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Concerning the Existence of Classical Solutions to the Stokes System. On the Minimal Assumptions Problem

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Abstract. In this notes we consider the stationary Stokes system in a bounded, connected, three-dimensional smooth domain, with homogeneous Dirichlet boundary condition. Proofs also apply to the n-dimensional case, and to other boundary conditions, like Navier-slip ones. We say here that a solution is *classical* if all derivatives appearing in the equations are continuous up to the boundary. It is well known, for long time, that solutions of the Stokes system are classical if the external forces belong to the Hőlder space $C^{0,\lambda}(\overline{\Omega})$. It is also well known that, in general, solutions are not classical in the presence of continuous external forces. Hence, a very challenging problem is to find Banach spaces, strictly containing the Hőlder spaces $C^{0,\lambda}(\overline{\Omega})$, such that solutions to the Stokes problem corresponding to forces in the above space are classical. We prove this result for external forces in a suitable functional space, denoted $\mathbf{C}_*(\overline{\Omega})$, introduced in references Beirão da Veiga (On the solutions in the large of the two-dimensional flow of a non-viscous incompressible fluid, 1982) and Beirão da Veiga (J Differ Equ 54(3):373–389, 1984) in connection with the Euler equations.

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1. Introduction and Main Results

We begin to introduce some notation. Ω is an open, bounded, connected set in $\mathbb{R}^3$, locally situated on one side of its boundary Γ. We assume that Γ is of class $C^{2,\lambda}(\overline{\Omega})$, for some λ, $0 < \lambda \leq 1$.

$C(\overline{\Omega})$ denotes the Banach space of all real continuous functions defined in $\overline{\Omega}$, with norm

$$\| f \| \equiv \sup_{x \in \Omega} | f(x) |.$$

We also need the classical spaces $C^1(\overline{\Omega})$ and $C^2(\overline{\Omega})$, with norm

$$\| u \|_1 \equiv \| u \| + \sum_{i=1}^{n} \| \partial_i u \| , \quad \| u \|_2 \equiv \| u \| + \sum_{i,j=1}^{n} \| \partial_{ij} u \| ,$$

respectively. Further, for each $\lambda \in (0, 1]$, we define de semi-norm

$$[f]_{0,\lambda} \equiv \sup_{x,y \in \Omega\,;\, x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\lambda} , \tag{1.1}$$

and consider the Hőlder space $C^{0,\lambda}(\overline{\Omega}) \equiv \{ f \in C(\overline{\Omega}) : [f]_{0,\lambda} < \infty \}$, normed by

$$\| f \|_{0,\lambda} = \| f \| + [f]_{0,\lambda}.$$

Clearly, $C^{0,1}(\overline{\Omega})$ is the space of Lipschitz continuous functions in $\overline{\Omega}$. Further, $C^\infty(\overline{\Omega})$ denotes the set of all restrictions to $\overline{\Omega}$ of indefinitely differentiable functions in $\mathbb{R}^3$.

Boldface symbols refer to vectors, vector spaces, and so on. Components of a generic vector $\boldsymbol{u}$ is indicated by u_i, with similar notation for tensors. Norms in function spaces, whose elements are vector fields, are defined in the usual way by means of the corresponding norms of the components.

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The quantities c, $c_0\,, c_1, \ldots,$ denote positive constants depending at most on Ω. For simplicity, we may use the same symbol c to denote different constants.

In what follows we consider the Stokes system (see, for instance, [8,11,19])

$$\begin{cases} -\Delta\,\boldsymbol{u} + \nabla\,p = \boldsymbol{f} \quad \text{in} \quad \Omega\,, \\ \nabla\cdot\,\boldsymbol{u} = 0 \quad \text{in} \quad \Omega\,, \\ \boldsymbol{u} = 0 \quad \text{on} \quad \Gamma. \end{cases} \tag{1.2}$$

If $\boldsymbol{f} \in \boldsymbol{C}(\overline{\Omega})$, this problem has a unique solution $(\boldsymbol{u},\, p) \in \boldsymbol{C}^1(\overline{\Omega}) \times C(\overline{\Omega})$, p up to a constant. Furthermore, the solution is given by

$$u_i(x) = \int_\Omega G_{i\,j}(x,\,y)\, f_j(y)\,dy\,, \quad p(x) = \int_\Omega g_j(x,\,y)\, f_j(y)\,dy\,, \tag{1.3}$$

where $\boldsymbol{G}$ and $\boldsymbol{g}$ are respectively the Green's tensor and vector associated with the above boundary value problem. Furthermore, the following estimates hold.

$$\begin{gathered} |\,G_{i\,j}(x,\,y)\,| \leq \frac{C}{|x-y|}\,, \\ \Big|\,\frac{\partial\,G_{i\,j}(x,\,y)}{\partial\,x_k}\,\Big| \leq \frac{C}{|x-y|^2}\,, \quad |\,g_j(x,\,y)\,| \leq \frac{C}{|x-y|^2}\,, \\ \Big|\,\frac{\partial^2\,G_{i\,j}(x,\,y)}{\partial\,x_k\,\partial\,x_l}\,\Big| \leq \frac{C}{|x-y|^3}\,, \quad |\,\frac{\partial\,g_j(x,\,y)}{\partial\,x_k}\,| \leq \frac{C}{|x-y|^3}\,, \end{gathered} \tag{1.4}$$

where the positive constant C depends only on Ω. A detailed treatment of the above properties can be found, for instance, in chapter 3 of the classical Ladyzhenskaya's famous treatise [11], where the author gives a quite complete overview on the classical theory of hydrodynamical potentials (due to Lichtenstein [12], and to Odqvist [14]). Furthermore, the author shows how to construct the Green functions $\boldsymbol{G}$ and $\boldsymbol{g}$ (Sect. 4, Chapter 3). The estimates (1.4) are contained in equations (46) and (47). They may also be found in Solonnikov's paper [15]; see also [7], [8] section IV.6, and [20]. The estimates (1.4) are a particular case of a set of much more general results, due to many authors. See, for instance, [2] and [16].

