Università degli Studi di Pisa

Corso di laurea magistrale in Matematica

A mathematical model on REM-NREM cycle

Tesi di laurea magistrale

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Introduction

During the last century the application of mathematical sciences to biological problems increased a lot. Mathematicians, physicists and biologists began to study different biological situations, human reactions, cellular interactions and found their cooperation really useful to represent realistically many mechanisms.
In this work I refer to some mathematical models of brain system neural interactions; in particular I pay attention to two different neural groups, whose interaction produces the alternation of REM and NON REM phases during human sleep.
I start from the model of Hobson and McCarley (1975); they represented through a Lotka-Volterra system the interaction between two neural groups involved in REM and NREM cycle.
Let $x(t)$ be the level of discharge activity in cells that promote the REM phase; let $y(t)$ be the level of discharge activity in cells that inhibit it; and let a, b, c, and d be positive constants. These terms are related by the Lotka-Volterra system:

$$\begin{cases} \frac{dx}{dt}(t) = ax(t) - bx(t)y(t) \\ \frac{dy}{dt}(t) = -cy(t) + dx(t)y(t), \end{cases} \tag{1}$$

As it is well known, the system periodically goes back to its initial state and this is a non-realistic scenario.
In a second step I use the recent interaction model of Kuramoto to obtain another system for the neurons activity and I generalize it to:

$$\begin{cases} \frac{d\theta}{dt} = \omega_1 + g(\theta - \phi) \\ \frac{d\phi}{dt} = \omega_2 - g(\theta - \phi) \end{cases} \tag{2}$$

where g is a nonlinear function, ω_j, $j = 1, 2$ are the frequencies of the neurons and θ and ϕ are the "phases" of neurons of the two different neural groups.
Finally, after a chapter of "mathematical tools", I introduce a diffusive model for this interaction, starting from the previous Kuramoto model. So I study the solutions, as well as their qualitative behavior, of the system

$$\begin{cases} \frac{\partial\theta}{dt} - \Delta\theta = \omega + g(\theta - \phi) \\ \frac{\partial\phi}{dt} - \Delta\phi = \omega - g(\theta - \phi) \end{cases} \tag{3}$$

where g is a nonlinear function with appropriate properties.
In the conclusive chapter I observe that my work is based on scientific data and it potentially leaves room to different applications.

Chapter 1

Biological overview

In this first chapter, I will give a sketch on basic biological background needed to realize this work.
I begin to describe neurons, their functions and their interaction and then I will say something on the structure of the sleep and on the brain neurons role in the alternation of REM and NREM phases.

1.1 The Neurons

The central nervous system [CNS] is composed entirely of two kinds of specialized cells: neurons and glia.
Neurons are the basic information processing structures in the CNS. The function of a neuron is to receive INPUT "information" from other neurons, to process that information, then to send "information" as OUTPUT to other neurons. Synapses are connections between neurons through which "information" flows from one neuron to another. Hence, neurons process all of the "information" that flows within, to, or out of the CNS. Processing many kinds of information requires many types of neurons; there may be as many as 10,000 types of them. Processing so much information requires a lot of neurons. "Best estimates" indicate that there are around 200 billion neurons in the brain alone!
Glia (or glial cells) are the cells that provide support to the neurons: not only glia provide the structural framework that allows networks of neurons to remain connected, they also attend to the brain's various "house" keeping functions (such as removing debris after neuronal death).
Since our main interest lies in exploring how information processing occurs in the brain, we are going to ignore glia. But before we see how neurons process information (and what that means), we need to know a few things about the structure of neurons.

1.1.1 Structure of neurons

Neurons have various morphologies depending on their functions. We can see an example in the following figure:

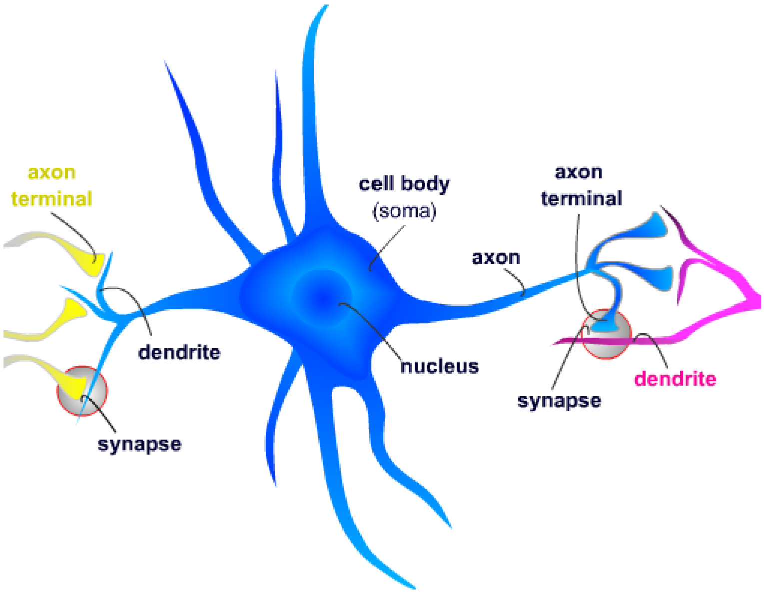


Figure 1.1: The structure of a neuron.

The signal is picked up by the neuron via the dendrites, branched structure extending less than one millimeter. Then the soma, also said the body of the neuron, deals with the processing of the signal. We can imagine the soma as an object approximately spherical having a diameter less than $70\mu m$ that may or may not release an electrical signal which propagates along the axon towards the other neurons. The axon is presented as a long protuberance with a diameter of few μm and it is connected to the dendrites of one or more neurons. A neuron may have many thousands of dendrites, but it will have only one axon. The fourth distinct part of a neuron lies at the end of the axon, the axon terminals. These are the structures that contain neurotransmitters. Neurotransmitters are the chemical medium through which signals flow from one neuron to the next at chemical synapses.

1.1.2 Neuronal signaling

To support the general function of the nervous system, neurons have evolved unique capabilities for intracellular signaling (communication within the cell) and intercellular signaling (communication between cells). To achieve long distance, rapid communication, neurons have evolved special abilities for sending electrical signals (action potentials) along axons. This mechanism, called **conduction**, is how the cell body of a neuron communicates with its own terminals via the axon. Communication between neurons is achieved at synapses by the process of **neurotransmission**.

1.1.3 Action potentials

To begin conduction, an action potential is generated near the cell body portion of the axon. An action potential is an electrical signal very much like the electrical signals in electronic devices. But whereas an electrical signal in an electronic device occurs because electrons move along a wire, an electrical signal in a neuron occurs because ions move across the neuronal membrane. Ions are electrically charged particles.
The protein membrane of a neuron acts as a barrier to ions. Ions move across the membrane through ion channels that open and close due to the presence of neurotransmitters. When the concentration of ions on the inside of the neuron changes, the electrical property of the membrane itself changes. Normally, the membrane potential of a neuron,i. e. the difference in voltage between the inside and the outside of the cell's membrane, rests as -70 millivolts. (Fig. 1.2 on the top)
At this resting potential, the neuron is said *polarized* and its ion channels (protein structures) are closed.

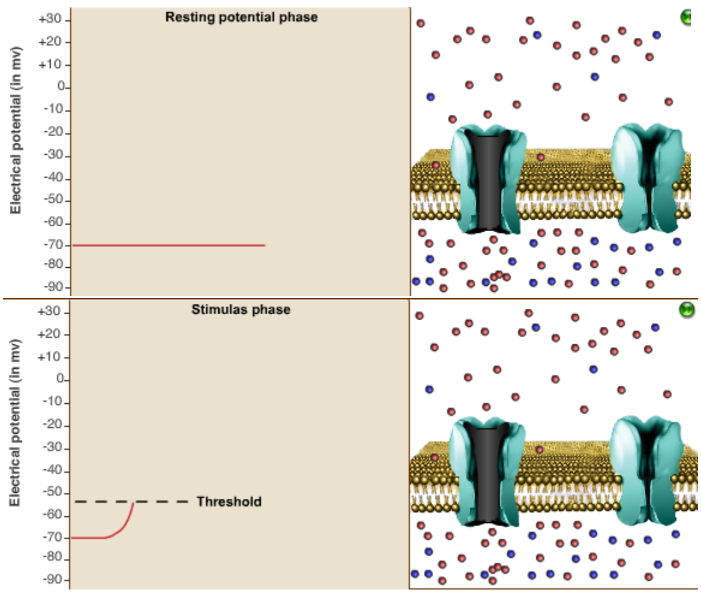


Figure 1.2: Resting potential phase.

When a neuron is stimulated as a result of neurotransmission the mem-

brane potential becomes slightly more positive: it partially *de-polarizes* (Fig. 1.2 on the bottom.) When this depolarization reaches a point of no return called a threshold, a large electrical signal is generated (Fig. 1.3 on the top). This is the *action potential*, also called spike. During complete depolarization, sodium channels open and sodium ions rush in. The sodium channels then close.
The potassium channels now open and some of the potassium ions, repelled by the positive charge inside, move to the outside (and the membrane's potential begins to return normal). This phase is said *repolarization* and then the potassium channels close (Fig. 1.3 on the bottom). Before the membrane potential stabilizes, there is a small undershoot in the membrane potential and the neuron cannot fire another action potential.

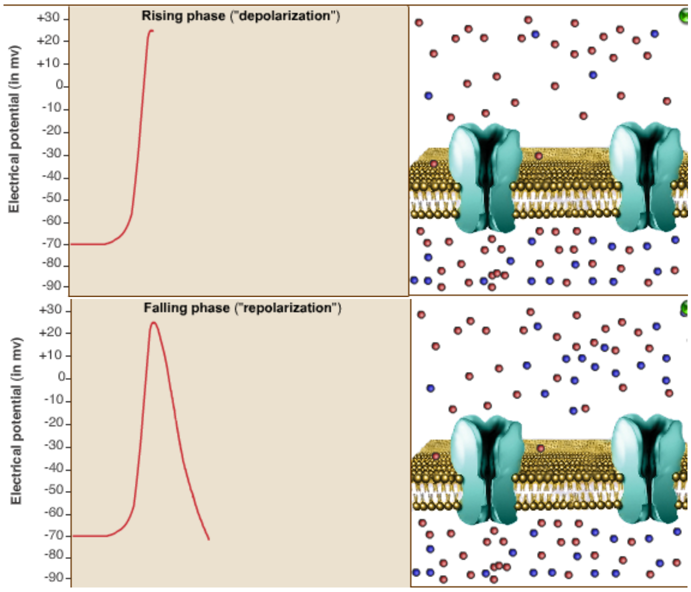


Figure 1.3: The action potential.

After this refractory period, the neuron is now ready to fire another action potential. Once fired, an action potential quickly (moving at rates up to 150 meters) spreads along the membrane of the axon like a wave until it reaches its axon terminals and via synapses with other neurons, axon terminal is where chemical neurotrasmission begins.

1.1.4 Neurotransmission

Neurotransmission (or synaptic transmission) is communication between neurons as accomplished by the movement of chemicals or electrical signals across a synapse. For any interneuron, its function is to receive INPUT

”information” from other neurons through synapses, to process that information, then to send ”information” as OUTPUT to other neurons through synapses. Consequently, an interneuron cannot fulfill its function if it is not connected to other neurons in a network. A network of neurons (or neural network) is merely a group of neurons through which information flows from one neuron to another. The image below represents a neural network.

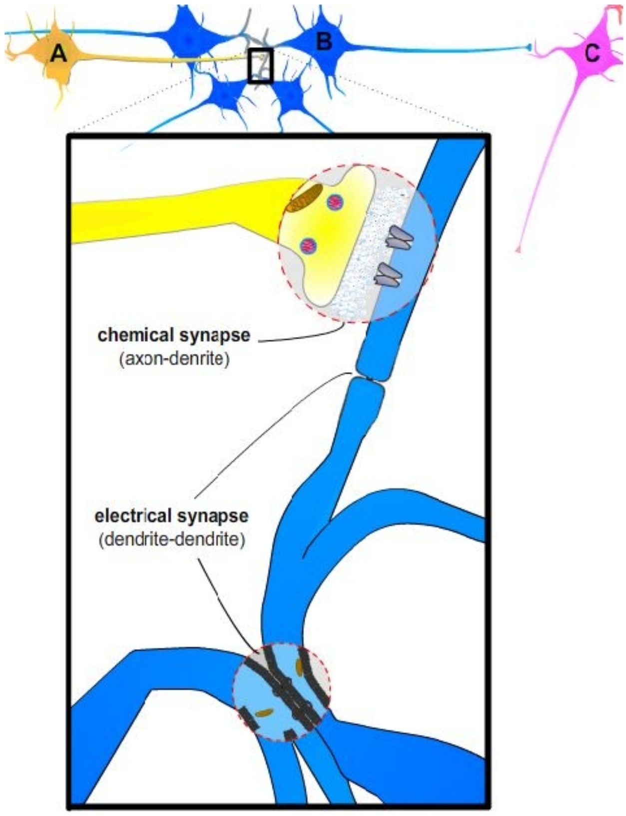


Figure 1.4: An example of neural network.

”Information” flows between the blue neurons through **electrical synapses**. ”Information” flows from yellow neuron A, through blue neuron B, to pink neuron C via **chemical synapses** (Fig. 1.4).

An electrical synapses between two neurons occurs when a gap junction fuses the membranes of a pair of dendrites. Gap junctions permit changes in the electrical properties of one neuron to effect the other (through a direct exchange of ions), so the two neurons essentially behave as one. Electrical neurotransmission is the process where an impulse (*synaptic potential*) in one neuron will cause a synchronous impulse in the other (Fig. 1.5).

Chemical neurotransmission occurs at chemical synapses. In chemical neurotransmission, the presynaptic neuron and the postsynaptic neuron are separated by a small gap, the synaptic cleft. The synaptic cleft is filled

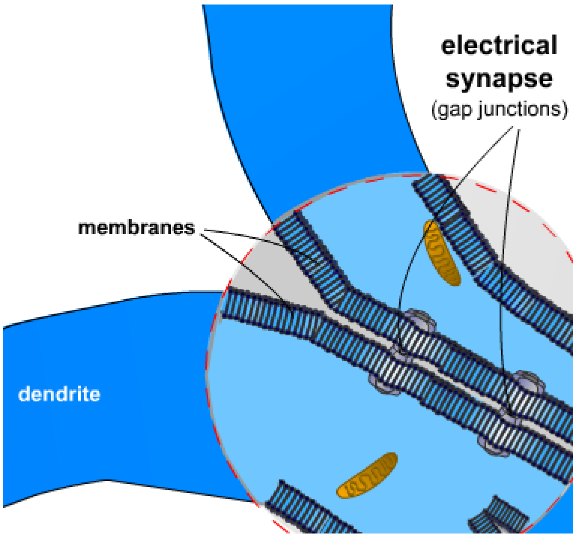


Figure 1.5: Electrical synapse.

with extracellular fluid (the fluid bathing all the cells in the brain). Although very small, typically on the order of a few nanometers, the synaptic cleft creates a physical barrier for the electrical signal carried by one neuron to be transferred to another neuron (Fig. 1.6). In electrical terms, the synaptic cleft would be considered a "short" in an electrical circuit. Chemical neurotransmission requires releasing neurotransmitter into the synaptic cleft before a synaptic potential can be produced as INPUT to the other cell. Neurotransmitter acts like a chemical messenger, linking the action potential of one neuron with a synaptic potential in another.

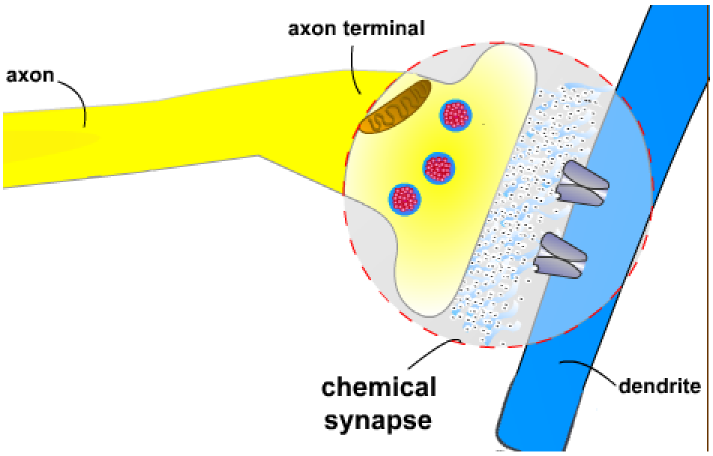


Figure 1.6: Chemical synapse.

Resuming, a typical neuron has thousands of INPUT synapses. The

synaptic potentials produced through those synapses determine whether it fires an action potential as OUTPUT.
When the sum of all its synaptic potential equals or exceeds the threshold, the neuron will fire an action potential along its axon.
The action potential will travel until it reaches the chemical synapse at its axon terminal (OUTPUT). Once there, it will trigger release of its own neurotransmitter, which will cause a synaptic potential in the new postsynaptic neuron.

From the functional point of view, the synapses are classified in two types: the **excitatory synapses** which enable impulses in the nervous system to be spread and the **inhibitory** ones, which cause their attenuation.
During an excitatory synapse an action potential in a presynaptic neuron increases the probability of an action potential occurring in a postsynaptic cell.
Inhibitory synapses, on the other hand, cause the neurotransmitters in the postsynaptic membrane to depolarize, decreasing its likelihood of the firing action potential.
Chemical synapses are the most prevalent and are the main player involved in excitatory synapses.

1.2 Dynamic neuron

In this little section I mention some different models made to simulate neural behaviors and their spiking.
Scientists investigated how the types of currents determine neuronal dynamics. We divide all currents into two major classes: amplifying and resonant, with the persistent $Na+$ current $I_{Na,p}$ and the persistent $K+$ current I_K being the typical examples of the former and the latter, respectively. Since there are tens of known currents, purely combinatorial argument implies that there are millions of different electrophysiological mechanisms of spike generation. They showed that any such mechanism must have at least one amplifying and one resonant current. Some mechanisms have one resonant and one amplifying current.
So, many models focused on system of equations where the variables involved are these currents.
They made correspondences between resting, excitable, and periodic spiking activity to a stable equilibrium or limit cycle, respectively, of a dynamic system.

However, in the first section, we saw that in electrical neurotransmission an impulse (*synaptic potential*) from one neuron causes a synchronous impulse in the other.
So like any other kind of physical, chemical, or biological oscillators, such neurons can synchronize and exhibit collective behavior that is not intrinsic to any individual neuron. For example, partial synchrony in cortical networks is believed to generate various brain oscillations, such as the alpha and gamma EEG rhythms. Increased synchrony may result in pathological types of activity, such as epilepsy.
Depending on the circumstances, synchrony can be good or bad, and it is important to know what factors contribute to synchrony and how to control it. There are various methods of reduction of coupled oscillators to simple phase models. The reduction method and the exact form of the phase model depend on the type of coupling (i.e., whether it is pulsed, weak, or slow).

