Building exotic manifold
PART II

- Existence of exotic $\mathbb{R}^4$
  - Positive 0-Whitehead doubles of knots;
- A conjecture of Zeeman;

<u>Ref.</u>: Akbulut's book Ch. 9.2, 9.3, 9.4.

Exotic $\mathbb{R}^4$

<u>Tool</u>: A knot which is top. slice but not smoothly slice.

<u>TOP.</u>:

<u>Thm.</u>: A knot K is top. slice iff it has trivial Alexander polynomial.

<u>Prop.</u>: The 0-Whitehead double of a knot has trivial Alexander polynomial.

SMOOTH:

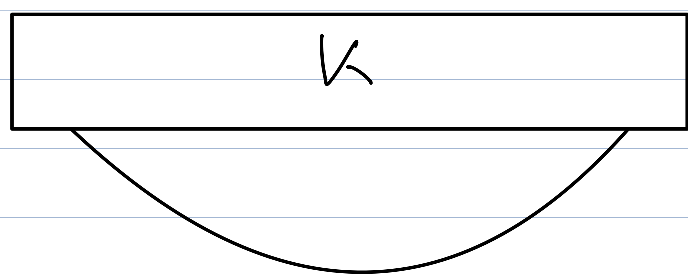
<u>Recall</u>: If a knot K has $TB(K) \geq 0$, then K is not smoothly slice.

Thm.: Let K be a legendrian knot with $TB(K) \geq 0$. Then we can find a legendrian diagram for $Wh_0^+(K)$ such that

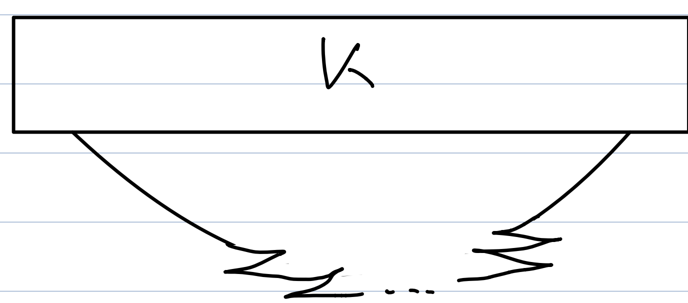
$$TB(Wh_0^+(K)) = 1.$$

Proof.:

Consider the knot K:



1° step) We add cusps to K in order to obtain $TB(K)=0$.



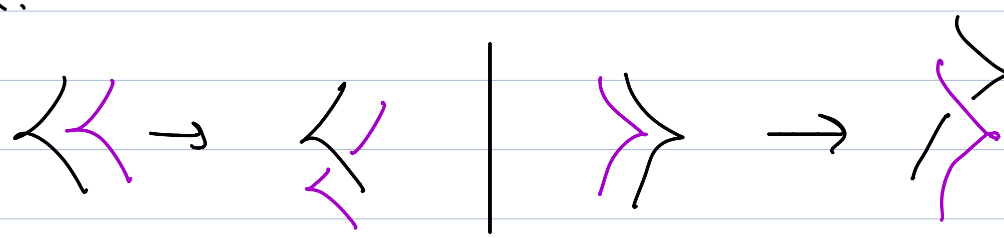
$TB(K)=0$.

2° step) Consider K' which is the blackboard framing pushoff of K.

$$lk(K', K) = w(K)$$

$$TB(K) = w(K) - c(K) = 0 \Rightarrow w(K) = c(K).$$

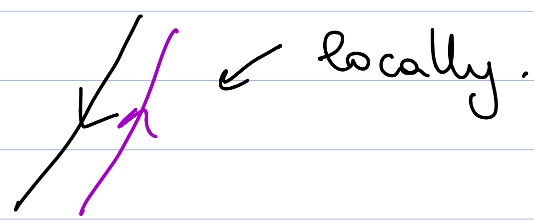
Modify K' near the cusps in order to obtain a 0-framing pushoff of K.



