1 - \tilde{p}_x(z)\bigr]$;

(ii) $\bigl[f(z) - f_y(z)\bigr]/\bigl[1 - \tilde{p}_x(z)\bigr]^{1/2}$;

(iii) $d\bigl(\tilde{f}_y\bigr)_z(\nu_x)$;

(iv) $\bigl[1 - \tilde{p}_x(z)\bigr]^{1/2} d\bigl(f - f_y\bigr)_z(\nu_x)$;

(v) $d\bigl(\tilde{f}_y\bigr)_z(\tau_x)/\bigl[1 - \tilde{p}_x(z)\bigr]^{1/2}$;

(vi) $d\bigl(f - f_y\bigr)_z(\tau_x)$.

Furthermore, the functions (i) *and* (iii) *have restricted K-limit α at x, and the maps* (ii), (iv) *and* (v) *have K-limit* 0 *at x.*

The rest of this section is devoted to the proof of this theorem. The idea is first to use the Julia lemma proved in chapter 2.4 to show that the maps (i)-(vi) are bounded in K-regions, next to prove that they have the stated limit along a restricted special curve — usually $t \mapsto \varphi_x(t)$ —, and finally to invoke Theorem 2.7.13. From now on, we shall assume the hypotheses of Theorem 2.7.14, without mentioning them anymore.

We begin showing that f has K-limit at x:

Proposition 2.7.15: *f has K-limit $y \in \partial D$ at x.*

Proof: Let $y \in \partial D$ be given by Theorem 2.4.16. We already know that f has non-tangential limit y at x; we claim that f has K-limit y at x.

Fix $M > 1$, and let $\{z_\nu\} \subset K_{z_0}(x, M)$ be a sequence converging to x. Then, by Lemma 2.7.1.(vii), for every $R > 0$ we eventually have $z_\nu \in E_{z_0}(x, R)$, and so, by Theorem 2.4.16, for every $R > 0$ we eventually have $f(z_\nu) \in E_{z_0}(y, \alpha R)$. This implies that every limit point of the sequence $\{f(z_\nu)\}$ should belong to $\bigcap_{R>0} \overline{E_{z_0}(y, \alpha R)}$. But this intersection is exactly $\{y\}$, by Theorem 2.4.14; hence $f(z_\nu) \to y$, and the assertion follows, **q.e.d.**

To deal with function (i), we need the following lemma:

Lemma 2.7.16: (i) *For every $R > 0$ we have*

$$p_x\bigl(E_{z_0}(x, R)\bigr) = E_{z_0}(x, R) \cap \varphi_x(\Delta) = \varphi_x\bigl(E^\Delta(1, R)\bigr);$$

(ii) *for every $R > 0$ we have $(\tilde{f}_y \circ \varphi_x)\bigl(E^\Delta(1, R)\bigr) \subset E^\Delta(1, \alpha R)$.*

Proof: (i) This follows from $p_x \circ \varphi_x = \varphi_x$ and Proposition 2.7.8.(i).

(ii) Indeed

$$(\tilde{f}_y \circ \varphi_x)\bigl(E^\Delta(1, R)\bigr) \subset \tilde{p}_y\Bigl(f\bigl(E_{z_0}(x, R)\bigr)\Bigr) \subset \tilde{p}_y\bigl(E_{z_0}(y, \alpha R)\bigr) = E^\Delta(1, \alpha R),$$

by Proposition 2.7.8.(i), Theorem 2.4.16 and (i), **q.e.d.**

Corollary 2.7.17: *We have*

$$\lim_{t\to 1}\frac{1-\tilde f_y\big(\varphi_x(t)\big)}{1-t}=\lim_{t\to 1}(\tilde f_y\circ\varphi_x)'(t)=\alpha.$$

Proof: We have

$$\liminf_{\zeta\to 1}\big[\omega(0,\zeta)-\omega\big(0,\tilde f_y\big(\varphi_x(\zeta)\big)\big)\big]=\tfrac12\log\alpha. \tag{2.7.24}$$

Indeed, (2.7.23) immediately yields an inequality, and the reverse inequality follows from Lemma 2.7.16.(ii) and Proposition 1.2.6.

Hence the assertion follows from (2.7.24) and the classical Julia-Wolff-Carathéodory Theorem 1.2.7, **q.e.d.**

Since $t\mapsto\varphi_x(t)$ is a restricted special x-curve and $t=\tilde p_x\big(\varphi_x(t)\big)$, it remains to show that the function (i) is bounded in K-regions:

Proposition 2.7.18: *Take $M>1$. Then for all $z\in K_{z_0}(x,M)$ we have*

$$\left|\frac{1-\tilde f_y(z)}{1-\tilde p_x(z)}\right|\le 2\alpha M^2.$$

Proof: Let $z\in K_{z_0}(x,M)$, and set

$$\tfrac12\log R=\log M-k_D(z_0,z).$$

Clearly, $z\in E_{z_0}(x,M)$. Then, by Theorem 2.4.16, $f(z)\in E_{z_0}(y,\alpha R)$; in particular, by Lemma 2.7.16.(i), $f_y(z)\in E_{z_0}(y,\alpha R)$, and thus

$$\lim_{t\to 1}\big[k_D\big(f_y(z),\varphi_y(t)\big)-\omega(0,t)\big]-k_D\big(z_0,f_y(z)\big)<\log(\alpha R),$$

for $-k_D\big(z_0,f_y(z)\big)<\frac12\log(\alpha R)$. Now, $f_y=\varphi_y\circ\tilde f_y$; hence

$$\begin{aligned}\log(\alpha R)&>\lim_{t\to 1}\big[\omega\big(\tilde f_y(z),t\big)-\omega(0,t)\big]-\omega\big(0,\tilde f_y(z)\big)\\&=\tfrac12\log\frac{|1-\tilde f_y(z)|^2}{1-|\tilde f_y(z)|^2}-\tfrac12\log\frac{1+|\tilde f_y(z)|}{1-|\tilde f_y(z)|}=\log\frac{|1-\tilde f_y(z)|}{1+|\tilde f_y(z)|}.\end{aligned}$$

Since $\log R=\log M^2-2\,k_D(z_0,z)$, we have

$$\begin{aligned}\log\frac{|1-\tilde f_y(z)|}{1+|\tilde f_y(z)|}&<\log\alpha+\log M^2-2\,k_D(z_0,z)\le\log(\alpha M^2)-2\,k_D\big(z_0,p_x(z)\big)\\&=\log(\alpha M^2)-\log\frac{1+|\tilde p_x(z)|}{1-|\tilde p_x(z)|},\end{aligned}$$

and so

$$\log\left|\frac{1-\tilde f_y(z)}{1-\tilde p_x(z)}\right|\le\log\frac{|1-\tilde f_y(z)|}{1-|\tilde p_x(z)|}<\log\left(\alpha M^2\frac{1+|\tilde f_y(z)|}{1+|\tilde p_x(z)|}\right)\le\log(2\alpha M^2),$$

q.e.d.

