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Hugo Beirão da Veiga, Petr Kaplický and Michael Růžička

Abstract. This article is concerned with the global regularity of weak solutions to systems describing the flow of shear thickening fluids under the homogeneous Dirichlet boundary condition. The extra stress tensor is given by a power law ansatz with shear exponent $p \geq 2$. We show that, if the data of the problem are smooth enough, the solution u of the steady generalized Stokes problem belongs to $W^{1,(np+2-p)/(n-2)}(\Omega)$. We use the method of tangential translations and reconstruct the regularity in the normal direction from the system, together with anisotropic embedding theorem. Corresponding results for the steady and unsteady generalized Navier–Stokes problem are also formulated.

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Keywords. Generalized Newtonian fluids, shear dependent viscosity, regularity up to the boundary.

1. Introduction

This article is concerned with systems describing motions of shear thickening fluids, which in steady situations are governed by the following system in Ω

$$\begin{aligned} -\operatorname{div} \mathcal{S}(Du) + \delta\,[\nabla u]u + \nabla \pi &= f, \\ \operatorname{div} u &= 0, \end{aligned} \tag{1.1}$$

equipped with Dirichlet boundary conditions on $\partial\Omega$

$$u = 0, \tag{1.2}$$

where $u : \Omega \to \mathbb{R}^n, \pi : \Omega \to \mathbb{R}$ $\delta = 0$ or $\delta = 1$, and $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a bounded domain with an $C^{2,1}$ boundary. Here we have denoted the velocity field by u and the pressure by π. The symbol

$$D\,u = \frac{1}{2}\,(\nabla u + \nabla u^{\,T}),$$

denotes the symmetric velocity gradient, $S(Du)$ the extra stress tensor and $([\nabla u]u)_i = \sum_{j=1}^n u_j \partial_j u_i$, $i = 1, \dots, n$ the convective term. Typical prototypes of extra stress tensors are

$$S(D) = (1 + |D|^2)^{\frac{p-2}{2}} D \quad \text{or} \quad S(D) = (1 + |D|)^{p-2} D. \tag{1.3}$$

We recall that a fluid is called shear thickening if $p > 2$ and shear thinning if $p < 2$.

We are interested in proving boundary regularity of weak solutions (u, π) of the problem (1.1) describing flows of shear thickening fluids. It is generally accepted that the regularity of weak solutions in this case is easier to obtain than in the case of a shear thinning fluid, where the convective term interacts with the elliptic term and may spoil the regularity of the solution. Certainly, in the case of a shear thickening fluid, the convective term is not the basic obstacle and, at least in the interior, the local regularity of weak solutions may be obtained. However, as soon as the boundary comes into the play, the question of regularity becomes more delicate, because there arise problems connected with the structure of the elliptic term, namely that it depends on the symmetric part of the gradient only, and with the presence of pressure in the equation. This may result in the loss of regularity of the flow of shear thickening fluids in comparison to the flow of Newtonian fluids.

We also investigate time dependent versions of the above problem in $I \times \Omega$ namely

$$\begin{aligned} \partial_t u - \operatorname{div} \mathcal{S}(Du) + [\nabla u]u + \nabla \pi = f, \\ \operatorname{div} u = 0, \end{aligned} \tag{1.4}$$

under the above Dirichlet boundary conditions (1.2) on $I \times \partial\Omega$, where $u : I \times \Omega \to \mathbb{R}^n$, $\pi : I \times \Omega \to \mathbb{R}$ and $I = (0, T)$, $T > 0$, is a nontrivial time interval. Since our results are local with respect to time we do not specify the initial condition.

The extra stress tensor $\mathcal{S}$ is assumed to possess *p-structure* with $p \geq 2$. More precisely we make the following assumptions on $\mathcal{S}$.

Assumption 1 (*extra stress tensor*). We assume that the extra stress tensor $\mathcal{S} \colon \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}_{\text{sym}}$ belongs to $C^0(\mathbb{R}^{n\times n}, \mathbb{R}^{n\times n}_{\text{sym}}) \cap C^1(\mathbb{R}^{n\times n} \setminus \{0\}, \mathbb{R}^{n\times n}_{\text{sym}})$, where $\mathbb{R}^{n\times n}_{\text{sym}} := \{A \in \mathbb{R}^{n\times n} \mid A = A^\top\}$, and satisfies $\mathcal{S}(0) = 0$ and $\mathcal{S}(A) = \mathcal{S}(A^{\text{sym}})$, where $A^{\text{sym}} := \frac{1}{2}(A + A^\top)$. Moreover, we assume that $\mathcal{S}$ has p-structure, i.e., there exists $p \in (1, \infty)$ such that

$$\sum_{i,j,k,l=1}^{n} \partial_{kl}\mathcal{S}_{ij}(A) B_{ij} B_{kl} \geq c\, (1 + |A^{\text{sym}}|)^{p-2}\, |B^{\text{sym}}|^2, \tag{1.5a}$$

$$|\partial_{kl}\mathcal{S}_{ij}(A)| \leq C\, (1 + |A^{\text{sym}}|)^{p-2} \tag{1.5b}$$

is satisfied for all $A, B \in \mathbb{R}^{n\times n}$ with $A^{\text{sym}} \neq 0$.

This assumptions is motivated by the typical prototypes for the extra stress tensor given in (1.3). We refer the reader to [11,26,40,43] for a more detailed discussion leading to Assumption 1.

Closely related to the extra stress tensor $\mathcal{S}$ with p-structure is the function $\mathcal{F} \colon \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}_{\text{sym}}$ defined through

$$\mathcal{F}(A) := (1 + |A^{\text{sym}}|)^{\frac{p-2}{2}}\, A^{\text{sym}}. \tag{1.6}$$

There is a close relation of the quantities $\mathcal{S}$ and $\mathcal{F}$ to Orlicz spaces and N-functions and we refer the reader to [11,45] for a detailed description (cf. Sect. 2). Since in the following we shall insert into $\mathcal{S}$ and $\mathcal{F}$ only symmetric tensors, we can drop in the above formulas the superscript "sym" and restrict the admitted tensors to symmetric ones.

Before formulating our main results we want to note that the system (1.4) is nowadays classical. It was proposed by Ladyzhenskaya in [36–38] as a modification of the Navier–Stokes system. Simultaneously Lions [39] suggested a similar system, however with the elliptic term depending on the full gradient. Since that time much work has been done concerning existence of the weak solutions to the systems (1.4) and (1.1) and their qualitative properties. We refer, without any ambition of completeness, to [9,11,13,14,25–27,31–33,40,41,44,48]. In spite of the fact that the system was extensively studied there are still many open problems, especially concerning the regularity of weak solutions.

The main results of the present paper contribute to the field of regularity properties of weak solutions to the variants of the problems (1.4) and (1.1).

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain with $C^{2,1}$ boundary and let $f \in L^2(\Omega)$. Then the unique weak solution $u \in W^{1,p}_0(\Omega)$ of problem (1.1), with $\delta = 0$, and (1.2) satisfies*

$$u \in W^{1,q}(\Omega), \quad \mathcal{F}(Du) \in W^{1,\frac{2q}{p+q-2}}(\Omega), \tag{1.7}$$

for

$$q = \frac{np + 2 - p}{n - 2}$$

if $n \geq 3$, and for all $q < +\infty$, if $n = 2$.

Theorem 1.2. *Let $\mathcal{S}$ satisfy Assumption 1 with $p \geq \max\{2, (3n)/(n+2)\}$. Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain with $C^{2,1}$ boundary, and let $f \in L^2(\Omega)$. Then, for $p \in [3, \infty) \cup (n/2, \infty)$, any weak solution $u \in W^{1,p}_0(\Omega)$ of problem (1.1), (1.2), with $\delta = 1$, satisfies (1.7), where q is as above.*

Remark 1.1. (i) In the interior and in tangential directions we get better regularity properties in the previous theorems. More precisely we get (cf. [42], (3.14), (3.13))

$$\mathcal{F}(Du) \in W^{1,2}_{\text{loc}}(\Omega), \quad \xi\partial_\tau \mathcal{F}(Du) \in L^2(\Omega),$$

where ξ is some cut off function with support near the boundary $\partial\Omega$ and the tangential derivative ∂_τ is defined locally by (2.1).

(ii) Since $p \geq 2$ it follows that (cf. (3.21))

$$|\nabla^2 u| \leq c\,|\nabla \mathcal{F}(Du)|.$$

Theorem 1.3. *Let $\mathcal{S}$ satisfy Assumption 1 with $p > (3n+2)/(n+2)$. Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain with $C^{2,1}$ boundary, $I = (0,T)$, $T > 0$, and let $f \in W^{1,2}_{\text{loc}}(I, L^2(\Omega))$. Then any weak solution u of the problem (1.4), and (1.2) satisfies*

$$\begin{aligned} u \in W^{1,\infty}_{\text{loc}}(I, L^2(\Omega)), &\quad \mathcal{F}(Du) \in W^{1,2}_{\text{loc}}(I, L^2(\Omega)) \\ u \in L^\infty_{\text{loc}}(I, W^{1,q}(\Omega)), &\quad \mathcal{F}(Du) \in L^\infty_{\text{loc}}(I, W^{1,\frac{2q}{p+q-2}}(\Omega)), \end{aligned} \tag{1.8}$$

where q is as above.

