GMT 19/20 Lecture 25 4/6/20

Setting: $f : \mathbb{R}^d \to \mathbb{R}^n$

Given ω, k-form on $\mathbb{R}^n$, $f^\# \omega$ is defined by (same reg. of f req.)

$$f^\# \omega(x) := (d_x f)^\# \cdot \omega(f(x)) .$$

Then

$$\underbrace{\| f^\# \omega(x) \|}_{\text{comass norm}} \leq \underbrace{\| d_x f \|}_{\text{operator norm}}^k \underbrace{\| \omega(f(x)) \|}_{\text{comass norm}}$$

In particular

(1) $$\| f^\# \omega \|_\infty \leq (\mathrm{Lip}(f))^k \| \omega \|_\infty$$

Moreover

(2) $$d(f^\# \omega) = f^\# (d\omega) .$$

If T is a k-current in $\mathbb{R}^d$, then $f_\# T$ is the k-current in $\mathbb{R}^m$ given by

$$\langle f_\# T, \omega \rangle := \langle T, f^\# \omega \rangle$$

k-form in $\mathbb{R}^m$

Some regularity of f is required, and also that f is PROPER.

Latter ass. is not needed if T has compact support.

! I will assume that T has compact support for the rest of this lecture.

Then (1) gives

assume $\mathbb{M}(T) < +\infty$
$T = \tau\mu, |\tau| = 1$
$f \in C^1$

$$\langle f_\# T, \omega \rangle = \langle T, f^\# \omega \rangle = \int \langle f^\# \omega, \tau \rangle \, d\mu$$

and

$$|\langle f_\# T, \omega \rangle| \leq \int \| f^\# \omega(x) \| \cdot \| \tau(x) \| \, d\mu(x) \leq \int \| d_x f \|^k \, \| \omega(x) \| \, \| \tau(x) \| \, d\mu(x)$$

and

$$M(f_\# T) \leq \int \|d_x f\|^k \, d\mu$$

$$\leq \Big(\sup_{x\in \operatorname{spt}(\mu)} \|d_x f\|\Big)^k \|\mu\| \quad M(T)$$

$$\leq (\operatorname{Lip}(f))^k \, M(T)$$

From (2) we get

$$\partial(f_\# T) = f_\# (\partial T).$$

Push-forward of a rectif. current.

Let $T = [E, \tau, m]$ with E k-rectif. & bounded

Let $\tilde{E} := f(E)$ ($\tilde{E}$ is k-rectif. !!!)

Let $\tilde{\tau}$ be any orientation of $\tilde{E}$. Then

$$(3) \qquad \boxed{f_\# T = [\underset{\substack{\shortparallel \\ f(E)}}{\tilde{E}}, \tilde{\tau}, \tilde{m}]}$$

where $\tilde{m}$ is given by

$$(4) \qquad \underset{\substack{\cap \\ \tilde{E} = f(E)}}{\tilde{m}(y)} := \sum_{x \in f^{-1}(y) \cap E} \pm m(x) \quad \text{for } \mathcal{H}^k\text{-a.e. } y \in \tilde{E}$$

where the sign $\pm$ is $+$ if

$d_x f : T_x E \to T_y \tilde{E}$ preserves orient.

and is $-$ otherwise.

(cf. def. of <u>degree</u> of a map)

<u>Rems</u> (Assume for simplicity that $\mathcal{H}^k(E) < +\infty$)

(1) Recall the area formula:

$$\int_{y \in \tilde{E}} \#(f^{-1}(y) \cap E) \, d\mathcal{H}^k(y) = \int_E J_T f(x) \, d\mathcal{H}^k(x) < +\infty$$

Then $f^{-1}(y) \cap E$ is finite for $\mathcal{H}^k$-a.e. y and the def. of $\tilde{m}(y)$ is well-posed

(2) Let $S := \{ x \in E : d_x f : T_x E \to T_{f(x)} \tilde{E}$ is NOT surjective $\}$

Then by the area formula above $\mathcal{H}^k(f(S)) = 0 \Rightarrow$ for $\mathcal{H}^k$-a.e. $y \in \tilde{E}$ and every $x \in f^{-1}(y) \cap E$, $d_x f$ is surjective (= maximal rank)

