GMT 19/20 lecture 17, 15/5/2020

Def. A cone $\mathcal{C} = \mathcal{C}(x, V, \alpha)$ is approx. tangent to a set E at x if

$V \in Gr(n,d)$, $\alpha \in (0, \frac{\pi}{2})$

$$\mathcal{H}^d(E \cap B(x,r) \cap \mathcal{C}^c) = o(r^d)$$

as $r \to 0$

Theorem 4 Let $E \subset \mathbb{R}^n$

Assume that at every $x \in E$:

(i) there exists an app. tangent cone $\mathcal{C}_x = \mathcal{C}(x, V(x), \alpha(x))$

(ii) $\Theta^d_*(E,x) > 0$.

Then E can be covered by countably many d-dimensional Lipschitz graphs

($\Rightarrow$ E is d-rectif.)

Lemma Take E as in Th. 4.
Fix $\alpha \in (0, \frac{\pi}{2})$, $V \in Gr(n,d)$, $\delta, r_0 > 0$ and let E_{α,V,δ,r_0} be the subset of all $x \in E$ s.t.

(i) $\mathcal{H}^d(E \cap B(x,r)) \geq \delta r^d \quad \forall r \leq r_0$

(ii) $\mathcal{H}^d(E \cap B(x,r) \cap \mathcal{C}^c(x,\alpha,V)) \leq \delta' r^d \quad \forall r \leq r_0$

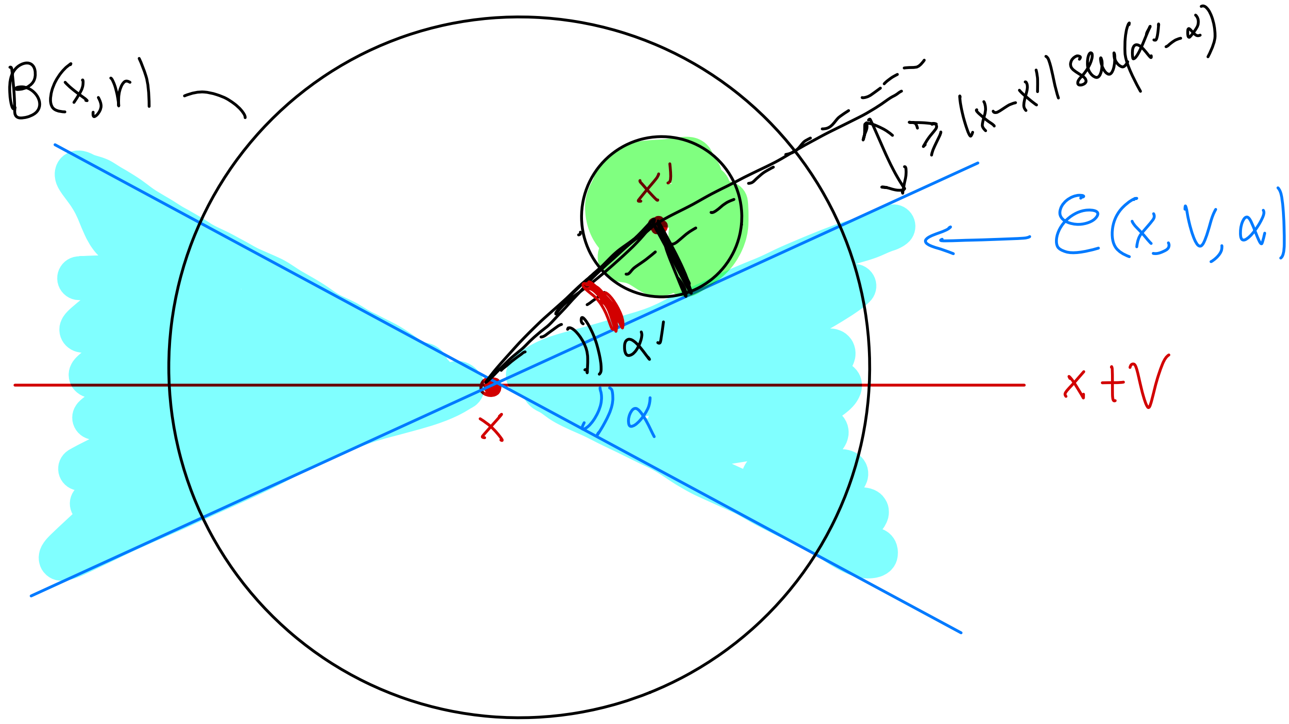
where $\delta' := \delta \left(\frac{\sin(\alpha'-\alpha)}{3}\right)^d \quad \alpha' := \frac{1}{2}\left(\frac{\pi}{2} - \alpha\right)$

Assume $F \subset E_{\alpha,V,\delta,r_0}$ and $\operatorname{diam}(F) \leq \frac{r_0}{2}$.
Then F is contained in the graph of a Lipschitz map $g : V \to V^\perp$
(and $\operatorname{Lip}(g) \leq \tan\alpha'$ where $\alpha' = \frac{1}{2}\left(\frac{\pi}{2} + \alpha\right)$)

Proof It suffices to show that $\forall\, x, x' \in F$ then $x' \in \mathcal{C}(x, V, \alpha')$

$\boxed{\text{this implies that } F \text{ is the graph of a Lipschitz map } g : \pi_V(F) \to V^\perp.}$

By contrad., assume $x' \notin \mathcal{E}(x, V, \alpha')$.