Many types of physical, chemical, and biological oscillators share an astonishing feature: they can be described by a single phase variable θ. In the context of tonic spiking, the phase is usually taken to be the time since the last spike.
Many models pay attention to neurons as biological oscillators. Most of them reduces the interaction between n neurons with phases ϕ_i $(i = n)$, to a system of the form:

$$\frac{d}{dt}\phi_i = \epsilon\omega_i + \epsilon\sum_{j\neq i}^{n} H_{i,j}(\phi_i - \phi_j) \quad \epsilon > 0. \tag{1.1}$$

where $\omega_i = H_{i,i}(0)$ describes a constant frequency deviation from the free-running oscillation and $H_{i,j}$ correspond to a gap-junction coupling of oscillators.
If we think of two neurons, we can describe their interaction with the system:

$$\begin{cases} \frac{d\phi_1}{dt} = \omega_1 - +H_1(\phi_2 - \phi_1) \\ \frac{d\phi_2}{dt} = \omega_2 - +H_1(\phi_1 - \phi_2), \end{cases}$$

In general, determining the stability of equilibria is a difficult problem.
Ermentrout (1992) found a simple sufficient condition. Namely, if

- $a_{ij} = H'_{ij}(\phi_j - \phi_i) \geq 0$,
- the directed graph defined by the matrix $A = (a_ij)$ is connected (each oscillator is influenced by every other oscillator),

then the equilibrium ϕ (the vector $\phi = (\phi_1, ..., \phi_n)$ such that (1.1) is satisfied) is neutrally stable and the correspondent limit cycle is asympotically

stable.
Another sufficient condition was found by Hoppensteadt and Izhikevich (1997). It states that if (1.1) satisfies

- $\omega_1 = ... = \omega_n = \omega$
- $H_{ij}(-\chi) = -H_{ij}(\chi)$, where $\chi = \phi_j - \phi_i$

for all i, j, then the network dynamics converge to a limit cycle.
However we pay attention to Kuramoto's theory (1975), with $H(\chi) = \sin \chi$. Kuramoto, a physicist in the Nonlinear Dynamics group at Kyoto University, solved the problem for N interacting "smooth" oscillators. They continuously interact by accelerating or decelerating and the modification of the speed of each oscillator is a function of the current position of all others. If the given conditions are met they eventually synchronize. The interaction also depends on a constant K that represents the"strength" of the communication.
According to Kuramoto, the angular speed of the $i-th$ oscillator is modified in this way:

$$\frac{d\phi_i}{dt} = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\phi_j - \phi_i).$$

1.3 REM-ON and REM-OFF phases

The aim of this section is to analyze the biological mechanisms which underlies the alternation of sleep and wake, and of REM and NREM phases.
If we want to define sleep, we can say that it:

- is a naturally-occurring state;
- is periodic and recurring;
- involves both the mind and the body;
- involves the temporary suspension of consciousness;
- involves the relaxation and inactivity of muscles.

Our nightly sleep is made up of several sleep cycles, each of which is composed of several different sleep stages, and the physiological and neurological differences between the two main types of sleep, NREM and REM are almost as profound as the differences between sleep and wakefulness (or, for that matter, night and day).
The sleep-wake cycle, is regulated by two separate biological mechanisms in

the body, which interact together and balance each other. This model, first posited by the Swiss sleep researcher Alexander Borbely in the early 1980s, is often referred to as the two-process model of sleep-wake regulation. The two processes are:

- circadian rhythm, also known as **Process C**, the regulation of the body's internal processes and alertness levels which is governed by the internal biological or circadian clock;

- sleep-wake homeostasis, or **Process S**, the accumulation of hypnogenic (sleep-inducing) substances in the brain, which generates a homeostatic sleep drive.

Both of these processes are influenced to some extent by the genes of the individual. In addition, various external factors can also have a direct or indirect effect on an individual's sleep-wake cycle.

The different types and stages of sleep can be best identified using *polysomnography*, which simultaneously measures several body functions such as brain wave activity (electroencephalogram or EEG), eye movement (electrooculogram or EOG), muscle activity (electromyogram or EMG), respiration, heart rhythm, etc. A simplified summary of these results can be combined into a graph called a hypnogram, which gives a useful visual cross-section of sleep patterns and sleep architecture.
The EEG is an extracellular recording obtained using macroelectrodes placed on the scalp, and it measures the electrical activity of cortical neurons of the area underlying the electrodes. More in detail, an EEG records the extracellular ionic current flow associated with the summed activity of many hundreds of thousands of neurons, located under the recording electrodes. The frequencies of the potentials recorded from the surface of the scalp of a normal human typically vary from 1 to 30 Hz, and the amplitudes typically range from 20 to $100\mu V$. Although the frequency characteristics of EEG potential are extremely complex and the amplitude may vary considerably over a short time interval, a few dominant frequency bands and amplitudes are typically observed.
We distinguish the following EEG rhythms:

- β waves, with frequencies $> 15Hz$, correspond to activated cerebral cortex, during states of vigilance;

- α waves, with frequencies $8-11Hz$, associated with a state of relaxed wakefulness;

- θ waves, with frequencies di $3,5-7,5Hz$, recorded during some sleep stages especially during light sleep;

- δ waves, with low frequencies ($< 3,5Hz$) and large amplitudes, correspond to deep states of sleep.

There are two main broad types of sleep, each with its own distinct physiological, neurological and psychological features: rapid eye movement (REM) sleep and non-rapid eye movement (NREM) sleep, the latter of which can in turn be divided into three or four separate stages.
NREM sleep is also called "synchronized sleep" because during this phase slow, regular, high-voltage, synchronized EEG waves are recorded. On the other hand in REM sleep, the brain is highly activated, and EEG waves are desynchronized, much as they are in waking, so REM sleep is called "desynchronized sleep"

NREM sleep consists of four separate stages (stage1, stage 2, stage 3, stage 4), which are followed in order upwards and downwards as sleep cycles progress.

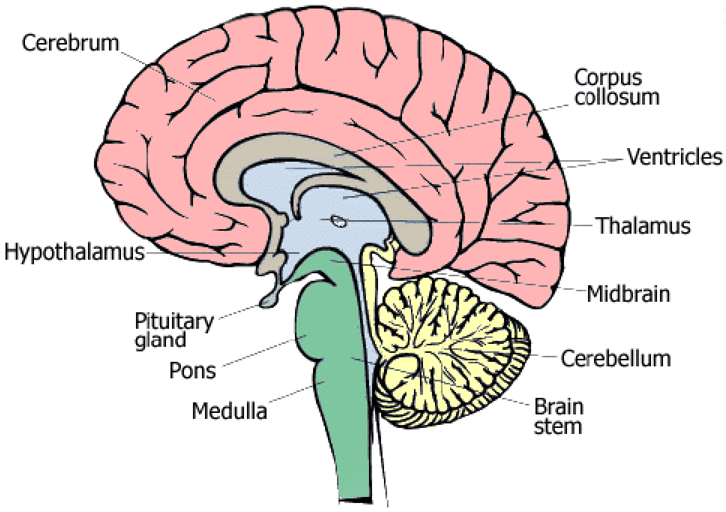
- **Stage 1** is the stage between wakefulness and sleep, in which the muscles are still quite active and the eyes roll around slowly and may open and close from time to time. During this stage are recorded α waves and θ waves.
- **Stage 2** is the first unequivocal stage of sleep, during which muscle activity decreases still further and conscious awareness of the outside world begins to fade completely. Brain waves during stage 2 are mainly in the θ wave range, but in addition stage 2 is also characterized by K-complexes, short negative high voltage peaks, followed by a slower positive complex, and then a final negative peak.
- **Stage 3** is also known as deep or delta or slow-wave sleep (SWS), and during this period the sleeper is even less responsive to the outside environment, essentially cut off from the world and unaware of any sounds or other stimuli. This stage is characterized by δ brain waves and by some sleep spindles, short bursts of increased brain activity last maybe half a second.
- **Stage 4** is characterized essentially by δ brain waves, for 20-40 minutes.

REM sleep occurs in cycles of about 90-120 minutes throughout the night, and it accounts for up to $20 - 25\%$ of total sleep time in adult humans, although the proportion decreases with age. In particular, REM sleep dominates the latter half of the sleep period, especially the hours before waking, and the REM component of each sleep cycle typically increases as the night goes on.
As the name suggests, it is associated with rapid (and apparently random)

side-to-side movements of the closed eyes, a phenomenon which can be monitored and measured by a technique called electrooculography (EOG). This eye motion is not constant (tonic) but intermittent (phasic). It is still not known exactly to what purpose it serves, but it is believed that the eye movements may relate to the internal visual images of the dreams that occur during REM sleep, especially as they are associated with brain wave spikes in the regions of the brain involved with vision (as well as elsewhere in the cerebral cortex).

Brain activity during REM sleep is largely characterized by low-amplitude mixed-frequency brain waves, quite similar to those experienced during the waking state: θ waves, α waves and even the high frequency β waves more typical of high-level active concentration and thinking.

We may now ask ourselves something more about the factors which cause REM/NREM alternation. Since late 1950s, it was believed that the brain-system played an important role in the sleep-wake cycle. Giuseppe Moruzzi and his his team were the first to prove the existence of different populations of neurons whose activity was required for the regulation of wakefulness and sleep. These groups were located in various part of reticular formation (a part of the midbrain, which is shown in figure ref7), but the physiologists were not able to explain how the alternation process worked.



brain.png

Figure 1.7: Human brain

Some steps further were made by other scientists: as illustrated in the next Chapter, a first mathematical model of REM/NREM cycle was developed in late 1970's by Hobson and McCarley. After this, some other followed, but much of the sleep mechanism is to be discovered yet.

Before analyzing Hobson and McCarley model and a recent neuronal model that can influence REM/NREM cycle, it is important to say something else on sleep-wake cycle.

The body's built-in circadian clock, which is centered in the hypothalamus organ in the brain, is the main mechanism that controls the timing of sleep, and is independent of the amount of preceding sleep or wakefulness. This internal clock is coordinated with the day-night / light-dark cycle over a 24-hour period, and regulates the body's sleep patterns, feeding patterns, core body temperature, brain wave activity, cell regeneration, hormone production, and other biological activities.
But circadian rhythms alone are not sufficient to cause and regulate sleep. There is also an inbuilt propensity toward sleep-wake homeostasis, which is balanced against the circadian element. Sleep-wake homeostasis is an internal biochemical system that operates as a kind of timer or counter, generating a homeostatic sleep drive or pressure to sleep and regulating sleep intensity. It effectively reminds the body that it needs to sleep after a certain time, and it works quite intuitively: the longer we have been awake, the stronger the desire and need to sleep becomes, and the more the likelihood of falling asleep increases; the longer we have been asleep, the more the pressure to sleep dissipates, and the more the likelihood of awakening increases.
The interaction between the two processes can visualized graphically as follows:

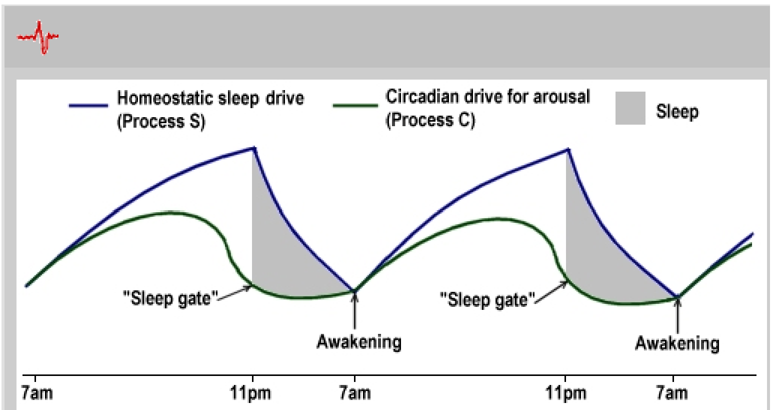


Figure 1.8: Sleep-wake regulation: interaction between the homeostatic sleep drive (Process S) and the circadian drive for arousal (Process C)

While homeostatic sleep drive typically increases throughout the day, effectively making a person more and more sleepy as the day goes on, it is countered and moderated by the circadian drive for arousal, at least until late evening, when the circadian clock slackens off its alerting system and be-

gins sleep-inducing melatonin production instead. This opens the so-called "sleep gate" (marked by the point in the diagram above where the homeostatic sleep drive is at its greatest distance above the circadian drive for arousal). The exact way in which this occurs is still not fully understood, but the recent neuronal group theory of sleep theorizes that individual groups of neurons in the brain enter into a state of sleep after a certain threshold of activity has been reached, and that, once enough groups of neurons are in this sleep state, the whole organism falls asleep.
During the night, while sleep is actually being experienced, the homeostatic sleep drive rapidly dissipates, and circadian-regulated melatonin production continues. In the early morning, melatonin secretion stops and the circadian alerting system begins to increase its activity again. Eventually, the point is reached where the circadian drive for arousal begins to overcome the homeostatic sleep drive (marked by the point in the diagram above where the two curves meet), triggering awakening, and the process begins all over again.

Chapter 2

Mathematical model

In a general Lotka-Volterra model, if we suppose that there is not a competition between the two groups involved (preys and predators), we obtain:

$$\begin{cases} \frac{dx}{dt}(t) = ax(t) - f(x(t), y(t)) \\ \frac{dy}{dt}(t) = -cy(t) + g(x(t), y(t)), \end{cases}$$

where $x(t)$ and $y(t)$ indicate the prey and predator populations at the instant t.
Their derivatives indicate the growth rate of the populations; the functions f and g express the dynamics of the populations. The constants a and c are positive and represent the decreasing and growth rates of the two populations in absence of interaction.

We analyze the reciprocal interaction model (proposed by R. McCarley and A. Hobson), which explains REM-NREM alternation as a result of the antagonist role played by two neuronal populations, $FTG - neurons$ and $LC - neurons$.
The most important features of the discharge time course of FTG are the periodically occurring peaks of discharge activity, each of which corresponds to a desynchronized (REM) sleep episode.
The process of transition to high discharge levels in desynchronized sleep in FTG neurons is of exponential order, so we can think of a self-excitation.
The activity from FTG to LC cells which is postulated to utilize acetylcholine is excitatory.
Connections from LC to FTG and from LC to LC cells are revealed by the presence of norepinephrine containing varicosities in each area; these synapses are assumed to utilize norepinephrine as a neurotransmitter and to be inhibitory.
The mathematical form of terms describing the influence of each population on itself is suggested by evidence that the rate of change of activity levels

in the FTG population is proportional to the current level of activity, and we propose that the same is true for the LC population, but with a negative sign because the recurrent feedback is inhibitory. The highly nonsinusoidal nature of FTG activity suggests that nonlinear FTG-LC interaction is to be expected. We model this effect by the simplest form of nonlinearity, the product of activities in the two populations; according with the reasonable physiological postulate that the effect of an excitatory or inhibitory input to the two populations will be proportional to the current level of discharge activity. Let $x(t)$ be the level of discharge activity in FTG cells and let $y(t)$ be the level of discharge activity in LC cells; and let a, b, c, and d be positive constants. These terms are related by the Lotka-Volterra system:

$$\begin{cases} \frac{dx}{dt}(t) = ax(t) - bx(t)y(t) \\ \frac{dy}{dt}(t) = -cy(t) + dx(t)y(t), \end{cases} \tag{2.1}$$

In our model, the FTG (excitatory) cells are analogous to the prey population, and the LC (inhibitory) cells are analogous to the predator population. These equations and more complicated variants have been extensively studied and the behavior of their solutions has been well documented, although no explicit solution in terms of elementary functions is available.

For the simple model and the parameters used here, there is a periodic solution. The equilibrium points of (2.1) are in $(0,0)$ and in $z = (\frac{c}{d}, \frac{a}{b})$. If we study the jacobian matrix in $(0,0)$, we deduce that the origin is a saddle point, so it is unstable. The eigenvalues of the Jacobian in z are pure imaginary so we have no information about the stability.

If we draw the two lines

$$y = \frac{a}{b}, \qquad x = \frac{c}{d},$$

we divide the first quadrant of the coordinate plane into four sectors.

In each sector the sign of $\dot{x}$ and $\dot{y}$ and is constant. The positive half-lines of the x-axis and y-axis are trajectories with the origin as limit set. The other solutions $(x(t), y(t))$ turn around the point z crossing the four sectors. We want to find a Lyapunov function $H(x,y) = F(x) + G(y)$ so that:

$$\dot{H} = \frac{dF}{dx}\dot{x} + \frac{dG}{dy}\dot{y} = x\frac{dF}{dx}(a - by) + y\frac{dG}{dy}(dx - c) \leq 0$$

We obtain $\dot{H} = 0$ if

$$\frac{x\frac{dF}{dx}}{dx - c} = \frac{y\frac{dG}{dy}}{by - a} = k \quad (k \text{ constant}).$$

So with $k = 1$, we have

$$\frac{dF}{dx} = d - \frac{c}{x} \Rightarrow F(x) = dx - c\log x + A, \quad A \in \mathbb{R}$$
$$\frac{dG}{dy} = b - \frac{a}{y} \Rightarrow G(y) = by - a\log y + B. \quad B \in \mathbb{R}$$

Finally the function

$$H(x,y) = dx - c\log x + by - a\log y$$

defined for $x > 0$, $y > 0$, is constant along the solutions of (2.1).
If we study the sign of $\frac{\partial H}{dx}$ and $\frac{\partial H}{dy}$ we can show that z is an absolute minimum point, so that $H(\cdot) - H(z)$ is a Lyapunov function for z and the point z is a stable equilibrium.
Now, through the following theorem we have a description of the type of solutins of the system (2.1).

Theorem 2.1. The trajectories of the system (2.1) which are different from the equilibrium point z and from the positive axis are closed orbits.

The system periodically returns to its initial state, with possibly very large oscillations.

2.1 Neurons as biological oscillators

If we think of FTF-cells and LC-cells as oscillators, we can use some physical model to study their interaction.
As seen in the first chapter, according to Kuramoto, the angular speed of the $i-th$ oscillator is modified in this way:

$$\frac{d\phi_i}{dt} = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\phi_j - \phi_i).$$

So we can describe two coupled oscillators (with phases θ and ϕ) through the following phase model:

$$\begin{cases} \frac{d\theta}{dt} = \omega_1 + A\sin(\theta - \phi) \\ \frac{d\phi}{dt} = \omega_2 + A\sin(\phi - \theta) \end{cases} \tag{2.2}$$

where ω_1 and ω_2 are the constant frequency deviations from the free-running oscillation and A is a costant, because the alternation of REM-NREM cycles depends from a symmetric activation of the two neuronal groups (FTG and LC).
Later on, we will to study also the general case

$$\begin{cases} \frac{d\theta}{dt} = \omega_1 + g(\theta - \phi) \\ \frac{d\phi}{dt} = \omega_2 - g(\theta - \phi) \end{cases} \tag{2.3}$$

where g is a nonlinear, locally Lypschitz function such that there is $A \in \mathbb{R}$ such that $g(u) \sim Au$ in a small neighborhod of $u = 0$.

