An overview of local dynamics in several complex variables

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Marco Abate
University of Pisa, Italy
http://www.dm.unipi.it/~abate

Local dynamics

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- Dynamics about a fixed point

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- Stable set (iterates do not escape)

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- Stable set (iterates do not escape)
- Conjugacy helps

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(it is holomorphically linearizable).

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$0 < |\lambda| < 1$: attracting; $|\lambda| > 1$: repelling

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Böttcher (1904):

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Écalle, Voronin (1981):

(very complicated) holomorphic classification

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if $\lambda\in B$ (full-measure subset of S^1) then all

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Cremer-Yoccoz (1927, 1988):
if $\lambda\notin B$ (dense uncountable subset of S^1) then

$$f(z) = \lambda z+z^2$$

is not holomorphically linearizable.

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Perron-Hadamard Stable Manifold Theorem (≥1928):

E^s: sum of gen. eigenspaces of attracting eigenvalues

E^u: sum of gen. eigenspaces of repelling eigenvalues

Then there are complex manifolds $W^{s/u}$

tangent to $E^{s/u}$ at the origin such that

$f^k(z) \to O$ iff $z \in W^s$ and $f^{-k}(z) \to O$ iff $z \in W^u$

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Grobman-Hartman (1959-60):

f is locally topologically conjugated to Az

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- Small divisors prevent holomorphic linearization

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$$f(z,w)=(w/2-z^2,z)$$

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Topological, holomorphic, formal classifications: wide open, as well as local dynamics. (Some results by Hubbard, Favre–Jonsson).

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- A. (2001): if $n = 2$ and O isolated fixed point, then there is a Fatou flower for f.

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- Pöschel (1986): partial linearization results

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Convergence of a Poincaré-Dulac normal form?

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