The study of minimal, *explicit* conditions on $\boldsymbol{f}$ in order to guarantee the continuity up to the boundary of all derivatives appearing in Eq. (1.2) is a classical, and extremely natural problem. It is well known, even in the simplest scalar case $\Delta\,u = f$ in Ω, $u = 0$ on Γ, that $f \in C(\overline{\Omega})$ does not imply $\nabla^2\,u \in C(\overline{\Omega})$. On the contrary, Hőlder continuity is sufficient, since $\nabla^2\,u \in C^{0,\,\lambda}(\overline{\Omega})$ if $f \in C^{0,\,\lambda}(\overline{\Omega})$. The above picture leads us to look for Banach spaces $\boldsymbol{C}_*(\overline{\Omega})$ satisfying

$$\boldsymbol{C}^{0,\,\lambda}(\overline{\Omega}) \subset \boldsymbol{C}_*(\overline{\Omega}) \subset \boldsymbol{C}(\overline{\Omega})\,, \tag{1.5}$$

with strict inclusions, and such that the solutions $(\boldsymbol{u},\, p)$ to problem (1.2) satisfy $\nabla^2\,\boldsymbol{u} \in \boldsymbol{C}(\overline{\Omega})$, and $\nabla\,p \in \boldsymbol{C}(\overline{\Omega})$.

The following is the main result of our paper. For the definition of $\boldsymbol{C}_*(\overline{\Omega})$ see the next section.

Theorem 1.1. *For every $\boldsymbol{f} \in \boldsymbol{C}_*(\overline{\Omega})$ the solution $(\boldsymbol{u},\, p)$ to the Stokes system (1.2) belongs to $\boldsymbol{C}^2(\overline{\Omega}) \times C^1(\overline{\Omega})$. Moreover, there is a constant c_0, depending only on Ω, such that the estimate*

$$\|\,\boldsymbol{u}\,\|_2 + \|\nabla\,p\,\| \leq c_0\,\|\,\boldsymbol{f}\,\|_*\,, \quad \forall\,\boldsymbol{f} \in \boldsymbol{C}_*(\overline{\Omega})\,, \tag{1.6}$$

holds.

Remark 1.1. Estimates (1.4) are the key tool in the proof of Theorem 1.1. Since they also hold under boundary conditions other than Dirichlet's, like, for instance, Navier-slip ones (see [17,18], and [6]), Theorem 1.1 continues to hold in these cases as well. It should be not particularly difficult to extend Theorem 1.1 to any solution to the corresponding Navier-Stokes equations, in the case of dimension less or equal to 3. For arbitrarily large dimensions the situation changes dramatically, as explained in the deep introduction of reference [9]

Remark 1.2. We introduced the functional space $\boldsymbol{C}_*(\overline{\Omega})$ in [4] and [5], where we proved the existence of a classical solution $\boldsymbol{u} \in C(\mathbb{R}; \boldsymbol{C}^1(\overline{\Omega}))$ to the initial-boundary value problem for the 2-D Euler equations with data $\boldsymbol{u}(0) = \boldsymbol{u}_0 \in \boldsymbol{C}_*(\overline{\Omega})$ and $\boldsymbol{f} \in L^1(\mathbb{R}; \boldsymbol{C}_*(\overline{\Omega}))$. In the above references the main properties of this functional space were already stated. Actually, results and proofs published in [5] were previously presented in a preparatory and more complete manuscript [3]. In fact, all results given in the following sections were already proved in [3], the only difference being that, instead of the Stokes system (1.2) considered here, we dealt with boundary value problems for second order elliptic (scalar) equations. The proofs concerning the latter problem remained unpublished until now. In fact, in reference [5] we merely stated Theorem 1.1 for the particular problem $-\Delta u = f$ in Ω, $u = 0$ on Γ (see Theorem 4.5 in [5]), since this result was sufficient to treat the 2-D Euler equations, which, at that time, was our focus. Furthermore, just after the statement of the above theorem, we claimed that we were able to prove the result for uniformly elliptic second-order equations under regular boundary value conditions. Here we turn back to the unpublished proof of this last result, by adapting it to the Stokes system (1.2).

Concerning the 2-D Euler equations, we remark that our results and proofs have been rediscovered, after many years, by other authors.

Remark 1.3. In [3] we have also introduced a second functional space $B_*(\overline{\Omega})$, strictly containing $C_*(\overline{\Omega})$, for which some of the regularity results proved for elliptic boundary value problems, and for the 2-D Euler equations, still apply by replacing $C_*(\overline{\Omega})$ by $B_*(\overline{\Omega})$. More precisely, the Theorem 3.1 holds for data in $B_*(\overline{\Omega})$. However we do not know whether Theorem 1.1 holds under this condition. In the final section of this paper we define $B_*(\overline{\Omega})$ and give some comment on this, and related, matter.

2. The Banach Space $\boldsymbol{C}_*(\overline{\Omega})$. Definition and Main Properties

To define and study the vector field case $\boldsymbol{C}_*(\overline{\Omega})$, it is clearly sufficient to consider the scalar case $C_*(\overline{\Omega})$. Algebraic properties and norms are defined by appealing to the single components.

We set

$$I(x;\, r) = \{\, y : |y - x| \leq r \,\}, \quad \Omega(x;\, r) = \Omega \cap I(x;\, r), \quad \Omega_c(x;\, r) = \Omega - \Omega(x;\, r).$$

For $f \in C(\overline{\Omega})$, and each $r > 0$ we set

$$\omega_f(r) \equiv \sup_{x,\, y \in \Omega\,;\, 0 < |x - y| \leq r} |\, f(x) - f(y)\,|. \tag{2.1}$$

Clearly, $\omega_{(f+g)}(r) \leq \omega_f(r) + \omega_g(r)$, and $|\,\omega_f(r) - \omega_g(r)\,| \leq \omega_{(f-g)}(r)$.

