GMT 19/20 Lecture 15 8/5/20

Properties of (and remarks on) rectif. & p.unrect. sets (continued)

- $\mathcal{H}^d(E) = 0 \;\not\Longrightarrow\;$ E can be covered by countably many Lipschitz images of sets in $\mathbb{R}^d$

$\exists$ K compact in $\mathbb{R}^2$ s.t. $\dim_{\mathcal{H}}(K) = 0$ but K cannot be covered by countably many Lipschitz images of $\mathbb{R}$.

Q. Characterize K compact in $\mathbb{R}^2$ s.t. K is covered by 1 curve of finite length $\longleftrightarrow$ Travelling Salesman Problem. due P. Jones

Thus E_0 in the def. of rectif. sets CANNOT be removed

- We cannot replace surfaces of class C^1 in the def. of rectif. sets by surface of class C^2 or even $\mathcal{C}^{1,\alpha}$ $\alpha > 0$.

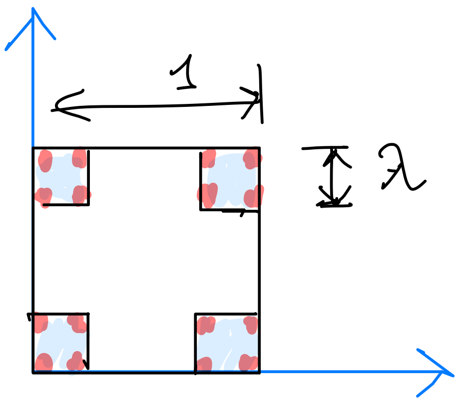
C^1 functions do NOT have the Lusin property with functions of class $C^{1,\alpha}$, $\alpha > 0$.

$\exists\, f \in C^1([0,1])$ s.t. $\forall \alpha$ $\forall g \in \mathcal{C}^{1,\alpha}([0,1])$
$\{x : f(x) = g(x)\}$ is $\mathcal{L}^1$-null.

- Fix $d \in (0,2]$. Then there exists K compact in $\mathbb{R}^2$ s.t. $\dim_H(K) = d$ and K is 1-purely unrect.

(rectifiability is not related to dimension)

fix $0 < \lambda < \frac{1}{2}$



K is a self-sim. fractal associated to 4 similarities with scaling factor λ

Then $d = \dim(K) = \dfrac{\log 4}{\log(1/\lambda)}$ (any number $d \in (0,2)$)

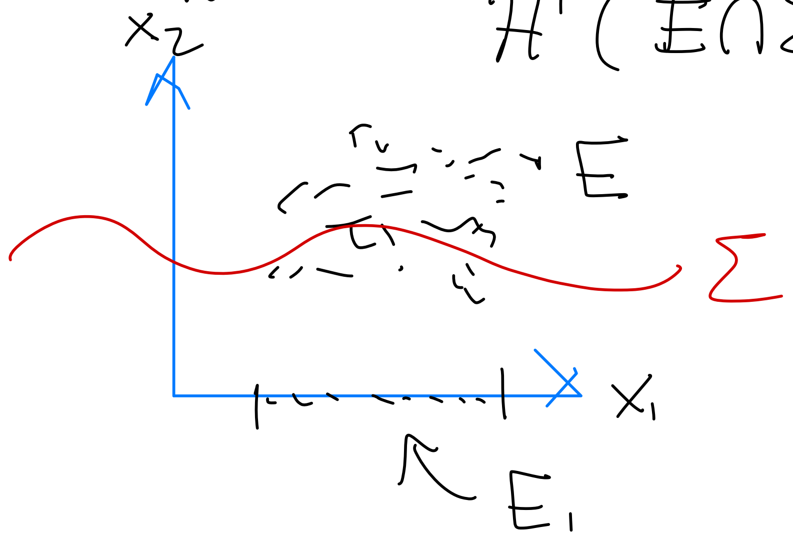
K is 1-p.u. because of

Lemma Given $E \subset \mathbb{R}^2$ with projections E_1, E_2 on the axes, if $|E_1| = |E_2| = 0$ then E is 1-p.u.

Proof

Step 1: if Σ is the graph of $g : \mathbb{R} \to \mathbb{R}$ of class C^1, $x_2 = g(x_1)$

then $\mathcal{H}^1(E \cap \Sigma) = 0$



$$\mathcal{H}^1(E \cap \Sigma) = \int_{P_1(E\cap\Sigma)} \sqrt{1+\dot{g}^2}\, dx_1$$

$$\leq \int_{P_1(E) = E_1} \sqrt{1+\dot{g}^2}\, dx_1 = 0$$

because $|P_1(E)| = 0$
$\overset{\|}{E_1}$

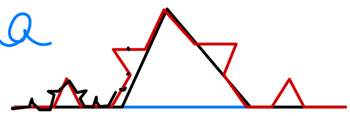
Step 2: same if Σ = graph of g, $x_1 = g(x_2)$

Step 3: every curve Σ of class C^1 can be covered by C^1 graphs.