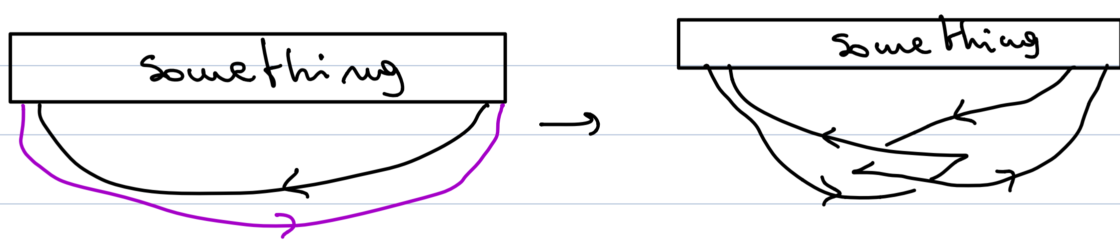
With this modification K' is the 0-framing pushoff of K. We call it K_0'.

$TB(K_0') = 0$. $\qquad lk(K, K_0') = 0$.

Note: I can change orientation on K_0' preserving the properties.



3° step) Obtain $Wh_+^0(K)$ and compute its TB



Compute $\tau B(Wh_0^+(K))$.

Inside $\boxed{\text{something}}$ I have
$w(\text{box}) - c(\text{box}) = 0$.

Below the box I have $+2-1=1$.

In the end I find

$$\tau B(Wh_0^+(K)) = 1.$$

$\square$

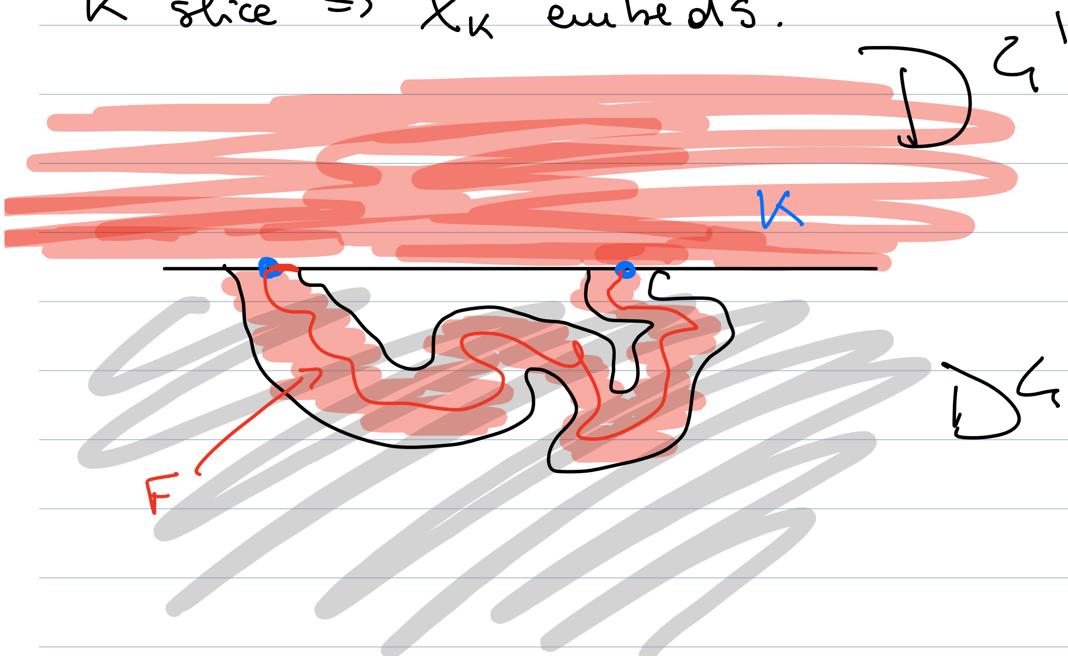
<u>Corollary</u>: All the iterated positive 0-Whitehead doubles of the trefoil are top. slice but not smoothly slice.

Using the tool to build exotic $\mathbb{R}^4$.

Lemma: Let K be a knot in $S^3 = \partial D^4$. Let X_K be D^4 with a 2-handle attached to K with 0-framing (knot trace). K is smoothly (resp. top.) slice iff X_K embeds in $\mathbb{R}^4$ smoothly (resp. topologically).