As in [4], we use the notation

$$[\, f\,]_* = [\, f\,]_{*,\,R} \equiv \int_0^R \omega_f(r)\, \frac{dr}{r}. \tag{2.2}$$

It is worth noting that R may be replaced by any positive constant δ. In fact, if $0 < \delta < R$, one has (with obvious notation)

$$[\, f\,]_{*,\,\delta} \leq [\, f\,]_{*,\,R} \leq [\, f\,]_{*,\,\delta} + 2 \left(\log \frac{R}{\delta} \right) \|\, f\,\|. \tag{2.3}$$

It follows that norms, obtained by addition of $\|\, f\,\|$ (see [2.5]), are equivalent. In particular, we will use the symbol $[\, f\,]_*$ also to denote the quantity $[\, f\,]_{*,\,\delta}$, where $\delta > 0$ is related to the geometry of Ω.

In the literature, the condition

$$\int_0^\delta \omega_f(r)\, \frac{dr}{r} < +\infty$$

is called *Dini's continuity condition*, see [10], equation (4.47). In [10], problem 4.2, it is remarked that if f satisfies Dini's condition in the whole space $\mathbb{R}^n$, then its Newtonian potential is a C^2 function in $\mathbb{R}^n$.

Definition 2.1. We set

$$C_*(\overline{\Omega}) \equiv \{ f \in C(\overline{\Omega}) : [f]_* < \infty \}. \tag{2.4}$$

It is immediate to verify that $C_*(\overline{\Omega})$ is a linear space, and that the quantity $[\cdot]_*$ is a semi-norm. We endow $C_*(\overline{\Omega})$ with the norm

$$\| f \|_* \equiv [f]_* + \| f \|. \tag{2.5}$$

Note that

$$[f]_* \leq \frac{R^\lambda}{\lambda} [f]_{0,\lambda}, \tag{2.6}$$

for each $f \in C^{0,\lambda}(\overline{\Omega})$, where $\lambda \in (0, 1]$.

The following simple remark looks quite significant.

Remark 2.1. Alternatively, we may define $C_*(\overline{\Omega})$ in a totally equivalent form, by replacing in Eq. (2.4) the quantity $|x - y| \leq r$ simply by $|x - y| = r$.

Next we show that the normed space $C_*(\overline{\Omega})$ is complete.

Theorem 2.1. *The space $C_*(\overline{\Omega})$ endowed with the norm $\| f \|_*$ is a Banach space.*

We start by proving the following partial result.

Proposition 2.1. *Let f_n be a Cauchy sequence in $C_*(\overline{\Omega})$, and let f be the uniform limit of this sequence. Then, $f \in C_*(\overline{\Omega})$. Moreover,*

$$[f]_* \leq \lim_{n \to +\infty} [f_n]_*, \tag{2.7}$$

where the existence of the limit is part of the statement.

Proof. Assume that

$$\lim_{m,n \to \infty} \| f_n - f_m \|_* = 0.$$

Clearly, there is a (unique) $f \in C(\overline{\Omega})$ such that $f_n \to f$, uniformly in $\overline{\Omega}$. In particular,

$$| \omega_{f_n}(r) - \omega_f(r) | \leq 2 \| f_n - f \|.$$

So, for each $r \in (0, R)$, the pointwise limit

$$\omega_f(r) = \lim_{n \to +\infty} \omega_{f_n}(r) \tag{2.8}$$

exists. It is worth noting that (2.8) follows merely from the uniform convergence.

On the other hand

$$\left| \int_0^R \omega_{f_n}(r) \frac{dr}{r} - \int_0^R \omega_{f_m}(r) \frac{dr}{r} \right| = | [f_n]_* - [f_m]_* | \leq [f_n - f_m]_*.$$

This shows that the limit in Eq. (2.9) below exists, since the sequence of integrals is a Cauchy sequence in $\mathbb{R}$. By appealing to (2.8) and to Fatou's lemma, we prove that

$$\int_0^R \omega_f(r)\,\frac{dr}{r} \leq \lim_{n\to\infty} \int_0^R \omega_{f_n}(r)\,\frac{dr}{r}. \tag{2.9}$$

□

We are now in a position to prove Theorem 2.1.

Proof. By replacing the couple $(f_n,\, f)$ by $(f_n - f,\, 0)$, we assume, without loosing generality, that $f = 0$.

Let $\epsilon > 0$ be given. Fix an integer n_ϵ such that $[\, f_n - f_m\,]_* < \epsilon$ whenever $m,\, n \geq n_\epsilon$, and choose a real $R_\epsilon \in (0,\, R\,]$ such that

$$\int_0^{R_\epsilon} \omega_{f_{n_\epsilon}}(r)\,\frac{dr}{r} < \epsilon.$$

Further, let m_ϵ be an integer such that

$$\|\, f_n \,\| < \Big(2\,\log(R/R_\epsilon)\Big)^{-1}\epsilon\,,$$

for all $n \geq m_\epsilon$.

If $n > \max\{\, n_\epsilon,\, m_\epsilon\,\}$ one has

$$[\, f_n\,]_* \leq \int_0^{R_\epsilon} \omega_{(f_n - f_{n_\epsilon})}(r)\,\frac{dr}{r} + \int_0^{R_\epsilon} \omega_{f_{n_\epsilon}}(r)\,\frac{dr}{r} + \int_{R_\epsilon}^{R} \omega_{f_n}(r)\,\frac{dr}{r}.$$

From the above estimates, it readily follows that $[\, f_n\,]_* \leq 3\,\epsilon$. Hence, $f_n \to 0$ in $C_*(\overline{\Omega})$. □

Note that if a sequence f_n converges uniformly to some $f \in C(\overline{\Omega})$, and $[\, f_n\,]_* \leq k$, then $f \in C_*(\overline{\Omega})$, and $[\, f\,]_* \leq k$.

Theorem 2.2. *The embedding*

$$C_*(\overline{\Omega}) \subset C(\overline{\Omega})$$

is compact.