Going back to (2.2), we can suppose $\omega_1 = \omega_2$ because the two neurons are similar (because they are brain system neurons) and we obtain:

$$\frac{d(\theta + \phi)}{dt} = 2\omega$$

which implies the conservation law : $\theta + \phi = 2\omega t + k$.

In order to make null the nonlinear part, we suppose that $\sin(\theta - \phi) = 0$, in such a way θ and ϕ are synchronized:

$$\theta = \phi \text{ or } \theta = \phi + k\pi$$

and we deduce:

$$\begin{cases} \frac{d\theta}{dt} = \omega \\ \frac{d\phi}{dt} = \omega \end{cases}$$

and

$$\begin{cases} \theta(t) = \omega t + \theta_0 \\ \phi(t) = \omega t + \phi_0 \end{cases}$$

with $\theta_0 = \phi_0$.
So, we can start finding steady states of the form $\theta_s(t) = \omega t + \theta_0$ and $\phi_s(t) = \omega t + \phi_0$ and without loss of generality $\theta_0 = \phi_0 = 0$.
Generic solution of the system (2.2) with $\omega_1 = \omega_2 = \omega$, can be expressed by

$$\theta(t) = \theta_s(t) + \theta_r(t), \quad \phi(t) = \phi_s(t) + \phi_r(t),$$

where $\theta_s = \phi_s = \omega t$.
If we replace θ with $\theta_s + \theta_r$ in the first equation of (2.4), $\phi = \phi_s + \phi_r$ in the second and remind that $\frac{d\theta}{dt} = \frac{d\phi}{dt} = \omega$ we get:

$$\begin{cases} \frac{d\theta_r}{dt} = A\sin(\theta_r - \phi_r) \\ \frac{d\phi_r}{dt} = A\sin(\phi_r - \theta_r) \end{cases} \tag{2.4}$$

In order to have informations of the qualitative behavior of the solution, we linearize this system and we obtain

$$\begin{bmatrix} \dot{\theta}_r \\ \dot{\phi}_r \end{bmatrix} = A \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \theta_r \\ \phi_r \end{bmatrix}.$$

The eigenvalues of this matrix are 0 and $2A$, so we cannot use the Lyapunov exponents theory to determine the system stability.
If we turn to the case in which $A\sin(\theta - \phi)$ is replaced with $g(\theta - \phi)$, we find the system:

$$\begin{cases} \frac{d\theta}{dt} = \omega + g(\theta - \phi) \\ \frac{d\phi}{dt} = \omega - g(\theta - \phi) \end{cases} \tag{2.5}$$

As before, if we sum the two equations we get:

$$\theta(t) + \phi(t) - 2\omega t = k,$$

that implies we can start considering solutions of the form $\theta_s(t) = \phi_s(t) = \omega t$.
To study the asymptotic stability, we consider a small perturbation:

$$\theta_r = \theta - \omega t, \ \ \phi_r = \phi - \omega t.$$

We substitute in (2.5) and the system becomes:

$$\begin{cases} \frac{d\theta_r}{dt} = g(\theta_r - \phi_r) \\ \frac{d\phi_r}{dt} = -g(\theta_r - \phi_r) \end{cases} \tag{2.6}$$

We linearize:

$$\begin{cases} \frac{d\theta_r}{dt} = A(\theta_r - \phi_r) \\ \frac{d\phi_r}{dt} = -A(\theta_r - \phi_r) \end{cases}$$

and the eigenvalues of the associated matrix are 0 and $2A$; again we cannot use the Lyapunov exponents theory.

However we want to study the solution of the systems (2.4) and (2.6).
We remind that $\dot{\theta_r} + \dot{\phi_r} = 0$, so $\theta_r(t) + \phi_r(t)$ is a conserved quantity of the two systems.
Now we pay attention to the difference of the phases $u = \theta_r - \phi_r$. With this definition, the system (2.4) becomes:

$$\dot{u} = 2A \sin u, \tag{2.7}$$

and, if we start from initial conditions $\theta(0) = \theta_0 + \epsilon_1$ and $\phi(0) = \phi_0 + \epsilon_2$, then $u(0) = \epsilon_1 - \epsilon_2$.
Similarly if we replace u in (2.6), we obtain:

$$\dot{u} = 2g(u) = 2Au + O(u^2) \tag{2.8}$$

with initial conditions $u(0) = \epsilon_1 - \epsilon_2$, where $\epsilon_1 = \theta_r(0)$ and $\epsilon_2 = \phi_r(0)$.
We would like to find some conditions on the costant A, from which we can deduce the stability or instability of these new systems.

2.1.1 Case $A < 0$.

We can prove that if $A < 0$, the solution $u(t)$ of (2.7) and (2.8) is asymptotically stable.
We use the foloowing theorem:

Theorem 2.2 (Asymptotic stability of the solitary wave)**.** Suppose g is a nonlinear, locally Lipschitz function such that there exists $A < 0$, satisfying $g(u) \sim Au$ in a small neighborhood of $u = 0$ and the system

$$\begin{cases} \frac{d\theta}{dt} = \omega + g(\theta - \phi) \\ \frac{d\phi}{dt} = \omega - g(\theta - \phi) \end{cases}$$

has initial condition $\theta(0) = \theta_0$, $\phi(0) = \phi_0$. Then there exists a sufficiently small $\epsilon > 0$ so that, if the initial conditions satisfy $|\theta_0 - \phi_0| \leq \epsilon$, the Cauchy problem associated with the system above has global solution

$$\theta(t) = \omega t + \frac{\theta_0 + \phi_0}{2} + v(t),$$

where

$$|v(t)| \leq C\epsilon e^{-2t}$$

for every $t > 0$.

Proof. As seen, if $\eta = \theta - \phi$ and $\xi = \theta + \phi$, then

$$\begin{cases} \frac{d\eta}{dt} = 2g(\eta) \\ \eta(0) = \theta_0 - \phi_0 \end{cases}$$

and

$$\begin{cases} \frac{d\xi}{dt} = 2\omega \\ \xi(0) = \theta_0 + \phi_0. \end{cases}$$

So,

$$\xi(t) = \theta_0 + \phi_0 + 2\omega t \quad \forall t \geq 0.$$

Since there exist $A > 0$, $k > 0$, $\gamma > 0$ such that

$$g(\eta) = A\eta + \sigma(\eta)$$

with $|\sigma(\eta)| \leq k|\eta|^2$ for $|\eta| \leq \gamma$, we have

$$\begin{cases} \frac{d\eta}{dt} = 2A(\eta) + 2\sigma(\eta) \\ \eta(0) = \theta_0 - \phi_0 \end{cases}$$

which implies, multiplying for e^{-2At},

$$\frac{d}{dt}(e^{-2At}\eta) = 2e^{-2At}\sigma(\eta)$$

therefore

$$e^{-2At}\eta(t) - (\theta_0 - \phi_0) = 2\int_0^t e^{-2As}\sigma(\eta(s))ds$$

and

$$\eta(t) = e^{2At}(\theta_0 - \phi_0) + 2\int_0^t e^{2A(t-s)}\sigma(\eta(s))ds. \tag{2.9}$$

If

$$N(\tau) = \sup_{t\in[0,\tau]}[|\eta(t)|e^{-2At}]$$

then, by (2.9), we have:

$$\begin{aligned} N(\tau) &\leq |\theta_0 - \phi_0| + 2\int_0^\tau e^{2As}kN(\tau)^2 ds \\ &= |\theta_0 - \phi_0| + \frac{1}{A}|e^{2A\tau-1}|KN(\tau)^2 \\ &\leq |\theta_0 - \phi_0| + \frac{k}{A}N(\tau)^2. \end{aligned}$$

We obtain:

$$\frac{k}{A}N(\tau)^2 - N(\tau) + |\theta_0 - \phi_0| \geq 0$$

which is satisfied by

$$N(\tau) \leq \frac{1 - \sqrt{1 - \frac{4k}{A}|\theta_0 - \phi_0|}}{2\frac{k}{A}}.$$

We have to exclude the values

$$N(\tau) \geq \frac{1 + \sqrt{1 - \frac{4k}{A}|\theta_0 - \phi_0|}}{2\frac{k}{A}} \geq \frac{A}{k},$$

because $N(0) = |\theta_0 - \phi_0|$ and, since $|\theta_0 - \phi_0|$ is sufficiently little, we can't accept $N(\tau) \geq \frac{A}{k}$.
Since $1 - \sqrt{1-x} \leq \frac{x}{2}$ for each $x > 0$, we obtain:

$$N(\tau) \leq \frac{1 - \sqrt{1 - \frac{4k}{A}|\theta_0 - \phi_0|}}{2\frac{k}{A}} \leq 2|\theta_0 - \phi_0|.$$

So, we can say that

$$\sup_{t>0} e^{-2At}|\eta(t)| \leq |\theta_0 - \phi_0| \quad |\theta_0 - \phi_0| < \epsilon < \frac{A}{k}.$$

Hence the solution is global and:

$$|\eta(t)| \leq e^{2At}|\theta_0 - \phi_0|$$

if $|\theta_0 - \phi_0| < \epsilon < \frac{A}{k}$. Finally we conclude that θ and ϕ satisfy

$$\begin{cases} \frac{d\theta}{dt}(t) = \frac{\eta(t)+\xi(t)}{2} = \frac{\theta_0+\phi_0}{2} + \omega t + v(t) \\ \frac{d\phi}{dt}(t) = \frac{\xi(t)-\eta(t)}{2} = \frac{\theta_0+\phi_0}{2} + \omega t - v(t) \end{cases}$$

where $|v(t)| \leq e^{2At}|\theta_0 - \phi_0|$.

2.1.2 Case $A > 0$.

Now we analyze the behavior of system

$$\begin{cases} \dot{u} = 2g(u) \\ u(0) = \epsilon \end{cases} \tag{2.10}$$

when $g(u) \sim Au$ as $u \to 0$ and $A > 0$.
More precisely, we suppose that $g : \mathbb{R} \to \mathbb{R}$ is a C^1 function such that:

$$\begin{cases} g(0) = g(B) = 0, \\ g > 0 \text{ in } (0, B), \\ g'(0) = A > 0, \\ g'(B) < 0 \end{cases} \tag{2.11}$$

Lemma 2.3. If $g : [0, B] \to \mathbb{R}$ is a C^1 function satisfying (2.11), then for every $\epsilon \in (0, B)$ we have

$$\int_\epsilon^B \frac{du}{g(u)} = +\infty.$$

Proof. In a neighborhood of B we have

$$g(u) = g'(u)(u - B) + o(B - u)$$

so that the integral must diverge.

We are going to apply this lemma in order to prove that, under assumption (2.11), the solution of (2.10) is global and satisfies

$$\lim_{t\to\infty} u(t) = B.$$

First of all, if $\epsilon \in (0, B)$, the solution of (2.10) is global: indeed there are two stationary solutions $u = 0$ and $u = B$, and for $0 < \epsilon < B$ the solution u of (2.10) must lie between 0 and B, so it cannot blow up at any $t > 0$.
Next, by separating variables we have

$$\int_0^T \frac{\dot{u}(t)}{g(u(t))} dt = T \quad \forall T > 0,$$

i. e.

$$\int_\epsilon^{u(t)} \frac{d\tau}{g(\tau)} = T \quad \forall T > 0. \tag{2.12}$$

Introduce now a primitive G of $\frac{1}{g}$, such that In the assumptions made in the last theorem, let $G(s)$ a function such that: $G(\epsilon) = 0$; then, by lemma 2.3,

$$\lim_{T \to B^-} G(T) = \lim_{T \to B^-} \int_\epsilon^T \frac{du}{g(u)} = +\infty$$

and also

$$\lim_{T \to 0^+} G(T) = -\infty.$$

Moreover G is strictly increasing, hence there exists $G^{-1} : \mathbb{R} \to]0, B[$, such that

$$\lim_{s \to \infty} G^{-1}(s) = B, \quad \lim_{s \to -\infty} G^{-1}(s) = 0.$$

In particular

$$G(u(T)) = \int_\epsilon^{u(T)} \frac{d\tau}{g(\tau)} = T \quad \forall T > 0,$$

i. e.

$$u(T) = G^{-1}(T) \quad \forall T > 0$$

and finally

$$\lim_{T \to B^-} u(T) = \lim_{T \to B^-} G^{-1}(T) = +\infty.$$

Moreover we can say something on the behavior of $B - u(t)$.
We want to prove that there exists $a > 0$ such that

$$\lim_{t \to \infty} e^{at}(B - u(t)) = 0.$$

Let us consider again the Taylor development at B:

$$\frac{du}{dt} = g(u) = g(B) + g'(B)(u - B) + o(u - B).$$

We have, for $0 < B - u < \delta$,

$$\frac{du}{dt} \geq |g'(B)||B - u| - \epsilon(B - u) = k(B - u)$$

where $k = |g'(B)| - \epsilon > 0$.
We set $v = B - u$. Then

$$\frac{dv}{dt} = -\frac{du}{dt} \leq -kv$$

which implies

$$e^{kt}\frac{dv}{dt} + ke^{kt}v \leq 0.$$

We deduce that:

$$\frac{d}{dt}(e^{kt}v(t)) \leq 0$$

and we conclude

$$v(t) \leq v(0)e^{-kt}$$

that is what we wanted to prove.

2.2 Kuramoto Model

The Kuramoto model has been the focus of extensive research and provides a system that can model synchronisation and desynchronisation in groups of coupled oscillators. Various modifications to the standard Kuramoto model have been made in order to enable it to be used as a model for alternative, specific applications.
The Kuramoto model considers a system of globally coupled oscillators, defined using the following equation:

$$\frac{d\phi_i}{dt} = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\phi_j - \phi_i),$$

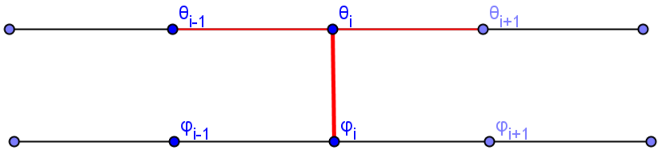
where ϕ_i is the phase of oscillator i, ω_i is the natural frequency of oscillator i, N is the total number of oscillators in the system and K is a constant referred to as the coupling constant.

The Kuramoto model and its corresponding analysis assumes the following to be true:

- All oscillators in the system are globally coupled.
- Individually, the oscillators are identical, except for possibly different natural frequencies ω_i.
- The phase response curve depends on the phase between two oscillators.
- The phase response curve has f a sinusoidal form.

Now we refer to the neuronal groups REM-ON as "neurons θ_i" and to the REM-OFF neurons as "ϕ_i" .
We use Kuramoto model to describe the interaction between neurons of the same group and between neurons of different groups. We suppose that their communication is of the following type:

We suppose, precisely, that the $i-th$ neuron $REM-ON$ with phase θ_i will be mostly influenced by its interaction with the $i+1-th$ and $i-1-th$ neurons REM-ON and with the $i-th$ neuron REM-OFF with phase ϕ_i.



So, from Kuramoto model, we deduce that the interaction between two neurons of different groups is described by following system:

$$\begin{cases} \frac{d\theta_i}{dt} = \omega + k_{i,1}[\sin(\theta_{i+1} - \theta_i) + \sin(\theta_{i-1} - \theta_i)] + k_{i,1}\sin(\phi_i - \theta_i) \\ \frac{d\phi_i}{dt} = \omega + k_{i,2}[\sin(\phi_{i+1} - \phi_i) + \sin(\phi_{i-1} - \phi_i)] + k_{i,2}\sin(\theta_i - \phi_i)] \end{cases}$$

where ω is the natural frequency of brain system neurons and $k_{i,j}$, $j = 1, 2$ are constants of interaction.
We remember that $\sin x = x + o(x^2)$ as $x \to 0$, so the system becomes:

$$\begin{cases} \frac{d\theta_i}{dt} \sim \omega + k_{i,1}[(\theta_{i+1} - \theta_i) + (\theta_{i-1} - \theta_i)] + k_{i,1}\sin(\phi_i - \theta_i) \\ \frac{d\phi_i}{dt} \sim \omega + k_{i,2}[(\phi_{i+1} - \phi_i) + (\phi_{i-1} - \phi_i)] + k_{i,2}\sin(\theta_i - \phi_i)] \end{cases} \tag{2.13}$$

Now, we recognize the discretization of the Laplacian operator $\Delta\theta \sim \theta_{i+1} - 2\theta_i + \theta_{i-1}$; passing to the limit, we obtain:

$$\begin{cases} \frac{\partial\theta}{dt} - \Delta\theta = \omega + A\sin(\theta - \phi) \\ \frac{\partial\phi}{dt} - \Delta\phi = \omega + A\sin(\phi - \theta). \end{cases} \tag{2.14}$$

where A is an appropriate constant. It is the diffusive model that we will study in the fourth chapter in its general form

$$\begin{cases} \frac{\partial\theta}{dt} - \Delta\theta = \omega + g(\theta - \phi) \\ \frac{\partial\phi}{dt} - \Delta\phi = \omega - g(\theta - \phi) \end{cases} \tag{2.15}$$

where g is a particular nonlinear function.

Chapter 3

Mathematical tools

In this chapter, we recall some mathematical concepts and theorems in order to study a more complex model linked with the previous analyzed.
We refer to $L^p(\Omega)$, $p \in (1, \infty)$ as the linear space of $p - th$ order integrable functions on Ω, which is a Banach space respect to the norm

$$||u||_{L^p(\Omega)} = \left(\int_\Omega |u(x)|^p dx \right)^{\frac{1}{p}}$$

By $C_0^\infty(\Omega)$ we denote the class of infinitely smooth functions in Ω with compact support.

3.1 Linear operators

In this section some results on linear operators are reported, most of them without proof. Let X, Y two normed spaces.

Definition 3.1. A map $A : X \to Y$ is a *linear operator* if $A(\alpha x + \beta z) = \alpha A(x) + \beta A(z)$,for all $x, z \in X$ and for all $\alpha, \beta \in \mathbb{R}$ (or $\alpha, \beta \in \mathbb{C}$).

We can write Ax instead of $A(x)$.

Definition 3.2. The *image* (or range) of A is the linear subspace of Y, $\mathcal{I}(A) = \{Ax : x \in X\}$. The *kernel* of A is the linear subspace of X, $\ker A = \{x \in X : Ax = 0\}$.

Definition 3.3. A linear operator $A : X \to Y$ is *bounded* if there exists $M \geq 0$ such that $\|Ax\|_Y \leq M\|x\|_X$, for every $x \in X$.

The set of bounded linear operator from X to Y is a vector space denoted by $\mathcal{L}(X, Y)$; if $X = Y$ we use $\mathcal{L}(X)$ instead of $\mathcal{L}(X, X)$.
The space $\mathcal{L}(X, Y)$ becomes a normed space with norm

$$\|A\|_{\mathcal{L}(X,Y)} = \sup_{\|u\|_X=1} \|Au\|_Y.$$

Theorem 3.4. Let X, Y normed spaces. If Y is a Banach space, then $\mathcal{L}(X, Y)$ is a Banach space.

