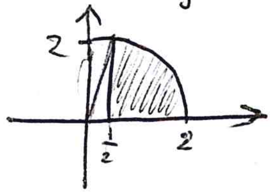
ANALISI 2 | Lu 16/12/19

Esempi sulle coordinate polari:

(1) $\iint_F \frac{y}{x^2+y^2}\,dx\,dy$, $F = \{(x,y): \ x^2+y^2 \le 4,\ x \ge \frac{1}{2},\ y \ge 0\}$.



L'insieme F è l'immagine di

$E = \{(r,\theta): \ r \le 2,\ r\cos\theta \ge \frac{1}{2},\ \sin\theta \ge 0\} =$

$= \{(r,\theta): \ 0 \le \theta \le \arccos\frac{1}{4},\ \frac{1}{2\cos\theta} \le r \le 2\}$,

il quale è un insieme normale del piano $\rho\theta$, rispetto all'asse θ. Si ha

$$\iint_F \frac{y}{x^2+y^2} = \int_0^{\arccos\frac{1}{4}} \left[\int_{\frac{1}{2\cos\theta}}^{2} \frac{r\sin\theta}{r^2}\, r\, dr \right] d\theta = \int_0^{\arccos\frac{1}{4}} \left(2 - \frac{1}{2\cos\theta}\right)\sin\theta\, d\theta =$$

$$= \left[-2\cos\theta + \frac{1}{2}\ln\cos\theta\right]_0^{\arccos\frac{1}{4}} = -\frac{1}{2} + 2 + \frac{1}{2}\ln 4 = \frac{3}{2} - \ln 2.$$

(2) Calcoliamo $\int_0^\infty e^{-x^2}dx$ col trucco seguente:

$$\left[\int_0^\infty e^{-x^2}dx\right]^2 = \left[\int_0^\infty e^{-x^2}dx\right]\cdot\left[\int_0^\infty e^{-y^2}dy\right] = \int_0^\infty \int_0^\infty e^{-x^2-y^2}dx\,dy =$$

(passando in coordinate polari)

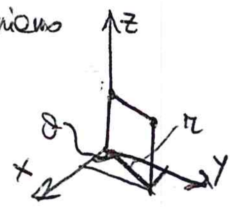
$$= \int_0^\infty \left[\int_0^{\pi/2} e^{-r^2} r\, d\theta\right] dr = \frac{\pi}{2}\int_0^\infty r e^{-r^2} dr = \frac{\pi}{4}\left[e^{-r^2}\right]_0^\infty = \frac{\pi}{4}.$$

Dunque $\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$ e $\int_{\mathbb{R}} e^{-x^2}dx = \sqrt{\pi}$.

Coordinate cilindriche in $\mathbb{R}^3$

Per $(x,y,z) \in \mathbb{R}^3$ poniamo

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z, \end{cases} \quad r \ge 0,\ \theta \in [0,2\pi],\ z \in \mathbb{R}. \text{ Si ha}$$



$$J_g(r,\theta,z) = \left| \det \begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \right| = r,$$

e, in analogia con il caso delle coordinate polari in $\mathbb{R}^2$, vale:

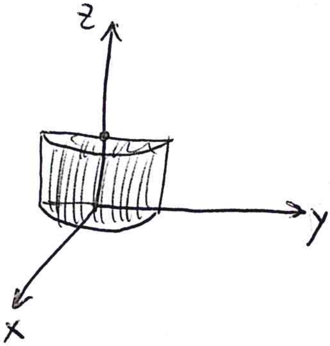
Proposizione Se f è integrabile su F, insieme misurabile di $\mathbb{R}^3$, posto
$$E = g^{-1}(F) = \{(r,\theta,z) : (r\cos\theta, r\sin\theta, z) \in F\}$$ si ha
$$\int_F f(x,y,z)\,dx\,dy\,dz = \int_E f(r\cos\theta, r\sin\theta, z)\, r\,dr\,d\theta\,dz. \qquad \square$$

Esempi (1) Solidi di rotazione: se $D \subseteq \{(x,y,z) : y=0,\ z \geq 0\}$ è misurabile, il ruotato di D attorno all'asse z è
$$F = \{(x,y,z) : (\sqrt{x^2+y^2}, 0, z) \in D\} = g(E),$$
ove
$$E = \{(r,\theta,z) : (r,0,z) \in D,\ \theta \in [0,2\pi]\}.$$
Si ha
$$m_3(F) = \int_E r\,dr\,d\theta\,dz = \int_D \int_0^{2\pi} r\,d\theta\,dr\,dz = 2\pi \int_D r\,dr\,dz = 2\pi \int_D x\,dx\,dz.$$
Quindi si ottiene il volume di F integrando per circonferenze orizzontali (quelle passanti per i punti (x,z) di D).