Proof. Let $f_n \in C_*(\overline{\Omega})$ be a sequence of functions such that

$$\|\, f_n \,\|_* \leq k\,,$$

for $n = 1,\, 2, \ldots$. Let $\epsilon > 0$ be given, set $\delta = R\, e^{-\frac{k}{\epsilon}}$, and assume that $|x - y\,| < \delta$.

Clearly,

$$\int_\delta^R \omega_{f_n}(r)\,\frac{dr}{r} \leq k.$$

Since $\omega_{f_n}(r)$ is a non-decreasing function of r, it follows that

$$\omega_{f_n}(\delta)\,\log\frac{R}{\delta} \leq k.$$

By tacking into account the definition of δ, it follows that $\omega_{f_n}(\delta) \leq \epsilon$, for each index n. This proves the equi-continuity required by Ascoli-Arzelà's theorem. □

Next we consider the problem of the extension of functions $f \in C_*(\overline{\Omega})$ outside their initial domain, without losing their basic properties. We set

$$\Omega_\delta \equiv \{x : dist(x,\, \Omega\,) < \delta\},$$

where $\delta > 0$. We show the following result.

Theorem 2.3. *There is a* $\delta > 0$ *such that the following holds. There is a linear continuous map* T *from* $C(\overline{\Omega})$ *to* $C(\overline{\Omega}_\delta)$ *, and from* $C_*(\overline{\Omega})$ *to* $C_*(\overline{\Omega}_\delta)$ *, such that* $T\, f(x) = f(x)$ *, for each* $x \in \overline{\Omega}$ *.*

Proof. It is well known that, for a sufficiently small positive δ, which depends only on Ω, we can construct a system of parallel surfaces Γ_r, $0 \leq r \leq \delta$, such that the surface Γ_r lies outside Ω, at a distance r from $\Gamma_0 = \Gamma$. For convenience, we set $R = \delta$ in Definition (2.2).

For each $x \in \overline{\Omega}_\delta - \Omega$, we denote by $\overline{x}$ the orthogonal projection of x upon the boundary Γ. We define the extension $T\, f = \widetilde{f}$ as follows.

$$\widetilde{f}(x) = \begin{cases} f(x)\,, & \text{if } x \in \overline{\Omega}\,, \\ f(\overline{x})\,, & \text{if } x \in \overline{\Omega}_\delta - \overline{\Omega}. \end{cases} \tag{2.10}$$

Since Γ is smooth and compact, the map $x \to \overline{x}$ is Lipschitz continuous. This leads to the existence of the positive constant k considered below. The map $Tf = \widetilde{f}$ is clearly linear continuous from $C(\overline{\Omega})$ to $C(\overline{\Omega}_\delta)$. Next we define, for each $r \in (0, \delta)$,

$$\omega_{\widetilde{f},\,\delta}(r) = \sup_{x,\, y \in \Omega_\delta;\, |x-y|< r} |\, \widetilde{f}(x) - \widetilde{f}(y)\, |\,,$$

and we show that

$$\omega_{\widetilde{f},\,\delta}(r) \leq \omega_f(k\, r). \tag{2.11}$$

Assume $|x - y| \leq \delta$. If $x \in \Omega_c$ and $y \in \overline{\Omega}$, then

$$|\, \widetilde{f}(x) - \widetilde{f}(y)\, | = |\, f(\overline{x}) - f(y)\, |\,, \quad \text{and} \quad |\overline{x} - y| \leq k\, |x - y|.$$

If $x,\, y \in \Omega_c$, then

$$|\, \widetilde{f}(x) - \widetilde{f}(y)\, | = |\, f(\overline{x}) - f(\overline{y})\, |\,, \quad \text{and} \quad |\overline{x} - \overline{y}| \leq k\, |x - y|.$$

Note that, in a neighborhood of a flat portion of Γ, one has $k = 1$. Equation (2.11) follows easily. Finally,

$$[\, f\,]_{*,\,\delta} = \int_0^\delta \omega_{\widetilde{f},\,\delta}(r)\, \frac{dr}{r} \leq \int_0^{k\,\delta} \omega_f(\rho)\, \frac{d\rho}{\rho} = [\, f\,]_{*,\,k\,\delta}.$$

□

We could also start the above proof by a preliminary localization procedure. We consider a suitable partition of unity ϕ_r, $r = 1, \ldots, M$, subordinated to an open covering $\mathcal{O}_r$ of Γ, $r = 1, \ldots, M$. Given $f \in C_*(\overline{\Omega})$, it is sufficient to prove the extension theorem for each single product $f_r = \phi_r\, f$, since products by regular functions ϕ preserve the $C_*(\overline{\Omega})$ estimates enjoyed by f. This is guaranteed by the following result.

Lemma 2.4. *If* ϕ *and* f *belong to* $C_*(\overline{\Omega})$ *then* $\phi\, f \in C_*(\overline{\Omega})$*. Furthermore,*

$$[\, \phi\, f\,]_* \leq \|\, \phi\, \|\, [\, f\,]_* + \|\, f\, \|\, [\, \phi\,]_*. \tag{2.12}$$

The following density theorem is a fundamental tool in the proof of Theorem 1.1.

Theorem 2.5. *The set* $C^\infty(\overline{\Omega})$ *is dense in* $C_*(\overline{\Omega})$ *.*

The proof of this result follows immediately from Theorem 2.3 together with the following Lemma.