Theorem 3.5. Let A a bijective linear operator in $\mathcal{L}(X, Y)$, then its inverse operator is in $\mathcal{L}(Y, X)$.

Let X be a Banach space and let $A : \mathcal{D}(\mathcal{A}) \to \mathcal{X}$ be a linear operator with domain $\mathcal{D}(\mathcal{A}) \subseteq \mathcal{X}$. Let I denote the identity operator on X. For any $\lambda \in \mathbb{C}$, let

$$A_\lambda = A - \lambda I;$$

λ is said to be a regular value if $R_\lambda(A)$, the inverse operator to A_λ:

1. exists;
2. is a bounded linear operator;
3. is defined on a dense subspace of X (and hence in all of X).

The resolvent set of A is the set of all regular values of A:

$$\rho(A) = \{\lambda \in \mathbb{C} | \lambda \text{ is a regular value of } A\}.$$

The spectrum of A is the set of all $\lambda \in \mathbb{C}$ for which the operator $A - \lambda I$ does not have a bounded inverse. In other words the spectrum is the complement of the resolvent set:

$$\sigma(A) = \mathbb{C} \setminus \rho(A).$$

If λ is an eigenvalue of A then the operator $A - \lambda I$ is not one-to-one, and therefore its inverse $(A - \lambda I)^{-1}$ is not defined. So the spectrum of an operator always contains all its eigenvalues, but is not limited to them.

Lemma 3.6. If $A \in \mathcal{L}(X)$ and $|\lambda| > \|A\|$, then $\lambda \in \rho(A)$ and $R_\lambda(A)$ can be expressed by a *Neumann* series:

$$R_\lambda(A) = -\frac{1}{\lambda} \sum_{n=0}^{\infty} \left(\frac{A}{\lambda}\right)^n.$$

The spectrum of a bounded operator A is always a closed, bounded and non-empty subset of the complex plane. The boundedness of the spectrum follows from the Neumann series expansion in λ; the spectrum $\sigma(A)$ is bounded by $||A||$. The bound $||A||$ on the spectrum can be refined somewhat. The convergence radius of Neumann series is

$$r = \limsup_{n\to\infty} \|A^n\|^{\frac{1}{n}} \leq \|A\|, \tag{3.1}$$

So $\sigma(A) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \leq r\}$ and we define the *spectral radius* as:

$$r(A) = \sup\{|\lambda| : \ \lambda \in \sigma(A)|\}.$$

It holds

$$\exists \lim_{n\to\infty} \|A^n\|^{1/n} = r(A).$$

Let $A \in \mathcal{L}(X)$. We can decompose $\sigma(A)$ as $\sigma(A) = \sigma_p(A) \cup \sigma_c(A) \cup \sigma_r(A)$, where these sets are disjoint and:

1. $\sigma_p(A) = \{\lambda \in \mathbb{C} : ker(A - \lambda I) \neq 0\}$ is the *point spectrum* and contains the eigenvalues of A;
2. $\sigma_c(A) = \{\lambda \in \mathbb{C} : (A - \lambda I)$ has a dense range but $(A - \lambda I)^{-1}$ is not bounded$\}$ is the *continuous spectrum*;
3. $\sigma_r(A) = \{\lambda \in \mathbb{C} : (A - \lambda I)$ has not dense range$\}$ is the *residual spectrum*.

3.2 Fourier transform in $L^2(\mathbb{R}^N)$

Let $f \in L^1(\mathbb{R}^N)$. The Fourier transform $\hat{f}$ of f is:

$$\hat{f}(\xi) = \int_{\mathbb{R}^N} e^{-i(x,\xi)} f(x) dm_N(x) \quad \xi \in \mathbb{R}^N \tag{3.2}$$

where (x, ξ) is the scalar product on $\mathbb{R}^N$. The operator $f \to \hat{f}$ is denoted by $\mathcal{F}$.

Proposition 3.7. The Fourier transform $\mathcal{F} : L^1(\mathbb{R}^N) \to L^\infty(\mathbb{R}^N)$ is a linear bounded operator; so:

$$\|\hat{f}\|_{L^\infty} \leq \|f\|_{L^1}.$$

It is also unitary.
Moreover for every $f \in L^1(\mathbb{R}^N)$, $\hat{f}$ is uniformly continuous on $\mathbb{R}^N$.

Remark 3.8. The Fourier transform $\hat{f}$ is C^1 and

$$D_j \hat{f}(\xi) = [\mathcal{F}(-ix_j f(x))](\xi). \tag{3.3}$$

Proposition 3.9. Let $f, g \in L^1(\mathbb{R}^N)$, we have:

$$\widehat{(f * g)}(\xi) = \hat{f}(\xi)\hat{g}(\xi) \quad \forall \xi \in \mathbb{R}^N.$$

Theorem 3.10 (Inverse of the Fourier transform)**.** If $f \in L^1(\mathbb{R}^N)$ is such that $\hat{f} \in L^1(\mathbb{R}^N)$, then

$$f = (2\pi)^{-N} \bar{\mathcal{F}}(\hat{f})$$

where $\bar{\mathcal{F}}(\hat{f}) = \int_{\mathbb{R}^N} e^{i(x,\xi)} \hat{f}(\xi) dm_N(\xi)$.

We are interested in the properties of Fourier transform in L^2. So we introduce the Schwartz space $S(\mathbb{R}^N)$ that is dense in $L^2(\mathbb{R}^N)$ and we see some properties of Fourier transform on this space.

Definition 3.11. The Schwartz space $S(\mathbb{R}^N)$ is defined by:

$$S(\mathbb{R}^N) = \{\phi \in C^\infty(\mathbb{R}^N) : x \to x^\alpha D^\beta \phi(x) \in L^\infty(\mathbb{R}^N) \ \forall \alpha, \beta \in \mathbb{N}^N\}$$

where $x^\alpha = x_1^{\alpha_1} \cdot ... \cdot x_N^{\alpha_N}$ and $D^\beta = D_1^{\beta_1} ... D_N^{\beta_N}$.

We define $|\beta| = \beta_1 + ... + \beta_N$.

Properties of $S(\mathbb{R}^N)$.

- It has a seminorm:

$$\mathcal{N}_p(\phi) = \sum_{|\alpha| \le p, |\beta| \le p} ||x^\alpha D^\beta \phi(x)||_{L^\infty} < \infty \quad \forall p \in \mathbb{Z};$$

- there exists a constant C_p such that

$$\sum_{|\alpha| \le p, |\beta| \le p} ||x^\alpha D^\beta \phi(x)||_{L^1} \le C_p \mathcal{N}_{p+N+1}(\phi) \quad \forall \phi \in S(\mathbb{R}^N);$$

- $\mathcal{F}$ maps $S(\mathbb{R}^N)$ in $S(\mathbb{R}^N)$ and there exists a constant K_p such that

$$\mathcal{N}_p(\hat{\phi}) \le K_p \mathcal{N}_{p+N+1}(\phi);$$

- $\mathcal{F}$ is an isomorphism and we have the inverse formula:

$$\phi(x) = (2\pi)^{-N} \int_{\mathbb{R}^N} \hat{\phi}(\xi) e^{i(x,\xi)} dm_N(\xi) = (2\pi)^{-N} \hat{\phi}(-x).$$

At the basis of these facts there are some properties of Fourier transform.

Proposition 3.12. Let $\phi \in S(\mathbb{R}^N)$; then

$$\widehat{D^\alpha \phi}(\xi) = i^{|\alpha|} \xi^\alpha \hat{\phi}(\xi) \quad \forall \xi \in \mathbb{R}^N$$

and

$$D^\alpha \hat{\phi}(\xi) = (-i)^{|\alpha|} \mathcal{F}(x^\alpha \phi(x))(\xi) \quad \forall \xi \in \mathbb{R}^N.$$

Proof. We obtain the first property integrating by parts $|\alpha|$ times in the integral that defines $\widehat{D^\alpha \phi}$ and this is possible because

$$\lim_{|x|\to\infty} |x^\alpha D^\beta \phi(x)| = 0 \quad \forall \alpha, \beta \in \mathbb{N}^N, \forall \phi \in S(\mathbb{R}^N).$$

The second property is an easy verification.

We have seen that $\mathcal{F}$ is an isomorphism from $(S(\mathbb{R}^N), \|\cdot\|_{L^2(\mathbb{R}^N)}$ into itself and, since $S(\mathbb{R}^N)$ is dense in $L^2(\mathbb{R}^N)$, it can be extended in a unique way to an isomorphism of $L^2(\mathbb{R}^N)$ into itself.

Theorem 3.13 (di Plancherel)**.** For all $f, g \in S(\mathbb{R}^N)$ we have the Parseval formula:

$$(\hat{f}, \hat{g})_{L^2(\mathbb{R}^N)} = (2\pi)^N (f, g)_{L^2(\mathbb{R}^N)}.$$

In particular:

$$\|\hat{f}\|_{L^2(\mathbb{R}^N)} = (2\pi)^{\frac{N}{2}} \|f\|_{L^2(\mathbb{R}^N)} \quad \forall f \in S(\mathbb{R}^N)$$

Corollary 3.14. The Fourier transform extends to a unique isomorphism of $L^2(\mathbb{R}^N)$ into itself. In particular we have

$$(\hat{f}, \hat{g})_{L^2(\mathbb{R}^N)} = (2\pi)^N (f, g)_{L^2(\mathbb{R}^N)} \quad \forall f, g \in L^2(\mathbb{R}^N)$$

and

$$f(x) = (2\pi)^{-N} \hat{\hat{f}}(-x) \text{q. o. in} \mathbb{R}^N \quad \forall f \in L^2(\mathbb{R}^N).$$

3.3 Sobolev spaces

Definition 3.15. Let $\Omega \subset \mathbb{R}^N$ be an open set and fix $m \in \mathbb{N}$,$p \in [1, \infty)$. Set

$$E^{m,p}(\Omega) = \left\{ u \in C^m(\Omega) : \left[\sum_{|\alpha| \le m} \int_\Omega |D^\alpha u|^p dx \right]^{\frac{1}{p}} = \|u\|_{m,p} < \infty \right\}$$

Remark 3.16. $E^{m,p}(\Omega)$ is a normed space with norm $\|\cdot\|_{m,p}$.

We remember that a completion of a metric space (X, d) is a pair consisting of a complete metric space (X^*, d^*) and an isometry $\phi : X \to X^*$ such that $\phi(X)$ is dense in X^*. Every metric space has a completion.

Definition 3.17. The Sobolev space $H^{m,p}(\Omega)$ is the completion of $E^{m,p}(\Omega)$ with respect to the norm $\|\cdot\|_{m,p}$.

Definition 3.18. Let $u \in L^p(\Omega)$. The function u is said to have a *strong derivative* in $L^p(\Omega)$ up to the order m if there exists $(u_k) \subset E^{m,p}(\Omega)$ such that $u_k \to u$ in $L^p(\Omega)$ and $(D^\alpha u_k)$ is a Cauchy sequence in $L^p(\Omega)$ for $1 \leq |\alpha| \leq m$. The functions $u^\alpha = \lim_{k\to\infty}(D^\alpha u_k)$ are the strong derivatives of u.

Remark 3.19. Strong derivatives are indipendent from the approximating sequences.

Proposition 3.20. $u \in H^{m,p}(\Omega) \Leftrightarrow u$ has strong derivatives in $L^p(\Omega)$ up to the order m.

Definition 3.21. Let $u \in L^1(\Omega)$ and let $\alpha \in \mathbb{N}_0^d$ a multi-index. The function u is said to have a *weak derivative* $v = D^\alpha u$ if there exists a function $v \in L^1(\Omega)$ such that

$$\int_\Omega u D^\alpha \phi dx = (-1)^{|\alpha|} \int_\Omega v \phi dx, \qquad \forall \phi \in C_0^\infty(\Omega)$$

The notion "weak derivative" suggests that it is a generalization of the classical concept of differentiability and that there are functions which are weakly differentiable, but not differentiable in the classical sense. We give an example.

Exemple 3.22. Let $d = 1$, and $\Omega = (-1, +1)$. The function $u(x) = |x|$, $x \in \Omega$ is not differentiable in the classical sense. However, it admits a weak derivative $D^1 u$ given by

$$D^1 u = \begin{cases} -1, & x < 0 \\ +1, & x > 0 \end{cases}$$

The proof is easy.

Definition 3.23. The Sobolev Space $W^{m,p}(\Omega)$ is the set of functions $u \in L^p(\Omega)$ with weak derivatives in $L^p(\Omega)$ up to the order m

Proposition 3.24. $W^{m,p}(\Omega)$ is a Banach space with the norm $||\cdot||_{m,p}$.

Proof. Let $(u_n) \subset W^{m,p}(\Omega)$ a Cauchy sequence with respect to $||\cdot||_{m,p}$. For each $\alpha \in \mathbb{N}^N$, with $|\alpha| \leq m$, there exist u and v_α in $\in L^p(\Omega)$ such that $u_n \to u$ and $D^\alpha u_n \to v_\alpha$ in $L^p(\Omega)$
For all n and for all $\phi \in C_0^\infty(\Omega)$

$$\int_\Omega u_n D^\alpha \phi dx = (-1)^\alpha \int_\Omega D^\alpha u_n \phi dx \;\; \forall \alpha \in \mathbb{N}^N, |\alpha| \leq m$$

and, for $n \to \infty$

$$\int_\Omega u D^\alpha \phi dx = (-1)^\alpha \int_\Omega v_\alpha \phi dx \;\; \forall \phi \in C_0^\infty(\Omega), |\alpha| \leq m$$

So, v_α is the $\alpha - th$ weak derivative of u and it is easy to see that $\|u_n - u\|_{m,p} \to 0$.

Remark 3.25. Weak and strong derivatives are unique.

Proof. If u_α and v_α are two $\alpha - th$ weak derivatives of u, then $\int_\Omega (u_\alpha - v_\alpha)\phi dx = 0$ forall $\phi \in C_0^\infty(\Omega)$. Let q the conjugate exponent to p, and let $\phi_n \subset C_0^\infty(\Omega)$ with $\phi_n \to \operatorname{sign}(u_\alpha - v_\alpha)|u_\alpha - v_\alpha|^{p-1}$ in $L^q(\Omega)$; for $n \to \infty$ we obtain $\int_\Omega |u_\alpha - v_\alpha|^p = 0$, so $u_\alpha = v_\alpha v$.

3.3.1 Characterization of Sobolev spaces

Usually $H^{m,p}(\Omega) \subset W^{m,p}(\Omega)$. We say that Ω is a bounded set with locally Lipschitz boundary if for every $x_0 \in \partial\Omega$ exists a neighborhood U of x_0 such that $U \cap \partial\Omega$ is graph of a Lipschitz function of $N-1$ variables. In this case, $H^{m,p}(\Omega) = W^{m,p}(\Omega)$. In this section we assume to be in this situation.

Proposition 3.26. Let $u \in W^{m,p}(\Omega)$ and $v \in C^m(\Omega) \cap W^{m,\infty}(\Omega)$ (i. e. $D^\alpha v$ are bounded in Ω for $|\alpha| \leq m$), then $u \cdot v$ is in $W^{m,p}(\Omega)$ and

$$D^\alpha(uv) = \sum_{|\beta| \leq |\alpha|} \binom{\alpha}{\beta} D^{\alpha-\beta} u D^\beta v$$

Definition 3.27. $W_0^{m,p}(\Omega) = \overline{C_0^\infty(\Omega)}$ in $\|\cdot\|_{m,p}$. In particular $W_0^{m,p}(\Omega)$ is complete.

Remark 3.28. If $\Omega = \mathbb{R}^N$ we have $W_0^{m,p}(\mathbb{R}^N) = W^{m,p}(\mathbb{R}^N)$, i. e. $C_0^\infty(\mathbb{R}^N)$ is dense in $W^{m,p}(\mathbb{R}^N)$.

Now we see some fundamental results.

Theorem 3.29 (Poincarè Inequality)**.** Let $\Omega \subset \mathbb{R}^N$ a bounded set. There exists $c = c(\Omega, m, p)$ such that

$$\sum_{|\alpha|=k} \|D^\alpha u\|_{L^p(\Omega)} \leq c \sum_{|\alpha|=m} \|D^\alpha u\|_{L^p(\Omega)} \quad \forall u \in W_0^{m,p}(\mathbb{R}^N), 0 \leq k \leq m$$

Theorem 3.30 (Sobolev theorem)**.** Let $\Omega \subset \mathbb{R}^N$ a bounded locally Lipschitz open, let $p \in [1, \infty[$. Then

- $p < N \Rightarrow W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$, $p^* = \frac{Np}{N-p}$
- $p = N \Rightarrow W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$, $\forall q < \infty$
- $p > N \Rightarrow W^{1,p}(\Omega) \hookrightarrow C^{0,\alpha}(\bar{\Omega})$, $\alpha = 1 - \frac{N}{p}$.

Theorem 3.31 (Rellich)**.** Moreover (see section 3.5 below)

- $p < N \Rightarrow Id : W^{1,p}(\Omega) \to L^q(\Omega)$, $q < p^*$ is compact;
- $p = N \Rightarrow Id : W^{1,p}(\Omega) \to L^q(\Omega)$, $q < \infty$ is compact;

- $p > N \Rightarrow Id : W^{1,p}(\Omega) \to C^{0,\alpha}(\bar{\Omega})$, $\alpha = 1 - \frac{N}{p}$ is compact.

Remark 3.32. Let $\Omega =]0, 2\pi[$ and consider S^1. We have $L^2(0, 2\pi) \simeq L^2(S^1)$ and $H^{1,2}(S^1) \subset H^{1,2}(0, 2\pi)$ (indeed $N = 1$, $p = 2 > 1$, so if $u \in H^{1,2}(S^1)$, then u continuous and periodic, whereas the elements of $H^{1,2}(0, 2\pi)$ are continuous but not necessarily periodic.)
If $u \in H^{1,2}(S^1)$, then $u' \in L^2(S^1)$. Let be u_k and u'_k the Fourier coefficients of u and u' in the system $\{e^{ikt}\}_{k\in\mathbb{Z}}$:

$$u_k = \frac{1}{2\pi} \int_0^{2\pi} u(t) e^{-ikt} dt, \qquad u'_k = \frac{1}{2\pi} \int_0^{2\pi} u'(t) e^{-ikt} dt,$$

then integrating by parts,

$$u'_k = iku_k \quad \forall k \neq 0, \;\; u'_0 = 0.$$

So, by Bessel equality ($\int_0^{2\pi} |u|^2 dt = 2\pi \sum_{k\in\mathbb{Z}} |u_k|^2 \; \forall u \in L^2(S^1)$), we obtain

$$u \in H^{1,2}(S^1) \Leftrightarrow \sum_{k\in\mathbb{Z}} k^2 |u_k|^2 < \infty$$

and

$$u \in L^2(S^1) \Leftrightarrow \sum_{k\in\mathbb{Z}} |u_k|^2 < \infty.$$

These properties allow to define **Sobolev fractional spaces** as follows:

$$H^{s,2}(S^1) = \{u \in L^2(S^1) : \sum_{k\in\mathbb{Z}} (1 + |k|^2)^s |u_k|^2 < \infty\}, \quad s \in]0, 1].$$

We observe that:

- if $0 < s < s' \leq 1$ then $H^{1,2}(S^1) \subset H^{s',2}(S^1) \subset H^{s,2}(S^1) \subset L^2(S^1)$
- if $s > \frac{1}{2}$ the functions of $H^{s,2}(S^1)$ are continuous because their Fourier series converges totally: indeed

$$\begin{aligned} \sum_{|k|\geq n}^{\infty} |u_k e^{ikt}| &= \sum_{|k|\geq n} \frac{|u_k| k^s}{k^s} \\ &\leq \left(\sum_{|k|\geq n} k^{2s} |u_k|^2 \right)^{\frac{1}{2}} \left(\sum_{|k|\geq n} \frac{1}{k^{2s}} \right)^{\frac{1}{2}} \\ &\leq \|u\|_{H^{s,2}} \left(\sum_{|k|\geq n} \frac{1}{k^{2s}} \right)^{\frac{1}{2}} \end{aligned} \tag{3.4}$$

  goes to zero as $n \to \infty$, since $2s > 1$.