Nel caso particolare in cui $D = \{(x,z) : a \leq z \leq b,\ 0 \leq x \leq f(z)\}$, con f continua e non negativa su $[a,b]$, si ritrova la formula che conosciamo:
$$m_3(F) = 2\pi \int_D x\,dx\,dz = 2\pi \int_a^b \int_0^{f(z)} x\,dx\,dz = \pi \int_a^b f(z)^2\,dz.$$

(2) $\int_F z(x^2+y^2)\,dx\,dy\,dz$, $F=\{(x,y,z):\ x^2+y^2\le 1,\ x\ge 0,\ 0\le z\le 1\}$.

Si ha $E=g^{-1}(F)=\{(r,\theta,z):\ 0\le r\le 1,\ |\theta|\le \frac{\pi}{2},\ 0\le z\le 1\}$ e dunque



$$\int_F z(x^2+y^2)\,dx\,dy\,dz =$$

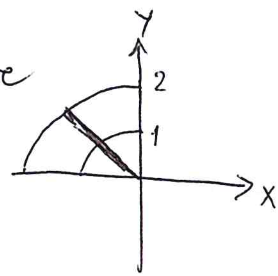
$$= \int_0^1 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^1 z r^3\,dr\,d\theta\,dz =$$

$$= \left[\frac{z^2}{2}\right]_0^1 \left[\frac{r^4}{4}\right]_0^1 \cdot \pi = \frac{\pi}{8}.$$

Esercizi

1. Riconsideriamo l'integrale $\int_E (x+y)\,dx\,dy$, ove

$$E=\{(x,y):\ 1\le x^2+y^2\le 4,\ x<0,\ y+x\ge 0\}.$$



In coordinate polari, E si esprime come

$$F=\{(r,\theta):\ 1\le r\le 2,\ \frac{\pi}{2}\le\theta\le\frac{3}{4}\pi\};$$

dunque si ha

$$\int_E (x+y)\,dx\,dy = \int_1^2 \int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} (r\cos\theta + r\sin\theta)\, r\,d\theta\,dr =$$

$$= \int_1^2 r^2\,dr \int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} (\cos\theta+\sin\theta)\,d\theta = \frac{7}{3}\left[\sin\theta-\cos\theta\right]_{\frac{\pi}{2}}^{\frac{3\pi}{4}} =$$

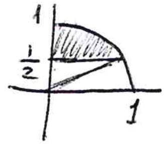
$$= \frac{7}{3}\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}-1\right) = \frac{7}{3}(\sqrt{2}-1).$$

2 • $\displaystyle\int_E x^3\,dx\,dy$, $\quad E = \{(x,y) : x \geq 0,\ y \geq \frac{1}{2},\ x^2+y^2 \leq 1\}$

Si ha $F = \{(r,\theta) : (r\cos\theta, r\sin\theta) \in E\} =$

$= \{(r,\theta) : \cos\theta \geq 0,\ r \leq 1,\ r\sin\theta \geq \frac{1}{2}\} =$

$= \{(r,\theta) : \frac{\pi}{6} \leq \theta \leq \frac{\pi}{2},\ \frac{1}{2\sin\theta} \leq r \leq 1\}$,



da cui

$$\int_E x^3\,dx\,dy = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cos^3\theta \int_{\frac{1}{2\sin\theta}}^{1} r^4\,dr\,d\theta =$$

$$= \frac{1}{5}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cos^3\theta \left(1 - \frac{1}{32\sin^5\theta}\right) d\theta = \quad [t = \sin\theta]$$

$$= \frac{1}{5}\int_{\frac{1}{2}}^{1} (1-t^2)\left(1 - \frac{1}{32t^5}\right) dt =$$

$$= \frac{1}{5}\int_{\frac{1}{2}}^{1} \left(1 - t^2 - \frac{1}{32}t^{-5} + \frac{1}{32}t^{-3}\right) dt =$$

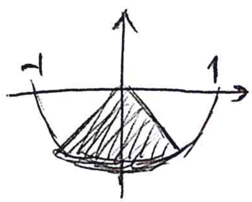
$$= \frac{1}{5}\left[t - \frac{t^3}{3} + \frac{1}{128}t^{-4} - \frac{1}{64}t^{-2}\right]_{\frac{1}{2}}^{1} =$$

$$= \frac{1}{5}\left[\left(1 - \frac{1}{3} + \frac{1}{128} - \frac{1}{64}\right) - \left(\frac{1}{2} - \frac{1}{24} + \frac{1}{8} - \frac{1}{16}\right)\right] = \frac{79}{672}.$$