Lemma 2.6. *Let* $f \in C_*(\Omega_\delta)$ *. There is family of functions* $f_\epsilon \in C^\infty(\overline{\Omega})$ *, convergent in the* $C_*(\overline{\Omega})$*-norm, as* $\epsilon \to 0$ *, to the restriction of* f *to* $\overline{\Omega}$ *.*

Proof. For each $\epsilon \in (0, \delta)$ define the Friedrich mollifiers

$$f_\epsilon(x) \equiv \int\limits_{|z|<\epsilon} j_\epsilon(z)\, f(x-z)\, dz\,, \quad \forall\, x \in \Omega_\delta\,, \tag{2.13}$$

where $j_\epsilon(z) \in C^\infty(\mathbb{R}^n)$ is defined in the standard way (see, for instance, [13], Chapter 2, section 1.3, or [1], section 1). In particular $j_\epsilon(z) = 0$ for $|z| \geq \epsilon$, and

$$\int j_\epsilon(z)\, dz = 1.$$

For convenience, assume in (2.13) that $f(x) = 0$ outside Ω_δ. It is well known that

$$\lim_{\epsilon \to 0} \| f_\epsilon - f \|_{C(\overline{\Omega})} = 0. \tag{2.14}$$

In the sequel we assume that $x, y \in \overline{\Omega}$, and $r \leq \delta$. From

$$(f_\epsilon - f)(x) = \int\limits_{|z|<\epsilon} j_\epsilon(z)\,(f(x-z) - f(x))\, dz$$

we show that the quantity

$$\omega_{f_\epsilon}(r) = \sup_{0<|x-y|\leq r;\, x,y\in\Omega} \Big| \int j_\epsilon(z) \big[f(x-z) - f(y-z) \big]\, dz \Big|$$

satisfies the estimate $\omega_{f_\epsilon}(r) \leq \omega_{f,\delta}(r)$. It follows that

$$\omega_{(f_\epsilon - f)}(r) \leq \omega_f(r) + \omega_{f_\epsilon}(r) \leq 2\,\omega_f(r;\delta). \tag{2.15}$$

On the other hand, by appealing to (2.14), one shows that

$$\lim_{\epsilon \to 0} \omega_{(f_\epsilon - f)}(r) = 0\,, \tag{2.16}$$

for each $r < \delta$. Hence, By Lebesgue's dominated convergence theorem,

$$\lim_{\epsilon \to 0} [f_\epsilon - f]_* = \lim_{\epsilon \to 0} \int_0^\delta \omega_{(f_\epsilon - f)}(r)\, \frac{dr}{r} = 0.$$

Note that the right hand side of (2.15) is integrable in $(0, \delta)$ with respect to dr/r. □

As for the proof of Theorem 2.3, it would be sufficient to prove the thesis for the single products $f_r = \phi_r\, f$.

3. Lipschitz Continuity of the First Order Derivatives

In this section we set $\boldsymbol{C}_*(\overline{\Omega}) = C_*(\overline{\Omega}) \times C_*(\overline{\Omega}) \times C_*(\overline{\Omega})$. We prove the following result.

Theorem 3.1. *Let $\boldsymbol{f} \in \boldsymbol{C}_*(\overline{\Omega})$, and let $\boldsymbol{u}$ be the solution to problem (1.2). Then the first order derivatives of the velocity $\boldsymbol{u}$, and the pressure p, are Lipschitz continuous in $\overline{\Omega}$. Furthermore, the estimate*

$$\| \boldsymbol{u} \|_{1,1} + \| p \|_{0,1} \leq c_0 \| \boldsymbol{f} \|_* \tag{3.1}$$

holds.

Proof. Let $x_0 \in \Omega$ be fixed, and define the auxiliary system

$$\begin{cases} -\Delta\, \boldsymbol{v}(x) + \nabla\, q(x) = \boldsymbol{f}(x_0) \quad \text{in} \quad \Omega\,, \\ \nabla \cdot \boldsymbol{v} = 0 \quad \text{in} \quad \Omega\,, \\ \boldsymbol{v} = 0 \quad \text{on} \quad \Gamma. \end{cases} \tag{3.2}$$

Note that the external force field is constant. Clearly, $\boldsymbol{v}$ and q are smooth. We start by showing that

$$\| \boldsymbol{v} \|_{1,1} + \| q \|_{0,1} \leq K \| \boldsymbol{f} \| , \tag{3.3}$$

where the constant K is independent of $\boldsymbol{f}(x_0)$.

Denote by $\boldsymbol{e}_1$ the constant vector field $\boldsymbol{e}_1 = (1, 0, 0)$ and, similarly, define $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$. Further, denote by $(\boldsymbol{v}^{(i)}, q^{(i)})$ the solution to problem (3.2) with $\boldsymbol{f}(x_0)$ replaced by $\boldsymbol{e}_i$. So,

$$\begin{cases} -\Delta \boldsymbol{v}^{(i)}(x) + \nabla q^{(i)}(x) = \boldsymbol{e}_i \quad \text{in} \quad \Omega , \\ \nabla \cdot \boldsymbol{v}^{(i)} = 0 \quad \text{in} \quad \Omega , \\ \boldsymbol{v}^{(i)} = 0 \quad \text{on} \quad \Gamma. \end{cases} \tag{3.4}$$

Define $K \geq 0$ by setting

$$K^2 = \sum_{i=1}^{3} \big(\| \boldsymbol{v}^{(i)} \|_{1,1}^2 + \| q^{(i)} \|_{0,1}^2 \big).$$

One has

$$\boldsymbol{v}(x) = \sum_{i=1}^{3} f_i(x_0)\, \boldsymbol{v}^{(i)}(x) , \quad \text{and} \quad q(x) = \sum_{i=1}^{3} f_i(x_0)\, q^{(i)}(x).$$

It readily follows that

$$\| \boldsymbol{v} \|_{1,1} + \| q \|_{0,1} \leq K \, |\boldsymbol{f}(x_0)| \leq K \, \| \boldsymbol{f} \| ,$$

as desired.