Ex: propose a generalization to arbitrary n and d.

- Note that also the von Koch curve is 1-p.u.

(but previous lemma does not apply here



- If E is $\mathcal{H}^d$-finite (or σ-finite)
  then $E = E_r \cup E_{pu}$ with
  E_r d-rectifiable.
  E_{pu} d-purely unrectif.
  (decomposition is unique up to $\mathcal{H}^d$-null sets)

Sketch of Proof: Let $\mathcal{F} = \{F \text{ } d\text{-rectif.}, F \subset E\}$

$\mathcal{F}$ is closed under countable union.

Then $\mathcal{F}$ has an element E_r which maximizes $\mathcal{H}^d$

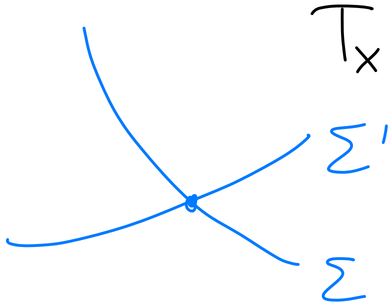
Note that $E \setminus E_r$ is p.u. $\square$

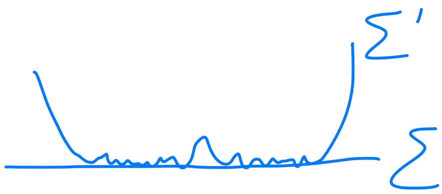
Tangent bundle (to a rectif. set)

Preliminary remarks

<u>Prop</u> Given $\Sigma, \tilde{\Sigma}$ d-dim. surface of class C^1 in $\mathbb{R}^n$ then

$$T_x\Sigma = T_x\tilde{\Sigma} \quad \text{for } \mathcal{H}^d\text{-a.e. } x \in \Sigma \cap \tilde{\Sigma}$$





Corollary of

<u>Lemma</u> Given $f, \tilde{f} : \Omega \text{ open} \subset \mathbb{R}^d \longrightarrow \mathbb{R}^{\cancel{n-d}}$ then $\nabla f(x) = \nabla\tilde{f}(x)$ for $\mathcal{L}^d$-a.e. x where $f(x) = \tilde{f}(x)$.

Borel

<u>Def</u> Given a set $E \subset \mathbb{R}^n$ and $V : \underset{E}{\overset{\cap}{x}} \longrightarrow \underset{Gr(n,d)}{\overset{\cap}{V(x)}}$

we say that V is a weak tangent bundle to E if $\forall\, \Sigma$ d-dim. surface of class C^1

$$T_x\Sigma = V(x) \quad \text{for } \mathcal{H}^d\text{-a.e. } x \in \Sigma \cap E$$

<u>Rem</u> If V exists, it is unique up to p.n. subset of E

Prop If E is d-rectifiable then it admits a weak tangent bundle
(which is unique up to $\mathcal{H}^d$-null sets)

Proof Write $E = \dot{\bigcup} E_i$ ← disjoint union
with $\mathcal{H}^d(E_0) = 0$, $E_i \subset \Sigma_i$ ← surface of class C^1

Set
$$V(x) = \begin{cases} T_x\Sigma_i & \text{if } x \in E_i\,,\ i>0. \\ \text{whatever} & \text{if } x \in E_0 \end{cases}$$

Take Σ C^1-surface: Then

because of Prop. above

$$V(x) = \cancel{T_x\Sigma_i} = T_x\Sigma \qquad \text{for } \mathcal{H}^d\text{-a.e. } x \in E_i \cap \Sigma$$

because of def di V

$$\Downarrow$$

$$V(x) = T_x\Sigma \quad \text{for } \mathcal{H}^d\text{-a.e. } x \in E \cap \Sigma.$$

NOTATION I write $T^W_x E$ for V.

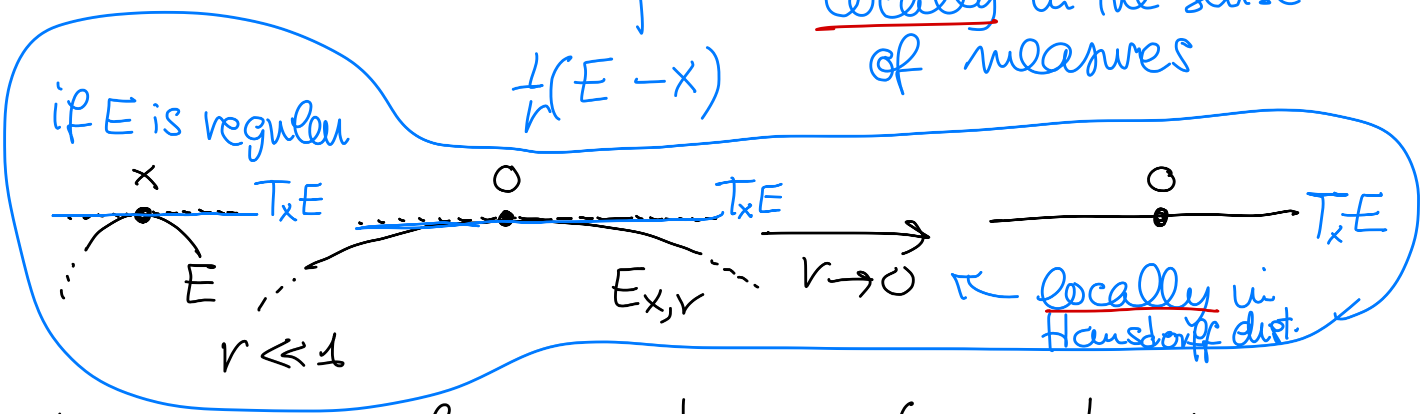
If E is not only rectif. but is also $\mathcal{H}^d$-locally finite, then we have stronger tangency properties.