Proof of (3) $\forall$ k-form $\omega \in \ldots$

$$\langle f_\# T, \omega \rangle := \langle T, f^\# \omega \rangle$$

$$= \int_E \langle f^\# \omega(x) , \underset{\substack{\| \\ \tau_1 \wedge \cdots \wedge \tau_k}}{\tau(x)} \rangle \, m(x) \, d\mathcal{H}^k(x)$$

$$= \int_E \langle \omega(f(x)) ; \underbrace{d_x f \cdot \tau_1(x) \wedge \cdots \wedge d_x f \cdot \tau_k(x)}_{\substack{\| \\ \pm \, J_T f(x) \cdot \tilde{\tau}(f(x))}} \rangle \, m \, d\mathcal{H}^k$$

same as in def. of $\tilde{m}$

$$= \int_E \langle \omega(f(x)) ; \tilde{\tau}(f(x)) \rangle \underbrace{(\pm m(x))}\underbrace{J_T f(x) \, d\mathcal{H}^k}$$

area formula

$$= \int_{f(E) = \tilde{E}} \langle \omega(y) ; \tilde{\tau}(y) \rangle \underbrace{\sum_{x \in f^{-1}(y) \cap E} \pm m(x)}_{\tilde{m}(y)} \, d\mathcal{H}^k(y)$$

$$= \langle [\tilde{E}, \tilde{\tau}, \tilde{m}] ; \omega \rangle .$$

Hence $f_\# T = [\tilde{E}, \tilde{\tau}, \tilde{m}]$. $\square$

Note that (3) implies that if T is rect. so is $f_\# T$, and if T is integral so is $f_\# T$.

Resums / Exercises

(1) let $f: \mathbb{R} \longrightarrow \mathbb{R}$, $f: x \longmapsto e^x$ smooth, but not proper

a) let $T := \sum_0^\infty \delta_{-n}$ 0-current

$\left(\langle T, \phi \rangle = \sum_0^\infty \phi(-n) \right)$

then $f_\# T$ should be $\sum_0^\infty \delta_{e^{-n}}$, but this is NOT a well-def. current.

Actually $f_\# T$ is NOT well-defined !!

b) let $T := [(-\infty, 0), e, 1]$ (rectif. 1-curr.)

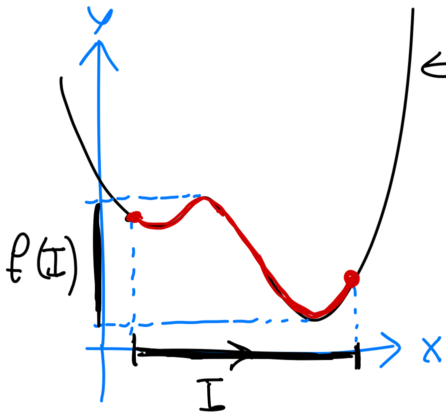
Then $f_\# T := [(0,1), e, 1]$

BUT $\partial(f_\# T) = \delta_1 - \delta_0 \neq f_\#(\partial T) = \delta_1$.

Note that $f: x \longmapsto e^x$, $f: \mathbb{R} \to (0, +\infty)$ is PROPER !

Then everything should work in a) and b) (check it out!!)

Ex 2



$\leftarrow y = f(x)$ $\quad f : \mathbb{R} \to \mathbb{R}$

let $T := [I, e, 1]$

What is $f_\# T$?

Ex 3

sphere in $\mathbb{R}^3$

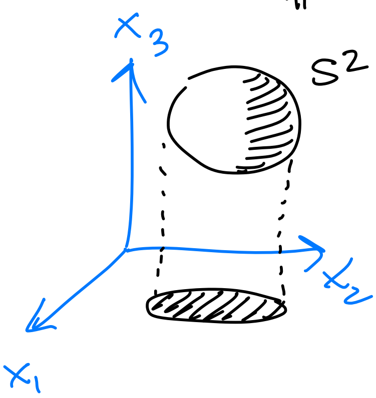
Let $T := T_{S^2} = [S^2, \tau_{S^2}, 1]$ in $\mathbb{R}^3$

rect. 2-current with compact support

Let $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$, $\quad x = (x_1, \dots, x_3) \longrightarrow (x_1, x_2)$

f smooth (not proper)

What is $f_\# T$? $\quad$ Should be $f_\# T = 0 \dots$



App. of push-forward

Th. 1 Let $T = \tau\mu$ be a <u>normal</u> k-current in $\mathbb{R}^d$.
$(|\tau| = 1)$ $\quad$ Then

$$\mu \ll \mathcal{I}^k \ll \mathcal{H}^k$$

integral geometric measure

in particular (if $T \neq 0$) $\dim_{\mathcal{H}}\left(\overset{\operatorname{supp} T}{\overset{\shortparallel}{\operatorname{supp}\mu}}\right) \geq k$.

Idea of proof (assume T has compact supp.)

I must show that $\underbrace{\mathcal{I}^k(E)=0}_{\substack{\Updownarrow \\ \mathcal{H}^k(P_V(E))=0 \\ \text{for a.e. } V\in G(n,k)}} \Rightarrow \mu(E)=0.$

Take V s.t. $\mathcal{H}^k(P_V(E))=0$.