Next we set

$$\boldsymbol{w}(x) \equiv \boldsymbol{u}(x) - \boldsymbol{v}(x) , \quad \text{and} \quad t(x) \equiv p(x) - q(x).$$

Clearly,

$$w_i(x) = \int_\Omega G_{ij}(x, y) \, (f_j(y) - f_j(x_0)) \, dy ,$$

and

$$t(x) = \int_\Omega g_j(x, y) \, (f_j(y) - f_j(x_0)) \, dy.$$

We start by considering the velocity. One has

$$\partial_k w_i(x) - \partial_k w_i(x_0) = \int_\Omega \big(\partial_k G_{ij}(x, y) - \partial_k G_{ij}(x_0, y) \big) (f_j(y) - f_j(x_0)) \, dy ,$$

where ∂_k stands for differentiation with respect to x_k and, clearly, $\partial_k w_i(x_0)$ means $\partial_k w_i(x)$ at point $x = x_0$. It follows that

$$| \partial_k w_i(x) - \partial_k w_i(x_0) | \leq \int_\Omega | \partial_k G_{ij}(x, y) - \partial_k G_{ij}(x_0, y) | \, | f_j(y) - f_j(x_0) | \, dy.$$

Set $\rho = |x - x_0|$. One has

$$\begin{aligned} | \partial_k w_i(x) - \partial_k w_i(x_0) | \leq & \int_{\Omega(x_0;\, 2\rho)} | \partial_k G_{ij}(x, y) - \partial_k G_{ij}(x_0, y) | \, | f_j(y) - f_j(x_0) | \, dy \\ & + \int_{\Omega_c(x_0;\, 2\rho)} | \partial_k G_{ij}(x, y) - \partial_k G_{ij}(x_0, y) | \, | f_j(y) - f_j(x_0) | \, dy \\ & \equiv I_1 + I_2. \end{aligned} \tag{3.5}$$

By appealing to (1.4), we show that

$$I_1 \leq 6 \| f \| \left(\int\limits_{\Omega(x_0;\, 2\,\rho)} \frac{C}{|x-y|^2}\, dy + \int\limits_{\Omega(x_0;\, 2\,\rho)} \frac{C}{|x_0-y|^2}\, dy \right)$$

$$\leq 6\, C \| f \| \left(\int\limits_{I(x;\, 3\,\rho)} \frac{dy}{|x-y|^2} + \int\limits_{I(x_0;\, 2\,\rho)} \frac{dy}{|x_0-y|^2} \right). \tag{3.6}$$

Hence

$$I_1 \leq c\,\rho \| f \| .$$

On the other hand, by appealing to the mean-value theorem and to (1.4), we get

$$|\, \partial_k\, G_{i\,j}(x,\, y) - \partial_k\, G_{i\,j}(x_0,\, y)\,| \leq C\,\rho\, |x' - y|^{-3} \leq C\,\rho\, 2^3\, |x_0 - y|^{-3},$$

for each $y \in \Omega_c(x_0;\, 2\,\rho)$, where the point x' belongs to the straight segment joining x_0 to x. Consequently,

$$I_2 \leq c\,\rho \int\limits_{\Omega_c(x_0;\, 2\,\rho)} \omega_{\boldsymbol{f}}(|y-x_0|)\, \frac{dy}{|x_0-y|^3} \leq c\,\rho \int\limits_{2\,\rho}^{R} \omega_{\boldsymbol{f}}(r)\, \frac{dr}{r}.$$

It follows that

$$I_2 \leq c\,\rho \| \boldsymbol{f} \|_{*}.$$

Next, by appealing to Eq. (3.9), and to the estimates proved above for I_1 and I_2, we show that

$$|\, \nabla\, \boldsymbol{w}(x) - \nabla\, \boldsymbol{w}(x_0)\,| \leq c\,\rho \| \boldsymbol{f} \|_{*}.$$

Consequently,

$$\begin{aligned} |\, \nabla \boldsymbol{u}(x) - \nabla\, \boldsymbol{u}(x_0)| &\leq |\, \nabla \boldsymbol{w}(x) - \nabla\, \boldsymbol{w}(x_0)\,| + |\, \nabla\, \boldsymbol{v}(x) - \nabla\, \boldsymbol{v}(x_0)\,| \\ &\leq c\,\rho \| \boldsymbol{f} \|_{*} + K\,\rho \| \boldsymbol{f} \|. \end{aligned}$$

So,

$$\frac{|\, \nabla\, \boldsymbol{u}(x) - \nabla\, \boldsymbol{u}(x_0)|}{|x-x_0|} \leq c \| f \|_{*}, \quad \forall x,\, x_0 \in \Omega,\, x \neq x_0. \tag{3.7}$$

Furthermore, by (1.4),

$$|\nabla\, \boldsymbol{u}(x)| \leq c \|\boldsymbol{f}\| \int\limits_{\Omega} \frac{C}{|\, x-y|^2}\, dy \leq c \| \boldsymbol{f} \|, \quad \forall\, x \in \Omega.$$

Hence

$$\|\nabla\, \boldsymbol{u}\| \leq c \| \boldsymbol{f} \|. \tag{3.8}$$

This equation, together with (3.11), proves (3.1) for the velocity $\boldsymbol{u}$.

Next we consider the pressure. As above,

$$|\, t(x) - t(x_0)\,| \leq \int\limits_{\Omega} |\, g_j(x,\, y) - g_j(x_0,\, y)\,|\, |\, f_j(y) - f_j(x_0)\,|\, dy.$$

Hence,

$$| t(x) - t(x_0) | \leq \int\limits_{\Omega(x_0;\, 2\,\rho)} | g_j(x,\, y) - g_j(x_0,\, y) | \, | f_j(y) - f_j(x_0) | \, dy \\ + \int\limits_{\Omega_c(x_0;\, 2\,\rho)} | g_j(x,\, y) - g_j(x_0,\, y) | \, | f_j(y) - f_j(x_0) | \, dy \equiv I_3 + I_4. \qquad (3.9)$$

By appealing to (1.4), we show that $I_3 \leq c\,\rho \| f \|$, since

$$I_3 \leq 2 \| f \| \left(\int\limits_{\Omega(x_0;\, 2\,\rho)} \frac{C}{|x-y|^2}\, dy + \int\limits_{\Omega(x_0;\, 2\,\rho)} \frac{C}{|x_0-y|^2}\, dy \right) \\ \leq 2\, C \| f \| \left(\int\limits_{I(x;\, 3\,\rho)} \frac{dy}{|x-y|^2} + \int\limits_{I(x_0;\, 2\,\rho)} \frac{dy}{|x_0-y|^2} \right). \qquad (3.10)$$