Remark 1.2. Note that Remark 1.1 also applies to Theorem 1.3. In particular we get

$$\mathcal{F}(Du) \in W^{1,2}_{\text{loc}}(I \times \Omega), \quad \xi\partial_\tau \mathcal{F}(Du) \in L^\infty_{\text{loc}}(I, L^2(\Omega)).$$

The main obstacle in the proofs of the above theorems is the boundary condition prescribed on the non-flat boundary $\partial\Omega$ together with the fact that the extra stress tensor $\mathcal{S}$ depends only on symmetric part of the velocity gradient. Moreover, the incompressibility of the flow (which results in the fact that weak solutions are divergence free) and in the appearance of the pressure term $\nabla\pi$ in the Eqs. (1.4) and (1.1), causes additional problems. Let us now briefly describe how we treat these difficulties in the case of the steady problem (1.1), with $\delta = 0$, and (1.2).

The regularity of the tangential derivatives $\partial_\tau u$ near the boundary, namely

$$\int_\Omega \xi|\partial_\tau \mathcal{F}(Du)|^2\,dx \simeq \int_\Omega \xi(1+|Du|)^{p-2}|D\partial_\tau u|^2\,dx < C \tag{1.9}$$

is obtained by the classical difference quotients method. As in [35,42], we appeal to translations parallel to the non-flat boundary $\partial\,\Omega$, hence without a previous change of coordinates, as done in [6] in order to straighten the boundary. In both cases, however, when deriving the main estimates there appear some terms which are not present if boundary is flat. In order to handle them we need $\partial\Omega \in C^{2,1}$.

The normal derivatives are restored from the Eqs. (1.4) by the method of [2] (cf. [3] for a simplified approach), where the same system on the half space is studied. In [6] the same ideas are applied in the non-flat boundary case. The main ingredient is that, due to the smoothness of $\partial\Omega$, it is possible to express the whole second gradient of u via the gradient of the tangential derivatives of u and π. To use (1.9) at this place, we have to overcome the problem that we do not know if Korn's inequality of the type

$$\int_\Omega (1+|Du|)^{p-2}|\nabla\partial_\tau u|^2\,dx \leq C\int_\Omega (1+|Du|)^{p-2}|D\partial_\tau u|^2\,dx$$

holds. This leads to a loss of regularity of normal derivatives. In spite of the fact that we improve the results from [2,6,8,42], a fundamental question remains open. Namely, if it is possible to show that

$$\int_\Omega |\nabla \mathcal{F}(Du)|^2\,dx \leq C.$$

Note that

$$\int_\Omega |\nabla \mathcal{F}(Du)|^2\,dx \simeq \int_\Omega (1+|Du|)^{p-2}|\nabla^2 u|^2\,dx.$$

To treat $\partial_\tau \pi$ we need the next important tool, the theorem about properties of the divergence operator, cf. [12] (cf. [2,3,6,8] for an alternative approach using Nečas' negative norm theorem), which allows to estimate the norm of $\partial_\tau \pi$ from (1.1). Also here we may set up an open question. Is it possible to get an estimate of some norm of $(1 + |Du|)^{(p-2)/2} \partial_\tau \pi$ and get $\left\|\nabla^2 u\right\|_2 \le C$?

The results in this article are new and improve all previous results. They are obtained by combining several methods from [2,8,35,42] with some new ideas. In [42] the problem (1.4) is studied in a three-dimensional domain. The main focus is put on the existence theory for p close to 2 to fill the previously existing gap between 2 and $\frac{11}{5}$. As a tool, regularity of the second derivatives of the weak solutions is used. It is obtained by the method of tangential differences provided $p \in [2, 3)$. The article [35] is devoted to the existence of a $W^{2,2+\epsilon}(\Omega)$ solution to the problem (1.1) in a two-dimensional domain for $p < 2$. We follow some parts of these papers to obtain regularity of tangential derivatives of Du and π. In [2,3] regularity of weak solutions of problems (1.4) and (1.1) are studied in the neighborhood of the flat portion of $\partial\Omega$. From these articles we learn how to reconstruct the normal derivatives from the tangential ones, see Sect. 3.2 below. In [8] the results proved in [3] are improved by appealing in particular to anisotropic embedding theorems (cf. [47]), since we have different information in different directions. The results in [3] have been extended in [6] to non-flat boundaries, an extension that also applies to the results in [8]. In [22] results between [3] and [8] are shown. In this paper, as in [6,35,42], we consider non-flat boundaries. In particular, we improve the regularity exponents obtained in [6] by finding a better balance between the two main terms that prevent optimal results. The treatment of the unsteady problem (1.4) is based on the results for the steady problem (1.1) and the very nice improvements of the time regularity of weak solutions in [14] compared to [42] (cf. [11,26]).

Concerning the shear thinning case, strongly related $W^{2,q}$ regularity results up to the boundary, under the boundary condition (1.2), are proved, for flat boundaries in [4,5,10], for cylindrical domains in [20,21], and for smooth arbitrary boundaries in [7]. Appeal to Troisi's anisotropic embedding theorems (instead of classical, isotropic, Sobolev embedding theorems), also used below, was introduced in [10]. In the forthcoming paper [23] the authors prove that, under a suitable smallness assumption on f, the solution u to the system (1.1), (1.3), under the boundary condition (1.2), belongs to $C^{1,\alpha} \cap W^{2,2}$, up to the boundary. Still in the shear-thinning case, $W^{2,q}-$ regularity results up to a flat or a polyhedral boundary, under no-stick boundary conditions, are proved in [28]. Here, the information about the normal derivative is obtained by testing the equation with second normal difference after suitable prolongation of the solution u outside Ω. However, under the Dirichlet boundary condition (1.2), we do not know a suitable prolongation of u. The above boundary condition was also included in [2], where $p > 2$.

Apart from the results mentioned above we know about some other investigations dealing with the boundary regularity of weak solutions to systems related to (1.4). Very interesting are the papers [29,30], where incompressibility and the pressure are dropped off the equation. In the article [46] the problem (1.1) is studied provided $p = n = 2$ by the method of straightening the boundary. In [15–19] very interesting, physical meaningful, problems are deeply studied. In [34] it has been shown that the unique weak solution u of (1.4) with (1.2) satisfies $u \in L^\infty(I, W^{2,2+\epsilon}(\Omega))$ provided $p \in [2, 4)$ and $n = 2$. The upper bound appears due to the lack of regularity of the second derivatives of the weak solution near the boundary. This type of results are interesting because they imply full regularity of the problems. We believe that the results presented here lead to the improvement of the result in [34] for all $p \ge 2$.

The paper is organized as follows. In Sect. 2 we recall the notation used throughout the paper. Moreover, we recall some basic facts related to difference quotients in tangential directions and to the extra stress tensor $\mathcal{S}$. In Sect. 3 we prove Theorem 1.1. In particular, we treat in detail the regularity in tangential directions in Sect. 3.1, and in normal directions in Sect. 3.2. Moreover, we prove some regularity properties of the pressure. In Sect. 4 we explain how the convective term can be incorporated into the scheme developed in Sect. 3. In the final section we use the previous results and a result from [14] to deal with the unsteady problem.

2. Preliminaries

In this article we use standard notation for Lebesgue spaces $(L^p(\Omega), \|\cdot\|_p)$, $1 \le p \le \infty$, and Sobolev spaces $(W^{k,p}(\Omega), \|\cdot\|_{k,p})$, $1 \le p \le \infty$, $k \in \mathbb{N}$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain with $C^{2,1}$ boundary. Moreover, we use the usual local versions of these spaces. Weak partial derivatives in spatial directions are denoted by ∂_i, $i = 1, \dots, n$. The tangential weak derivatives near the boundary are denoted by ∂_τ (cf. (2.1)). $W_0^{1,p}(\Omega)$ is the subspace of $W^{1,p}(\Omega)$ consisting of functions with zero trace. The dual space of $W_0^{1,p}(\Omega)$ is denoted by $W^{-1,p'}(\Omega)$, where p' is the dual exponent to p, i.e. $\frac{1}{p} + \frac{1}{p'} = 1$. The duality pairing between these spaces is denoted by $\langle \cdot, \cdot \rangle_{1,p}$. For the treatment of the unsteady problem we shall make use of the Bochner spaces $L^p(I, X)$ and $W^{k,p}(I, X)$, $1 \le p \le \infty$, $k \in \mathbb{N}$, where $(X, \|.\|_X)$ is some Banach space. Differentiation with respect to time is denoted by ∂_t.