3 • $\displaystyle\int_E y \ln\frac{|x|}{\sqrt{x^2+y^2}}\,dx\,dy$, $\quad E = \{(x,y) : x^2+y^2 \leq 1,\ y \leq -|x|\}$

In coordinate polari, $(x,y) \in E$ se e solo se

$r \leq 1$, $\frac{5}{4}\pi \leq \theta \leq \frac{7}{4}\pi$.



Quindi

$$\int_E y \ln \frac{|x|}{\sqrt{x^2+y^2}}\,dx\,dy = \int_{\frac{5}{4}\pi}^{\frac{7\pi}{4}} \int_0^1 r\sin\theta \ln|\cos\theta|\, r\,dr\,d\theta =$$

(per simmetria rispetto all'asse y)

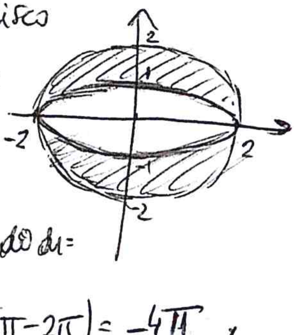
$$= 2\int_{\frac{3}{2}\pi}^{\frac{7\pi}{4}}\int_0^1 r^2 \sin\theta \ln\cos\theta\,dr\,d\theta = \frac{2}{3}\int_{\frac{3}{2}\pi}^{\frac{7\pi}{4}} \sin\theta\ln\cos\theta\,d\theta =$$

$[\cos\theta = t]$

$$= \frac{2}{3}\int_0^{\frac{1}{\sqrt2}} -\ln t\,dt = \frac{2}{3}\Big[-t\ln t + t\Big]_0^{\frac{1}{\sqrt2}} = \frac{1}{3\sqrt2}\ln 2 + \frac{\sqrt2}{3}.$$

4. $\int_F (x^2-2y^2)\,dx\,dy$, $\quad F=\{(x,y): x^2+y^2\le 4,\ \frac{x^2}{4}+y^2\ge 1\}$.

F è la parte del disco di raggio 2 che è esterna all'ellisse. Conviene calcolare l'integrale su tutto il disco D e sottrarre l'integrale su tutta l'ellisse E. Si ha



$$\int_D (x^2-2y^2)\,dx\,dy = \int_0^2\int_0^{2\pi} r^2(\cos^2\theta - 2\sin^2\theta)\, r\,d\theta\,dr =$$

$$= \left[\frac{r^4}{4}\right]_0^2 \int_0^{2\pi}(\cos^2\theta-2\sin^2\theta)\,d\theta = \frac{16}{4}(\pi-2\pi) = -4\pi,$$

mentre, usando coordinate ellittiche $\frac{x}{2}=r\cos\theta$, $y=r\sin\theta$,

$$\int_E (x^2-2y^2)\,dx\,dy = \int_0^1\int_0^{2\pi}(4r^2\cos^2\theta - 2r^2\sin^2\theta)\,2r\,d\theta\,dr =$$

$$= \int_0^1 r^3\,dr \cdot \int_0^{2\pi}(8\cos^2\theta - 4\sin^2\theta)\,d\theta = \frac{1}{4}\cdot(8\pi-4\pi)=\pi.$$

Dunque l'integrale vale: $-4\pi-\pi=-5\pi$.

5. $\int_D z(x+y^2)\,dx\,dy\,dz$, $\quad D=\{(x,y,z):\ x^2+y^2\le z\le 2-\sqrt{x^2+y^2}\}$.

In coordinate cilindriche si ha $(x,y,z)\in D$ se e solo se

$r^2\le z\le 2-r$, e dunque $r^2\le 2-r$, cioè $0\le r\le 1$

(sarebbe $-2\le r\le 1$, ma r deve essere ≥ 0). Dunque

$$\int_D z(x+y^2)\,dx\,dy\,dz = \int_0^1\left[\int_0^{2\pi}\left[\int_{r^2}^{2-r} z(r\cos\theta+r^2\sin^2\theta)\,r\,dz\right]d\theta\right]dr=$$

$$=\cancel{\int_0^{2\pi}\cos\theta\,d\theta\left[\int_0^1\int_{r^2}^{2-r} zr^2\,dz\,dr\right]}+\int_0^{\pi}\sin^2\theta\,d\theta\left[\int_0^1\int_{r^2}^{2-r} zr^3\,dz\,dr\right]=$$

$$=\frac{\pi}{2}\int_0^1 r^3\left[(2-r)^2-r^4\right]dr=\frac{\pi}{2}\int_0^1 r^3(4-4r+r^2-r^4)\,dr=$$

$$=\frac{\pi}{2}\int_0^1\left[4r^3-4r^4+r^5-r^7\right]dr=\pi\cdot\frac{29}{240}.$$

