Extended Abstract

Concurrent Proof Net Construction

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The proof construction paradigm, proposed by Andreoli in [3], identifies formulas to instructions and proofs to states: *state* refers to the variable assignments of a programme, *instructions* means specification of a class of state transitions (expressing a change in the variable assignment). In the proof construction paradigm, proofs are taken to be "incomplete proofs" (i.e., with proper axioms). The branches of a proof identify threads of execution, and their open leaves (proper axioms) identify the state of the currently active threads of the execution. Naively, the kind of transition can be performed on these incomplete proofs is the "expansion" of open leaves by means of the gluing of new branches. Andreoli showed this construction process applies to any kind of proof (tree) in the (Gentzen like) sequent calculus style. But, in order to express concurrency, a sequent system is not an adequate tool. Proof nets of linear logic, introduced by Girard in [1] and [2], seem offer a better (de-sequentialized) solution. Roughly proof nets give a parallel representation of proofs: they are graphs, not trees.
In this work we study the construction of a proof net by concurrent agents (called *bipoles*). The proof construction is an incremental bottom-up process: the structure is built by the juxtaposition of bipoles. Now, whereas it is easy to check that the structure built concurrently by bipoles is a "proof structure" it is not so easy to guarantee that it is a "proof net", i.e. a sequentializable proof structure. In other words, checking the "correctness criterion", which is a topological property, is a task which may involve a large portion of the structure: this can generate conflict among the agent-bipoles. Here we show that this problem, at least in the unit-free mutliplicative and additive fragment of linear logic (called MALL) can be minimized, only exploiting topological property of MALL proof nets.
In the following we only give the main ingredient of our work: the notion of *bipolar proof net.*

Given a set of *elementary weights*, i.e. boolean variables (denoted by $p, q, ...$), a *weight* is a product (conjunction) of elementary weights p and of negation of elementary weights $\overline{p}$. We use ϵ_p to speak of p or $\overline{p}$. As a convention, we use 1 for the empty product and 0 for a product where p and $\overline{p}$ appear. We also replace $p.p$ by p. We say that the weight w *depends on* p when p or $\overline{p}$ appears in w.

A *link* consists of a node, labeled by a connective of MALL ($\otimes, \parr, \&, \oplus$), together with two disjoint sets of edges (or *places*), the set of *top edges* and the set of *bottom edges*, and with a polarity :
– *positive links*, of type $\otimes$ or $\oplus$, must have *at least one bottom edge*, and possibly no top edge;
– *negative links*, of type $\parr$ or $\&$, must have *exactly one bottom edge* (see Figure 1).
A link l_1 is *just below* a link l_2 if there exists a top place of l_1 which is bottom place of l_2 (the notion "*just above*" is defined in the symmetric way).

A *proof-structure* is a set π of links satisfying the following conditions:
– the set of top (resp. bottom) edges of two links are disjoint;
– we associate an elementary weight to each &-link called its *eigen weight*;

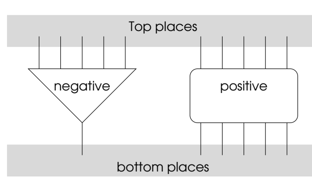

Figure 1:

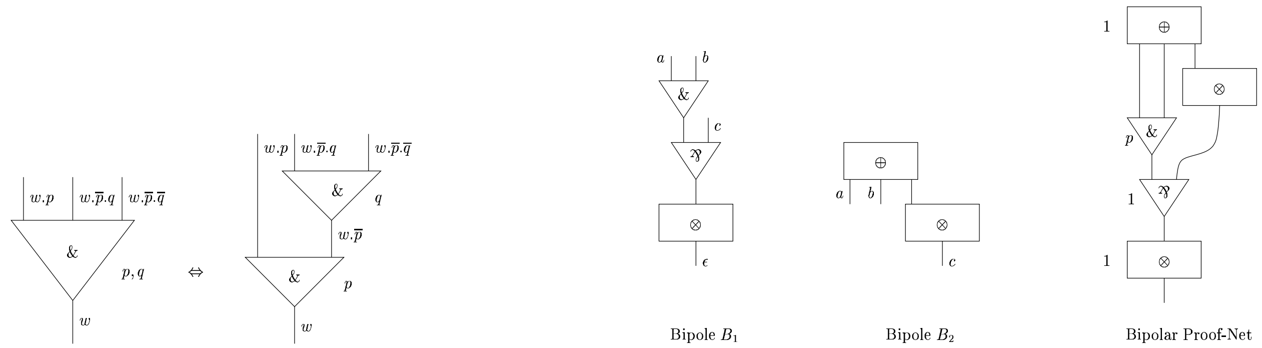

Figure 2:

– we associate a weight to each link (and its bottom edges) with the constraint that if two nodes have a common edge, they must have the same weight, except when:

1. the edge is a top place of a &-link then
   - (*binary case*) if w is the weight of a binary &-link and p is its eigen weight then w does not depend on p and its just above links must have weight $w.p$, resp., $w.\overline{p}$;
   - (n*-ary case*) if w is the weight of a generalized n-ary &-link, $n > 2$, and $b_1, ..., b_{n-1}$ are its eigen weights then w does not depends on $b_1, ..., b_{n-1}$ and its just above links must have weights, respectively, $w.w_1, ..., w.w_n$, with w_i, $1 \leq i \leq n$, a monomial depending only on $\epsilon_{b_1}, ..., \epsilon_{b_{n-1}}$; weights $w.w_1, ..., w.w_n$ are composed by simply iteration of the binary case (see the left side picture of of Figure 2).
2. the edge is a top or bottom edge of a $\oplus$-link, its weight w depends on an elementary weight p and there exists a different (bottom or top) edge of this $\oplus$-link whose weight w' depends also on p; in this case we take the (disjoint) sum of w, w'.

A proof-structure must also satisfy the following properties:
– a conclusion link, i.e. a link with only pending bottom edges, has weight 1;
– if w is the weight of a &-link with eigen weight p and w' is a weight depending on p and appearing in the proof structure then $w' \leq w$.

A *valuation* φ for a proof-structure π is a function from the set of the eigen weights of π into $\{0, 1\}$. Such a valuation can easily be seen as a function defined on the set of all the weights of π.

Given a valuation φ of a proof-structure π, the *slice* $\varphi(\pi)$ is the graph obtained from π by keeping only the links with weights w such that $\varphi(w) = 1$ and the edges just below a kept node.

Given a valuation φ of a proof-structure π, a switch $\mathcal{S}$ of π is defined as a non oriented graph constructed with the links and the edges of $\varphi(\pi)$ with the modifications:
– for each ⅋-link we keep only one top edge;
– for each &-link L we erase the top edge appearing in $\varphi(\pi)$ and we add an edge, called *jump edge*, from a link depending on L to L (this may change nothing).

A proof-structure π is is called *bipolar* when any edge occurring at the top of some positive link also occurs at the bottom of some negative link in π e vice-versa; moreover we say that π is a *bipole* when none negative link is just below a positive one, and it contains at least a positive link. Examples of bipolar structures are given on the right side of in Figure 2. A bipolar proof-structure can be seen as an union of disjoint bipoles: this decomposition is unique.

Bipolar Proof Net – A bipolar proof-structure is a *proof-net* if all its switches are acyclic and connected.

References

[1] Girard, J.-Y. Linear Logic. *Theoretical Computer Science* 50:1-102, 1987.

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[3] Andreoli, J.-M. Focussing and Proof Construction. *Ann. of Pure and App. Logic*, 107(1):131-163, 2001.