If $u \in \mathbb{R}^m$ and $v \in \mathbb{R}^n$ then $u \otimes v \in \mathbb{R}^{m\times n}$ and $(u \otimes v)_{ij} := u_i v_j$. If $m = n$ then $u \otimes_S v := \frac{1}{2}(u \otimes v + (u \otimes v)^\top)$. We do not use any special symbol for the scalar products of vectors and tensors, e.g., $\mathcal{S}(Du)B = \mathcal{S}_{ij}(Du)D_{ij}u$, where the summation convention over repeated Latin indices from 1 to n was used, alike in the rest of the article. We use universal constants $c, C > 0$, which may vary in different occurrences. We write $f \simeq g$ if there exist positive constants c and C such that $cf \le g \le Cf$.

Since $\partial\Omega \in C^{2,1}$, for each point $P \in \partial\Omega$ there are local coordinates such that in these coordinates we have $P = 0$ and $\partial\Omega$ is locally described by a $C^{2,1}$-function $a_P : B_R^{n-1}(0) \to B_R^1(0)$, where $B_R^k(0)$ is the k-dimensional ball with center 0 and radius $R > 0$ (which is small enough and will be fixed later) with the following properties

$$x \in \Omega_P := \Omega \cap (B_R^{n-1}(0) \times B_R^1(0)) \Leftrightarrow x_n > a_P(x_1, \dots, x_{n-1}),$$
$$\nabla a_P(0) = 0, \quad |\nabla a_P| < R \quad \text{on } B_R^{n-1}(0).$$

As $\partial\Omega$ is compact there exists a finite set of points $\Gamma \subset \partial\Omega$ and an open set $\Omega_0 \subset\subset \Omega$ such that

$$\Omega \subset \Omega_0 \cup \bigcup_{P\in\Gamma} \Omega_P.$$

To this covering of Ω we construct a partition of unity $\{\xi_0, \xi_P, P \in \Gamma\}$ such that for all $P \in \Gamma$ the sets $\operatorname{spt}\xi_P$ and $\partial\Omega_P \setminus \partial\Omega$ have positive distance $\operatorname{dist}(\operatorname{spt}\xi_P, \partial\Omega_P \setminus \partial\Omega) \ge h_0$, for some suitable small $h_0 > 0$.

Let us fix some $P \in \Gamma$ and write for simplicity $\xi = \xi_P$, $a = a_P$. Moreover, we use the notation $x = (x', x_n)$ and denote by $e^i, i = 1, \dots, n$ the standard basis in $\mathbb{R}^n$. For $h \in (0, h_0)$, $\alpha \in \{1, \dots, n-1\}$, and a function φ with $\operatorname{spt}\varphi \subset \operatorname{spt}\xi$ we define positive and negative tangential translations by

$$\varphi_\tau(x', x_n) := \varphi\left(x' + he^\alpha, x_n + a(x' + he^\alpha) - a(x')\right)$$
$$\varphi_{-\tau}(x', x_n) := \varphi\left(x' - he^\alpha, x_n + a(x' - he^\alpha) - a(x')\right)$$

and tangential differences through

$$d_\tau\varphi := \varphi_\tau - \varphi, \qquad d_\tau^-\varphi := \varphi_{-\tau} - \varphi.$$

One easily checks (cf. [42, Section 3]) that

$$h^{-1}d_\tau\varphi \to \partial_\tau\varphi := \partial_\alpha\varphi + \partial_\alpha a\,\partial_n\varphi \qquad \text{as } h \to 0 \tag{2.1}$$

almost everywhere in $\operatorname{spt}\xi$ if $\varphi \in W^{1,1}(\Omega)$. Moreover, for all $1 < q < \infty$, all $\varphi \in W_0^{1,q}(\Omega)$, and all sufficiently small h we have

$$\|h^{-1}d_\tau\varphi\|_{q,\operatorname{spt}\xi} \le c(a)\|\nabla\varphi\|_q. \tag{2.2}$$

Conversely, if $\|h^{-1}d_\tau\varphi\|_{q,\operatorname{spt}\xi} \le C$ for all sufficiently small h, then we get

$$\|\partial_\tau\varphi\|_{q,\operatorname{spt}\xi} \le C. \tag{2.3}$$

Now we formulate some auxiliary lemmas related to these objects. The first lemma clarifies the non commutativity of tangential translations and tangential differences with partial derivatives. For simplicity we denote $\nabla a := (\partial_1 a, \dots, \partial_{n-1}a, 0)$ and use also a_τ and $a_{-\tau}$ with obvious meaning.

Lemma 2.1. *Let* spt $\varphi \subset$ spt ξ. *Then*

$$\begin{aligned} \nabla d_\tau \varphi &= d_\tau \nabla \varphi + (\partial_n \varphi)_\tau \otimes d_\tau \nabla a, \\ D d_\tau \varphi &= d_\tau D\varphi + (\partial_n \varphi)_\tau \otimes_S d_\tau \nabla a, \\ \operatorname{div} d_\tau \varphi &= d_\tau \operatorname{div}\varphi + (\partial_n \varphi)_\tau d_\tau \nabla a. \end{aligned}$$

For strictly related formulae, see the lemma 6.3, and equations (6.12) and (6.13), in [6].

The second lemma is devoted to the relation between tangential differences and tangential translations.

Lemma 2.2. *Let* spt $\varphi \subset$ spt ξ. *Then*

$$(d_\tau^- \varphi)_\tau = -d_\tau \varphi, \quad (d_\tau \varphi)_{-\tau} = -d_\tau^- \varphi.$$

The discrete version of the partial integration formula will be often used.

Lemma 2.3. *Let* spt $g \cup$ spt $f \subset$ spt ξ. *Then*

$$\int_\Omega f g_{-\tau}\, dx = \int_\Omega f_\tau g\, dx.$$

Consequently, $\int_\Omega f d_\tau g\, dx = \int_\Omega (d_\tau^- f) g\, dx$.

Also the discrete product rule will be used.

Lemma 2.4. *Let* spt $g \cup$ spt $f \subset$ spt ξ. *Then*

$$d_\tau(fg) = f_\tau\, d_\tau g + (d_\tau f)\, g.$$

In the following sections the tangential differences of the extra stress tensor $\mathcal{S}$ will play a crucial role. Before we derive various formulas related to it we will clarify the role of $\mathcal{F}$ and its relation to differences of $\mathcal{S}$. The following lemma is proved in [24, Lemma 2.1] (cf. [45, Lemma 6.16]).

Lemma 2.5. *Let* $\mathcal{S}$ *satisfy Assumption* 1 *with* $p \in (1,\infty)$ *and let* $\mathcal{F}$ *be defined by* (1.6). *Then for all* A, $B \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$ *there holds*

$$\begin{aligned} (\mathcal{S}(A) - \mathcal{S}(B))(A-B) &\simeq |A-B|^2(1+|B|+|A|)^{p-2} \\ &\simeq |\mathcal{F}(A) - \mathcal{F}(B)|^2 \end{aligned} \tag{2.4}$$

$$|\mathcal{S}(A) - \mathcal{S}(B)| \simeq |A-B|(1+|B|+|A|)^{p-2}, \tag{2.5}$$

with constants depending only on p *and the constants in Assumption* 1.

Using this lemma we easily see

$$\begin{aligned} |d_\tau \mathcal{S}(Du)| &\le C\,(1+|Du|+|(Du)_\tau|)^{p-2}|d_\tau Du| \\ &\le C\,|d_\tau \mathcal{F}(Du)|\,(1+|Du|+|(Du)_\tau|)^{\frac{p-2}{2}}, \end{aligned} \tag{2.6}$$

$$\begin{aligned} d_\tau \mathcal{S}(Du)\, d_\tau Du &\simeq |d_\tau \mathcal{F}(Du)|^2 \\ &\simeq (1+|Du|+|(Du)_\tau|)^{p-2}|d_\tau Du|^2. \end{aligned} \tag{2.7}$$

These relations, Lemma 2.1 and (2.2) yield

$$\int_\Omega \xi^2 d_\tau \mathcal{S}(Du) D(d_\tau u)\, dx \ge c\, \|\xi d_\tau \mathcal{F}(Du)\|_2^2 - C(\nabla^2 a)\, h^2\, \|1+|\nabla u|\|_p^p. \tag{2.8}$$

All assertions from this section may be proved by easy manipulation with definitions and we drop their proofs.

We conclude this section by the following anisotropic embedding theorem.

Theorem 2.1. *Let $P \in \Gamma$, $F \in W^{1,1}(\Omega_P)$ with $\operatorname{spt} F \subset \operatorname{spt} \xi_P$. Let all tangential derivatives $\partial_\tau F$ satisfy $\partial_\tau F \in L^\kappa(\Omega_P)$, for some $\kappa > 1$, and let $\partial_n F \in L^\mu(\Omega_P)$, for some $\mu > 1$. Let $1/\mu + (n-1)/\kappa \geq \omega$ for some $\omega > 1$. Then $F \in L^\nu(\Omega_P)$ with ν given by $1/\nu = 1/n\,((n-1)/\kappa + 1/\mu - 1)$ and*

$$\|F\|_\nu \leq C\left(\|\partial_\tau F\|_\kappa + \|F\|_\kappa\right)^{\frac{n-1}{n}} \left(\|\partial_n F\|_\mu + \|F\|_\mu\right)^{\frac{1}{n}}.$$

The constant $C > 0$ depends on ω but not on μ, κ, ν.