Corollary 2.7.19: *We have*

$$K'\text{-}\lim_{z\to x} \frac{1-\tilde f_y(z)}{1-\tilde p_x(z)} = \alpha.$$

Proof: Corollary 2.7.17, Proposition 2.7.18 and Theorem 2.7.13, **q.e.d.**

Before attacking map (ii), we need precise information about the behavior of the function K_{z_0} under the projection p_x:

Proposition 2.7.20: *For every $x \in \partial D$ there exists $c_x > 0$ such that*

$$\forall z \in \overline{D} \qquad c_x\|z - p_x(z)\|^2 \le K_{z_0}(z)^2 - K_{z_0}\big(p_x(z)\big)^2. \tag{2.7.25}$$

Proof: We begin checking the consistency of (2.7.25). It is clear that $K_{z_0}(z) \ge K_{z_0}\big(p_x(z)\big)$; we claim that $K_{z_0}(z) = K_{z_0}\big(p_x(z)\big)$ iff $z = p_x(z)$, that is iff $z \in \varphi_x(\overline{\Delta})$. Indeed, assume that $z \in \overline{D}$ is such that $K_{z_0}(z) = K_{z_0}\big(p_x(z)\big)$. There are two cases:

(a) $z \in \partial D$. Then $K_{z_0}(z) = 1$; so $p_x(z) \in \partial D$ too and, by (2.6.31), this implies $z = p_x(z) \in \varphi_x(\partial\Delta)$.

(b) $z \in D$. Let $\psi = \tilde p_x \circ \varphi_z$. Clearly, $\psi(0) = 0$; since $K_{z_0}\big(\varphi_z(\zeta)\big) = |\zeta|$ and $K_{z_0}\big(p_x(z)\big) = |\tilde p_x(z)|$, we have

$$\big|\psi\big(\tilde p_z(z)\big)\big| = |\tilde p_x(z)| = K_{z_0}\big(p_x(z)\big) = K_{z_0}(z) = |\tilde p_z(z)|;$$

therefore, by Schwarz's lemma, $\psi(\zeta) = e^{i\theta}\zeta$ for some $\theta \in \mathbf{R}$ and, in particular,

$$p_x \circ \varphi_z(\partial\Delta) = \varphi_x(\partial\Delta) \subset \partial D.$$

By (2.6.31) it follows that $\varphi_z(\partial\Delta) = \varphi_x(\partial\Delta)$, and so $z \in \varphi_x(\Delta)$ and $z = p_x(z)$.

So (2.7.25) is consistent. Now we want to prove it for $z \in \partial D$, where it becomes

$$c_x\|z - p_x(z)\|^2 \le 1 - K_{z_0}\big(p_x(z)\big)^2. \tag{2.7.26}$$

Theorems 2.3.51 and 2.3.52 yield the existence of a constant $c'_x > 0$ depending only on x and z_0 such that

$$c'_x\, d\big(\varphi_x(\zeta), \partial D) \le 1 - |\zeta|$$

for all $\zeta \in \overline{\Delta}$. Since $p_x(z) \in \varphi_x(\overline{\Delta})$, it follows that

$$\forall z \in \partial D \qquad 1 - K_{z_0}\big(p_x(z)\big)^2 \ge 1 - |\tilde p_x(z)| \ge c'_x\, d\big(p_x(z), \partial D\big). \tag{2.7.27}$$

On the other hand, if $p_x(z) \ne z$ by definition $\big(z - p_x(z)\big)/\big\|z - p_x(z)\big\|$ is orthogonal to $\varphi^*_x\big(\tilde p_x(z)\big)$, and therefore it is close to a vector tangent to ∂D when z is close to $\varphi_x(\partial\Delta)$. Since $p_x(z)$ is close to ∂D iff z is close to $\varphi_x(\partial\Delta)$ (and thus to $p_x(z)$ itself), the strong convexity of D yields the existence of a constant $c''_x > 0$ independing of z such that

$$\forall z \in \partial D \qquad d(p_x(z), \partial D) \ge c''_x\|z - p_x(z)\|^2, \tag{2.7.28}$$

and (2.7.26) follows from (2.7.27) and (2.7.28).

Finally, assume $z \in D$, and define $F\colon \overline{\Delta} \to \mathbf{C}^{n+1}$ by

$$F(\zeta) = \left(\sqrt{c_x}\left[\varphi_z(\zeta) - p_x\left(\varphi_z(\zeta)\right)\right], \tilde{p}_x\left(\varphi_z(\zeta)\right)\right).$$

Then (2.7.26) says that

$$\forall \tau \in \partial\Delta \qquad\qquad \|F(\tau)\|^2 \le 1.$$

Hence (note that $F(0) = 0$) we have $F(\Delta) \subset B^{n+1}$ and $\|F(\zeta)\|^2 \le |\zeta|^2$ for all $\zeta \in \Delta$. In particular,

$$c_x\|z - p_x(z)\|^2 + K_{z_0}\left(p_x(z)\right)^2 = \left\|F\left(\tilde{p}_z(z)\right)\right\|^2 \le |\tilde{p}_z(z)|^2 = K_{z_0}(z)^2,$$

q.e.d.