Def Assume $E \subset \mathbb{R}^n$ is $\mathcal{H}^d$-locally finite and $V \in Gr(n,d)$. We say that V is an *approximate* tangent space to E at x if

$$\mathcal{H}^d \llcorner E_{x,r} \longrightarrow \mathcal{H}^d \llcorner V$$

$E_{x,r} = \frac{1}{r}(E - x)$

locally in the sense of measures



that is $$\int_{E_{x,r}} g \, d\mathcal{H}^d \longrightarrow \int_V g \, d\mathcal{H}^d \quad \forall g \in C_c(\mathbb{R}^d)$$

Rem If V exists it is unique, and denoted by $T_x^a V$

Prop If E is d-rectifiable & $\mathcal{H}^d$-locally finite then $T_x^w E$ is the approximate tangent space to E at x for $\mathcal{H}^d$-a.e. x.

Proof Write $E = \bigcup_i E_i$ with

$\mathcal{H}^d(E_0) = 0$, $E_i \subset \Sigma_i$ C^1-surface for $i > 0$.

I fix $i > 0$ and prove that

$$\mathcal{H}^d \llcorner E_{x,r} \xrightarrow[(\dots)]{} \mathcal{H}^d \llcorner T_x \Sigma_i \qquad \text{for } \mathcal{H}^d\text{-a.e. } x \in E_i$$

Indeed

$$(E)_{x,r} = \underset{\Sigma}{\underset{\cup}{(\Sigma_{\cancel{i}})_{x,r}}} \setminus (\Sigma_{\cancel{i}} \setminus E)_{x,r} \cup (E \setminus \Sigma_{\cancel{i}})_{x,r}$$

then

$$\mathcal{H}^d \llcorner E_{x,r} = \mathcal{H}^d \llcorner \Sigma_{x,r} - \mathcal{H}^d \llcorner (\Sigma \setminus E)_{x,r} + \mathcal{H}^d \llcorner (E \setminus \Sigma)_{x,r}$$

$$\mathcal{H}^d \llcorner \Sigma_{x,r} \xrightarrow{r \to 0} \mathcal{H}^d \llcorner T_x \Sigma \quad \text{for } x \in \Sigma$$

$$\mathcal{H}^d \llcorner (\Sigma \setminus E)_{x,r} \to 0 \quad \text{for } \mathcal{H}^d\text{-a.e. } x \in E_i$$

$$\mathcal{H}^d \llcorner (E \setminus \Sigma)_{x,r} \to 0 \quad \text{for } \mathcal{H}^d\text{-a.e. } x \in E_i$$

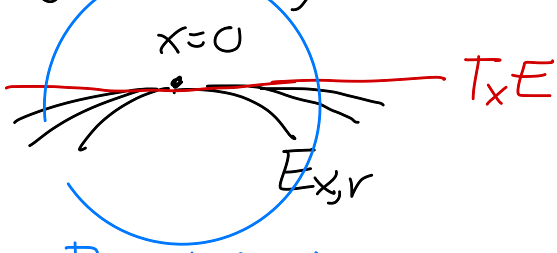
the mass on every Ball $B_R = B(0,R)$ converge to 0

(i) $\mathcal{H}^d \llcorner \Sigma_{x,r} \longrightarrow \mathcal{H}^d \llcorner T_x\Sigma \quad \forall x \in \Sigma$

locally in the sense of measures

Intuitively natural

$\boxed{\text{but check the details!}}$!



$B_R = B(0,R)$

(ii) $$M_{B_{\cancel{R}\,1}}\left(\mathcal{H}^d \llcorner (\Sigma \setminus E)_{x,r}\right) = \mathcal{H}^d\left((\Sigma \setminus E)_{x,r} \cap B_{\cancel{R}\,1}\right)$$

proof for R=1 only

$$= \frac{\mathcal{H}^d(\Sigma \setminus E \cap B(x,r))}{r^d}$$

$$\xrightarrow[r \to 0]{} \Theta^d(\Sigma \setminus E, x) \cdot \alpha_d = 0$$

for $\mathcal{H}^d$-a.e. $x \notin \Sigma \setminus E$

in partic. for $\mathcal{H}^d$-a.e. $x \in E$:

(iii) similar to (ii)

$\square$