Proof. The theorem is proved in [47] provided the boundary is flat. The case of the non-flat boundary can be converted to the previous case by the coordinate transformation $\Phi : (x', x_n) \to (x', x_n + a(x'))$. Note that defining $G := F \circ \Phi$ it holds $\partial_\alpha G = (\partial_\alpha F + \partial_\alpha a \partial_n F) \circ \Phi = \partial_\tau F \circ \Phi$ for $\alpha \in \{1, \ldots, n-1\}$, and $\partial_n G = (\partial_n F) \circ \Phi$. Since the Jacobian of the transformation Φ is one, we can use [47, Theorem 4.3] for G and the reverse transformation to Φ to get the statement of the theorem. □

3. Steady p-Stokes Problem

Let us start with the definition of weak solutions and some remarks about its existence. We refer to the problem (1.1), with $\delta = 0$, (1.2), and $\mathcal{S}$ satisfying Assumption 1 as the *steady p-Stokes* problem. Note that in this section the universal constants $c, C > 0$ do not depend on f.

Definition 3.1. Let $\mathcal{S}$ satisfy Assumption 1 with $p \geq 2$ and let $f \in W^{-1,p'}(\Omega)$. A weak solution of the steady p-Stokes problem is a function $u \in W_0^{1,p}(\Omega)$, with $\operatorname{div} u = 0$ such that for all $\varphi \in W_0^{1,p}(\Omega)$ with $\operatorname{div} \varphi = 0$ holds

$$\int_\Omega \mathcal{S}(Du) D\varphi \, dx = \langle f, \varphi \rangle_{1,p}. \tag{3.1}$$

The existence of a weak solution of the steady p-Stokes problem is easily obtained using the Galerkin method together with the monotone operator theory. A weak solution always satisfies the a priori estimate

$$\|Du\|_p^p \leq C\, \|f\|_{-1,p'}^{p'}. \tag{3.2}$$

Uniqueness of weak solutions follows from the strict monotonicity of $\mathcal{S}$, which is consequence of Assumption 1. To the weak solution u there exists an associated pressure $\pi \in L^{p'}(\Omega)$, with $\int_\Omega \pi \, dx = 0$, satisfying for all $\varphi \in W_0^{1,p}(\Omega)$

$$\int_\Omega \mathcal{S}(Du) D\varphi + \pi \operatorname{div} \varphi \, dx = \langle f, \varphi \rangle_{1,p} \tag{3.3}$$

and

$$\|\pi\|_{p'} \leq C\, \|f\|_{-1,p'}, \tag{3.4}$$

as it is proved in [1, Theorem 2.8]. Similarly, the following lemma holds.

Lemma 3.1. *Let $f \in W^{-1,\frac{q}{p-1}}(\Omega)$ and let u be a weak solution of the steady p-Stokes problem satisfying $Du \in L^q(\Omega)$, where $p \leq q \leq (2n)(p-1)/(n-2)$. Then the associated pressure π belongs to $L^{\frac{q}{p-1}}(\Omega)$ and satisfies*

$$\|\pi\|_{\frac{q}{p-1}} \leq C\left(1 + \|Du\|_q^{p-1} + \|f\|_{-1,\frac{q}{p-1}}\right). \tag{3.5}$$

Here we have used the growth property of $\mathcal{S}$ (cf. (2.5)), namely that for $A \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$ holds

$$|\mathcal{S}(A)| \leq C\,(1 + |A|)^{p-2} |A|. \tag{3.6}$$

Remark 3.1. Note that for $f \in L^2(\Omega)$ and $p \geq 2$ we get $f \in W^{-1,p'}(\Omega)$. Also note that $L^2(\Omega) \hookrightarrow W^{-1,\frac{q}{p-1}}(\Omega) \hookrightarrow W^{-1,p'}(\Omega)$ if $p \leq q \leq 2n(p-1)/(n-2)$, so we can substitute the norm of f in (3.5) by $\|f\|_2$. The parameter q is chosen such that the information to be obtained on ∇u is better than the one from the energy estimate (3.2), and that the conditions on q do not impose any additional restriction on p.

3.1. Regularity in Tangential Directions

Fix $P \in \Gamma$ and let $\xi = \xi_P$, $a = a_P$, Ω_P, $\alpha \in \{1, \ldots, n-1\}$, $h \in (0, h_0)$ be as in Sect. 2. We consider the test function φ in the weak formulation (3.3) in the form $\varphi = d_\tau^-(\xi\psi)$ with $\psi \in C_0^\infty(\Omega)$ to get, with the help of Lemmas 2.1–2.4,

$$\begin{aligned}\int_\Omega & \xi d_\tau \mathcal{S}(Du)D\psi + (d_\tau\pi)\xi \operatorname{div}\psi \, dx \\ &= -\int_\Omega \mathcal{S}(D)\left((\partial_n(\xi\psi))_{-\tau} \otimes_S d_\tau^- \nabla a\right) + \pi(\partial_n(\xi\psi))_{-\tau} d_\tau^- \nabla a \, dx \\ &\quad - \int_\Omega d_\tau \mathcal{S}(Du)\nabla\xi \otimes_S \psi + (d_\tau\pi)\nabla\xi\psi \, dx + \int_\Omega f d_\tau^-(\xi\psi)\, dx. \end{aligned} \tag{3.7}$$

Due to the fact that $u \in W_0^{1,p}(\Omega)$, $\pi \in L^{p'}(\Omega)$ and the density of $C_0^\infty(\Omega)$ in $W_0^{1,p}(\Omega)$, Eq. (3.7) remains valid for all $\psi \in W_0^{1,p}(\Omega)$ and we can set $\psi = \xi d_\tau u$. Note that $\xi d_\tau u$ has zero trace on Ω_P. We get an information from the first term on the left-hand side of (3.7). By (2.8), $(2.6)_2$, Young's and Korn's inequalities, (2.2), and (3.2) it may be estimated from below by

$$\int_\Omega \xi d_\tau \mathcal{S}(Du)D(\xi d_\tau u)\, dx \geq c\, \|\xi d_\tau \mathcal{F}(Du)\|_2^2 - C\, h^2\, \|1 + |\nabla u|\|_p^p .$$

The second term on the left-hand side of (3.7) we treat with the help of Lemma 2.1 and the fact that $\operatorname{div} u = 0$. This leads to

$$\int_\Omega (d_\tau\pi)\xi \operatorname{div}(\xi d_\tau u)\, dx = \int_\Omega (d_\tau\pi)\xi(\nabla\xi d_\tau u + \xi(\partial_n u)_\tau d_\tau \nabla a)\, dx.$$

Using these two relations we obtain from (3.7)

$$\begin{aligned} c\, &\|\xi d_\tau \mathcal{F}(Du)\|_2^2 \\ &\leq C \int_\Omega |d_\tau\pi|\xi\left(|d_\tau u||\nabla\xi| + h|(\partial_n u)_\tau|\right)\, dx + \int_\Omega f d_\tau^-(\xi^2 d_\tau u)\, dx \\ &\quad - \int_\Omega \mathcal{S}(Du)(\partial_n(\xi^2 d_\tau u))_{-\tau} \otimes_S d_\tau^- \nabla a\, dx - \int_\Omega \pi(\partial_n(\xi^2 d_\tau u))_{-\tau} d_\tau^- \nabla a\, dx \\ &\quad - \int_\Omega \xi d_\tau \mathcal{S}(Du)\nabla\xi \otimes_S d_\tau u\, dx - \int_\Omega \xi d_\tau \pi \nabla\xi d_\tau u\, dx + C\, h^2\, \|1 + |\nabla u|\|_p^p \\ &= I_1 + \cdots + I_6 + C\, h^2\, \|1 + |\nabla u|\|_p^p . \end{aligned} \tag{3.8}$$

Let us now gradually estimate all integrals on the right-hand side. Using (2.2), $(2.6)_2$, and Young's inequality we get

$$\begin{aligned} |I_1 + I_6| &\leq C\, h\, \|\xi d_\tau \pi\|_{p'}\, \|\nabla u\|_p\,, \\ |I_5| &\leq \varepsilon\, \|\xi d_\tau \mathcal{F}(Du)\|_2^2 + C_\varepsilon\, h^2\, \|1 + |\nabla u|\|_p^p . \end{aligned}$$

Next, by (2.2) and Korn's inequality we get

$$|I_2| \le C \, \|f\|_2 \left\|D(\xi^2 d_\tau u)\right\|_2 \, h.$$

Note that from (2.7) it follows $|d_\tau Du| \le C \, |d_\tau \mathcal{F}(Du)|$. Using this, Lemma 2.1 and $p \ge 2$ we finally obtain

$$|I_2| \le \varepsilon \, \|\xi d_\tau \mathcal{F}(Du)\|_2^2 + C \, h^2 \left(\|f\|_2^2 + \|1 + |\nabla u|\|_p^p \right),$$

where C also depends on $\nabla^2 a$ and on $\varepsilon > 0$. Thus, it remains to estimate the terms I_3 and I_4. Since these estimates are similar we present it only for I_3. To this end we write using Lemma 2.1

$$(\partial_n(\xi^2 d_\tau u))_{-\tau} = (\xi^2 d_\tau \partial_n u + 2\xi \partial_n \xi d_\tau u)_{-\tau}$$

to get by Lemmas 2.2 and 2.3

$$I_3 = \int\limits_\Omega \mathcal{S}((Du)_\tau) d_\tau \nabla a \otimes_S (\xi^2 d_\tau \partial_n u + 2\xi \partial_n \xi d_\tau u) \, dx =: I_{31} + I_{32}.$$