For the case $p = 2$, we use the notation $H^s(\Omega)$ instead of $H^{s,2}(\Omega)$ for $s \in \mathbb{R}$. So in the last observation we have $H^{s,2}(S^1) = H^s(S^1)$.
In the case of $\Omega = \mathbb{R}^N$ we can introduce the **Sobolev spaces** $H^s(\mathbb{R}^N)$, also in another way, through the distribution theory.
Briefly, we see the principal definitions to define the space $H^s(\mathbb{R}^N)$.
Firstly we note that, if $\Omega \subset \mathbb{R}^N$ and $u \in L^1_{loc}(\Omega)$, the expression

$$(f, \phi) = \int_\Omega f(x)\phi(x) \quad \phi \in C_0^\infty(\Omega)$$

is meaningful.

Definition 3.33. Let $s \in \mathbb{R}$, let $\Omega \subset \mathbb{R}^N$ an open set.
A linear form u is a **distribution** on Ω if there exist $p \in \mathbb{Z}$ and a constant c such that, for all compact $K \subset \Omega$:

$$|(u, \phi)| \leq c \sum_{|\alpha| \leq p} \|D^\alpha \phi\|_{L^\infty(K)} \quad \forall \phi \in C_0^\infty(\Omega).$$

The vectorial space of the distributions on Ω is denoted by $\mathcal{D}'(\Omega)$

Definition 3.34. Let $u \in \mathcal{D}'(\mathbb{R}^N)$. We say that $u \in S'(\mathbb{R}^N)$ if there exist $p \in \mathbb{N}$ and $C \geq 0$ such that

$$|(u, \phi)| \leq C\mathcal{N}_p(\phi) \quad \forall \phi \in C_0^\infty(\mathbb{R}^N).$$

Properties of $S'(\mathbb{R}^N)$

- $\{u_j\} \subset S'(\mathbb{R}^N)$, $u_j \to u$ in $S'(\mathbb{R}^N)$ if we have

$$\lim_{j\to\infty} (u_j, \phi) = (u, \phi) \quad \forall \phi \in S(\mathbb{R}^N);$$

- If $u \in S'(\mathbb{R}^N)$ then all its derivatives are in $S'(\mathbb{R}^N)$;
- If $u_j \to u$ in $S'(\mathbb{R}^N)$ then $D^\alpha u_j \to D^\alpha u$ in $S'(\mathbb{R}^N)$;
- If $u \in S'(\mathbb{R}^N)$ then its Fourier transform $\hat{u}$ is defined by

$$(\hat{u}, \phi) = (u, \hat{\phi}) \quad \forall \phi \in S(\mathbb{R}^N).$$

Finally we can introduce the spaces $H^s(\mathbb{R}^N)$.

Definition 3.35. Let $s \in \mathbb{R}$ and $u \in \mathcal{D}'(\mathbb{R}^N)$, then $u \in H^s(\mathbb{R}^N)$ if

1. $u \in S'(\mathbb{R}^N)$
2. $\hat{u} \in L^1_{loc}(\mathbb{R}^N)$

3.
$$\int_{\mathbb{R}^N} (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 d\xi < \infty.$$

Remark 3.36. It is equivalent to say that $u \in H^s(\mathbb{R}^N)$ if $(1+|\xi|^2)^{\frac{s}{2}}|\hat{u}(\xi)| \in L^2(\mathbb{R}^N)$.

Proposition 3.37. The spaces $H^s(\mathbb{R}^N)$ are Hilbert spaces with the scalar product:
$$(u, v) = \int_{\mathbb{R}^N} (1 + |\xi|^2)^s \overline{\hat{v}(\xi)} \hat{u}(\xi) d\xi$$
and this scalar product induces the norm:
$$\|u\|_{H^s} = \|(1 + |\xi|^2)^{\frac{s}{2}} \hat{u}\|_{L^2}$$

In the proof of this proposition it is important to note that the map $\psi : H^s \to L^2$ such that
$$\psi(u) = (1 + |\xi|^2)^{\frac{s}{2}} \hat{u(\xi)}$$
is a surjective isometry. So $H^s(\mathbb{R}^N)$ is complete because $L^2(\mathbb{R}^N)$ is complete too.

Proposition 3.38. If $s \geq s'$ then $H^s(\mathbb{R}^N) \subset H^{s'}(\mathbb{R}^N)$

Proof. We know that $(1+|\xi|^2)^{\frac{s}{2}} \hat{u(\xi)} \in L^2(\mathbb{R}^N)$. Now we want to prove that
$$(1 + |\xi|^2)^{s'} |\hat{u}(\xi)|^2 \in L^2(\mathbb{R}^N).$$
But this is true. In fact:
$$(1 + |\xi|^2)^{s'} |\hat{u}(\xi)|^2 = [(1 + |\xi|^2)^{\frac{s'}{2}} |\hat{u}(\xi)|][(1 + |\xi|^2)^{s' - \frac{s}{2}} |\hat{u}(\xi)|]$$
The two terms are in $L^2(\mathbb{R}^N)$, so by Holder inequality we conclude.

Theorem 3.39. The spaces $H^s(\mathbb{R}^N)$ is an algebra for $s > \frac{n}{2}$.

In the case $N = 1$, the case that we will use in the next chapter, we have similarly:

Proposition 3.40. If $v, w \in H^s(S^1)$ with $s > \frac{1}{2}$, then $v \cdot w \in H^s(S^1)$.

3.4 Unbounded closed operators.

We can generalize the results of the first section on linear operators to unbounded operators, on a Hilbert space, for simplicity.

3.4.1 Unbounded operators

We consider a complex Hilbert space H. The scalar product will be denoted by (u, v). We take the convention that the scalar product is antilinear with respect to the second argument.
We recall a well known result for Hilbert spaces.

Theorem 3.41 (Riesz's Theorem)**.** Let $u \to F(u)$ a linear continuous form on H. Then there exists a unique $w \in H$ such that

$$F(u) = (u, w) \quad \forall u \in H$$

There is a similar version with antilinear maps.

Theorem 3.42. Let $u \to F(u)$ an antilinear continuous form on H. Then there exists a unique $w \in H$ such that

$$F(u) = (w, u) \quad \forall u \in H$$

If a linear operator $T : u \to Tu$, is defined on a subspace H_0 of H, then H_0 is denoted by $D(T)$ and is called the domain of T.
T is bounded if it is continuous from $D(T)$ (with the topology induced by the topology of H) into H. When $D(T) = H$, we recover the notion of linear continuous operators on H.
When $D(T)$ is not equal to H, we shall always assume that:

$$D(T) \text{ is dense in } H \tag{3.5}$$

Note that, if T is bounded, then it admits a unique continuous extension to H. In this case the generalized notion is not interesting. We are mainly interested in extensions of this theory and will consider unbounded operators. The point is to find a natural notion replacing this notion of boundedness. This is the object of the next definition.

Definition 3.43. The operator T is called **closed** if the graph $G(T)$ of T is closed in $H \times H$.

We recall that

$$G(T) = \{(x, y) \in H \times H,\ x \in D(T),\ y = Tx\} \tag{3.6}$$

Equivalently, we can say:

Definition 3.44 (Closed operator)**.** Let T be an operator on H with (dense) domain $D(T)$. We say that T is closed if the conditions

- $u_n \in D(T)$
- $u_n \to u \in H$

- $Tu_n \to v \in H$

imply

- $u \in D(T)$
- $v = Tu$.

Theorem 3.45 (Closed graph theorem)**.** If X, Y are Hilbert spaces and T is a closed operator, $T : X \to Y$, then $T \in \mathcal{L}(X, Y)$.

3.4.2 Adjoints

When we have an operator T in $\mathcal{L}(H)$, it is easy to define the Hilbertian adjoint T^* by the identity:

$$(T^*u, v) = (u, Tv) \quad \forall u \in H, \forall v \in H. \tag{3.7}$$

The map $v \to (u, Tv)$ defines a continuous antilinear map on H and can be expressed, using Riesz's Theorem, by the scalar product by an element which is called T^*u. The linearity and the continuity of T^* is then easily proved using (3.7). Let us now give the definition of the adjoint of an unbounded operator.

Definition 3.46. If T is an unbounded operator on H whose domain $D(T)$ is dense in H, we first define the domain of T^* by

$$D(T^*) = \{u \in H : \exists k_u \geq 0 : \ |(u, Tv)| \leq k_u \|u\|_H, \ \forall v \in D(T)\}. \tag{3.8}$$

The map $v \mapsto (u, Tv)$ is then extensible to an antilinear continuous form on H.
Using Riesz' Theorem, there exists $f \in H$ such that

$$(f, v) = (u, Tv) \quad \forall u \in D(T^*), \forall v \in D(T).$$

The uniqueness of f is a consequence of the density of $D(T)$ in H and we can then define T^*u by

$$T^*u = f.$$

Proposition 3.47. T^* is a closed operator.

Proof. Let (v_n) be a sequence in $D(T^*)$ such that $v_n \to v$ in H and $T^*v_n \to w^*$ in H for some pair (v, w^*). We would like to show that (v, w^*) belongs to the graph of T^*.
For all $u \in D(T)$, we have:

$$(Tu, v) = \lim_{n \to +\infty} (Tu, v_n) = \lim_{n \to +\infty} (u, T^*v_n) = (u, w^*) \tag{3.9}$$

Coming back to the definition of $D(T^*)$ we get from (3.9) that $v \in D(T^*)$ and $T^*v = w^*$.
This means that (v, w^*) belongs to the graph of T^*.

Exemple 3.48. We consider $T_0 = -\Delta$ with $D(T_0) = C_0^\infty(\mathbb{R}^m)$. This operator is not closed. For this, it is enough to consider some u in $H^2(\mathbb{R}^m)$ and not in $C_0^\infty(\mathbb{R}^m)$ and to consider a sequence $u_n \in C_0^\infty(\mathbb{R}^m)$ such that $u_n \to u$ in $H^2(\mathbb{R}^m)$. The sequence $(u_n, -\Delta u_n)$ is contained in $G(T_0)$ and converges in $L^2(\mathbb{R}^m) \times L^2(\mathbb{R}^m)$ to $(u, -\Delta u)$ which does not belong to $G(T_0)$.
Now, we denote $T_1 = -\Delta$ with $D(T_1) = H^2(\mathbb{R}^m)$. Let us show that:

$$T_0^* = T_1.$$

We have to prove that:

$$\begin{aligned} H^2(\mathbb{R}^m) &= D(T_0^*) \\ &= \{u \in L^2(\mathbb{R}^m) : |(u, T_0 v)_{L^2(\mathbb{R}^m)}| \le C_u \|v\|_{L^2(\mathbb{R}^m)} \forall v \in C_0^\infty(\mathbb{R}^m)\} \end{aligned}$$

($\subseteq$).
If $u \in H^2(\mathbb{R}^m)$, then

$$\begin{aligned} |(u, T_0 v)_{L^2(\mathbb{R}^m)}| &= \left| \int_{\mathbb{R}^m} u(x)(-\Delta v(x))dx \right| = \left| \int_{\mathbb{R}^m} (-\Delta u(x))v(x)dx \right| \\ &\le \|\Delta u\|_{L^2(\mathbb{R}^m} \|v\|_{L^2(\mathbb{R}^m} \quad \forall v \in C_0^\infty(\mathbb{R}^m)\} \end{aligned}$$

and we deduce that $u \in D(T_0^*)$, with $C_k = \|\Delta u\|_{L^2(\mathbb{R}^N)}$.
($\supseteq$).
If $u \in L^2(\mathbb{R}^m)$ is such that there exists C_u that satisfies:

$$\left| \int_{\mathbb{R}^m} u(x)(-\Delta v(x))dx \right| \le C_u \|v\|_{L^2(\mathbb{R}^m)} \quad \forall v \in C_0^\infty(\mathbb{R}^m)\},$$

then $-\Delta u \in L^2(\mathbb{R}^m)$ and

$$|(-\Delta u, v)| = |(u, -\Delta v)| \le C_u \|v\|_{L^2(\mathbb{R}^N)} \quad \forall v \in C_0^\infty(\mathbb{R}^m)\} \tag{3.10}$$

Hence, by Parseval equality, $|\xi|^2 \hat{u}(\xi) \in L^2(\mathbb{R}^m)$ which implies $(1+|\xi|^2)\hat{u}(\xi) \in L^2(\mathbb{R}^m)$, i. e. $u \in H^2(\mathbb{R}^m)$.
By proposition (3.47), we conclude that T_1 is a closed operator.

3.4.3 Symmetric and selfadjoint operators

Definition 3.49. We shall say that $T : D(T) \subseteq H \to H$ is **symmetric** if it satisfies

$$(Tu, v) = (u, Tv) \quad \forall u, v \in D(T).$$

Definition 3.50. We shall say that T is **selfadjoint** if $T^* = T$, i. e.

$$D(T) = D(T^*) \text{ and } Tu = T^*u, \quad \forall u \in D(T).$$

Proposition 3.51. A selfadjoint operator is closed.

This is immediate because T^* is closed.

Remark 3.52. If T is selfadjoint then $T + \lambda I$ is selfadjoint for any real λ.

3.5 Compact operators

Definition 3.53. Let X and Y be normed spaces. A linear operator $T : X \to Y$ is **compact** if it is continuous and transforms bounded sets of X into relatively compact sets of Y.

Now we see some important results about compact operators.

Proposition 3.54. Let X be a normed space and Y a Banach space. If $\{T_n\}_{n \in N^+}$ is a sequence of compact operators from X to Y such that $T_n \to T \in \mathcal{L}(X, Y)$, then T is a compact operator.

Proposition 3.55. If X is a normed space and $T \in \mathcal{L}(X)$ a injective compact operator, then

$$0 \in \rho(T) \Leftrightarrow \dim(X) < \infty$$

Corollary 3.56. If X is a normed space, $\dim(X) < \infty$ and $T \in \mathcal{L}(X)$ a injective compact operator, then $0 \in \sigma(T)$

Proposition 3.57. Let X and Y be normed spaces and $T \in \mathcal{L}(X, Y)$. If T is a compact operator, then T^* is compact too.
If T^* is a compact operators and Y is a Banach space, then T is compact.

In a Hilbert space, we have more interesting results.

Proposition 3.58. If H is a Hilbert space and $T \in \mathcal{L}(H)$, then

$$T \text{ is compact } \Leftrightarrow \forall \{x_n\} \subseteq H, \text{ with } x_n \rightharpoonup x \in H, \text{ we have } Tx_n \to Tx \in H$$

Proposition 3.59. If H is a Hilbert space and $T \in \mathcal{L}(H)$ is a selfadjoint operator, then:

- all eigenvalues of T are real;
- different eigenvalues relative to different eigenvectors are mutually orthogonal.

Theorem 3.60. Let H be a Hilbert space and let $T \in \mathcal{L}(H)$. T is a compact operator if and only if there exists a sequence of operators $T_n \subseteq \mathcal{L}(H)$, such that $\dim \mathcal{I}(T_n) < \infty$ for all $n \in \mathbb{N}$ and $T_n \to T$ in $\mathcal{L}(H)$.

Definition 3.61. A linear operator $T : H \to H$ is **positive** if $(Tu, u) \geq 0$ for all $u \in H$.

Theorem 3.62. Let H be a Hilbert space and let $T : H \to H$ be a linear operator which is compact, selfadjoint and positive, with $\mathcal{I}(T)$ dense in H. Then:

- the eigenvalues of T are real and positive;
- they are at most a countable infinity $\{\mu_k\}_{k\in\mathbb{N}^+}$;
- all eigenvalues have a finite multiplicity;
- $\mu_k \geq \mu_{k+1}$ and decrease to zero as $k \to \infty$;
- eigenvectors relative to different eigenvalues are orthogonal.

3.6 Semigroups of linear operators

Linear equations of mathematical physics can often be written in the abstract form

$$\begin{cases} u'(t) = Au(t), & t \in [0,T] \\ u(0) = x \end{cases} \tag{3.11}$$

where A is a linear, usually unbounded, operator defined on a linear subspace $D(A)$, the domain of A, of a Banach space E. typically a space of functions.

Exemple 3.63. Let D be an open domain in R^N with topological boundary ∂D. We consider the heat equation on $D \times [0,T]$

$$\begin{cases} \frac{\partial u}{dt}(t,\xi) = \Delta u(t,\xi), & t \in [0,T],\ \xi \in D \\ u(t,\xi) = 0, & t \in [0,T],\ \xi \in \partial D \\ u(0,\xi) = u_0(\xi), & \xi \in D. \end{cases} \tag{3.12}$$

For initial values $x = u_0 \in L^p(D)$ with $1 \leq p < \infty$, this problem can be rewritten in the abstract form (3.11) by taking $X = L^p(D)$ and defining A by

$$\begin{cases} D(A) = \{f \in W^{2,p}(D) : f = 0 \text{ su } \partial D\} = W^{2,p}(D) \cap W_0^{1,p}(D) \\ Af = \Delta f, \quad \forall f \in D(A). \end{cases}$$

The idea is now that instead of looking for a solution $u : [0,T] \times D \to \mathbb{R}$ of (3.12) one looks for a solution $u : [0,T] \to L^p(D)$ of (3.11). To get an idea how this may be done we first take a look at the much simpler case where $X = R^N$ and $A : D(A) = X \to X$ is represented by a $(N \times N)$-matrix. In that case, the unique solution of (3.12) is given by

$$u(t) = e^{tA}u_0, \quad t \in [0,T],$$

where $e^{tA} = \sum_{n=0}^{\infty} \frac{t^n A^n}{n!}$. The matrices e^{tA} may be thought of as "solution operators" mapping the initial value u_0 to the solution $e^{tA}u_0$ at time t. Clearly, $e^{0A} = I, e^{tA}e^{sA} = e^{(t+s)A}$, and $t \to e^{tA}$ is continuous. We generalise these properties to infinite dimensions as follows.

Let X be a real or complex Banach space.