This is what we need for:

Proposition 2.7.21: *The map $(f - f_y)/(1 - \tilde{p}_x)^{1/2}$ is K-bounded.*

Proof: By Propositions 2.7.20 and 2.7.18 there is a constant $c > 0$ such that for every $z \in K_{z_0}(x, M)$ we have

$$\frac{\|f(z) - f_y(z)\|^2}{|1 - \tilde{p}_x(z)|} \le c\,\frac{K_{z_0}\left(f(z)\right) - K_{z_0}\left(f_y(z)\right)}{|1 - \tilde{p}_x(z)|} \le 2\alpha c M^2\,\frac{K_{z_0}\left(f(z)\right) - K_{z_0}\left(f_y(z)\right)}{|1 - \tilde{f}_y(z)|}.$$

But now $|1 - \tilde{f}_y(z)| \ge 1 - |\tilde{f}_y(z)| = 1 - K_{z_0}\left(f_y(z)\right)$; hence for every $z \in K_{z_0}(x, M)$ we have

$$\frac{\|f(z) - f_y(z)\|^2}{|1 - \tilde{p}_x(z)|} \le 2\alpha c M^2\,\frac{K_{z_0}\left(f(z)\right) - K_{z_0}\left(f_y(z)\right)}{1 - K_{z_0}\left(f_y(z)\right)} \le 2\alpha c M^2,$$

q.e.d.

The proof of the existence of the limit requires

Lemma 2.7.22: *We have*

$$\lim_{t\to 1}\left[k_D\left(z_0, \varphi_x(t)\right) - k_D\left(z_0, f_y\left(\varphi_x(t)\right)\right)\right] = \tfrac{1}{2}\log\alpha; \tag{2.7.29}$$

$$\lim_{t\to 1}\left[k_D\left(z_0, \varphi_x(t)\right) - k_D\left(z_0, f\left(\varphi_x(t)\right)\right)\right] = \tfrac{1}{2}\log\alpha; \tag{2.7.30}$$

$$\lim_{t\to 1}\left[k_D\left(z_0, f\left(\varphi_x(t)\right)\right) - k_D\left(z_0, f_y\left(\varphi_x(t)\right)\right)\right] = 0. \tag{2.7.31}$$

Proof: By (2.7.23) it follows that

$$\begin{aligned}\liminf_{t\to1}\tfrac12\log\frac{1-\bigl|\tilde f_y\bigl(\varphi_x(t)\bigr)\bigr|}{1-t}&=\liminf_{t\to1}\bigl[k_D\bigl(z_0,\varphi_x(t)\bigr)-k_D\bigl(z_0,f_y\bigl(\varphi_x(t)\bigr)\bigr)\bigr]\\&\ge\liminf_{t\to1}\bigl[k_D\bigl(z_0,\varphi_x(t)\bigr)-k_D\bigl(z_0,f\bigl(\varphi_x(t)\bigr)\bigr)\bigr]\\&\ge\tfrac12\log\alpha;\end{aligned}$$

on the other hand, by Corollary 2.7.17

$$\limsup_{t\to1}\tfrac12\log\frac{1-\bigl|\tilde f_y\bigl(\varphi_x(t)\bigr)\bigr|}{1-t}\le\lim_{t\to1}\tfrac12\log\frac{\bigl|1-\tilde f_y\bigl(\varphi_x(t)\bigr)\bigr|}{1-t}=\tfrac12\log\alpha,$$

and (2.7.29) follows. Furthermore,

$$\limsup_{t\to1}\bigl[k_D\bigl(z_0,\varphi_x(t)\bigr)-k_D\bigl(z_0,f\bigl(\varphi_x(t)\bigr)\bigr)\bigr]\le\lim_{t\to1}\bigl[\omega(0,t)-\omega\bigl(0,\tilde f_y\bigl(\varphi_x(t)\bigr)\bigr)\bigr]=\tfrac12\log\alpha,$$

and (2.7.30) is proved. Finally, (2.7.31) is a trivial consequence of (2.7.29) and (2.7.30), **q.e.d.**

Hence

Proposition 2.7.23: *We have*

$$K'\text{-}\lim_{z\to x}\frac{f(z)-f_y(z)}{\bigl(1-\tilde p_x(z)\bigr)^{1/2}}=0.$$

Proof: By Theorem 2.7.13 and Proposition 2.7.21 it suffices to show that

$$\lim_{t\to1}\frac{f\bigl(\varphi_x(t)\bigr)-f_y\bigl(\varphi_x(t)\bigr)}{(1-t)^{1/2}}=0.$$

Using Proposition 2.7.20 and Corollary 2.7.17 we are reduced to proving that

$$\lim_{t\to1}\frac{K_{z_0}\bigl(f\bigl(\varphi_x(t)\bigr)\bigr)-K_{z_0}\bigl(f_y\bigl(\varphi_x(t)\bigr)\bigr)}{1-\tilde f_y\bigl(\varphi_x(t)\bigr)}=0.$$

Since $|1-\tilde f_y(z)|\ge1-|\tilde f_y(z)|=1-K_{z_0}\bigl(f_y(z)\bigr)$, it suffices to show that

$$\lim_{t\to1}\frac{1-K_{z_0}\bigl(f\bigl(\varphi_x(t)\bigr)\bigr)}{1-K_{z_0}\bigl(f_y\bigl(\varphi_x(t)\bigr)\bigr)}=1.$$

Indeed, for all $z\in D$

$$k_D\bigl(z_0,f(z)\bigr)-k_D\bigl(z_0,f_y(z)\bigr)=\tfrac12\log\frac{1+K_{z_0}\bigl(f(z)\bigr)}{1+K_{z_0}\bigl(f_y(z)\bigr)}-\tfrac12\log\frac{1-K_{z_0}\bigl(f(z)\bigr)}{1-K_{z_0}\bigl(f_y(z)\bigr)},$$

and the assertion follows from (2.7.31), **q.e.d.**

Now we can start to deal with the differential of f. As usual, we need several preliminary lemmas. First of all, we should discuss the behavior of the Kobayashi metric $\kappa_D(z;v)$ near the boundary, beginning with

Lemma 2.7.24: *Let $D \subset\subset \mathbf{C}^n$ be a strongly convex domain. Then there are a neighbourhood U of ∂D and a constant $c_0 > 0$ such that for every $z \in D \cap U$ and v_1, $v_2 \in \mathbf{C}^n$ we have*

$$\kappa_D(z; v_1 + v_2) \leq c_0 \big[\kappa_D(z; v_1) + \kappa_D(z; v_2)\big].$$