From (3.6), and (2.2) we obtain

$$|I_{32}| \le C \, h^2 \, \|1 + |\nabla u|\|_p^p,$$

where C also depends on $\nabla^2 a$. For the treatment of I_{31} we have to use Lemmas 2.2 and 2.3 again. Due to this we need $\partial\Omega \in C^{2,1}$. We obtain proceeding similarly as in the estimates of the terms I_5 and I_{32}

$$\begin{aligned} |I_{31}| = \left| \int\limits_\Omega d_\tau^- \left(\xi^2 \mathcal{S}((Du)_\tau) d_\tau \nabla a\right) \otimes_S \partial_n u \, dx \right| \le \Big| \int\limits_\Omega \big[\xi^2 d_\tau \mathcal{S}(Du) d_\tau \nabla a + \mathcal{S}(Du) d_\tau (\nabla a) d_\tau^- \xi^2 \\ - \mathcal{S}(Du)(\xi^2)_{-\tau} (d_\tau \nabla a + d_\tau^- \nabla a) \big] \otimes_S \partial_n u \, dx \Big| \le \varepsilon \, \|\xi d_\tau \mathcal{F}(Du)\|_2^2 + C \, h^2 \, \|1 + |\nabla u|\|_p^p, \end{aligned}$$

where the constant C depends on $\|\nabla^2 a\|_{1,\infty}$ and ε. Recall that $\pi \in L^{p'}(\Omega)$. Proceeding similarly as for I_3 we get

$$|I_4| \le C \left(h \, \|\xi d_\tau \pi\|_{p'} \, \|\nabla u\|_p + h^2 \left(\|\pi\|_{p'}^{p'} + \|1 + |\nabla u|\|_p^p \right) \right).$$

Finally, combining all estimates above, (3.2), (3.4), and choosing $\varepsilon > 0$ sufficiently small we get

$$\|\xi d_\tau \mathcal{F}(Du)\|_2^2 \le C \left(h \, \|\xi d_\tau \pi\|_{p'} \, \|\nabla u\|_p + h^2 \left(1 + \|f\|_2^2 \right) \right). \tag{3.9}$$

In order to get regularity in the tangential directions we have to estimate $\|\xi d_\tau \pi\|_{p'}$. In view of the next subsection we will prove in the next lemma a slightly more general estimate than needed here.

Lemma 3.2. *Let u be the weak solution of the steady p-Stokes problem and let π be the associated pressure. Assume that for some $q \ge p$, $1 \le S \le 2q/(p+q-2)$ we have*

$$K := \|1 + |Du|\|_q^{p-1} + \|\pi\|_S + \|f\|_S < \infty. \tag{3.10}$$

Then

$$\|\xi d_\tau \pi\|_S \le C \left(\|\xi d_\tau \mathcal{F}(Du)\|_2 \, \|1 + |Du|\|_q^{\frac{p-2}{2}} + h \, K \right).$$

Proof. We solve an auxiliary problem

$$\begin{aligned} \operatorname{div} \psi &= |\xi d_\tau \pi|^{S-2} \xi d_\tau \pi - \fint\limits_\Omega |\xi d_\tau \pi|^{S-2} \xi d_\tau \pi \, dx \quad \text{in spt}\, \xi, \\ \psi &= 0 \quad \text{at } \partial(\operatorname{spt} \xi \cap \Omega). \end{aligned}$$

By the Bogovskii theorem [12] such ψ exists and satisfies

$$\|\psi\|_{1,S'} \le C \, \|\xi d_\tau \pi\|_S^{S-1}. \tag{3.11}$$

After prolongation by zero outside $\operatorname{spt}\xi$, ψ is a suitable test function in (3.7) due to (3.10), the growth property of $\mathcal{S}$ (cf. (3.6)), and $q \geq p$ (hence $q \geq S$). By inserting ψ into (3.7) we get from the second term on the left-hand side using $\pi \in L^{p'}(\Omega)$

$$\int_\Omega (d_\tau \pi)\xi \operatorname{div}\psi \, dx = \|\xi d_\tau \pi\|_S^S + \fint_\Omega |\xi d_\tau \pi|^{S-2}\xi d_\tau \pi \, dx \int_\Omega \pi d_\tau^- \xi \, dx \geq \|\xi d_\tau \pi\|_S^S - C\, h \,\|\xi d_\tau \pi\|_S^{S-1}\, \|\pi\|_1 .$$

The first term on the left-hand side can be estimated, using $(2.6)_2$, Hölder's inequality and (3.11), by

$$C\, \|\xi d_\tau \mathcal{F}(Du)\|_2 \, \|\xi d_\tau \pi\|_S^{S-1} \left(1 + \|Du\|\right)_{(p-2)\frac{S}{2-S}}^{\frac{p-2}{2}} .$$

The other terms are treated easily. Thus we arrive at

$$\begin{aligned} \|\xi d_\tau \pi\|_S^S &\leq C\, \|\xi d_\tau \pi\|_S^{S-1} \left[\|\xi d_\tau \mathcal{F}(Du)\|_2 \left(1 + \|Du\|_{(p-2)\frac{S}{2-S}}^{\frac{p-2}{2}} \right) + h \left(\|f\|_S + \|\mathcal{S}(Du)\|_S + \|\pi\|_S \right) \right] \\ &\leq C\, \|\xi d_\tau \pi\|_S^{S-1} \left[\|\xi d_\tau \mathcal{F}(Du)\|_2 \left(1 + \|Du\|_q^{\frac{p-2}{2}} \right) + h \left(\|f\|_S + (1 + \|Du\|_q)^{p-1} + \|\pi\|_S \right) \right], \end{aligned}$$

where we used the growth property (3.6) of $\mathcal{S}, p \leq q, (p-2)S/(2-S) \leq q, (p-1)S \leq q$. This inequality yields the assertion. □

Now we come back to (3.9). Using $f \in L^2(\Omega)$, (3.2), (3.4), and Remark 3.1 we can choose $q = p$, $S = p'$ in Lemma 3.2. Thus we obtain from (3.9) and Young's inequality

$$\|\xi d_\tau \mathcal{F}(Du)\|_2^2 \leq C\, h^2 \left(1 + \|f\|_2^2 \right). \tag{3.12}$$

Note that $\nabla u \in L^p(\Omega)$ implies $\mathcal{F}(Du) \in L^2(\Omega)$. Consequently we obtain from the properties of the tangential differences (cf. (2.3))

$$\|\xi \partial_\tau \mathcal{F}(Du)\|_2^2 \leq C(1 + \|f\|_2^2). \tag{3.13}$$

A similar procedure can be done in the interior of Ω for all differences. This leads to (cf. [42, section 3]) $u \in W^{2,2}_{\mathrm{loc}}(\Omega)$. In particular we know that

$$\|\xi_0 \nabla \mathcal{F}(Du)\|_2^2 \leq C(1 + \|f\|_2^2), \tag{3.14}$$

and that the system (1.1), with $\delta = 0$, holds almost everywhere in Ω.

3.2. Regularity in Normal Direction

In this section we obtain new information about the regularity of u from Eq. (1.1) and the information about tangential regularity (3.13).