On the other hand, by the mean-value theorem and (1.4), we get

$$| \partial_k\, g_j(x,\, y) - \partial_k\, g_j(x_0,\, y) | \leq C\,\rho\, |x' - y|^{-3} \leq C\,\rho\, 2^3\, |x_0 - y|^{-3},$$

for each $y \in \Omega_c(x_0;\, 2\,\rho)$, where the point x' belongs to the straight segment joining x_0 to x. Consequently,

$$I_4 \leq c\,\rho \int\limits_{\Omega_c(x_0;\, 2\,\rho)} \omega_{\boldsymbol{f}}(|y - x_0|)\, \frac{dy}{|x_0 - y|^3} \leq c\,\rho \int_{2\,\rho}^{R} \omega_{\boldsymbol{f}}(r)\, \frac{dr}{r} \leq c\,\rho \| \boldsymbol{f} \|_*.$$

By appealing to (3.9), and to the estimates proved for I_3 and I_4, we show that

$$| t(x) - t(x_0) | \leq c\,\rho \| \boldsymbol{f} \|_*.$$

Consequently,

$$| p(x) - p(x_0) | \leq | t(x) - t(x_0) | + | q(x) - q(x_0) | \\ \leq c\,\rho \| \boldsymbol{f} \|_* + K\,\rho \| \boldsymbol{f} \|.$$

So,

$$\frac{| p(x) - p(x_0) |}{|x - x_0|} \leq c \| f \|_*, \quad \forall x,\, x_0 \in \Omega,\, x \neq x_0. \qquad (3.11)$$

Further, by (1.4),

$$| p(x) | \leq c \| \boldsymbol{f} \| \int\limits_{\Omega} \frac{C}{|x-y|^2}\, dy \leq c \| \boldsymbol{f} \|, \quad \forall\, x \in \Omega.$$

Hence

$$\| p \| \leq c \| \boldsymbol{f} \|. \qquad (3.12)$$

This equation, together with (3.11), proves (3.1) for the pressure p. □

4. Proof of Theorem 1.1

Due to Theorem 3.1, it is sufficient to show that the second order derivatives of $\boldsymbol{u}(x)$ and the first order derivatives of $p(x)$ exist and are continuous in $\overline{\Omega}$.

By Theorem 2.5, there exists a sequence of vector fields $\boldsymbol{f}_m \in \boldsymbol{C}^\infty(\overline{\Omega})$ such that $\boldsymbol{f}_m \to \boldsymbol{f}$ in $\boldsymbol{C}_*(\overline{\Omega})$. Consider the solutions $(\boldsymbol{u}^m, p^m)$ of problems

$$\begin{cases} -\Delta \boldsymbol{u}^m(x) + \nabla p^m(x) = \boldsymbol{f}^m(x) \quad \text{in} \quad \Omega, \\ \nabla \cdot \boldsymbol{u}^m = 0 \quad \text{in} \quad \Omega, \\ \boldsymbol{u}^m = 0 \quad \text{on} \quad \Gamma. \end{cases} \tag{4.1}$$

It is well know that $\boldsymbol{u}^m \in \boldsymbol{C}^2(\overline{\Omega})$ and $p^m \in C^1(\overline{\Omega})$. Furthermore, by applying the result stated in Theorem 3.1 to the system

$$\begin{cases} -\Delta(\boldsymbol{u}^m - \boldsymbol{u}^n) + \nabla(p^m - p^n) = \boldsymbol{f}^m - \boldsymbol{f}^n \quad \text{in} \quad \Omega, \\ \nabla \cdot (\boldsymbol{u}^m - \boldsymbol{u}^n) = 0 \quad \text{in} \quad \Omega, \\ \boldsymbol{u}^m - \boldsymbol{u}^n = 0 \quad \text{on} \quad \Gamma, \end{cases}$$

we show that $\| \partial^2_{ij} \boldsymbol{u}^m - \partial^2_{ij} \boldsymbol{u}^n \| + \| \partial_i p^m - \partial_i p^n \| \leq c_0 \| \boldsymbol{f}_m - \boldsymbol{f}_n \|_*$. This proves that $\partial^2_{ij} \boldsymbol{u}^m$ and $\partial_i p^m$ are Cauchy sequences in $\boldsymbol{C}(\overline{\Omega})$ and $C(\overline{\Omega})$, respectively. Hence, there are elements $\widetilde{u}_{ij}, \widetilde{p}_i \in C(\overline{\Omega})$ such that the sequences $\partial^2_{ij} \boldsymbol{u}^m$ and $\partial_i p^m$ are uniformly convergent in $\overline{\Omega}$ to $\widetilde{u}_{ij}$ and $\widetilde{p}_i$ respectively. on the other hand, from (1.2), (4.1), and (3.8), it follows that $\| \partial_i \boldsymbol{u}^m - \partial_i \boldsymbol{u} \| \leq c \| \boldsymbol{f}_m - \boldsymbol{f} \|$. Hence $\partial_i \boldsymbol{u}^m$ converges uniformly in $\overline{\Omega}$ to $\partial_i \boldsymbol{u}$. The above picture shows that the second order derivatives $\partial^2_{ij} \boldsymbol{u}$ must coincide with the continuous functions $\widetilde{u}_{ij}$, for $i, j = 1, 2, 3$. A similar argument shows that the first order derivatives $\partial_i p$ are given by $\widetilde{p}_i$, for $i = 1, 2, 3$.

5. The Space $B_*(\overline{\Omega})$

As already remarked at the end of the introduction, we wonder whether there are functional spaces $B_*(\overline{\Omega})$ such that the inclusions $C_*(\overline{\Omega}) \subset B_*(\overline{\Omega}) \subset C(\overline{\Omega})$ are proper, and Theorem 1.1 still holds with $C_*(\overline{\Omega})$ replaced by $B_*(\overline{\Omega})$. Having this goal in mind, in [3] we have defined and study a functional space $B_*(\overline{\Omega})$ as follows. Set

$$\omega_f(x; r) = \sup_{y \in \Omega(x; r)} | f(x) - f(y) |, \tag{5.1}$$

fix a positive real $\delta > 0$, and define semi-norms

$$p_x(f) \equiv \int_0^\delta \omega_f(x; r) \, \frac{dr}{r}, \tag{5.2}$$

for each $x \in \overline{\Omega}$. Note that continuity of f at single points x follows necessarily from finiteness of the integral in Eq. (5.1). To avoid unnecessary technical arguments, we assume that $f \in C(\overline{\Omega})$.