Then $T_V := (P_V)_\# T$ is a k-normal current in $V \simeq \mathbb{R}^k$.

Then $T_V = [\mathbb{R}^k, e, u]$ with $u \in BV_{loc}(\mathbb{R}^k)$

Then $T_V(P_V(E)) = \langle T_V, 1_{P_V(E)} \cdot dx \rangle = 0$

$\overset{?}{\Rightarrow}$ $T(E) = \int_E \tau \, d\mu = 0$

as a vector measure

$\overset{?}{\Rightarrow}$ $\mu(E)=0$ $\square$

Application 2: Homothopy formula

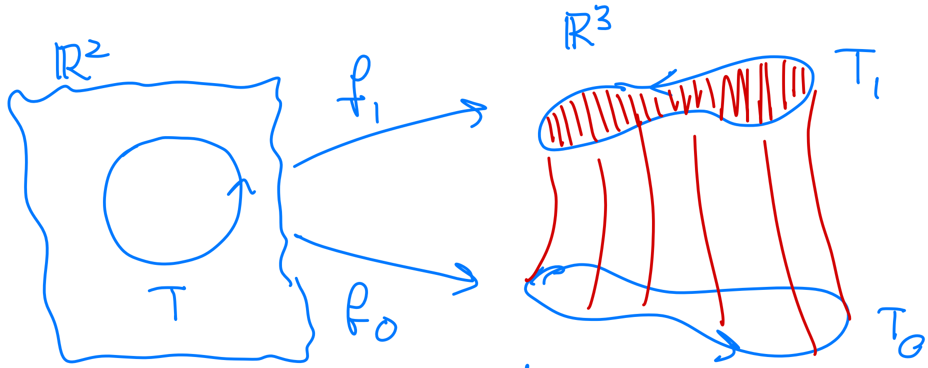
T is a k-current in $\mathbb{R}^d$ with compact support and $\partial T = 0$.

Let $f_0, f_1 : \mathbb{R}^d \to \mathbb{R}^m$ be homothopic maps, i.e., $\exists\ F : [0,1]\times\mathbb{R}^d \to \mathbb{R}^m$ with $F(0,x)=f_0(x)$, $F(1,x)=f_1(x)$.

(same regularity of f_0, f_1, F is needed!)

Let $T_0 := (f_0)_\# T, \quad T_1 := (f_1)_\# T.$

Then
$$T_1 - T_0 = \partial S$$



where $\quad S := F_\# (\underbrace{I \times T}_{k+1 \text{ current in } \mathbb{R}\times\mathbb{R}^k})$

where $\quad I := T_{[0,1]} = [[0,1], e, 1]$ 1- current in $\mathbb{R}$

Proof
$$\begin{aligned} \partial S &= \partial F_\# (I \times T) \\ &= F_\# (\partial (I \times T)) \\ &= F_\# (\partial I \times T - I \times \underbrace{\cancel{\partial T}}_{=0}) \\ &= F_\# ((\delta_1 - \delta_0) \times T) \\ &= F_\# (\delta_1 \times T) - F_\# (\delta_0 \times T) = T_1 - T_0 . \end{aligned}$$

$\square$

Rems

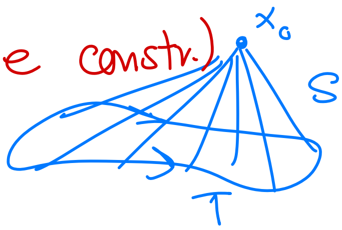
(1) If $\mathbb{M}(T)$ (we can take f_0, f_1, F of class C^1) then S has finite mass

(2) If T is rectif./integral then S is rectifiable/integral.

Application of homothopy formula.

Th. If T is a k-current in $\mathbb{R}^d$ ($k \geq 1$) with $\partial T = 0$ ($\mathbb{M}(T) < +\infty$ & compact support)(T rect./integ.) then $T = \partial S$ ($\mathbb{M}(S) < +\infty$ & compact supp.)(S rect./int.)

Proof (cone constr.)



Take $x_0 \in \mathbb{R}^d$, take $F : [0,1] \times \mathbb{R}^d \to \mathbb{R}^d$

s.t. $F(1,x) = x$ ($= f_1(x)$) & $F(0,x) = x_0$ ($= f_0(x)$)

e.g. $F(t,x) = tx + (1-t)x_0$.

Take $S := F_\#(I \times T)$.

Then $\partial S = \underbrace{(f_1)_\# T}_{T} - \underbrace{(f_0)_\# T}_{0} = T$. $\square$

Ex Constancy lemma + pushforward of rectif. currents + homothopy formula

$\Downarrow$

Theory of degree for maps between oriented manifolds.