As mentioned above, second derivatives of u exist almost everywhere in Ω and the system (1.1), with $\delta = 0$, holds almost everywhere in Ω. We use the first $n-1$ equations in (1.1), with $\delta = 0$, and the equation $\operatorname{div} u = 0$ to obtain information about the whole second gradient of u (cf. [42, section 3], [2, proof of Lemma 4.5] and [6, section 10]). Using the definition of $\partial_\tau u$ in (2.1) and $\operatorname{div} u = 0$ we get for all $i, j, k = 1, \ldots, n, (j, k) \neq (n, n)$

$$|\partial_j \partial_k u_i| \le |\nabla \partial_\tau u| + |\nabla a||\nabla^2 u|, \tag{3.15}$$

$$|\partial_n \partial_n u_n| = \left| \sum_{i=1}^{n-1} \partial_i \partial_n u_i \right| \le C\, \big(|\nabla \partial_\tau u| + |\nabla a||\nabla^2 u|\big). \tag{3.16}$$

It remains to reconstruct only $\partial_n^2 u_j$, $j = 1, \dots, n-1$. Expressing them from first $n-1$ equations of (1.1), with $\delta = 0$, we get for all $i \in \{1, \dots, n-1\}$ (without summation convention).

$$-\sum_{l=1}^{n-1} \partial_{nl} \mathcal{S}_{in}(Du) \partial_n^2 u_l + \partial_i \pi = f_i + \sum_{j=1}^{n-1} \partial_j \mathcal{S}_{ij}(Du) + \partial_{nn} \mathcal{S}_{in}(Du) \partial_n^2 u_n$$
$$+ \sum_{k=1}^{n-1} \sum_{l=1}^{n} \partial_{kl} \mathcal{S}_{in}(Du) \partial_n \partial_k u_l =: (RHS). \tag{3.17}$$

We know from the properties of $\mathcal{S}$ (cf. (3.6)), (3.15), and (3.16) that

$$|(RHS)| \le C\, \big(|f| + (1 + |Du|)^{p-2} |\nabla \partial_\tau u| + |\nabla a|(1 + |Du|)^{p-2} |\nabla^2 u|\,\big).$$

Multiplying i-th equation of (3.17) by $\partial_n^2 u_i$, summing over $i = 1, \dots, n-1$, using (1.5a), $\partial_i \pi = \partial_\tau \pi - \partial_i a \partial_n \pi$, adding to this (3.16) multiplied by $(1 + |Du|)^{p-2}$, we get

$$(1 + |Du|)^{p-2} |\partial_n^2 u|^2 \le C\, \big(|\partial_\tau \pi| + (1 + |Du|)^{p-2} |\nabla \partial_\tau u| + |f|$$
$$+ |\nabla a|\, \big(|\partial_n \pi| + (1 + |Du|)^{p-2} |\nabla^2 u|\big)\,\big)\, |\partial_n^2 u|.$$

We divide this by $|\partial_n^2 u|$, notice that from n-th equation of (1.1), with $\delta = 0$, follows

$$|\partial_n \pi| \le C\, \big(|f| + (1 + |Du|)^{p-2} |\nabla^2 u|\big)$$

and get

$$(1 + |Du|)^{p-2} |\partial_n^2 u| \le C\, \big(|\partial_\tau \pi| + (1 + |Du|)^{p-2} |\nabla \partial_\tau u| + |f|$$
$$+ |\nabla a|(1 + |Du|)^{p-2} |\nabla^2 u|\,\big)\,.$$

Adding to this (3.15) multiplied by $(1 + |Du|)^{p-2}$, and finally choosing $R > 0$ small enough in Sect. 2, namely $R < 1/(4C)$, to absorb the term with $|\nabla a|$ into the left-hand side, we get ([8], (6.6))

$$(1 + |Du|)^{p-2} |\nabla^2 u| \le C\, \big(|\partial_\tau \pi| + (1 + |Du|)^{p-2} |\nabla \partial_\tau u| + |f|\big). \tag{3.18}$$

Following [3], we set

$$r(q) = \frac{2q}{(p + q - 2)}.$$

Let us assume for a while

$$\nabla u \in L^q(\Omega), \quad p \le q \le (p-1) \frac{2n}{n-2}. \tag{3.19}$$

Since $L^2(\Omega) \hookrightarrow W^{-1, \frac{q}{p-1}}(\Omega)$ we obtain from Lemma 3.1 that $\pi \in L^{\frac{q}{p-1}}(\Omega)$ and

$$\|\pi\|_{\frac{q}{p-1}} \le C \left(\|f\|_2 + \|1 + |\nabla u|\|_q^{p-1} \right).$$

Consequently the assumptions of Lemma 3.2 are satisfied with $S = r(q)$, since for $q \ge p$ we have $r(q) \le q/(p-1)$. Lemma 3.2 and (3.12) thus imply

$$\|\xi d_\tau \pi\|_{r(q)} \le C\, h \left(\|1 + |\nabla u|\|_q^{p-1} + \|f\|_2^{\frac{2}{p}(p-1)} \right),$$

which by the properties of the tangential differences yields

$$\|\xi \partial_\tau \pi\|_{r(q)} \le C \left(\|1 + |\nabla u|\|_q^{p-1} + \|f\|_2^{\frac{2}{p}(p-1)} \right). \tag{3.20}$$

Note that $2q/(p+q-2) \leq 2$. In view of (3.20) we would like to take the $L^{\frac{2q}{p+q-2}}$-norm of (3.18) after multiplication with ξ, because this would give us an estimate of the left-hand side in terms of $\|\nabla u\|_q$ and $\|f\|_2$ only, provided $\left\|\xi(1+|Du|)^{p-2}|\nabla \partial_\tau u|\right\|_{r(q)}$ is also finite. One has

$$|\nabla \mathcal{F}(Du)| \simeq (1+|Du|)^{\frac{p-2}{2}} |\nabla Du| \tag{3.21}$$

and

$$|\partial_\tau \mathcal{F}(Du)| \simeq (1+|Du|)^{\frac{p-2}{2}} |D\partial_\tau u|. \tag{3.22}$$

See (4.17), (4.20) in [2] or (52) in [8]. Unfortunately, this information together with (3.13) is not sufficient to estimate $\left\|\xi(1+|Du|)^{p-2}|\nabla \partial_\tau u|\right\|_{r(q)}$ using Hölder's inequality, since it is not clear how to bound $\nabla \partial_\tau u$ by $D\partial_\tau u$. Nevertheless we can extract some suboptimal information for $\nabla \partial_\tau u$ from (3.13). Using (3.22), $p \geq 2$, and Korn's inequality we get from (3.13)

$$c\,\|\xi \nabla \partial_\tau u\|_2 - C\,\|\nabla u\|_2 \leq \|\xi D\partial_\tau u\|_2 \leq \|\xi \partial_\tau \mathcal{F}(Du)\|_2 < \infty. \tag{3.23}$$

In order to estimate the right-hand side of (3.18) we proceed as follows. We divide the second term in (3.18) by $(1+|Du|)^\beta$ for some suitable $\beta > 0$, multiply by ξ, and estimate by Hölder's and Korn's inequalities

$$\begin{aligned}\left\|\xi(1+|Du|)^{p-2-\beta}|\nabla \partial_\tau u|\right\|_{r(q)} &\leq \|\xi \nabla \partial_\tau u\|_2 \,\||1+|Du|\||^{p-2-\beta}_{(p-2-\beta)\frac{2q}{p-2}} \\ &\leq \|\xi \partial_\tau \mathcal{F}(Du)\|_2 \,\|1+|Du|\|^{p-2-\beta}_{(p-2-\beta)\frac{2q}{p-2}}\end{aligned} \tag{3.24}$$

Requiring now $2q(p-2-\beta)/(p-2) = q$ yields $\beta = (p-2)/2$. Taking the $L^{r(q)}$-norm of (3.18) multiplied by $\xi(1+|Du|)^{-(p-2)/2}$ we get using (3.20), (3.24) with (3.13), and $f \in L^2(\Omega)$, also using $\nabla^2 u \simeq \nabla Du$, and (3.21) that

$$\|\xi \nabla \mathcal{F}(Du)\|_{r(q)} \leq \|\xi(1+|Du|)^{\frac{p-2}{2}}|\nabla^2 u|\|_{r(q)} \leq C\left(\|1+|\nabla u|\|_q^{p-1} + \|f\|_2^{\frac{2}{p}(p-1)}\right). \tag{3.25}$$

Theorem 2.1 with $F = \xi \mathcal{F}(Du)$, $\partial_\tau F \in L^2(\Omega)$ by (3.13) and (3.2), $\partial_n F \in L^{r(q)}(\Omega)$ by (3.25) and (3.2) yields

$$\|\xi \mathcal{F}(Du)\|_\nu \leq C\left(\|1+|\nabla u|\|_q^{p-1} + \|f\|_2^{\frac{2}{p}(p-1)}\right)^{\frac{1}{n}} (1+\|f\|_2)^{\frac{n-1}{n}} \tag{3.26}$$

with

$$\frac{1}{\nu} = \frac{1}{n}\left(\frac{n-1}{2} + \frac{1}{r(q)} - 1\right) \quad \text{which is} \quad \nu = \frac{2nq}{q(n-2)+p-2}.$$

Recalling that $\{\xi_P : P \in \Gamma\}$ was a partition of unity we obtain from (3.26), using also definition of $\mathcal{F}$, and Korn's inequality

$$\|\nabla u\|_{\frac{pnq}{q(n-2)+p-2}} \leq C\,\|\mathcal{F}(Du)\|^{\frac{2}{p}}_{\frac{2nq}{q(n-2)+p-2}} \leq K\,\|\nabla u\|_s^\alpha + K, \tag{3.27}$$

where we defined for suitable $C > 1$

$$\alpha := \frac{2(p-1)}{pn}, \quad K := C\,(1+\|f\|_2)^{\frac{4}{pp'}}. \tag{3.28}$$

Since, $q < nqp/(q(n-2)+p-2)$ if $q < (np+2-p)/(n-2)$ ($q < +\infty$ if $n = 2$) we improved the information (3.19) in this case. Moreover, since now

$$\|\nabla u\|_q \leq C\|\nabla u\|_{\frac{pnq}{q(n-2)+p-2}} < +\infty$$

and $\alpha < 1$ depends only on p and n we get by Young's inequality from (3.27)

$$\|\nabla u\|_{\frac{pnq}{q(n-2)+p-2}} \leq CK^{\frac{1}{1-\alpha}}. \tag{3.29}$$

As we always know $\nabla u \in L^p(\Omega)$ we may start the procedure above (with $q = p$ in (3.19)) to get (3.29) for all $q \in (1, (np + 2 - p)/(n - 2))$.