Proof: SMOOTH CAT.:

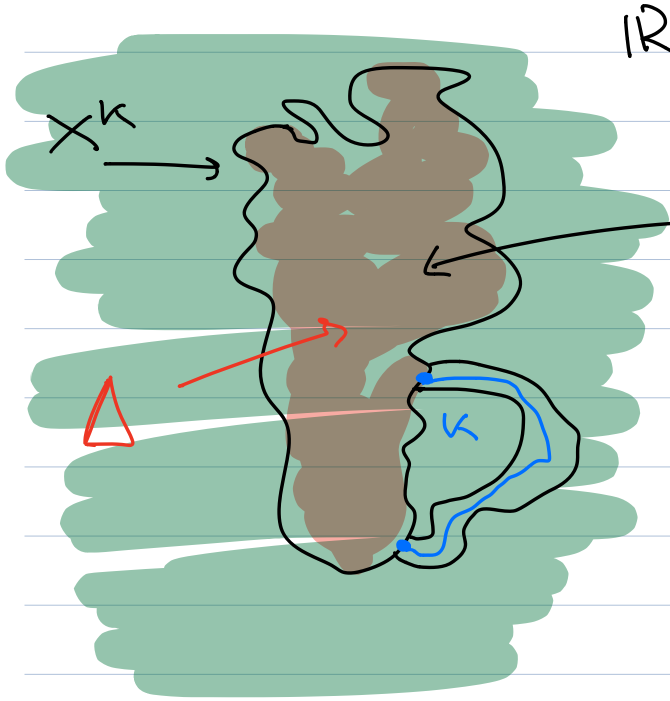
K slice $\Rightarrow$ X_K embeds.



Consider $D^{4'} \cup \underline{N(F)} \cong X_K \subseteq \mathbb{R}^4$.

X_K embeds $\Rightarrow$ K is slice.

$\mathbb{R}^4 \subseteq S^4$



It is a $D^4 \subseteq S^4$
$\Rightarrow$ its complement is a D^4.

Look at K in the boundary of the complement of Δ.

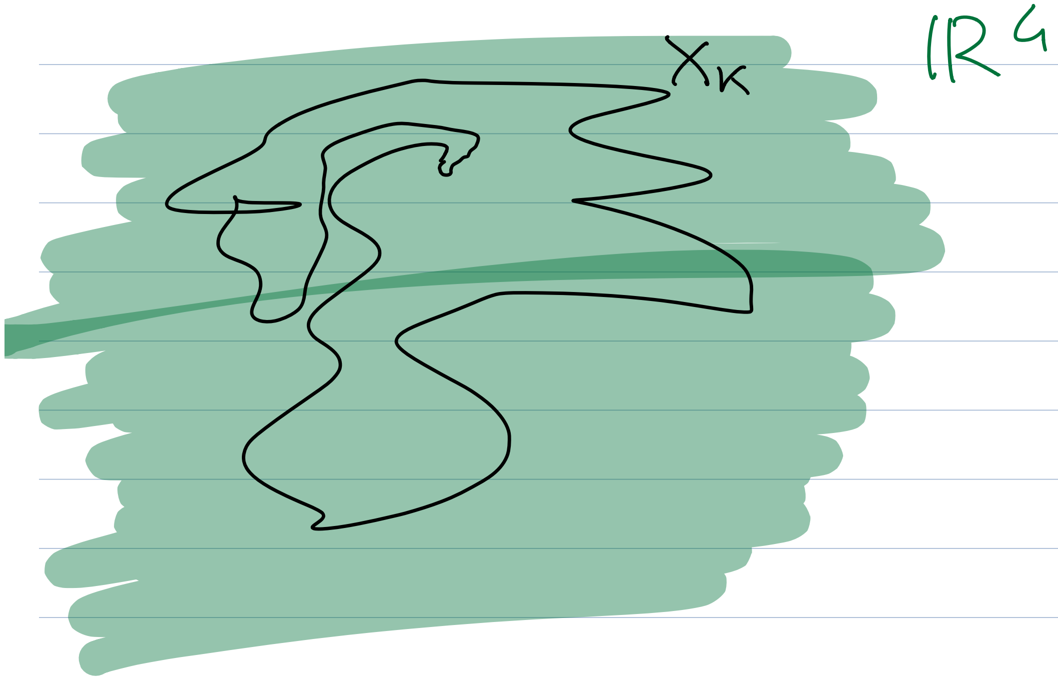
K is a knot in S^3 and the core of the 2-handle is a convenient slice disk for K.
$\Rightarrow$ K is slice.

$\square$

Consider X_K $\quad K = Wh_0^+(\text{trefoil})$.

X_K embeds top. in $\mathbb{R}^4$ via f.

Consider $f(X_K)$ and push forward the smooth str. of X_K on it.



Extend the smooth structure on $f(X_K)$ to $\mathbb{R}^4$.
This smooth structure cannot be the standard one.

$\square$