Definition 3.64. A family $S = S(t)_{t>0}$ of bounded linear operators acting on a Banach space X is called a C^0-semigroup if the following three properties are satisfied:

1. $S(0) = I$;
2. $S(t)S(s) = S(t+s)$ for all $t, s \geq 0$;
3. $\lim_{t\downarrow 0} ||S(t)x - x|| = 0$ for all $x \in X$.

Remark 3.65. In general a family $S = S(t)_{t>0}$ of linear operators acting on a Banach space X is called a **semigroup** if satisfies the first two properties of the precedent definition.

The infinitesimal generator, or briefly the generator, of S is the linear operator A with domain $D(A)$ defined by

$$D(A) = \left\{ x \in X : \exists \lim_{t\downarrow 0} \frac{1}{t}(S(t)x - x) \right\}$$

and

$$Ax = \lim_{t\downarrow 0} \frac{1}{t}(S(t)x - x), \;\; x \in D(A).$$

Remark 3.66. If A generates the C^0- semigroup $(S(t))_{t>0}$, then $A - \mu$ generates the C^0-semigroup $(e^{-\mu t}S(t))_{t>0}$.

Proposition 3.67. Let S be a C^0-semigroup on X. There exist constants $M > 1$ and $\omega \in \mathbb{R}$ such that $||S(t)||_{\mathcal{L}(X)} \leq Me^{\omega t}$ for all $t \geq 0$.

Proposition 3.68. Let S be a C^0- semigroup on X with generator A.

- For all $x \in X$ the orbit $t \to S(t)x$ is continuous for $t \geq 0$.
- For all $x \in D(A)$ and $t \geq 0$, $S(t)x \in D(A)$ and $AS(t)x = S(t)Ax$.
- For all $x \in X$, $\int_0^t S(s)xds \in D(A)$ and

$$A \int_0^t S(s)xds = S(t)x - x.$$

  If $x \in D(A)$, than both sides are equal to $\int_0^t S(s)Axds$.

- The generator A is a closed and densely defined operator.

- For all $x \in D(A)$ and $t \geq 0$, the orbit $t \to S(t)x$ is continuously differentiable and

$$\frac{d}{dt}S(t)x = AS(t)x = S(t)Ax.$$

Definition 3.69. A classical solution of (3.11) is a continuous function $u : [0,T] \to X$ which belongs to $C^1((0,T];X) \cap C((0,T], D(A))$ and satisfies $u(0) = x$ and $u'(t) = Au(t)$ for all $t \in (0,T]$.

Corollary 3.70. For initial values $x \in D(A)$ the problem (3.11) has a unique classical solution, which is given by $u(t) = S(t)x$ and, by (3.67), we have

$$\|u(t)\|_X \leq Me^{\omega t}\|u_0\|_X.$$

Proof. The last proposition proves that $t \to u(t) = S(t)x$ is a classical solution. Suppose that $t \to v(t)$ is another classical solution. It is easy to check that the function $s \to S(t-s)v(s)$ is continuous on $[0,t]$ and continuously differentiable on $(0,t)$ with derivative

$$\frac{d}{dt}S(t-s)v(s) = -AS(t-s)v(s) + S(t-s)v'(s) = 0$$

where we used that v is a classical solution. Thus, $s \to S(t-s)v(s)$ is constant on every interval $[0,t]$. Since $v(0) = x$ it follows that $v(t) = S(t-t)v(t) = S(t-0)v(0) = S(t)x = u(t)$.

Theorem 3.71. Let S be a C^0- semigroup on the Banach space X and let $\omega \in \mathbb{R}$, $M \geq 0$ be as in (3.67) such that:

$$\|S(t)\|_{\mathcal{L}(X)} \leq Me^{\omega t} \quad \forall t \geq 0.$$

If $(A, D(A))$ is the generator of $(S(t))_{t\geq 0}$, then:

- if there exists $\lambda \in \mathbb{C}$ such that $R(\lambda)x = \int_0^t e^{-\lambda t}S(t)xdt$, $x \in X$, is a well defined and bounded operator from X to X, then $\lambda \in \rho(A)$ and $R(\lambda, A) = R(\lambda)$;
- if $Re\lambda > \omega$, then $\lambda \in \rho(A)$ and $R(\lambda, A) = R(\lambda)$;
- for all $\lambda \in \mathbb{C}$, such that $Re\lambda > \omega$,

$$\|R(\lambda, A)\|_{\mathcal{L}(X)} \leq \frac{M}{Re\lambda - \omega}.$$

Remark 3.72. The expression

$$R(\lambda, A)x = \int_0^\infty e^{-\lambda t} S(t) x dt, \quad x \in X,$$

is the **integral representation of the resolvent**. The integral exists as improper Riemann integral:

$$R(\lambda, A)x = \lim_{T \to \infty} \int_0^T e^{-\lambda t} S(t) x dt.$$

Theorem 3.73 (Hille - Yosida)**.** Let $(A, D(A)$ be a linear operator on the Banach space X. Let $\omega \in \mathbb{R}, \ M \geq 0$, the following statements are equivalent:

- A is the generator of a C^0- semigroup $(S(t))_{t\geq 0}$ such that $\|T(t)\|_{\mathcal{L}(X)} \leq Me^{\omega t}$ for all $t \geq 0$.
- A is closed, $\overline{D(A)} = X$, $(\omega, \infty) \subset \rho(A)$ and

$$\|R(\lambda, A)\|^n_{\mathcal{L}(X)} \leq \frac{M}{(\lambda - \omega)^n} \quad \forall \lambda > \omega, n \in \mathbb{N}. \tag{3.13}$$

Moreover, if one of these conditions is verified, then $\{Re\lambda > \omega\} \subset \rho(A)$ and

$$\|R(\lambda, A)\|^n_{\mathcal{L}(X)} \leq \frac{M}{(Re\lambda - \omega)^n} \tag{3.14}$$

for every $\lambda \in \mathbb{C}$ such that $Re\lambda > \omega$ and $n \in \mathbb{N}$.

3.7 Sectorial operators and analytic semigroups

Let us first define the main objects of our study.

Definition 3.74. For $\theta \in (\frac{\pi}{2}, \pi)$, $\omega \in \mathbb{R}$, consider the sector

$$S_{\theta,\omega} = \{\lambda \in \mathbb{C} \setminus \{\omega\} : |\arg(\lambda - \omega)| < \theta\}.$$

Let X be a Banach space and let $A : D(A) \subset X \to X$ be a linear operator, with not necessarily dense domain.

Definition 3.75. A is said to be **sectorial** if there are constants $\omega \in \mathbb{R}$, $\theta \in (\frac{\pi}{2}, \pi)$, $M > 0$ such that

$$\begin{cases} \rho(A) \supset S_{\theta,\omega}, \\ \|R(\lambda, A)\|_{\mathcal{L}(X)} \leq \frac{M}{|\lambda - \omega|} \quad \lambda \in S_{\theta,\omega}. \end{cases} \tag{3.15}$$

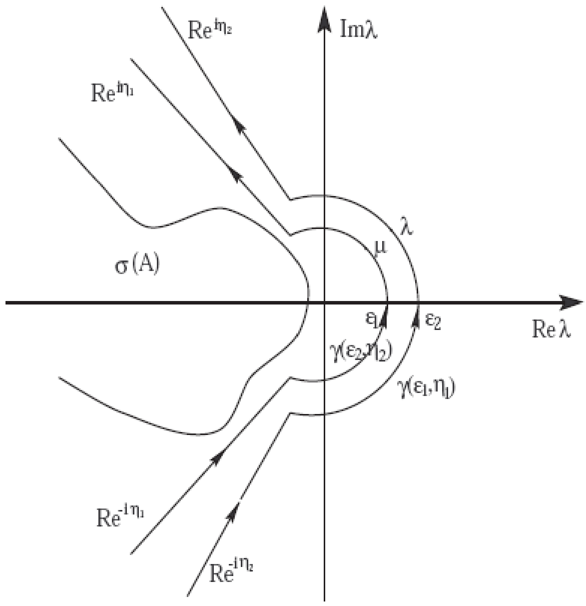


Figure 3.1: Resolvent set and spectrum of a sectorial operator.

The fact that the resolvent set of A is not empty implies that A is closed, so that $D(A)$, endowed with the graph norm:

$$\|x\|_{D(A)} = \|x\|_X + \|Ax\|_X$$

is a Banach space.
For every $t > 0$, (3.15) allows us to define a linear bounded operator e^{tA} in X, by means of the Dunford integral

$$e^{tA} = \frac{1}{2\pi i}\int_{\omega+\gamma_{r,\eta}} e^{t\lambda}R(\lambda, A)d\lambda, \tag{3.16}$$

where $r > 0$, $\eta \in (\frac{\pi}{2}, \theta)$ and $\gamma_{r,\eta}$ is the curve

$$\{\lambda \in \mathbb{C} : |\arg(\lambda)| = \eta,\ ||\lambda| \geq r\} \cup \{\lambda \in \mathbb{C} : |\arg(\lambda)| \leq \eta,\ ||\lambda| = r\}$$

oriented counterclockwise.
We also set:

$$e^{0A}x = x \quad \forall x \in X. \tag{3.17}$$

Since the function $\lambda \to e^{tA}R(\lambda, A)$ is holomorphic in $S_{\theta,\omega}$, the definition of e^{tA} is independent of the choice of η and r.

Definition 3.76. Let $A : D(A) \to X$ be a sectorial operator. The family $\{e^{tA} : t \geq 0\}$ is said to be the analytic semigroup generated by A in X.

Definition 3.77. A semigroup $S(t)$ is said analytic if the function $t \to S(t)$ is analytic in $(0, \infty)$ with values in $\mathcal{L}(X)$.

3.7.1 Basic Properties of e^{tA}

Let $A : D(A) \to X$ be a sectorial operator and let e^{tA} be the analytic semigroup generated by A.

1. $e^{tA}x \in D(A^k)$ for each $t > 0$, $x \in X$, $k \in \mathbb{N}$. If $x \in D(A^k)$, then

$$A^k e^{tA} x = e^{tA} A^k x, \quad \forall t \geq 0.$$

2. $e^{tA} e^{sA} = e^{(t+s)}A$, for every $t, s \geq 0$.

3. There are constants $M_0, M_1, M_2, ...,$ such that

$$\begin{cases} \|e^{tA}\|_{\mathcal{L}(X)} \leq M_0 e^{\omega t}, \quad t > 0, \\ \|t^k (A - \omega I)^k e^{tA}\|_{\mathcal{L}(X)} \leq M_k e^{\omega t}, \quad t > 0, \end{cases} \tag{3.18}$$

where ω is the constant of the assumption (3.15).

4. The function $t \to e^{tA}$ belongs to $C^\infty(]0, +\infty[, \mathcal{L}(X))$ and

$$\frac{d^k}{dt^k} e^{tA} = A^k e^{tA}, \quad t > 0.$$

Now we state a sufficient condition to be a sectorial operator.

Proposition 3.78. Let $A : D(A) \subset X \to X$ be a linear operator such that $\rho(A)$ contains a half plane $\{\lambda \in \mathbb{C} : Re\lambda \geq \omega\}$, and

$$\|\lambda R(\lambda, A)\|_{\mathcal{L}(X)} \leq M, \quad Re\lambda \geq \omega$$

with $\omega \in \mathbb{R}$, $M > 0$. Then A is sectorial.

Proposition 3.79. If A is sectorial with dense domain in X, then

$$\lim_{t \to 0} e^{tA} x = x \quad \forall x \in X,$$

so $\{e^{tA}\}_{t \geq 0}$ is a C^0-semigroup.

3.8 Laplacian operator

We consider the Laplacian operator:

$$\Delta = \partial^2_{x_1} + ... + \partial^2_{x_n}, \quad n \geq 1;$$

- It is symmetric and positive.
- As seen in the example (3.48), $-\Delta$ with $D(-\Delta) = H^2(\mathbb{R}^m)$ is a closed operator.

- It is a sectorial operator.
  Indeed let us consider the equation

$$zu(x) - \Delta u(x) = f(x), \quad x \in \mathbb{R}^m, \tag{3.19}$$

where $f \in L^2(\mathbb{R}^m)$ and $z \in \mathbb{C}$, $z = a + ib$ with $a > 0$.
Taking the Fourier transform, we obtain

$$z\hat{u} + |\xi|^2 \hat{u} = \hat{f}$$

which implies

$$\hat{u} = \frac{\hat{f}}{z + |\xi|^2}. \tag{3.20}$$

By (3.19) we deduce that $u = R(z, \Delta)f$; since $a = Rez > 0$,

$$\begin{aligned} \|zR(z,\Delta)f\|_{L^2} &= \|zu\|_{L^2} = |z|\|\hat{u}\|_{L^2} \leq |z| \left\| \frac{\hat{f}}{z + |\xi|^2} \right\|_{L^2} \\ &\leq \|\hat{f}\|_{L^2} = \|f\|_{L^2}. \end{aligned}$$

With this result we are in the assumptions of the proposition (3.78), so we conclude that the Laplacian is sectorial.

Chapter 4

A diffusive model

In this chapter we try to connect the discharge activity of *REM-on* and *REM-off* neurons to their position in a bounded set. We want to generalize the phase models (2.2) and (2.3), studied in the first chapter, analyzing the case $\theta = \theta(t, x)$ and $\phi = \phi(t, x)$, where the variable x represents the neuron position.
We study the diffusive model that we obtain at the end of the first chapter. We start from Kuramoto model to model the interaction between two neurons of different groups, REM-ON and REM-OFF neurons, and we obtain a discretization of Laplacian operator; so we arrived to the following phase model:

$$\begin{cases} \frac{\partial \theta}{\partial t} - \Delta\theta = \omega + A\sin(\theta - \phi) \\ \frac{\partial \phi}{\partial t} - \Delta\phi = \omega - A\sin(\theta - \phi) \end{cases} \tag{4.1}$$

where Δ is the Laplacian operator.
We set

$$D(\Delta) = \{f : S^1 \to \mathbb{R} | f(0) = f(2\pi) \text{ and } \sum_{k\in\mathbb{Z}} |\hat{f}(k)|^2(1+k^2)^2 < \infty\} = H^2(S^1)$$

We generalize (4.1) to the system:

$$\begin{cases} \frac{\partial \theta}{\partial t} - \Delta\theta = \omega + g(\theta - \phi) \\ \frac{\partial \phi}{\partial t} - \Delta\phi = \omega - g(\theta - \phi) \end{cases} \tag{4.2}$$

where g is a nonlinear function with appropriate properties.
Let us see some general results.

Lemma 4.1. Let $f(t,x) = \sum_{k\in\mathbb{Z}} f_k(t)e^{ikx}$ in $H^1([0,\infty) \times S^1)$. The system

$$\begin{cases} \frac{\partial \theta}{\partial t}(t,x) - \Delta\theta(t,x) = f(t,x) \\ \theta(0) = 0 \end{cases} \tag{4.3}$$

has the solution

$$\theta(t,x) = \sum_{k\in\mathbb{Z}} \left(\int_0^t e^{-k^2(t-s)} f_k(s,x)ds \right) e^{ikx}.$$

Proof. We look for a solution in the following form:

$$\theta(t,x) = \sum_{k\in\mathbb{Z}} \theta_k(t) e^{ikx}.$$

If we put it in the equation of the system (4.3) and we differentiate termwise, obtaining

$$\frac{\partial\theta}{\partial t}(t) - \Delta\theta(t,x) = \sum_{k\in\mathbb{Z}} \left(\frac{\partial\theta_k}{dt}(t) + k^2\theta_k(t) \right) e^{ikx} = \sum_{k\in\mathbb{Z}} f_k(t,x) e^{ikx}.$$

So,

$$\begin{cases} \frac{\partial\theta_k}{dt}(t) + k^2\theta_k(t) = f_k(t,x) \\ \theta_k(0) = 0 \end{cases}$$

which is solved by

$$\theta_k(t) = \int_0^t e^{-k^2(t-s)} f_k(s,x)ds.$$

So we obtain:

$$\theta(t,x) = \sum_{k\in\mathbb{Z}} \left(\int_0^t e^{-k^2(t-s)} f_k(s,x)ds \right) e^{ikx}$$

and we can really differentiate termwise, since $f \in H^1((0,\infty)\times S^1)$. By definition of Fourier coefficient, we have

$$\begin{aligned} \theta(t,x) &= \frac{1}{2\pi} \sum_{k\in\mathbb{Z}} \int_0^t \left(\int_0^{2\pi} e^{-k^2(t-s)} f(s,y) e^{-iky} dy \right) e^{ikx} ds \\ &= \frac{1}{2\pi} \int_0^t \int_0^{2\pi} \sum_{k\in\mathbb{Z}} e^{-k^2(t-s)} f(s,y) e^{-iky} e^{ikx} dyds \\ &= \int_0^t \int_0^{2\pi} G(t,s,x,y) f(s,y) dyds \end{aligned}$$

where x and y are in S^1, $t \geq s$ and

$$G(t,s,x,y) = \sum_{k\in\mathbb{Z}} e^{-k^2(t-s)} e^{-iky} e^{ikx}.$$

The series converges with all its derivatives if $t > s > 0$. So θ satisfies the differential equation in $((0,\infty)\times S^1)$. Moreover if $f \in C(\overline{(0,\infty)\times S^1})$ then θ is continuous.

Lemma 4.2. Let $\chi(x) \in L^2(S^1)$,

$$\begin{cases} \frac{\partial\theta}{dt}(t,x) - \Delta\theta(t,x) = 0 \\ \theta(0,x) = \chi(x) \end{cases} \tag{4.4}$$

has the solution

$$\theta(t,x) = \sum_{k\in\mathbb{Z}} e^{-k^2 t}\chi_k(x)e^{ikx}.$$

Proof. By separation of variables, we obtain this solution.
It converges in $L^2(S^1)$ for every fixed t and satisfies (4.2) in $(0,\infty) \times S^1$.
Moreover if $\sum_{k\in\mathbb{Z}} |\chi_k| < \infty$, $\theta(t,x)$ is also continuous in $[0,\infty) \times S^1$.

Remark 4.3. We have

$$\theta = e^{\Delta t}\chi,$$

and

$$[\theta(t,\cdot)]_k = (e^{\Delta t}\chi)_k = e^{-k^2 t}\chi_k.$$

Lemma 4.4. Let $f \in H^1([0,\infty) \times S^1)$, $\chi \in L^2(S^1)$. The system

$$\begin{cases} \frac{\partial\theta}{dt}(t,x) - \Delta\theta(t,x) = f(t,x) \\ \theta(0,x) = \chi(x) \end{cases} \tag{4.5}$$

has the solution

$$\theta(t,x) = \sum_{k\in\mathbb{Z}} \left(e^{-k^2 t}\chi_k + \int_0^t e^{-k^2(t-s)} f_k(s)ds \right) e^{ikx}.$$

Proof.
We conclude using the results of two last lemmas and the linearity of the equations involved.

We can prove these results also using semigroups properties.

Remark 4.5. By a simple integration by parts twice, it follows that $-\Delta$ is a selfadjoint operator in $L^2(S^1)$.