Proof: Let U be a tubular neighbourhood of ∂D, and choose $R > r > 0$ so that for every $x \in \partial D$ the euclidean ball $B_R(x)$, respectively $B_r(x)$, tangent to ∂D at x of radius R, respectively r, contains D, respectively is contained in $D \cap U$. Take $z \in D \cap U$, and let $x \in \partial D$ be such that $\|z - x\| = d(z, \partial D)$. Then for every v_1, $v_2 \in \mathbf{C}^n$ we have

$$\kappa_D(z; v_1 + v_2) \leq \kappa_{B_r(x)}(z; v_1 + v_2) \leq \kappa_{B_r(x)}(z; v_1) + \kappa_{B_r(x)}(z; v_2),$$

because $\kappa_{B_r(x)}(z; \cdot)$ is a norm on $\mathbf{C}^n$. Now, by the choice of x,

$$\forall v \in \mathbf{C}^n \qquad \kappa_{B_r(x)}(z; v) \leq \frac{R}{r}\, \kappa_{B_R(x)}(z; v);$$

therefore

$$\kappa_D(z; v_1 + v_2) \leq \frac{R}{r}\big[\kappa_{B_R(x)}(z; v_1) + \kappa_{B_R(x)}(z; v_2)\big] \leq \frac{R}{r}\big[\kappa_D(z; v_1) + \kappa_D(z; v_2)\big],$$

and we are done, **q.e.d.**

In Theorem 2.3.70 we have shown that $\kappa_D(z; \nu_x)$ is of the same order as $d(z, \partial D)^{-1}$ as $z \to x$, and that $\kappa_D(z; \tau_x)$ is of the same order as $d(z, \partial D)^{-1/2}$ as $z \to x$ non-tangentially. We need a refinement of these facts:

Proposition 2.7.25: (i) *$\kappa_D(z; \nu_x)\, d(z, \partial D)$ is bounded in every K-region $K_{z_0}(x, M)$;*
(ii) *$\kappa_D(z; \tau_x)^2 d(z, \partial D)$ is bounded in every K-region $K_{z_0}(x, M)$, uniformly in $\|\tau_x\|$.*

Proof: (i) This immediately follows from Theorem 2.3.70.

(ii) Since D has C^2 boundary, we can extend differentiably the outer unit normal vector field $\mathbf{n}$ to a neighbourhood U of ∂D, exactly as in section 2.3.6. Then if $z \in U \cap D$ and $v \in \mathbf{C}^n$, there is a well-defined splitting $v = v_T(z) + v_N(z)$, where $v_N(z) = (v, \mathbf{n}_z)\mathbf{n}_z$ and $v_T(z) = v - v_N(z)$. In Theorem 2.3.70 we have shown that

$$\lim_{z \to x} \kappa_D\big(z; (\tau_x)_N(z)\big) d(z, \partial D) = 0, \tag{2.7.32}$$

and

$$\lim_{z \to x} \kappa_D\big(z; (\tau_x)_T(z)\big)^2 d(z, \partial D) = \tfrac{1}{2} L_{D,x}(\tau_x, \tau_x), \tag{2.7.33}$$

uniformly in $\|\tau_x\|$, where $L_{D,x}$ is the Levi form of D at x.

To prove the assertion, we can restrict our attention to $z \in K_{z_0}(x,M)$ close enough to x. For those points, Lemma 2.7.24 yields $c_1 > 0$ such that

$$\begin{aligned}\kappa_D(z;\tau_x)^2 &\le c_1\big[\kappa_D\big(z;(\tau_x)_T(z)\big) + \kappa_D\big(z;(\tau_x)_N(z)\big)\big]^2\\ &\le 2c_1\big[\kappa_D\big(z;(\tau_x)_T(z)\big)^2 + \kappa_D\big(z;(\tau_x)_N(z)\big)^2\big].\end{aligned}$$

So by (2.7.33) it suffices to show that $\kappa_D\big(z;(\tau_x)_N(z)\big)^2 d(z,\partial D)$ is bounded in $K_{z_0}(x;M)$; by (2.7.32) we can reduce ourselves to showing that $(\tau_x)_N(z)/d(z,\partial D)^{1/2}$ is bounded in all K-regions.

Now let $B \supset D$ be an euclidean ball tangent to ∂D at x; up to a linear isomorphism we can assume $B = B^n$. Since $\overline{K_{z_0}(x,M)} \cap \partial D = \{x\}$, and $K_{z_0}(x,M)$ is contained in $E_{z_0}(x,M^2)$ which is, by Corollary 2.6.49, strongly convex near x, it follows that $d(z,\partial D)$ is of the same order as $d(z,\partial B)$ as z goes to x within $K_{z_0}(x,M)$. So it remains to show there exists $C > 0$ such that

$$\forall z \in K_{z_0}(x,M) \qquad \frac{\|(\tau_x)_N(z)\|}{d(z,\partial B)^{1/2}} = \frac{\|(\tau_x)_N(z)\|}{(1-\|z\|)^{1/2}} \le C.$$

The strong convexity of D implies

$$\left\|\mathbf{n}_z - \frac{z}{\|z\|}\right\|^2 \le c_2(1-\|z\|)$$

for some $c_2 > 0$ and for all $z \in D$ sufficiently close to x. Hence

$$\left\|(\tau_x)_N(z) - \left(\tau_x, \frac{z}{\|z\|}\right)\frac{z}{\|z\|}\right\|^2 \le 4\|\tau_x\|^2\left\|\mathbf{n}_z - \frac{z}{\|z\|}\right\|^2 \le 4c_2\|\tau_x\|^2(1-\|z\|).$$

Therefore it suffices to show that $|(\tau_x,z)|^2/(1-\|z\|)$ is bounded for $z \in K_{z_0}(x,M)$ close enough to x. But indeed, recalling that $(\tau_x,x) = 0$ because $\tau_x \in T_x^{\mathbf{C}}(\partial B^n)$, we have

$$\begin{aligned}|(\tau_x,z)|^2 = |(\tau_x, z-(z,x)x)|^2 \le \|\tau_x\|^2\|z-(z,x)x\|^2 &< \|\tau_x\|^2(1-|(z,x)|^2)\\ &\le 2\|\tau_x\|^2|1-(z,x)| \le \frac{2\|\tau_x\|^2 M}{\varepsilon_0}(1-\|z\|),\end{aligned}$$

where $\varepsilon_0 > 0$ is given by Proposition 2.7.6, **q.e.d.**

We need two more lemmas:

Lemma 2.7.26: *We have*

$$\forall z \in K_{z_0}(x,M) \qquad k_D(z_0,z) - k_D\big(z_0,p_x(z)\big) \le \log M. \tag{2.7.34}$$

Proof: For any $\varepsilon > 0$ there is $t < 1$ such that

$$\begin{aligned}\varepsilon + \log M &> k_D\big(z,\varphi_x(t)\big) - \omega(0,t) + k_D(z_0,z)\\ &\ge k_D\big(p_x(z),\varphi_x(t)\big) - k_D\big(z_0,\varphi_x(t)\big) + k_D(z_0,z)\\ &\ge k_D(z_0,z) - k_D\big(z_0,p_x(z)\big),\end{aligned}$$

and (2.7.34) follows, **q.e.d.**

Lemma 2.7.27: *Take $M_1 > M > 1$ and set $r = (M_1 - M)/(M_1 + M) < 1$. For every $z \in K_{z_0}(x, M)$ let $\sigma_z \in \mathrm{Hol}(\Delta, D)$ denote a complex geodesic such that $\sigma_z(0) = z$; then*

$$\sigma_z(\Delta_r) \subset K_{z_0}(x, M_1).$$

Proof: Let $\delta = \frac{1}{2}\log(M_1/M) > 0$; then $\zeta \in \Delta_r$ iff $\omega(0,\zeta) < \delta$. Then

$$\begin{aligned}\lim_{t\to 1}&\big[k_D\big(\sigma_z(\zeta), \varphi_x(t)\big) - \omega(0,t)\big] + k_D\big(z_0, \sigma_z(\zeta)\big)\\ &\le 2\,k_D\big(\sigma_z(\zeta), z\big) + \lim_{t\to 1}\big[k_D\big(z, \varphi_x(t)\big) - \omega(0,t)\big] + k_D(z_0, z)\\ &< 2\,\omega(0,\zeta) + \log M < \log M_1\end{aligned}$$

for all $z \in K_{z_0}(x, M)$ and $\zeta \in \Delta_r$, **q.e.d.**

Now we can show that the components of the differential of f (with appropriate weights) are bounded in K-regions:

Proposition 2.7.28: *The map $d(\tilde f_y)_z(\nu_x)$ is K-bounded.*

Proof: For all $z \in K_{z_0}(x, M)$ let $\sigma_z\colon \Delta \to D$ be the unique (Corollary 2.6.30) complex geodesic such that $\sigma_z(0) = z$ and $\sigma_z'(0) = \nu_x/\kappa_D(z;\nu_x)$. By Lemma 2.7.27, for every $M_1 > M > 1$ there exists $r > 0$ independent of z such that $\sigma_z(\overline{\Delta_r}) \subset K_{z_0}(x, M_1)$ for all $z \in K_{z_0}(x, M)$.

Now Cauchy's formula yields

$$d(\tilde f_y)_z(\nu_x) = \kappa_D(z;\nu_x)\frac{d}{d\zeta}(\tilde f_y \circ \sigma_z)(0) = \frac{\kappa_D(z;\nu_x)}{2\pi i}\int\limits_{|\lambda|=r}\frac{\tilde f_y\big(\sigma_z(\lambda)\big)}{\lambda^2}\,d\lambda.$$

If we replace $\tilde f_y\big(\sigma_z(\lambda)\big)$ by $\tilde f_y\big(\sigma_z(\lambda)\big) - 1$, the integral does not change and we get

$$d(\tilde f_y)_z(\nu_x) = \frac{1}{2\pi}\int\limits_{-\pi}^{\pi}\frac{\tilde f_y\big(\sigma_z(re^{i\theta})\big) - 1}{\tilde p_x\big(\sigma_z(re^{i\theta})\big) - 1}\cdot\frac{\tilde p_x\big(\sigma_z(re^{i\theta})\big) - 1}{\tilde p_x(z) - 1}\cdot\frac{\kappa_D(z;\nu_x)(\tilde p_x(z) - 1)}{re^{i\theta}}\,d\theta.$$

If $z \in K_{z_0}(x, M)$, the first factor in the integrand is bounded, thanks to Lemma 2.7.27 and Proposition 2.7.18. For the second factor, Lemma 2.7.27 together with Proposition 2.7.8.(ii) yield

$$\left|\frac{1 - \tilde p_x\big(\sigma_z(re^{i\theta})\big)}{1 - \tilde p_x(z)}\right| \le M_1\,\frac{1 - K_{z_0}\big(p_x\big(\sigma_z(re^{i\theta})\big)\big)}{1 - K_{z_0}\big(p_x(z)\big)}.$$

On the other hand,

$$\begin{aligned}\tfrac{1}{2}\log\left[\frac{1 - K_{z_0}\big(p_x\big(\sigma_z(re^{i\theta})\big)\big)}{2\big[1 - K_{z_0}\big(p_x(z)\big)\big]}\right] &\le \big|k_D\big(z_0, p_x(z)\big) - k_D\big(z_0, p_x\big(\sigma_z(re^{i\theta})\big)\big)\big|\\ &\le k_D\big(p_x(z), p_x\big(\sigma_z(re^{i\theta})\big)\big)\\ &\le k_D\big(z, \sigma_z(re^{i\theta})\big) = \omega(0, r);\end{aligned}$$

hence

$$\forall z \in K_{z_0}(x,M) \qquad \left| \frac{1-\tilde{p}_x\bigl(\sigma_z(re^{i\theta})\bigr)}{1-\tilde{p}_x(z)} \right| \le 2M_1 \frac{1+r}{1-r}. \tag{2.7.35}$$

To estimate the third factor, note that, by Proposition 2.7.25.(i) and Theorem 2.3.52, there exists $c_0 > 0$ such that

$$\kappa_D(z;\nu_x)|1-\tilde{p}_x(z)| \le c_0 M \, \frac{1-K_{z_0}\bigl(p_x(z)\bigr)}{1-K_{z_0}(z)}$$

for every $z \in K_{z_0}(x,M)$. But Lemma 2.7.26 yields

$$\tfrac{1}{2}\log\left[\frac{1-K_{z_0}\bigl(p_x(z)\bigr)}{2\bigl(1-K_{z_0}(z)\bigr)} \right] \le k_D(z_0,z) - k_D\bigl(z_0,p_x(z)\bigr) \le \log M,$$

and thus

$$\forall z \in K_{z_0}(x,M) \qquad \kappa_D(z;\nu_x)|1-\tilde{p}_x(z)| \le 2c_0 M^3, \tag{2.7.36}$$

q.e.d.