Let $r := 2|x'-x|$, $\rho := |x-x'| \cdot \sin(\alpha'-\alpha)$

$= \frac{r}{2} \sin(\alpha'-\alpha)$

then (since diam $F \leq \frac{r_0}{2}$)

$r \leq r_0$

$\rho = \frac{r}{2} \sin(\alpha'-\alpha) < \frac{r}{2}$

$B(x', \rho) \subset B(x, \frac{r}{2}+\rho) \subset B(x, r)$

$B(x', \rho) \subset B(x, r) \setminus \mathcal{E}(x, V, \alpha)$ (*)

Then

by (*) and ass. (ii) ; by (i)

$$\delta' r^d \geq \mathcal{H}^d(E\cap B(x',\rho)) \geq \delta \rho^d$$
$$= \delta\left(\frac{\sin(\alpha'-\alpha)}{2}\right)^d r^d$$

Contradiction!

because $\delta' < \delta\left(\frac{\sin(\alpha'-\alpha)}{2}\right)^d$. $\square$

Proof of Theorem 4

We write E as countable union of subsets that satisfy the ass. of the lemma!!

More precisely $\forall r_0$ let $\mathcal{Y}_{r_0}$ a countable family of sets D with $\mathrm{diam}(D) \le \frac{r_0}{2}$ which cover $\mathbb{R}^n$

$\forall m$ let $\mathcal{G}_m$ be a finite set in $G(n,d)$ s.t. $\forall \tilde{V} \in G(n,d)$ $\exists V \in \mathcal{G}_m$ with $\tilde{V} \subset \mathcal{C}(0,V,\frac{1}{m})$. I claim that

$$E = \bigcup_{\substack{m=1,2,\ldots \\ m'=1,2,\ldots}} \bigcup_{V\in\mathcal{G}_m} \underbrace{\bigcup_{D\in\mathcal{Y}_{1/m'}} \left(E_{\alpha,V,\delta,r_0} \cap D\right)}_{E_{\alpha,V,\delta,r_0}}$$

$$\alpha = \frac{\pi}{2}-\frac{1}{m}, \quad \delta = \frac{1}{m}, \quad r_0 = \frac{1}{m'}$$

take $x \in E$, I show that $\exists\, m, m', V$

s.t. $x \in E_{\alpha, V, \delta, r_0}$ with $\alpha = \frac{\pi}{2} - \frac{1}{m}$, $\delta = \frac{1}{m}$, $r_0 = \frac{1}{m'}$

$C_x = \mathcal{C}(x, \alpha(x), V(x))$

I choose m s.t.

$$\Theta^d_*(E, x) > \frac{1}{m} \quad \& \quad \alpha(x) \le \frac{\pi}{2} - \frac{2}{m}$$

$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

$$\boxed{\mathcal{H}^d(E \cap B(x,r)) \geq \frac{r^d}{m}}$$

for r ~~small enough~~ for $r \le \frac{1}{m'}$

$\exists\, V \in G_{d\,m}$ s.t. $V(x) \subset \mathcal{C}(V, \frac{1}{m})$

$$\Downarrow$$

$$\mathcal{C}_x = \mathcal{C}(x, V(x), \alpha(x)) \subset \mathcal{C}\Big(x, V, \underbrace{\tfrac{\pi}{2} - \tfrac{1}{m}}_{\alpha}\Big)$$

$$\Downarrow$$

$$\mathcal{H}^d\left(E \cap B(x,r) \cap \mathcal{C}^c\left(x, V, \tfrac{\pi}{2} - \tfrac{1}{m}\right)\right) = o(r^d)$$

$$\Downarrow$$

$$\boxed{\mathcal{H}^d\left(E \cap B(x,r) \cap \mathcal{C}^c\left(x, V, \tfrac{\pi}{2} - \tfrac{1}{m}\right)\right) \le \delta' r^d}$$

for r ~~small enough~~ for $r \le \frac{1}{m'}$

Choose m' s.t.

$\square$

Final details/remarks on rectifiable sets.

1. About Th. 4.

$\forall\, n, d$ there exists a compact set $K \subset \mathbb{R}^n$ s.t. $\mathcal{H}^d(K) > 0$ BUT $\Theta^d_*(K, x) = 0$ for $\mathcal{H}^d$ a.e. $x \in K$.

This means that the ass. on $\Theta^d_*(E, x)$ in Th. 4 cannot be removed for free.