Remark 4.6. The operator Δ generates an analytic semigroup in $L^2(S^1)$.
Indeed, from the equation

$$\lambda u - \Delta u = f$$

with $Re\lambda > 0$, we get, multiplying by an integrating by parts:

$$\lambda\|u\|^2_{L^2} + \|\nabla u\|^2_{L^2(S^1)} = (f,u) \le \|f\|_{L^2(S^1)}\|u\|_{L^2(S^1)},$$

so that $\|u\|_{L^2(S^1)} \le \frac{\|f\|_{L^2(S^1)}}{|\lambda|}$.
Hence

$$\|e^{\Delta t}\|_{L^2(S^1)} \le 1 \quad \forall t \ge 0.$$

From remark 4.3 and lemma 4.4 we obtain the following lemma.

Lemma 4.7. Let $f \in H^1([0,\infty) \times S^1)$, $\chi \in L^2(S^1)$. The system (4.5) has the solution

$$\theta(t,x) = e^{\Delta t}\chi(x) + \int_0^t e^{\Delta(t-s)} f(s) ds.$$

Therefore, from the system

$$\begin{cases} \frac{\partial \theta}{\partial t} - \Delta\theta = \omega + f(\theta - \phi) \\ \frac{\partial \phi}{\partial t} - \Delta\phi = \omega - f(\theta - \phi) \end{cases} \tag{4.6}$$

we obtain:

$$\begin{cases} \theta(t) = e^{\Delta t}\theta_0 + \int_0^t e^{\Delta(t-s)}[\omega + f(\theta(s) - \phi(s))]ds \\ \phi(t) = e^{\Delta t}\phi_0 + \int_0^t e^{\Delta(t-s)}[\omega - f(\theta(s) - \phi(s))]ds \end{cases} \tag{4.7}$$

From the case of a general function f, let us return to our initial system (4.2), with g a nonlinear function,

$$\begin{cases} \frac{\partial \theta}{\partial t} - \Delta\theta = \omega + A(\theta - \phi) + h(\theta - \phi) \\ \frac{\partial \phi}{\partial t} - \Delta\phi = \omega - A(\theta - \phi) - h(\theta - \phi) \end{cases} \tag{4.8}$$

where we assume that $g(u) = A(u) + h(u)$, where $A \in \mathbb{R}$ and

$$\begin{cases} h \in C^1(S^1) \\ |h(u)| \leq c_1|u|^2 \text{ for } u \in H^s(S^1) \\ |h'(u)| \leq c_2|u| \text{ for } u \in H^s(S^1). \end{cases} \tag{4.9}$$

We rewrite the system as:

$$\begin{cases} \frac{\partial \theta}{\partial t} - \Delta\theta - \omega - A(\theta - \phi) = h(\theta - \phi) \\ \frac{\partial \phi}{\partial t} - \Delta\phi - \omega + A(\theta - \phi) = -h(\theta - \phi). \end{cases} \tag{4.10}$$

If we set

$$\begin{cases} \theta(t) = \omega t + \tilde{\theta}(t) \\ \phi(t) = \omega t + \tilde{\phi}(t) \end{cases}$$

we obtain:

$$\begin{cases} \frac{\partial \tilde{\theta}}{\partial t} - \Delta\tilde{\theta} - A(\tilde{\theta} - \tilde{\phi}) = h(\tilde{\theta} - \tilde{\phi}) \\ \frac{\partial \tilde{\phi}}{\partial t} - \Delta\tilde{\phi} + A(\tilde{\theta} - \tilde{\phi}) = -h(\tilde{\theta} - \tilde{\phi}). \end{cases}$$

Summing the two equation of the system, we obtain

$$\frac{\partial(\tilde{\theta}+\tilde{\phi})}{\partial t} - \Delta(\tilde{\theta}+\tilde{\phi}) = 0.$$

Hence

$$\tilde{\theta}+\tilde{\phi} = e^{\Delta}t(\tilde{\theta}_0+\tilde{\phi}_0)$$

which is a bonded function.
Subtracting, we obtain

$$\frac{\partial(\tilde{\theta}-\tilde{\phi})}{\partial t} - \Delta(\tilde{\theta}-\tilde{\phi}) - 2A(\tilde{\theta}-\tilde{\phi}) = 2h(\tilde{\theta}-\tilde{\phi}). \tag{4.11}$$

We recall $u = \tilde{\theta}-\tilde{\phi}$ and the last equation becomes

$$\frac{\partial u}{\partial t} - \Delta u - 2Au = 2h(u), \tag{4.12}$$

and we add the initial conditions:

$$\begin{cases} \theta(0,x) = \theta_0(x), & x \in S^1 \\ \phi(0,x) = \phi_0(x), & x \in S^1. \end{cases}$$

4.1 Case $A < 0$.

Let us study equation (4.12) with $A < 0$.

Theorem 4.8. [Asymptotic stability] Consider the system:

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) - \Delta u(t,x) - 2Au(t,x) = 2h(u(t,x)), & (t,x) \in [0,\infty[\times S^1 \\ u(0,x) = u_0(x), & x \in S^1 \end{cases} \tag{4.13}$$

with the assumption (4.9) on h and $A < 0$.
If $\|u_0\|_{H^s(S^1)} < \epsilon$, there exists a unique solution $u \in C([0,+\infty), H^s(S^1))$ satisfying

$$u(t,x) = e^{-Gt}u(0,x) + 2\int_0^t e^{-G(t-\tau)}h(u(\tau,x))d\tau,$$

where $G = -\Delta - 2A$, and

$$\|u(t,x)\|_{H^s(S^1)} \leq ce^{-\delta t}\|u_0\|_{H^s(S^1)}$$

for some $\delta > 0$.

Before proving this result, we make a remark.

Remark 4.9. The operator G is sectorial and, in fact,

$$e^{-Gt}x = e^{2At}e^{\Delta t}x \quad \forall x \in L^2(S^1).$$

This follows by observing that:

$$R(\lambda, -G) = R(\lambda - 2A, \Delta)$$

and using the integral representation (3.16). In particular

$$\|e^{-Gt}\|_{L^2(S^1)} = e^{2At}\|e^{\Delta t}\|_{L^2(S^1)} \leq e^{2At}.$$

Proof.
We want to prove that:

$$\|e^{-Gt}f\|_{H^s(S^1)} \leq e^{2At}\|f\|_{H^s(S^1)}. \tag{4.14}$$

Indeed, for $f \in H^s(S^1)$, we have:

$$\begin{aligned}
\|e^{-Gt}f\|_{H^s(S^1)} &= e^{2At}\|e^{-\Delta t}f\|_{H^s(S^1)} \\
&= e^{2At}\left(\sum -k \in \mathbb{Z}|1+k^2|^s|e^{-\Delta t}f|_k^2\right)^{\frac{1}{2}} \\
&= e^{2At}\left(\sum -k \in \mathbb{Z}|1+k^2|^s e^{k^2 t}|f_k|^2\right)^{\frac{1}{2}} \\
&\leq e^{2At}\left(\sum -k \in \mathbb{Z}|1+k^2|^s|f_k|^2\right)^{\frac{1}{2}} \\
&\leq e^{2At}\|f\|_{H^s(S^1)}.
\end{aligned} \tag{4.15}$$

We have also:

$$\|h(u)\|_{H^s} \leq L\|u\|^2_{H^s(S^1)}(S^1) \tag{4.16}$$

for a positive constant L, as a consequence of 3.40 and (4.9).
Now we write the solution of (4.22) is

$$u(t) = e^{-Gt}u_0 + 2\int_0^t e^{-G(t-\tau)}h(u(\tau))d\tau \tag{4.17}$$

We introduce the Banach space:

$$\mathcal{B} = \{u : [0,\infty[\to H^s(S^1) : \sup_{t\geq 0} e^{\delta t}\|u(t,\cdot)\|_{H^s(S^1)} < \infty\}$$

where δ is a constant and $s > \frac{1}{2}$.
It is easy to prove that this space is complete in relation to the norm:

$$\|u\|_{\mathcal{B}} = \sup_{t\geq 0} e^{\delta t}\|u(t,\cdot)\|_{H^s(S^1)}.$$

By this definition, we deduce the following property:

$$\|u(t,\cdot)\|_{H^s(S^1)} \leq e^{-\delta t}\|u\|_{\mathcal{B}}. \tag{4.18}$$

We take δ such that $\delta \leq \frac{|A|}{2}$.
Now we define the map:

$$\begin{aligned} \phi\colon \mathcal{B}_\epsilon &= \{u \in \mathcal{B} : \|u\|_B \leq p\epsilon\} \to \mathcal{B} \\ u &\mapsto e^{-Gt}u(0) + 2\int_0^t e^{-G(t-\tau)}h(u(\tau))d\tau, \end{aligned} \tag{4.19}$$

where $p > 1$.

To conclude the proof of the theorem, we shaw that the map ϕ is a contraction in the complete metric space (B_ϵ, d) where d is the metric induced by $\|\cdot\|_B$.
To use this result, we have to check two assumptions:

1. $\phi(u) \in \mathcal{B}_\epsilon$ for every $u \in B_\epsilon$;
2. If u_1, u_2 are in $\mathcal{B}_\epsilon$ then

$$\|\phi(u_1) - \phi(u_2)\|_{\mathcal{B}} \leq \frac{1}{2}\|u_1 - u_2\|_{\mathcal{B}}$$

We begin from the first point.

Lemma 4.10. If $\mathcal{B}$ is the Banach space just introduced and $\phi : \mathcal{B}_\epsilon \to \mathcal{B}$, as above, there exists $\epsilon_0 > 0$ such that $\phi(\mathcal{B}_\epsilon) \subset \mathcal{B}_\epsilon$, for all $\epsilon \in (0, \epsilon_0)$.

Proof
By (4.14), (4.16) and (4.18), we have:

$$\begin{aligned} 2\left\|\int_0^t e^{-G(t-\tau)}h(u(\tau))d\tau\right\|_{H^s(S^1)} &\leq 2\int_0^t e^{-2|A|(t-\tau)}\|h(u(\tau))\|_{H^s(S^1)}d\tau \\ &\leq 2L\int_0^t e^{-2|A|(t-\tau)}\|u(\tau)\|^2_{H^s(S^1)}d\tau \\ &\leq 2L\int_0^t e^{-2|A|(t-\tau)}e^{-2\delta\tau}\|u(\tau)\|^2_{\mathcal{B}}d\tau \end{aligned}$$

Now, we use that $\delta \leq \frac{|A|}{2}$,

$$
\begin{aligned}
2\left\| \int_0^t e^{-G(t-\tau)} h(u(\tau)) d\tau \right\|_{H^s(S^1)} &\leq 2L\|u(t)\|_{\mathcal{B}}^2 \int_0^t e^{-4\delta(t-\tau)} e^{-2\delta\tau} d\tau \\
&\leq Le^{-4\delta t}\|u\|_{\mathcal{B}}^2 \int_0^t 2e^{2\delta\tau} d\tau \\
&\leq \frac{L}{\delta} e^{-2\delta t}\|u\|_{\mathcal{B}}^2
\end{aligned}
$$

So, by (4.14), we have:

$$
\begin{aligned}
\|\phi(u(t))\|_{H^s(S^1)} &\leq e^{-2|A|t}\|u_0\|_{H^s(S^1)} + \frac{L}{\delta} e^{-2\delta t}\|u\|_{\mathcal{B}}^2 \\
&\leq e^{-4\delta t}\|u_0\|_{H^s(S^1)} + \frac{L}{\delta} e^{-2\delta t}\|u\|_{\mathcal{B}}^2 \\
&\leq e^{-\delta t}\left(\|u_0\|_{H^s(S^1)} + \frac{L}{\delta}\|u\|_{\mathcal{B}}^2 \right)
\end{aligned}
$$

We initially supposed $\|u_0\|_{H^s(S^1)} \leq \epsilon$; hence $\|u\|_{\mathcal{B}} \leq p\epsilon$. So:

$$
\|\phi(u(t))\|_{H^s(S^1)} \leq e^{-\delta t}[\epsilon + \frac{L}{\delta}\|u\|_{\mathcal{B}}^2] \leq e^{-\delta t}[\epsilon + \frac{L}{\delta}(p\epsilon)^2]
$$

and a fortiori

$$
\|\phi(u)\|_{\mathcal{B}} \leq \epsilon + \frac{L}{\delta} p^2 \epsilon^2.
$$

If we take $\epsilon_0 = \frac{\delta(p-1)}{Lp^2}$, the lemma is proved.

For the second point we consider:

$$
\phi(u_1) - \phi(u_2) = 2\int_0^t e^{-G(t-\tau)}[h(u_1(\tau)) - h(u_2(\tau))] d\tau.
$$

Since $H^s(S^1)$ is an algebra for $s > \frac{1}{2}$, and the function h satisfies (4.8),

$$
\|h(u_1) - h(u_2)\|_{H^s(S^1)} \leq c_2\|u_1 - u_2\|_{H^s(S^1)}(\|u_1\|_{H^s(S^1)} + \|u_2\|_{H^s(S^1)}). \quad (4.20)
$$

Now, proceeding as in the precedent lemma,

$$
\|\phi(u_1) - \phi(u_2)\|_{H^s(S^1)} \leq c_1 \int_0^t e^{-2|A|(t-\tau)} e^{-2\delta\tau}\|u_1 - u_2\|_{\mathcal{B}}(\|u_1\|_{\mathcal{B}} + \|u_2\|_{\mathcal{B}}) d\tau \quad (4.21)
$$

and, as above, we conclude:

$$\begin{aligned}\|\phi(u_1)-\phi(u_2)\|_B &\le c_2\|u_1-u_2\|_{\mathcal{B}}(\|u_1\|_{\mathcal{B}}+\|u_2\|_{\mathcal{B}})\\ &\le 2c_2p\epsilon\|u_1-u_2\|_{\mathcal{B}}\\ &\le \frac{1}{2}\|u_1-u_2\|_{\mathcal{B}}\end{aligned}$$

provided we choose ϵ such that $2c_2p\epsilon \le \frac{1}{2}$.

We have just proved that ϕ has a fixed point, so we have exactly:

$$u(t) = \phi(u(t)).$$

By (4.18), we deduce the asymptotic stability with respect to the norm of $H^s(S^1)$ and theorem 4.8 is proved with $\delta \le \frac{|A|}{2}$.

Finally, we have shown that:

- $\tilde{\theta}(t,x) + \tilde{\phi}(t,x) = e^{\Delta t}(\tilde{\theta}_0(x) + \tilde{\phi}_0(x))$;
- $\|\tilde{\theta}(t,x) - \tilde{\phi}(t,x)\|_{H^s(S^1)} \le c\epsilon e^{-\delta t}$;

where

$$\begin{cases}\theta(t,x)+\phi(t,x) = \tilde{\theta}(t,x)+\tilde{\phi}(t,x)+2\omega t\\ \theta(t,x)-\phi(t,x) = \tilde{\theta}(t,x)-\tilde{\phi}(t,x) = u(t,x).\end{cases}$$

We deduce that

$$\begin{cases}\theta(t,x) = \frac{1}{2}[u(t,x)+\tilde{\theta}(t,x)+\tilde{\phi}(t,x)+2\omega t]\\ \phi(t,x) = \frac{1}{2}[-u(t,x)+\tilde{\theta}(t,x)+\tilde{\phi}(t,x)+2\omega t];\end{cases}$$

which implies that the two phases have the same asymptotic behavior when $t \to \infty$.
We have found the same result as in the model without diffusion in the first chapter.

4.2 Case $A > 0$.

Before studying the system

$$\begin{cases}\frac{\partial u}{\partial t}(t,x) - \Delta u(t,x) - 2Au(t,x) = 2h(u(t,x)) \quad (t,x)\in[0,\infty[\times S^1\\ u(0,x) = u_0(x) \quad x\in S^1\end{cases} \tag{4.22}$$

with the assumption (4.9) and $A > 0$, we need some results.

Proposition 4.11 (Maximum principle)**.** Let $u(t,x)$ be a regular function in $[0,T] \times S^1$ such that

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) - \Delta u(t,x) \geq 0 \quad (t,x) \in [0,T[\times S^1 \\ u(0,x) \geq 0 \quad x \in S^1 \end{cases} \tag{4.23}$$

Then

$$u(t,x) \geq 0 \quad (t,x) \in [0,T[\times S^1.$$

Proof. Let $\epsilon > 0$ and

$$v(t,x) = u(t,x) + \epsilon t.$$

We have

$$\begin{cases} \frac{\partial v}{\partial t}(t,x) - \Delta v(t,x) = \frac{\partial u}{\partial t}(t,x) - \Delta u(t,x) + \epsilon > 0 \quad (t,x) \in [0,T[\times S^1 \\ v(0,x) = u(0,x) \geq 0 \quad x \in S^1 \end{cases}$$

Let (t_0, x_0) be a minimum of the function v in $[0,T[\times S^1$, then $(t_0,x_0) \in [0,T[\times S^1$ or $t_0 = 0, x_0 \in S^1$.
In the first case

$$\frac{\partial v}{\partial t}(t,x) < 0 \quad \text{ and } \quad v_{x,x}(t_0,x_0) = \Delta v(t_0,x_0) \geq 0$$

so

$$\frac{\partial v}{\partial t}(t_0,x_0) - \Delta v(t_0,x_0) \leq 0$$

but this is not possible.
Therefore $t_0 = 0$ and

$$v(t,x) \geq v(0,x) = \min_{[0,T[\times S^1} v \geq 0.$$

So

$$u(t,x) = v(t,x) - \epsilon t \geq -\epsilon T \quad \forall (t,x) \in [0,T[\times S^1$$

and, since ϵ is a free parameter

$$u(t,x) \geq 0$$

in $[0,T[\times S^1$.