Proposition 2.7.29: *The map* $[1-\tilde{p}_x(z)]^{1/2} d(f-f_y)_z(\nu_x)$ *is* K*-bounded.*

Proof: For all $z \in K_{z_0}(x,M)$ let $\sigma_z \colon \Delta \to D$ be as in the proof of Proposition 2.7.28. Then Cauchy's formula yields

$$d(f-f_y)_z(\nu_x) = \frac{\kappa_D(z;\nu_x)}{2\pi i} \int\limits_{|\lambda|=r} \frac{f\bigl(\sigma_z(\lambda)\bigr) - f_y\bigl(\sigma_z(\lambda)\bigr)}{\lambda^2}\, d\lambda,$$

that is

$$\begin{aligned}
&[1-\tilde{p}_x(z)]^{1/2} d(f-f_y)_z(\nu_x) \\
&\quad = \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} \frac{f\bigl(\sigma_z(re^{i\theta})\bigr) - f_y\bigl(\sigma_z(re^{i\theta})\bigr)}{\bigl[1-\tilde{p}_x\bigl(\sigma_z(re^{i\theta})\bigr)\bigr]^{1/2}} \left[\frac{1-\tilde{p}_x\bigl(\sigma_z(re^{i\theta})\bigr)}{1-\tilde{p}_x(z)} \right]^{1/2} \frac{\kappa_D(z;\nu_x)\bigl(1-\tilde{p}_x(z)\bigr)}{re^{i\theta}}\, d\theta,
\end{aligned}$$

and the assertion follows from Lemma 2.7.27, Proposition 2.7.21, (2.7.35) and (2.7.36), **q.e.d.**

Proposition 2.7.30: *The function* $[d(\tilde{f}_y)_z(\tau_x)]/[1-\tilde{p}_x(z)]^{1/2}$ *is* K*-bounded, uniformly in* $\|\tau_x\|$.

Proof: For every $z \in K_{z_0}(x,M)$ let $\sigma_z \colon \Delta \to D$ be the unique complex geodesic such that $\sigma_z(0) = z$ and $\sigma_z'(0) = \tau_x/\kappa_D(z;\tau_x)$. Then

$$\frac{d(\tilde{f}_y)_z(\tau_x)}{[1-\tilde{p}_x(z)]^{1/2}} = \frac{-1}{2\pi} \int\limits_{-\pi}^{\pi} \frac{1-\tilde{f}_y\bigl(\sigma_z(re^{i\theta})\bigr)}{1-\tilde{p}_x\bigl(\sigma_z(re^{i\theta})\bigr)} \cdot \frac{1-\tilde{p}_x\bigl(\sigma_z(re^{i\theta})\bigr)}{1-\tilde{p}_x(z)} \cdot \frac{\kappa_D(z;\tau_x)[1-\tilde{p}_x(z)]^{1/2}}{re^{i\theta}}\, d\theta,$$

and the assertion follows from Lemma 2.7.27, Proposition 2.7.18, (2.7.35), Theorem 2.3.52 and Proposition 2.7.25.(ii), **q.e.d.**

Proposition 2.7.31: *The map $d(f-f_y)_z(\tau_x)$ is K-bounded, uniformly in $\|\tau_x\|$.*

Proof: Let $\sigma_z\colon\Delta\to D$ be as in the proof of Proposition 2.7.30. Then

$$\begin{aligned}&d(f-f_y)_z(\tau_x)\\&\quad=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{f\bigl(\sigma_z(re^{i\theta})\bigr)-f_y\bigl(\sigma_z(re^{i\theta})\bigr)}{\bigl[1-\tilde p_x\bigl(\sigma_z(re^{i\theta})\bigr)\bigr]^{1/2}}\left[\frac{1-\tilde p_x\bigl(\sigma_z(re^{i\theta})\bigr)}{1-\tilde p_x(z)}\right]^{1/2}\frac{\kappa_D(z;\tau_x)[1-\tilde p_x(z)]^{1/2}}{re^{i\theta}}\,d\theta,\end{aligned}$$

and the assertion follows from Lemma 2.7.27, Proposition 2.7.21, (2.7.35), Theorem 2.3.52 and Proposition 2.7.25.(ii), **q.e.d.**

So it remains to show that the maps (iii), (iv) and (v) have the stated restricted K-limits. The first case is quite easy:

Proposition 2.7.32: *We have*

$$K'\text{-}\lim_{z\to x} d(\tilde f_y)_z(\nu_x)=\alpha.$$

Proof: By Proposition 2.7.28 and Theorem 2.7.13, it suffices to show

$$\lim_{t\to 1} d(\tilde f_y)_{\varphi_x(t)}(\nu_x)=\alpha.$$

We know, by Propositions 2.7.28 and 2.7.30, that $\|d(\tilde f_y)_z\|$ is bounded in K-regions; hence the assertion follows from Corollary 2.7.17 and the fact that $\varphi_x'(t)\to\nu_x$ as $t\to 1$, **q.e.d.**

The next case requires again an integral representation, and a different perturbation of the usual x-curve $t\mapsto\varphi_x(t)$:

Proposition 2.7.33: *We have*

$$K'\text{-}\lim_{z\to x}\,[1-\tilde p_x(z)]^{1/2}d(f-f_y)_z(\nu_x)=0.$$

Proof: First of all we claim that

$$\lim_{t\to 1}\,(1-t)^{1/2}d(f-f_y)_{\varphi_x(t)}\bigl(\varphi_x'(t)\bigr)=0. \tag{2.7.37}$$

Choose $\varepsilon\in(0,1)$ and for every $t\in(0,1)$ let $\sigma_t\colon\Delta_\varepsilon\to D$ be defined by

$$\sigma_t(\zeta)=\varphi_x\bigl(t+\zeta(1-t)\bigr).$$

Clearly, $\sigma_t(0)=\varphi_x(t)$ and $\sigma_t'(0)=(1-t)\varphi_x'(t)$. Moreover, for all $\zeta\in\Delta_\varepsilon$ we have

$$\frac{|1-t-\zeta(1-t)|}{1-|t+\zeta(1-t)|}=\frac{(1-t)|1-\zeta|}{1-|1-(1-t)(1-\zeta)|}\le\frac{1+\varepsilon}{1-\varepsilon};$$

hence $\sigma_t(\overline{\Delta_\varepsilon}) \subset K_{z_0}(x,M)$ for all $M > (1+\varepsilon)/(1-\varepsilon)$. In particular, for all $\theta \in \mathbf{R}$ the x-curve $t \mapsto \sigma_t(\varepsilon e^{i\theta})$ is special and M-restricted.