If $n = 2$ the statement of the theorem follows from (3.27) and (3.25), since $\{\xi_P : P \in \Gamma\}$ is partition of the unity.

Let us now consider the case $n > 2$ and follow [8, Proof of Theorem 2.3]. We note that the constant K in the estimate (3.29) depends on the norm of the embedding $W^{1,r(q)}(\Omega) \hookrightarrow L^{2nq/((q(n-2)+p-2)}(\Omega)$ which can be estimated uniformly with respect to $s \in (1, (np + 2 - p)/(n - 2))$, since $n > 2$. Thus the estimate (3.29) is uniform with respect to $q \in (1, (np+2-p)/(n-2))$ and independent of $\|\nabla u\|_q$. Hence, the estimate (3.29) remains valid also for $q = (np + 2 - p)/(n - 2)$. Repeating the procedure starting from (3.19) with this q one concludes the proof. Indeed, (1.7) then follows from (3.27) and (3.25), since $\{\xi_P : P \in \Gamma\}$ is a partition of the unity.

4. Steady p-Navier–Stokes Problem

This section is devoted to the study of regularity of the system (1.1), with $\delta = 1$, (1.2), and $\mathcal{S}$ satisfying Assumption 1. We refer to this problem as the *steady p-Navier–Stokes* problem. Since the main obstacle, namely the interaction of non-flat boundary with the nonlinear elliptic term was overcome in the previous sections, we will concentrate only on how to deal with the convective term. Precisely, we show for which values of the growth parameter p the diffusive part of the system dominates the convective one.

We will consider only the case $p \geq \max\{2, (3n)/(n + 2)\}$ because then it is allowed to test the weak formulation of the problem by the weak solution itself.

Definition 4.1. Let $\mathcal{S}$ satisfy Assumption 1 with $p \geq \max\{2, (3n)/(n + 2)\}$ and let $f \in W^{-1,p'}(\Omega)$. A weak solution of the steady p-Navier–Stokes problem is a function $u \in W^{1,p}_0(\Omega)$, with $\operatorname{div} u = 0$ such that for all $\varphi \in W^{1,p}_0(\Omega)$ with $\operatorname{div} \varphi = 0$ it holds

$$\int_\Omega \mathcal{S}(Du)D\varphi + ([\nabla u]u)\varphi \, dx = \langle f, \varphi\rangle_{1,p}. \tag{4.1}$$

The existence of a weak solution of the steady p-Navier–Stokes problem is easy to obtain by the Galerkin method together with monotone operator theory and a compactness result (cf. [39]). A weak solution always satisfies the estimate

$$\|Du\|_p^p \leq C \, \|f\|_{-1,p'}^{p'}. \tag{4.2}$$

For $f \in L^2(\Omega)$ and $p \geq 2$ we always have $f \in W^{-1,p'}(\Omega)$. To the weak solution u there exists an associated pressure $\pi \in L^{p'}(\Omega)$, with $\int_\Omega \pi \, dx = 0$, satisfying for all $\varphi \in W^{1,p}_0(\Omega)$

$$\int_\Omega \mathcal{S}(Du)D\varphi + ([\nabla u]u)\varphi + \pi \operatorname{div} \varphi \, dx = \langle f, \varphi\rangle_{1,p} \tag{4.3}$$

and

$$\|\pi\|_{p'} \leq C \left(\|f\|_{-1,p'} + \|f\|_{-1,p'}^{\frac{2}{p-1}} \right). \tag{4.4}$$

Now we would like to prove Theorem 1.2. Clearly, since $[\nabla u]u \in L^2(\Omega)$ for $p \geq (4n)/(n + 2)$ we can view the convective term as an additional right-hand side. Thus Theorem 1.1 immediately yields Theorem 1.2 for such p's. In particular the case $n = 2$ is completely covered. Thus we restrict ourselves in the following to the case $n \geq 3$ and $p < (4n)/(n + 2)$. Note that this in particular implies $p < n$. Another important observation is that it is enough to show Theorem 1.2 only for $q \in (1, (np + 2 - p)/(n - 2))$ since then from the obtained regularity follows $[\nabla u]u \in L^2(\Omega)$ and the statement of the theorem in the case $q = (np + 2 - p)/(n - 2)$ follows from Sect. 3.

Let us assume that

$$p \geq \max\left\{2, \frac{3n}{n+2}\right\} \qquad \text{and} \qquad p \in \left[3, \frac{4n}{n+2}\right) \bigcup \left(\frac{n}{2}, \frac{4n}{n+2}\right). \tag{4.5}$$

In order to prove Theorem 1.2 in this case we follow the procedure from Sects. 3.1 and 3.2. As the computations are very similar, we mention only the differences.

4.1. Regularity in Tangential Direction

When deriving estimates in tangential directions for Eq. (1.1), with $\delta = 1$, we have to deal also with convective term. In Eq. (3.7) there appears additionally the term

$$\int_\Omega \left([(\nabla u)_\tau] d_\tau u + [\nabla d_\tau u] u - [(\partial_n u)_\tau \otimes d_\tau \nabla a] u\right) \xi \psi \, dx.$$

Setting $\psi = \xi d_\tau u$ the second integral reduces due to $\operatorname{div} u = 0$ and we have to estimate

$$\int_\Omega \xi^2 \left([(\nabla u)_\tau] d_\tau u\right) d_\tau u \, dx - \int_\Omega \xi |d_\tau u|^2 [\nabla \xi] u + \xi^2 \left([(\partial_n u)_\tau \otimes d_\tau \nabla a] u\right) d_\tau u \, dx.$$

If $p \geq 3$ we easily get that all these integrals are bounded by Ch^2. Thus we obtain as in Sect. 3.1 that

$$\|\xi \partial_\tau \mathcal{F}(Du)\|_2^2 \leq C. \tag{4.6}$$

Let now $p \geq \max\{2, (3n)/(n+2)\}$ be such that $p \in (n/2, 3)$. Using (2.2), Sobolev's embedding theorem, Korn's inequality, Hölder's inequality, and proceeding as in the treatment of the term I_2 in Sect. 3.1 we can estimate the second integral above by

$$\begin{aligned} C\, h \, \|Du\|_p^2 \, \|\xi d_\tau u\|_{\frac{2n}{n-2}} &\leq C\, h \, \|Du\|_p^2 \, \|D(\xi d_\tau u)\|_2 \\ &\leq C\, h^2 \left(\|Du\|_p^3 + \|Du\|_p^4\right) + \varepsilon \, \|\xi d_\tau \mathcal{F}(Du)\|_2^2 \end{aligned}$$

as long as $p \geq 4n/(n+4)$. One easily checks that this is true for our choice of p we are just considering. Note that the last term can be absorbed into the left-hand side of (3.8). The first integral above can be estimated with Hölder's inequality by

$$C \, \|\nabla u\|_p \, \|\xi d_\tau u\|_{2p'}^2 \, .$$

As $p < 2p' < 2n/(n-2)$ if $n/2 < p < 3$ we can interpolate the last term between $L^p(\Omega)$ and $L^{\frac{2n}{n-2}}(\Omega)$. After that we proceed as in the above treatment of the second integral and obtain

$$C \, \|\nabla u\|_p \, \|\xi d_\tau u\|_{2p'}^2 \leq C\, h^2 \left(\|Du\|_p^3 + \|Du\|_p^{\frac{p(6-n)}{2p-n}}\right) + \varepsilon \, \|\xi d_\tau \mathcal{F}(Du)\|_2^2$$

Again the last term can be absorbed into the left-hand side of (3.8). Proceeding now as in Sect. 3.1 we again derive (4.6).

4.2. Regularity in Normal Direction

In this section we follow Sect. 3.2 and reconstruct regularity in normal directions from the previous information. We define

$$g := f - [\nabla u] u$$

and get (3.18) with f replaced by g. We have to consider only such p's which satisfy (4.5), since for $p \geq (4n)/(n+2)$ Theorem 1.2 is already proved. Thus, now $g \notin L^2(\Omega)$.

We assume for a while that (3.19) holds true. In this case Lemma 3.1 yields $\pi \in L^{\frac{q}{p-1}}(\Omega)$. Indeed, as discussed in Remark 3.1 we have $f \in L^2(\Omega) \hookrightarrow W^{-1,\frac{q}{p-1}}(\Omega)$. Assumption (3.19) for $q < n$ implies

$|u|^2 \in L^{\frac{nq}{2(n-q)}}(\Omega)$ by Sobolev's embedding theorem. Consequently $[\nabla u]u = \operatorname{div}(u \otimes u) \in W^{-1,\frac{q}{p-1}}(\Omega)$, because $nq/(2(n-q)) \geq q/(p-1)$ for $p \geq (3n)/(n+2)$. For $s \geq n$ we have $|u|^2 \in L^r(\Omega)$ for all $r < \infty$ and we immediately obtain $[\nabla u]u \in W^{-1,\frac{q}{p-1}}(\Omega)$.