2. The following (not hard to prove) Th. holds:

If E is $\mathcal{H}^d$-locally finite & for $\mathcal{H}^d$-a.e. $x \in E$ $\exists\, V(x) \in Gr(n,d)$ s.t. $\mathcal{C}(x, V(x), \alpha)$ is app. tang. to E at x $\forall\, \alpha > 0$,

Then E is d-rectif.

3| Characterization of rectifiable sets by projections.

Observe that if E is d-rectif. in $\mathbb{R}^d$ then $\mathcal{H}^d(\underbrace{\pi_V}_{\text{orthog. projection on } V}(E)) > 0$ for a.e. $V \in Gr(n,d)$

a.e.: w.r.t. Haar measure on $Gr(n,d)$

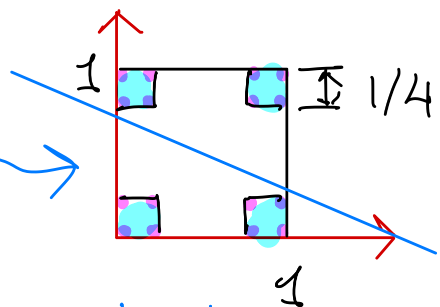
(if $d=1$, $\mathcal{H}^1(\pi_V(E)) = 0$ for all lines $V \in Gr(n,1)$ except 1)

Theorem (Besicovitch, $\boxed{\text{Federer}}$)

If E is $\mathcal{H}^d$-finite & d-pur. unrect. then

$$\mathcal{H}^d(\pi_V(E)) = 0 \quad \text{for a.e. } V.$$

Check by hand this case →

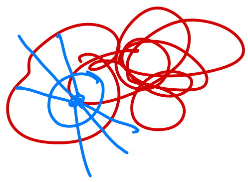


What about the von Koch curve (which is 1-pur. unrectif.)?

4 Th. If K compact $\subset X$, $\mathcal{H}^1(K) < +\infty$ and K is connected, then K is covered by a single continuous path $\gamma : [0,1] \to X$ ~~with finite length~~ Lipschitz

in particular K is rectifiable.

Moreover tangent lines are tangent in the classical sense, not just in the approx. sense.



5 Charact. of rectifiable sets by density.

Observe that if E is d-rectif. in $\mathbb{R}^n$ and $\mathcal{H}^d$-locally finite then

$$\Theta(E,x) = 1 \quad \text{at } \mathcal{H}^d\text{-a.e. } x$$

Conversely

Theorem (Besicovitch, Marstrand-Mattila, $\boxed{\text{Preiss}}$)

If E is $\mathcal{H}^d$-locally finite &

$\Theta_d(E,x)$ exists for $\mathcal{H}^d$-a.e. $x \in E$

Then $d \in \mathbb{N}$ and E is d-rectifiable.

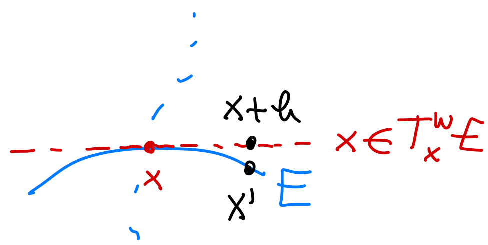
6 | Rademacher theorem and area formula for rectifiable sets.

Let E be d-rectif. set in $\mathbb{R}^m$, let $f : \mathbb{R}^m \to \mathbb{R}^n$ be Lipschitz.

Th (Rademacher)

For $\mathcal{H}^d$ a.e. $x \in E$, f is tangentially differentiable at x, that is, $\exists\ T : T_x^w E \to \mathbb{R}^n$ linear s.t.

$$f(x+h) = f(x) + Th + o(|h|) \quad \text{as } h \to 0$$
$$\forall h \in T_x^w E$$



$$f(x') = f(x) + T\big(\pi(x'-x)\big) + o(|x'-x|)$$

In particular T depends only on the restriction of f to E, T is unique, denoted by $d_x^{tau} f$

Sketch of Proof First: prove it for $E \subset \Sigma$, Σ d-dim. surface of class C^1

Second: general case.

Then for $\mathcal{H}^d$-a.e. $x \in E$ define the tangential Jacobian of f at x

$$J_{tan} f(x) := |\det(d_x^{tan} f)|$$
$$= \sqrt{\det\big(\nabla_{tan}^t f(x) \underbrace{\nabla_{tan} f(x)}_{n \times d \text{ matrix}}\big)}$$

<u>Th</u> (area formula)

Take E, f as above. Then

$$\underbrace{\int \#\left(f^{-1}(y) \cap E\right) d\mathcal{H}^d(y)}_{\substack{\mathcal{H}^d \text{ measure of } f(E) \\ \text{counting multiplicity}}} = \int_E J_{tan} f(x)\, d\mathcal{H}^d(x)$$

Sketch of proof

First case: $E \subset \Sigma \leftarrow$ d-dimensional surface of class C^1

(essentially proved few lectures ago)

Second case: general E.