Now write

$$(1-t)^{1/2} d(f-f_y)_{\varphi_x(t)}\bigl(\varphi_x'(t)\bigr)$$
$$= \frac{1}{2\pi(1-t)^{1/2}} \int_{-\pi}^{\pi} \frac{f\bigl(\sigma_t(\varepsilon e^{i\theta})\bigr) - f_y\bigl(\sigma_t(\varepsilon e^{i\theta})\bigr)}{(1-t-\varepsilon(1-t)e^{i\theta})^{1/2}} \cdot \frac{(1-t-\varepsilon(1-t)e^{i\theta})^{1/2}}{\varepsilon e^{i\theta}}\, d\theta$$
$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f\bigl(\sigma_t(\varepsilon e^{i\theta})\bigr) - f_y\bigl(\sigma_t(\varepsilon e^{i\theta})\bigr)}{\bigl[1-\tilde{p}_x\bigl(\sigma_t(\varepsilon e^{i\theta})\bigr)\bigr]^{1/2}} \cdot \frac{(1-\varepsilon e^{i\theta})^{1/2}}{\varepsilon e^{i\theta}}\, d\theta.$$

The second factor in the integrand is bounded, and the first factor converges punctually and boundedly to 0 as $t \to 1$, by Propositions 2.7.21 and 2.7.23; therefore (2.7.37) follows from the dominated convergence theorem.

Finally, we know, by Propositions 2.7.29 and 2.7.31, that $\bigl\|[1-\tilde{p}_x(z)]^{1/2} d(f-f_y)_z\bigr\|$ is bounded in K-regions; hence (2.7.37) implies

$$\lim_{t\to 1} (1-t)^{1/2} d(f-f_y)_{\varphi_x(t)}(\nu_x) = 0,$$

for $\varphi_x'(t) \to \nu_x$ as $t \to 1$, and Theorem 2.7.13 yields the assertion, **q.e.d.**

We have one function left, the hardest one. To dispose of this last case, ending the book, we shall use a trick similar to the one used in Step (e) of the proof of Theorem 2.2.29.

Proposition 2.7.34: *We have*

$$K'\text{-}\lim_{z\to x} \frac{d(\tilde{f}_y)_z(\tau_x)}{[1-\tilde{p}_x(z)]^{1/2}} = 0.$$

Proof: As usual, by Theorem 2.7.13 and Proposition 2.7.30 it suffices to show that

$$\lim_{t\to 1} \frac{d(\tilde{f}_y)_{\varphi_x(t)}(\tau_x)}{(1-t)^{1/2}} = 0. \tag{2.7.38}$$

We need some preparation. Consider the map $\Phi\colon \Delta \times \mathbf{C} \to \mathbf{C}^n$ given by

$$\Phi(\zeta,\eta) = \varphi_x(\zeta) + \eta\tau_x.$$

Clearly, $\Phi^{-1}(D) \cap (\mathbf{C} \times \{0\}) = \Delta$ and $\Phi^{-1}(D) \cap (\{\zeta\} \times \mathbf{C})$ is convex for all $\zeta \in \Delta$. Furthermore, since D is strongly convex, τ_x is tangent to ∂D at x and $t \mapsto \varphi_x(t)$ is transversal, there is an euclidean ball $B \subset \Phi^{-1}(D)$ of center $(t_0, 0)$ and radius $1-t_0$ for a suitable $t_0 \in (0,1)$.

Now define $\tilde{h}\colon B \to \Delta$ by

$$\tilde{h}(\zeta,\eta) = \tilde{f}_y\bigl(\Phi(\zeta,\eta)\bigr).$$

We note that $\tilde h(\zeta,0)=\tilde f_y\big(\varphi_x(\zeta)\big)$ and $\partial\tilde h(\zeta,0)/\partial\zeta=d(\tilde f_y)_{\varphi_x(\zeta)}(\tau_x)$. Hence we can write

$$\tilde h(\zeta,\eta)=\tilde f_y\big(\varphi_x(\zeta)\big)+\eta\, d(\tilde f_y)_{\varphi_x(\zeta)}(\tau_x)+o(|\eta|).$$

Set

$$h(\zeta,\eta)=\tilde f_y\big(\varphi_x(\zeta)\big)+\tfrac{1}{2}\eta\, d(\tilde f_y)_{\varphi_x(\zeta)}(\tau_x)=\tilde f_y\big(\varphi_x(\zeta)\big)+\eta(1-\zeta)^{1/2}g(\zeta),$$

where $g(\zeta)=d(\tilde f_y)_{\varphi_x(\zeta)}(\tau_x)/2(1-\zeta)^{1/2}$. Since h is the arithmetic mean of the first two partial sums of the power series expansion of $\tilde h$, h sends B into Δ. Furthermore, (2.7.38) is equivalent to $g(t)\to 0$ as $t\to 1$.