We set

$$\langle f \rangle_* = \sup_{x \in \overline{\Omega}} \int_0^\delta \omega_f(x; r) \, \frac{dr}{r} = \sup_{x \in \overline{\Omega}} p_x(f), \tag{5.3}$$

and define

$$B_*(\overline{\Omega}) \equiv \{ f \in C(\overline{\Omega}) : \langle f \rangle_* < +\infty \}. \tag{5.4}$$

The space $B_*(\overline{\Omega})$, endowed with $\| f \|^* \equiv \| f \| + \langle f \rangle_*$, is a normed linear space. Since

$$[f]_* = \int_0^\delta \sup_{x \in \overline{\Omega}} \omega_f(x; r) \, \frac{dr}{r}, \tag{5.5}$$

one has $\langle f \rangle_* \leq [f]_*$. The inclusion $C_*(\overline{\Omega}) \subset B_*(\overline{\Omega})$ is strict. We exhibit explicit functions which do not belong to $C_*(\overline{\Omega})$ (due to strong oscillations), but still belong to $B_*(\overline{\Omega})$.

In [3], we have shown that Theorem 3.1 holds with $C_*(\overline{\Omega})$ replaced by $B_*(\overline{\Omega})$. Finally, by adapting the proofs developed in [5], we may prove that solutions to the 2D-Euler equations, with initial data in $B_*(\overline{\Omega})$ and vanishing external forces, satisfy $\boldsymbol{u} \in C(\mathbb{R}; \boldsymbol{C}^{0,1}(\overline{\Omega}))$.

On the other hand, we did not succeed in proving or disproving Theorem 1.1 for data in $B_*(\overline{\Omega})$. Theorem 2.3 still holds, but the extension of Theorem 2.5 to $B_*(\overline{\Omega})$ could be false. This remains a challenging open problem. We will turn back to this matter in the forthcoming publication [3].

It is worth noting that other significant candidates could be obtained by replacing, in the definitions of $C_*(\overline{\Omega})$ or $B_*(\overline{\Omega})$, the quantity $\omega_f(x;r)$ defined in Eq. (5.1), by

$$\widetilde{\omega}_f(x;r) = \Big| f(x) - |\,\Omega(x;r)\,|^{-1} \int_{\Omega(x;r)} f(y)\,dy \Big|, \tag{5.6}$$

and similar variants.

References

[1] Agmon, S.: Lectures on Elliptic Boundary Value Problems, Van Nostrand Mathematical Studies. D.Van Nostrand Company, New York (1965)

[2] Agmon, S., Douglis, A., Nirenberg, L.: Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions II. Commun. Pure Appl. Math. **17**, 35–92 (1964)

[3] Beirão da Veiga H.: An overview on classical solutions to 2-D Euler equations and to elliptic boundary value problems, to be submitted to the "London Math. Soc. Lecture Notes"

[4] Beirão da Veiga H.: On the solutions in the large of the two-dimensional flow of a non-viscous incompressible fluid. MRC Technical Summary Report no. 2424, September (1982)

[5] Beirão da Veiga, H.: On the solutions in the large of the two-dimensional flow of a nonviscous incompressible fluid. J. Differ. Equ. **54**(3), 373–389 (1984)

[6] Beirão da Veiga, H., Berselli, L.C.: Navier-Stokes equations: Green's matrices, vorticity direction, and regularity up to the boundary. J. Differ. Equ. **246**, 597–628 (2009)

[7] Cattabriga, L.: Su un problema al contorno relativo al sistema di equazioni di Stokes. Rend. Mat. Sem. Padova **31**, 308–340 (1961)

[8] Galdi G.P.: An Introduction to the Mathematical Theory of the Navier-Stokes Equations: Linearized Steady Problems, Springer Tracts in Natural Philosophy, vol. 38, Second corrected printing, Springer, Berlin (1998)

[9] Galdi G.P.: Introduction to the mathematical theory of the Navier-Stokes equations, vol. 2, Springer, Berlin (1998)

[10] Gilbarg, D., Trudinger, N.S.: Elliptic Partial Differential Equations of Second Order. Springer, Berlin (1977)

[11] Ladyzenskaya, O.A.: The Mathematical Theory of Viscous Incompressible Flow. Gordon and Breach, New York (1969)

[12] Lichtenstein, L.: Űber einige Existenceprobleme der Hygrodynamik. Math. Z. **28**, 387–415 (1928)

[13] Nečas, J.: Les Methodes Directes en Theorie des Equations Elliptiques. Academia, Prague (1967)

[14] Odqvist, F.K.G.: Űber die Randwertaufgaben der Hydrodynamik Zåher Flűssigkeiten. Math. Z. **32**, 329–375 (1930)

[15] Solonnikov, V.A.: On estimates of Green's tensors for certain boundary value problems. Doklady Akad. Nauk. **130**, 128–131 (1960)

[16] Solonnikov V.A.: General boundary value problems for Douglis-Nirenberg elliptic systems.II. In: Proceedings of the Steklov Institute of Mathematics, vol. 92 (1966). English translation, American Mathematical Society (1968)

[17] Solonnikov, V.A.: On Green's matrices for elliptic boundary problem I. Trudy Mat. Inst. Steklov **110**, 123–170 (1970)

[18] Solonnikov, V.A.: On Green's matrices for elliptic boundary problem II. Trudy Mat. Inst. Steklov **116**, 187–226 (1971)

[19] Temam, R.: Navier-Stokes Equations. North-Holland, Amsterdam (1984)

[20] Valli, A.: On the integral representation of the solutions to the Stokes system. Rend. Mat. Sem. Padova **74**, 85–114 (1985)

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