In Sect. 3.2 we found that it would be optimal to take the $L^{r(q)}$-norm of (3.18) with f replaced by g. Thus we have to investigate to which space belongs g. For $q \geq (4n)/(n+2)$ we easily see that $g \in L^2(\Omega) \hookrightarrow L^{r(q)}(\Omega)$. For $q \in [p, (4n)/(n+2))$ follows $g \in L^{Q(q)}(\Omega)$ with $Q(q) := qn/(2n-q)$. For these q we have that $Q(q) \geq r(q)$ if and only if $q \geq n(6-p)/(n+2)$.

If $p \geq n(6-p)/(n+2)$, which is $p \geq 3n/(n+1)$, we know for all q in (3.19) that $Q(q) \geq r(q)$ and the proof can be concluded as in Sect. 3.2.

Let us consider the case $p < 3n/(n+1)$. We start with remark that for $n \geq 5$ there are no p's that satisfy simultaneously (4.5) and $p < 3n/(n+1)$. Note that if $q \geq n(6-p)/(n+2)$ we know that $g \in L^{r(q)}(\Omega)$ and we can conclude the proof as in Sect. 3.2. Consequently we have to consider only for $n = 3, 4$ such p and q that satisfy

$$p \in \left[2, \frac{3n}{n+1}\right) \bigcap \left(\frac{n}{2}, \frac{3n}{n+1}\right) \tag{4.7}$$

and

$$q \in \left[p, \frac{n(6-p)}{n+2}\right). \tag{4.8}$$

For these values of p and q we have $q < 4n/(n+2) < n$ and $Q(q) < r(q)$.

We want to show that for p and q satisfying (4.7) and (4.8) we obtain $Du \in L^{\hat{q}}(\Omega)$ with $\hat{q} \geq n(6-p)/(n+2)$. Then we can again conclude the proof as in Sect. 3.2. For that we proceed as follows. From the definition of $Q(q)$ we know that $g \in L^{Q(q)}(\Omega)$ and since $q/(p-1) \geq r(q) > Q(q)$ we also have $\pi \in L^{Q(q)}(\Omega)$. Now we apply Lemma 3.2 with $S = Q(q)$. Consequently we get

$$\|\xi \partial_\tau \pi\|_{Q(q)} \leq C. \tag{4.9}$$

Taking the $L^{Q(q)}$-norm of (3.18) divided by $(1+|Du|)^\beta$ with f replaced by g it remains to estimate by Hölders inequality, and (3.23)

$$\begin{aligned} \left\|\xi(1+|Du|)^{p-2-\beta}|\nabla \partial_\tau u|\right\|_{Q(q)} &\leq C\left(1 + \|\xi \nabla \partial_\tau u\|_2 \, \|1+|Du|\|_{(p-2-\beta)\frac{2Q(q)}{2-Q(q)}}\right) \\ &\leq C\left(1 + \|Du\|_{(p-2-\beta)\frac{2Q(q)}{2-Q(q)}}\right). \end{aligned}$$

Requiring now $2Q(q)(p-2-\beta)/(2-Q(q)) \leq s$ we compute $\beta \geq p - 4 + q(n+2)/(2n)$. We set

(a) $\beta = p - 4 + q(n+2)/(2n)$ is $q \leq 2(4-p)n/(n+2)$

(b) $\beta = 0$ otherwise.

In the case (a) it follows

$$\left\|(1+|Du|)^{3-q\frac{n+2}{2n}}\right\|_{1,Q(q)} + \left\|(1+|Du|)^{2-q\frac{n+2}{2n}}|\nabla^2 u|\right\|_{Q(q)} < \infty$$

and by embedding

$$\|Du\|_{\frac{q(6n-q(n+2))}{4(n-q)}} < \infty.$$

In the case (b) we get

$$\left\|(1+|Du|)^{p-1}\right\|_{1,Q(q)} + \left\|(1+|Du|)^{p-2}|\nabla^2 u|\right\|_{Q(q)} < \infty$$

and by embedding

$$\|Du\|_{\frac{(p-1)qn}{2(n-q)}} < \infty.$$

Since there exists $\beta_0 > 0$ such that

$$\min\left\{\frac{q(6n-q(n+2))}{4(n-q)}, \frac{(p-1)qn}{2(n-q)}\right\} - q \geq \beta_0$$

for p satisfying (4.7) and $q \in [p, n(6-p)/(n+2)]$ satisfying (4.8) we may iterate this process finitely many times to get

$$\|Du\|_{\frac{n(6-p)}{n+2}} < +\infty.$$

Finally, as it was already mentioned, the proof concludes as in Sect. 3.2.

5. Unsteady *p*-Navier–Stokes Problem in 3D

In this section we extend the results shown in the previous sections to any local-in-time weak solution of the evolutionary problem (1.4), (1.2) with $\mathcal{S}$ satisfying Assumption 1 considered on $Q := I \times \Omega$. We refer to this problem as the *unsteady p-Navier–Stokes problem*. Theorem 1.3 improves previous results stated in references [3,6,8,42].

We will consider only the case $p \geq (2+3n)/(n+2)$ because then it is allowed to test the weak formulation of the problem by the weak solution itself. We start with definition of weak solution.

Definition 5.1. Let $\mathcal{S}$ satisfy Assumption 1 with $p \geq (2+3n)/(n+2)$ and let $f \in L^{p'}(I, W^{-1,p'}(\Omega))$. We say that $u \in L^p_{\mathrm{loc}}(I, W^{1,p}_0(\Omega)) \cap L^\infty_{\mathrm{loc}}(I, L^2(\Omega))$ with $\operatorname{div} u = 0$ is a weak solution of the unsteady p-Navier–Stokes problem if for all $\varphi \in C^\infty_0(I, W^{1,p}_0(\Omega))$ with $\operatorname{div}\varphi = 0$ holds

$$\int_Q -u\partial_t\varphi + \mathcal{S}(Du)D\varphi + ([\nabla u]u)\varphi \, dx\, dt = \int_I \langle f, \varphi\rangle_{1,p}\, dt. \tag{5.1}$$

Note that the test function φ in (5.1) have compact support in time and thus we do not have to specify an initial condition. It was already proved in [38] that under suitable assumptions on the data there exists a global-in-time weak solution for $p \geq (2+3n)/(n+2)$. By a global-in-time weak solution we mean a function $u \in L^p(I, W^{1,p}_0(\Omega)) \cap L^\infty(I, L^2(\Omega))$ with $\operatorname{div} u = 0$, that satisfies for all $\varphi \in C^\infty(\overline{I}, W^{1,p}_0(\Omega))$ with $\operatorname{div}\varphi = 0$ and $\varphi(T) = 0$ Eq. (5.1) with $\int_\Omega u_0\varphi(0)\, dx$ added on the right-hand side. It was proved in [14] that if f is more regular in time also u has better regularity properties in time if $n = 3$. However, the method used there works also in other dimension than $n = 3$. We formulate the result in the following theorem.

Theorem 5.1. *Let $p > (2+3n)/(n+2)$. Let $\mathcal{S}$ satisfy Assumption 1 and $f \in W^{1,2}_{\mathrm{loc}}(I, L^2(\Omega))$. Then every weak solution u satisfies*

$$u \in W^{1,\infty}_{\mathrm{loc}}(I, L^2(\Omega)) \quad \mathcal{F}(Du) \in W^{1,2}_{\mathrm{loc}}(I, L^2(\Omega))$$

Proof. Detailed proof is given in [14, Corollary 1.3] for $n = 3$. If $n \neq 3$ the same method works. □

Remark 5.1. Since $W^{1,2}_{\mathrm{loc}}(I, L^2(\Omega)) \subset L^\infty_{\mathrm{loc}}(I, L^2(\Omega))$ the statement of Theorem 5.1 implies by definition of $\mathcal{F}(Du)$ that $u \in L^\infty_{\mathrm{loc}}(I, W^{1,p}_0(\Omega))$.

Now we would like to prove Theorem 1.3. Let u be a weak solution. By Theorem 5.1 and Remark 5.1 we have $u \in L^\infty_{\mathrm{loc}}(I, W^{1,p}_0(\Omega))$, $\partial_t u \in L^\infty_{\mathrm{loc}}(I, L^2(\Omega))$, i.e. choosing $J \subset\subset I$ there exists $C > 0$ that for a.e. $t \in J$

$$\|u(t)\|_{1,p} + \|\partial_t u(t)\|_2 \leq C.$$

Since for a.e. $t \in J$ also

$$-\operatorname{div}\mathcal{S}(Du(t)) + [\nabla u(t)]\, u(t) + \nabla\pi(t) = f(t) - \partial_t u(t) \tag{5.2}$$

in $W^{-1,p'}(\Omega)$ we have that $u(t)$ is a weak solution of (5.2) with (1.2) and we may apply Theorem 1.2 to get (1.8).

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