An Aristotelian way of counting infinite sets - Marco Forti[1]

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The naïve notion of "size" for collections should obey to the following principles, derived from the common practice of counting finitely many objects (let $\mathfrak{s}$ denote *size*, and $<$ the *natural linear ordering* of sizes):

AP (Aristotle's Principle) *If A is a proper subcollection of B then* $\mathfrak{s}(A) < \mathfrak{s}(B)$.

CP (Cantor's Principle) $\mathfrak{s}(A) = \mathfrak{s}(B) \iff$ *A is in 1–1 correspondence with B.*

A not much deeper inspection into our intuition of size brings us to introduce an operation of addition, according to the principle

SP (Sum Principle) $\mathfrak{s}(A) + \mathfrak{s}(B) = \mathfrak{s}(A \cup B) + \mathfrak{s}(A \cap B)$.

Long before the celebrated Galileo's remark, the first two principles revealed incompatible for infinite collections. (And this fact led Leibniz to assert the *impossibility of infinite numbers*). By relaxing AP to $\mathfrak{s}(A) \leq \mathfrak{s}(B)$, Cantor obtained its beautiful theory of cardinalities, but the awkward cardinal arithmetic, where $\alpha + \beta = \max(\alpha, \beta)$ whenever α is infinite, makes it unsuitable to a natural introduction of "infinitesimal" numbers. The question arises as to find a suitable weakening of CP that allows to maintain AP and to obtain, through SP and an analogous multiplicative[2] principle, a better-looking arithmetic of sizes.

Following a far-reaching idea of Benci's, a positive answer has been given in [1], but only for *countable "labelled" sets* (see also [2]). We generalize, in a "natural" way, the "numerosities" of [1] to *arbitrary sets of* (Von Neumann) *ordinals*, and then, after suitably labelling the universe, to *all sets*. The resulting notion of size satisfies, besides AP and SP, also the rightpointing arrow of CP, and the leftpointing arrow for a large class of "natural" correspondences. Moreover disjointness can be naturally strengthened to "well-spacedness" so as to define a product of sizes satisfying the following principle:

PP (Product Principle) *If A is a collection of well-spaced sets of ordinals, each of the same size of B, then* $\mathfrak{s}(A) \cdot \mathfrak{s}(B) = \mathfrak{s}(\bigcup A)$.

Moreover sum (and product) of sizes of ordinals are the *natural ones* (i.e. as Cantor's normal forms, or Conway's surreal numbers). In fact the arithmetic of sizes is best possible, in the sense that they *embed isomorphically into a nonstandard model* $^*\mathbb{N}$. Most wanted would be a converse of AP, namely that

$\mathfrak{s}(A) < \mathfrak{s}(B)$ *only if A has the same size of some proper subcollection of B*,

since then the sizes *in themselves* would constitute a nonstandard model $^*\mathbb{N}$. Following [1], this property can be obtained for *countable* sets, by using a *selective ultrafilter* on $\mathbb{N}$. At present, the attempts of extending this principle to arbitrary sets seem to affect more fundamental principles, like *linearity of the ordering* of sizes, or the *"rightpointing" Cantor principle.*

References

[1] V. Benci, M. Di Nasso - Numerosities of labelled sets: a new way of counting, *Adv. Math.* **173** (2003), 50–67.

[2] V. Benci, M. Di Nasso, M. Forti - The Eightfold Path to Nonstandard Analysis, in *Nonstandard Methods and Applications in Mathematics* (N.J. Cutland, M. Di Nasso, D.A. Ross, eds.), L.N. in Logic, A.S.L. (to appear).

[1] Joint research with Vieri Benci and Mauro Di Nasso.

[2] Multiplication of sizes cannot be simply defined through cartesian products, because, e.g., $\{0\} \times A$ might be a *proper subset* of A. *A fortiori*, the size of the *union of a disjoint family of sets of equal size* cannot be always equal to the product of the size of the family times that of any of its members.