Choose $\varepsilon>0$ and set $c=\alpha^2/\varepsilon^2(1-t_0)$. We wish to estimate

$$\limsup_{t\to 1}\big|g\big(t+ic(1-t)\big)\big|.$$

Set $\zeta_t=t+ic(1-t)$; it is easy to check that $(\zeta_t,0)\in B$ if $(1-t)\le 2(1-t_0)/(1+c^2)$. Moreover

$$(1-t_0)^2-|\zeta_t-t_0|^2>(1-t_0)(1-t)$$

if $(1-t)<(1-t_0)/(1+c^2)$; hence if t is sufficiently close to 1 we can find $\eta_t\in\mathbf{C}$ such that

$$(1-t_0)^2-|\zeta_t-t_0|^2>|\eta_t|^2>(1-t_0)(1-t) \tag{2.7.39}$$

and

$$\eta_t(1-\zeta_t)^{1/2}g(\zeta_t)\in\mathbf{R}. \tag{2.7.40}$$

In particular, $(\zeta_t,\eta_t)\in B$ if $(1-t)<(1-t_0)/(1+c^2)$. By definition,

$$|1-\zeta_t|=(1-t)\sqrt{1+c^2}\ge c\,(1-t);$$

hence (2.7.39) yields

$$|\eta_t(1-\zeta_t)^{1/2}g(\zeta_t)|\ge(1-t_0)^{1/2}c^{1/2}(1-t)|g(\zeta_t)|. \tag{2.7.41}$$

Now, $\zeta_t\in K^\Delta(1,2\sqrt{1+c^2})$ if $(1-t)<(1-t_0)/(1+c^2)$; hence, by Corollary 2.7.19

$$\frac{1-\tilde f_y\big(\varphi_x(\zeta_t)\big)}{1-\zeta_t}=\alpha+o(1)$$

as $t\to 1$, that is

$$\tilde f_y\big(\varphi_x(\zeta_t)\big)=1-\big(\alpha+o(1)\big)(1-ic)(1-t). \tag{2.7.42}$$

Putting together (2.7.40), (2.7.41) and (2.7.42) we get

$$1\ge\operatorname{Re}[h(\zeta,\eta)]\ge 1-\big(\alpha+o(1)\big)(1-t)+(1-t_0)^{1/2}c^{1/2}(1-t)|g(\zeta_t)|,$$

that is

$$|g(\zeta_t)|\le\frac{\alpha+o(1)}{(1-t_0)^{1/2}c^{1/2}}.$$

Therefore

$$\limsup_{t\to 1}\big|g\big(t+ic(1-t)\big)\big|\le\frac{\alpha}{(1-t_0)^{1/2}c^{1/2}}=\varepsilon.$$

Clearly the same estimate holds for $\zeta_t'=t-ic(1-t)$. Since, by Proposition 2.7.30, $|g(\zeta)|$ is bounded in the angular region bounded by these two lines, it follows that

$$\limsup_{t\to 1}|g(t)|\le\varepsilon.$$

Since ε is arbitrary, the assertion follows, **q.e.d.**

NOTES

As already discussed in the notes to chapter 2.2, the K-regions in B^n have been introduced by Korányi [1969] and Korányi and Stein [1968] to study the boundary behavior of harmonic and holomorphic functions in B^n. Their results were later generalized by Stein [1972] himself to a generic C^2 domain $D \subset\subset \mathbf{C}^n$, replacing Korányi regions by the so-called *admissible approach regions* $A(x, M)$ of *vertex* $x \in \partial D$ and *aperture* $M > 1$ defined by

$$A(x, M) = \bigl\{z \in D \bigm| |(z - x, \mathbf{n}_x)| < M\delta_x(z),\ \|z - x\|^2 < M\delta_x(z)\bigr\},$$

where

$$\delta_x(z) = \min\bigl\{d(z, \partial D),\ d(z, x + T_x\partial D)\bigr\};$$

note that $\delta_x(z) = d(z, \partial D)$ if D is convex. A substantially larger (and more complicated) family of admissible approach regions has been also defined by Čirka [1973].

Our definition of K-regions (taken from Abate [1988f]), was originally motivated by the successful definition of horospheres in bounded domains given in chapter 2.4; note that, by Propositions 2.7.4 and 2.7.6, in strongly convex domains our K-regions and the admissible approach regions have comparable shapes near the boundary. Another kind of approach regions involving invariant objects has been introduced by Cima and Krantz [1983]: if $x \in \partial D$ and $M > 1$, they defined the *K-admissible* (where K here stands for Kobayashi) *approach region* $\mathcal{A}(x, M)$ of *vertex* x and *aperture* $M > 1$ by

$$\mathcal{A}(x, M) = \{z \in D \mid \kappa_D(z; -\mathbf{n}_x) < M\};$$

again, by Theorem 2.3.70, in strongly pseudoconvex domains the shape of K-admissible approach regions near the boundary is similar to the shape of admissible approach regions.

Čirka [1973] has been the first one to sistematically study Lindelöf's theorems for domains in $\mathbf{C}^n$. He proved that if a bounded holomorphic function defined in a C^1 domain D has limit along a non-tangential x-curve for some $x \in \partial D$, then it has the same limit along a large class of x-curves depending on the position of ∂D with respect to complex submanifolds of $\mathbf{C}^n$ tangent to $T_x^{\mathbf{C}}(\partial D)$ at x; see also Zav'jalov and Drožžinov [1982], Khurumov [1983] and Dovbush [1987].

Another version of the Lindelöf theorem in $\mathbf{C}^n$ is due to Cima and Krantz [1983]. Let $D \subset\subset \mathbf{C}^n$ be a C^2 domain; for every $x \in \partial D$ and $M > 1$ denote by

$$\Gamma(x, M) = \{z \in D \mid \|z - x\| < M\, d(z, \partial D)\}$$

the cone of vertex x and aperture M. Then Cima and Krantz proved that if a bounded holomorphic function $f\colon D \to \mathbf{C}$ admits limit L along a non-tangential x-curve, then f has limit L along any x-curve σ such that

$$\lim_{t \to 1} k_D\bigl(\sigma(t), \Gamma(x, M)\bigr) = 0$$

for some $M > 1$. Actually, their result holds for normal holomorphic functions, which are not necessarily bounded.

Our version of the Lindelöf theorem is a very particular case of a large class of Lindelöf's theorems discussed in Abate [1988f]. The idea is that every time we have a device allowing the projection of x-curves into an analytic disk transversal to ∂D at x, then we can define restricted and special curves exactly as in section 2.7.2, and the argument used to prove Theorem 2.7.9 yields a Lindelöf's theorem.

A first tentative extension of the Julia-Wolff-Carathéodory theorem to bounded domains in $\mathbf{C}^2$ is due to Wachs [1940]. As discussed in the notes to chapter 2.2, after the preliminary version of Hervé [1963a], Rudin [1980] gave the final form to the Julia-Wolff-Carathéodory theorem in B^n. Our version, Theorem 2.7.14, is taken from Abate [1988f], as well as most of this chapter.