Theorem 4.12 (Comparison theorem)**.** Let v, w be regular functions in $[0,T] \times S^1$, such that

$$\begin{cases} \frac{\partial v}{\partial t}(t,x) - \Delta v(t,x) - f(v(t,x)) \geq 0, \quad (t,x) \in [0,T] \times S^1 \\ v(0,x) \geq \phi_0(x), \quad x \in S^1 \end{cases}$$

and

$$\begin{cases} \frac{\partial w}{\partial t}(t,x) - \Delta w(t,x) - f(w(t,x)) \leq 0, & (t,x) \in [0,T] \times S^1 \\ w(0,x) \leq \phi_0(x), & x \in S^1 \end{cases}$$

where ϕ_0 is continuous in S^1 and $f \in C^1(\mathbb{R})$ with $|f'(u)| \leq M_k$ for every $u \in [-k,k]$. Then

$$w(t,x) \leq v(t,x), \quad (t,x) \in [0,T] \times S^1.$$

Proof. We introduce the function

$$\gamma(t,x) = \begin{cases} \frac{f(v(t,x))-f(w(t,x))}{v(t,x))-w(t,x)} & \text{if } v(t,x) \neq w(t,x) \\ f'(v(t,x)) & \text{if } v(t,x) = w(t,x). \end{cases}$$

By Lagrange theorem, we have

$$|\gamma(t,x)| \leq M_k \quad (t,x) \in [0,T] \times S^1$$

where $k = \max\{\|v\|_\infty, \|w\|_\infty\}$. If we set $z = v - w$, then z satisfies:

$$\begin{cases} \frac{\partial z}{\partial t}(t,x) - \Delta z(t,x) = \frac{\partial v}{\partial t}(t,x) - \Delta v(t,x) - \frac{\partial w}{\partial t}(t,x) + \Delta w(t,x) & (t,x) \in [0,T] \times S^1 \\ v(0,x) \geq 0, & x \in S^1 \end{cases}$$

and we note that

$$\frac{\partial z}{\partial t}(t,x) - \Delta z(t,x) \geq f(v) - f(w) \geq -M(v-w) = -Mz, \quad (t,x) \in [0,T] \times S^1.$$

Moreover, if $y(t,x) = e^{Mt} z(t,x)$ then

$$\begin{cases} \frac{\partial y}{\partial t}(t,x) - \Delta y(t,x) = e^{Mt}(\frac{\partial z}{\partial t}(t,x) - \Delta z(t,x) + Mz) \geq 0, & (t,x) \in [0,T] \times S^1 \\ y(0,x) = z(0,x) \geq 0, & x \in S^1. \end{cases}$$

By the maximum principle, $y(t,x) \geq 0$ in $[0,T] \times S^1$, therefore $z(t,x) \geq 0$ in $[0,T] \times S^1$ and we deduce that $w(t,x) \leq v(t,x)$ in $[0,T] \times S^1$.

Corollary 4.13. If $g : \mathbb{R} \to \mathbb{R}$ such that

- $g \in C^1(\mathbb{R})$;
- $g(0) = g(B) = 0$ with $B > 0$;
- $g'(0) > 0$, $g'(B) < 0$, $g > 0$ in $(0, B)$.

and if u is a regular function such that

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) - \Delta u(t,x) = g(u), & (t,x) \in [0,T] \times S^1 \\ u(0,x) = u_0(x), & x \in S^1. \end{cases}$$

with $0 \leq u_0(x) \leq B$ for every x in S^1, then

$$0 \leq u(t,x) \leq B \quad \forall (t,x) \in [0,T] \times S^1$$

Proof. We apply the comparison theorem in two different cases:

- firstly we take $v \equiv u$ and $w \equiv 0$, so we obtain $u \geq 0$;
- then we take $v \equiv B$ and $w \equiv u$ so we obtain $u \leq B$.

Remark 4.14. The solution of the above corollary is defined in $[0, \infty) \times S^1$, with the assumption that $0 \leq u_0(x) \leq B$. In fact it cannot diverge to infinity because $0 \leq u(t, x) \leq B$.

Now, we consider the equation

$$\frac{\partial u}{\partial t}(t, x) - \Delta u(t, x) = g(u), \quad (t, x) \in [0, T] \times S^1 \tag{4.24}$$

with g as in (4.13).

Proposition 4.15. The equation above has the stationary solutions $u \equiv 0$ and $u \equiv B$ and:

1. the solution $u \equiv 0$ is unstable;
2. the solution $u \equiv B$ is asymptotically stable.

Proof. (1.) We write

$$g(u) = g'(0)u + h(u),$$

where $g'(0) = A > 0$, $|h(u)| \leq Lu^2$ and $|h'(u)| \leq L_1|u|$.
The equation becomes

$$\frac{\partial u}{\partial t}(t, x) - (\Delta + A)u(t, x) = h(u), \quad (t, x) \in [0, T] \times S^1$$

with $u_0(x) = u(0, x)$.
If we suppose that 0 is stable, then for every $\epsilon > 0$ there exist $\gamma, M > 0$ such that

$$\|u_0\|_\infty \leq \gamma \Rightarrow \|u(t, \cdot)\|_\infty \leq \epsilon \quad \forall t \geq M.$$

If we choose $\alpha \in (0, A)$, $\rho \in (0, \frac{A-\alpha}{L_1})$, we note that

$$\begin{cases} |u| \leq \rho \Rightarrow g'(u) = A + h'(u) \geq A - L_1|u| \geq A - L_1\rho \geq \alpha \\ 0 \leq u \leq \rho \Rightarrow g(u) = g(u) - g(0) = g'(\xi)u \geq \alpha u. \end{cases}$$

Now, let $\epsilon = \frac{1}{n}$, $n \in \mathbb{N}^+$. If we choose σ in $\left(1 - \frac{1}{np}, 1\right)$, so that $\rho(1-\sigma) < \frac{1}{n}$ and θ in $\big(0, \alpha(\frac{1}{\sigma} - 1)\big)$ so that $\sigma(\theta + \alpha) < \alpha$, we set

$$u_1(t, x) = \rho[1 - \sigma e^{-\theta t}].$$

We note that $u_1(0) = \rho(1-\sigma) < \frac{1}{n}$.
Obviously we have

$$0 < u_1(t,x) \le \rho, \quad (t,x) \in [0,T] \times S^1.$$

Moreover

$$\begin{aligned}
\frac{\partial u_1}{dt}(t,x) - \Delta u_1(t,x) - g(u_1) &= \rho\sigma\theta e^{-\theta t} - g(u_1) \\
&\le \rho\sigma\theta e^{-\theta t} - \alpha u_1 \\
&= \rho\sigma\theta e^{-\theta t} - \alpha\rho(1-\sigma e^{-\theta t}) \\
&= \rho\sigma(\theta+\alpha)e^{-\theta t} - \alpha\rho \\
&< \rho\alpha e^{-\theta t} - \rho\alpha = \rho\alpha(e^{-\theta t} - 1) < 0.
\end{aligned}$$

Now choose $u_0(x) \equiv \frac{1}{n}$ and let $u(t,x)$ the correspondent solution.
By the maximum principle

$$u_1(t,x) \le u(t,x);$$

So

$$\liminf_{t\to\infty} u(t,x) \ge \liminf_{t\to\infty} u_1(t,x) = \rho,$$

therefore

$$\|u\|_\infty \ge \rho$$

although $\|u_0\|_\infty = \frac{1}{n} < \gamma$, for all $n \ge n_\epsilon$. So, if we choose $\epsilon < \rho$ we have a contradiction with the stability condition.

(2.) Let $v = B - u$, with u a generic solution of the equation (4.24), such that $u(0,x) = u_0(x)$.
It follows that:

$$\begin{cases} \frac{\partial v}{\partial t}(t,x) - \Delta v(t,x) = -g(B-u), & (t,x) \in]0,\infty[\times S^1 \\ v(0,x) = B - u_0(x), & x \in S^1. \end{cases}$$

We note that:

$$g(B-v) = g(B) - g'(B)v + \phi(v),$$

where $g(B) = 0$, $g'(B) < 0$ and

$$\begin{cases} |\phi(v)| \le L|v|^2 \\ |\phi'(v)| \le L_1|v|. \end{cases}$$

Let us set

$$Y = \{u \in C([0,\infty) \times S^1) : \sup_{t\ge 0} e^{\delta t}\|u(t,\cdot)\|_\infty < \infty\},$$

and

$$\|u\|_Y = \sup_{[0,\infty)\times S^1} e^{\delta t}|u(t,x)|,$$

where $0 < \delta < |g'(B)|$, i. e. $\delta + g'(B) < 0$.
Let be $\Gamma : B_Y(0,r) \to C([0,\infty) \times S^1)$ a map defined by $\Gamma(v) = w$ such that:

$$\begin{cases} \frac{\partial w}{\partial t}(t,x) - \Delta w(t,x) - g'(B)w = \phi(v), \quad (t,x) \in]0,\infty[\times S^1 \\ w(0,x) = B - u_0(x), \quad x \in S^1. \end{cases}$$

Remark 4.16. We have:

$$\|e^{\Delta t} f\|_\infty \leq \|f\|_\infty. \tag{4.25}$$

Proof. It follows from the maximum principle: let $u(x,t) = (e^{\Delta t} f)(x)$. It satisfies

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) - \Delta u(t,x) = 0, \quad (t,x) \in]0,\infty[\times S^1 \\ u(0,x) = f(x), \quad x \in S^1. \end{cases}$$

So, if $m = \min_{x\in S^1} f(x)$, $v = m$ satisfies

$$\begin{cases} \frac{\partial v}{\partial t}(t,x) - \Delta v(t,x) = 0, \quad (t,x) \in]0,\infty[\times S^1 \\ v(0,x) \geq 0, \quad x \in S^1. \end{cases}$$

and, by the maximum principle, $v \geq 0$, i. e. $u \geq m$.
Similarly, if $M = \max_{x\in S^1} f(x)$, $w = M - u$ satisfies

$$\begin{cases} \frac{\partial w}{\partial t}(t,x) - \Delta w(t,x) = 0, \quad (t,x) \in]0,\infty[\times S^1 \\ w(0,x) \geq 0, \quad x \in S^1. \end{cases}$$

and, by the maximum principle, $v \geq 0$, i. e. $u \leq M$.
Therefore:

$$m \leq u(t,x) \leq M,$$

i. e. $\|u\|_\infty \leq \min\{|m|, |M|\} = \|f\|_\infty$, and this proves the Remark.

Now, from (4.25), if $v, z \in B_Y(0,r)$ we have:

$$\begin{aligned} e^{\delta t}|\Gamma(v)(t) - \Gamma(z)(t)| &= e^{\delta t}\left|\int_0^t e^{[\Delta+g'(B)](t-s)}[\phi(v(s,\cdot)) - \phi(z(s,\cdot))]ds\right| \\ &\leq e^{\delta t}\int_0^t e^{|-g'(B)|(t-s)}|\phi'(\xi(s,\cdot))||v(s,\cdot) - z(s,\cdot)|ds \\ &\leq \int_0^t e^{[\delta-g'(B)](t-s)}L_1 r e^{\delta s}|v(s,\cdot) - z(s,\cdot)|ds \\ &\leq \|v-z\|_Y L_1 r\frac{1}{|\delta + g'(B)|}. \end{aligned} \tag{4.26}$$

Therefore, if $r\frac{L_1}{|\delta+g'(B)|} \leq \frac{1}{2}$ then

$$\|\Gamma(v) - \Gamma(z)\|_Y \leq \frac{1}{2}\|v - z\|_Y.$$

Moreover, let $v \in B_Y(0,r)$, we have

$$\begin{aligned} e^{\delta t}|\Gamma(v)(t)| &\leq e^{\delta t}|\Gamma(v)(t) - \Gamma(0)(t)| + e^{\delta t}|\Gamma(0)(t)| \\ &\leq \frac{1}{2}\|v\|_Y + e^{\delta t}|e^{[\Delta+g'(B)]t}[B - u_0](x)| \\ &\leq \frac{1}{2}\|v\|_Y + 2\pi\|B - u_0\|_\infty \leq r \qquad (4.27) \end{aligned}$$

provided $\|B - u_0\|_\infty \leq r_0$ with $\frac{1}{2}r + 2\pi r_0 \leq r$, i . e. $r_0 \leq \frac{r}{4\pi}$.
So, if $r < \frac{|\delta+g'(B)|}{2L_1}$ and $\|B - u_0\|_\infty \leq \frac{r}{4\pi}$, Γ is a map from $B_y(0,r)$ in itself and is a contraction.
Therefore there exists a unique v in $B_Y(0,r)$ such that $v = \Gamma(v)$, i. e. satisfies:

$$\begin{cases} \frac{\partial v}{\partial t}(t,x) - \Delta v(t,x) = g(v), \quad (t,x) \in]0,\infty[\times S^1 \\ v(0,x) = B - u_0(x), \quad x \in S^1. \end{cases}$$

with $\|B - u_0\|_\infty \leq \frac{r}{4\pi}$. Moreover, as seen,

$$\|v\|_Y \leq \frac{1}{2}\|v\|_Y + 2\pi\|B - u_0\|_\infty$$

which implies $\|v\|_Y \leq 4\pi\|B - u_0\|_\infty$
So, if $\|B - u_0\|_\infty \leq \frac{r}{4\pi}$,

$$|B - u(t)| = |v(t)| \leq 4\pi e^{-\delta t}\|B - u_0\|_\infty.$$

Therefore $u \equiv B$ is asymptotically stable.

Finally, as in the case $A < 0$, we have

$$\begin{cases} \theta(t,x) = \frac{1}{2}[u(t,x) + \tilde{\theta}(t,x) + \tilde{\phi}(t,x) + 2\omega t] \\ \phi(t,x) = \frac{1}{2}[-u(t,x) + \tilde{\theta}(t,x) + \tilde{\phi}(t,x) + 2\omega t]; \end{cases}$$

but if $A > 0$ they asymptotically will have a difference of phase B, as in the non-diffusive model.

Chapter 5

Conclusion

Mathematical models generally analyze the temporal dynamics of the neuronal behavior; we also considered a spatial coordinate. This chapter justifies our analysis of the neuronal phases based also on on the localization of their electrical signals.
In fact during the last decays sleep has been investigated using functional neuroimaging.
The main technique used is positron emission tomography (PET), which shows the distribution of compounds labeled with positron-emitting isotopes. Moreover, recently, functional magnetic resonance imaging (fMRI) has also been used to study brain activity across the sleep-wake cycle. This technique measures the variations in brain perfusion related to neural activity, by assessing the blood oxygen level-dependent (BOLD) signal.
Functional brain imaging offers the opportunity to study the brain structures, at the cortical and subcortical levels (not easily accessible through standard scalp EEG recordings), that participate in the generation or propagation of cerebral rhythms of NREM and REM sleep. Recent studies using mainly EEG/ fMRI have successfully characterized the neural correlates of these phasic activities of sleep. These studies refine the description of brain function beyond the stages of sleep and provide new insight into the mechanisms of spontaneous brain activity in humans.
They have shown a decrease in brain activity during NREM sleep and a sustained level of brain function during REM sleep when compared to wakefulness, in addition to specifically segregated patterns of regional neural activity for each sleep stage.

PET and block-design fMRI (i.e., contrasting "blocks" of NREM sleep with "blocks" of waking) have consistently found a drop of brain activity during NREM sleep when compared to wakefulness.
Regionally, reductions of brain activity were located in subcortical (brainstem, thalamus, basal ganglia, basal forebrain) and cortical (prefrontal cor-

tex, anterior cingulate cortex, and precuneus) regions. These brain structures include neuronal populations involved in arousal and awakening, as well as areas which are among the most active ones during wakefulness.

Patterns of brain activity during REM sleep, as assessed by PET, are drastically different from patterns in NREM sleep.
Several brain structures enhance their activity during REM sleep compared to waking (pontine tegmentum, thalamus, basal forebrain, amygdala, hippocampus, anterior cingulate cortex, temporo-occipital areas) while others decrease (dorsolateral prefrontal cortex, posterior cingulate gyrus, precuneus, and inferior parietal cortex).

More recent neuroimaging studies have addressed the correlates of phasic neural events that build up the architecture of sleep stages. These studies are based on the assumption that brain activity during a specific stage of sleep is not constant and homogeneous over time, but is structured by spontaneous, transient, and recurrent neural processes.
The following figures show neural correlates of NREM sleep oscillations as demonstrated by PET.

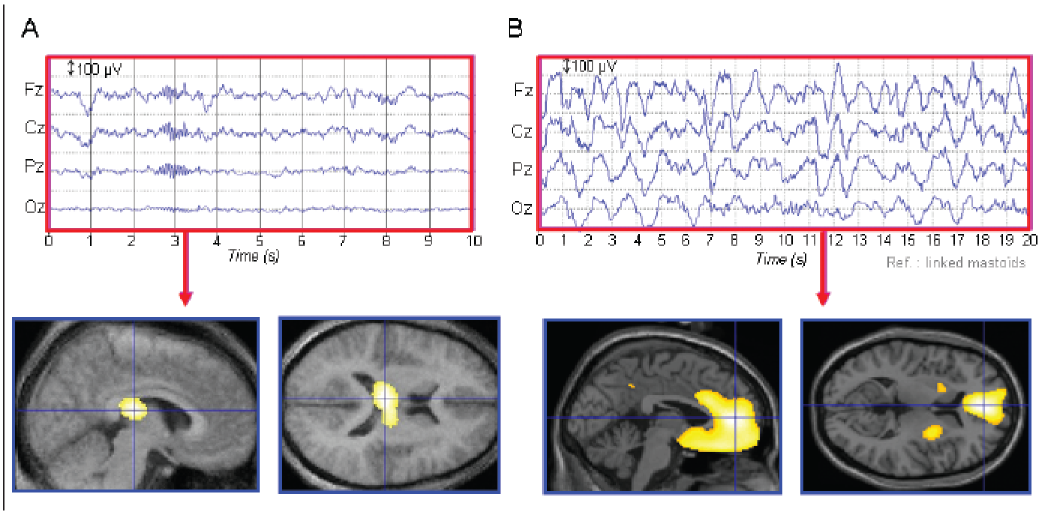


Figure *A* shows PET correlates of spindles.
The upper panel shows a (stage 2) NREM sleep epoch depicting a typical spindle on scalp EEG recording. Brain activity is averaged over the duration of PET acquisition ($\sim$ 1 min) within the NREM sleep epoch and correlated with sigma activity calculated for the corresponding period. The middle panel shows that the only significant correlation is located in the thalamus bilaterally.

Figure B shows PET correlates of slow waves.
The upper panel shows a (stage 4) NREM sleep epoch depicting typical slow waves on scalp EEG recording. Brain activity is averaged over the duration of PET acquisition ($\sim$ 1 min) within the NREM sleep epoch and correlated with delta activity calculated for the corresponding period. The middle panel shows the significant correlations located in anterior cingulate cortex, basal forebrain, striatum, insula, and precuneus. Similar results can be also reached by EEG and fMRI.
We conclude that these data support our study of the localization of the neural activity.

Bibliography

[1] Eugene M. Izhikevich, Dynamical Systems in Neuroscience: The Geometry of Excitability and Bursting, Massachusetts Institute of Technology, 2007.

[2] A. Lunardi, Analytic Semigroups and Optimal Regularity in Parabolic Problems, Birkhäuser, 1995.

[3] Paolo Acquistapace, Appunti di Analisi funzionale, http://www.dm.unipi.it/ acquistp/anafun.pdf .

[4] R. W. McCarley, J. A. Hobson, Neuronal Excitability Modulation over the Sleep Cycle: A Structural and Mathematical Model, Science. 1975 Jul 4, Vol. 189 no. 4196 pp.58-60.

[5] Thien Thanh Dang-Vu, Manuel Schabus, Martin Desseilles, Virginie Sterpenich, Maxime Bonjean, Pierre Maquet, Functional Neuroimaging Insights into the Physiology of Human Sleep, Sleep. Dec 1, 2010; 33(12): 1589-1603.

[6] J. A. Hobson, R. W. McCarley, P. W. Wyzinski, Sleep Cycle Oscillation: Reciprocal Discharge by Two Brainstem Neuronal Groups, Science. 1975 Jul 4, Vol. 189 no. 4196 pp. 55-58.