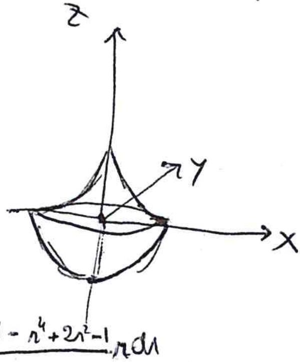
6. $\int_D \frac{z}{4-x^2-y^2}\,dx\,dy\,dz$, $\quad D=$ ruotato attorno all'asse z di

$E=\{(x,z):\ x^2-1\le z\le (x-1)^2,\ 0\le x\le 1\}$.

In coordinate polari D si descrive con

$\theta\in[0,2\pi]$, $\ r^2-1\le z\le (r-1)^2$,

e $r^2-1\le (r-1)^2$, cioè $0\le r\le 1$. Dunque



$$\int_D \frac{z}{4-x^2-y^2}\,dx\,dy\,dz=\int_0^{2\pi}\int_0^1\int_{r^2-1}^{(r-1)^2}\frac{zr}{4-r^2}\,dz\,dr\,d\theta=$$

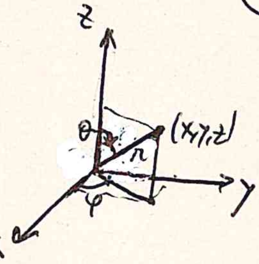
$$=2\pi\int_0^1\frac{r}{4-r^2}\left[\frac{z^2}{2}\right]_{(r^2-1)}^{(r-1)^2}dr=\pi\int_0^1\frac{r^4-4r^3+6r^2-4r+1-r^4+2r^2-1}{4-r^2}\,r\,dr$$

$$=\pi\int_0^1\frac{-4r^4+8r^3-4r^2}{4-r^2}\,dr=4\pi\int_0^1\frac{(r^2-4+4)(-r^2+2r-1)}{4-r^2}\,dr=$$

$$=4\pi\int_0^1\left[r^2-2r+1+\frac{4(4-r^2+2r-5)}{4-r^2}\right]dr=4\pi\left[\frac{r^3}{3}-r^2+5r-4\ln(4-r^2)-5\ln\frac{2+r}{2-r}\right]_0^1=$$

$$=4\pi\left[13/3-4\ln 3+8\ln 2\right].$$

Coordinate sferiche in $\mathbb{R}^3$



Per $(x,y,z)\in\mathbb{R}^3$ poniamo

$$g(r,\theta,\varphi): \begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta, \end{cases} \qquad r>0,\ \theta\in[0,\pi],\ \varphi\in[0,2\pi].$$

Si ha

$$J_g(r,\theta,\varphi) = \det\begin{pmatrix} \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\theta & -r\sin\theta & 0 \end{pmatrix} =$$

$$= \cos\theta\left[r^2\cos\theta\sin\theta\right] + r\sin\theta\left[r\sin^2\theta\right] = r^2\sin\theta.$$

Vale allora (essendo $|J_g| = r^2\sin\theta$):

Proposizione. Se $F\subseteq\mathbb{R}^3$ è misurabile e f è integrabile su F, posto $E = g^{-1}(F) = \{(r,\theta,\varphi) : (r\sin\theta\cos\varphi,\ r\sin\theta\sin\varphi,\ r\cos\theta)\in F\}$, si ha

$$\int_F f(x,y,z)\,dx\,dy\,dz = \int_E f(r\sin\theta\cos\varphi,\ r\sin\theta\sin\varphi,\ r\cos\theta)\, r^2\sin\theta\,dr\,d\theta\,d\varphi. \square$$

Esempi (1) $\displaystyle\int_F (x^2+y^2+z^2)\,dx\,dy\,dz$, $F = \{(x,y,z) : x^2+y^2+z^2\le 1,\ 0\le z\le\sqrt{x^2+y^2}\}$.

F è la parte di emisfero superiore che sta sotto il cono.