Amer. Math. Soc. Transl.
Volume **232**, 2014

On singular parabolic p-Laplacian systems under nonsmooth external forces. Regularity up to the boundary

H. Beirão da Veiga

To the memory of Olga A. Ladyzhenskaya, with warm recollections of friendship

Abstract. We study the regularity of the solutions to an initial-boundary value problem for a system of the p-Laplacian type in $n \geq 3$ space variables. External forces are square-integrable in the space-time cylinder. Under this natural assumption, the gradient of the solutions may be unbounded. As a consequence, roughly speaking, the ellipticity coefficient of the linearized parabolic equation may be not bounded from below by a positive constant. We show suitable integrability in the space-time cylinder, up to the boundary, for the second order space derivatives. The singular case is also covered.

1. Introduction and main result

The aim of this paper is to introduce a simple idea in the simplest form. Sometimes, to avoid technicalities, we will assume that solutions (which exist and are unique) are sufficiently smooth. So we left to the reader additional details concerning approximation theory.

In the sequel we consider the evolution problem

$$\begin{cases} \partial_t u - \nabla \cdot \big((\mu + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u \big) = f(t, x), & \text{in } (0, T) \times \Omega, \\ u = 0 \ \text{ on } (0, T) \times \partial\Omega, \\ u(0) = u_0 \ \text{ in } \Omega, \end{cases} \tag{1.1}$$

where $p \in (1, 2]$ (see (1.6) below), $T \in (0, \infty]$, and $\mu \geq 0$ are constants. Here u is an N-dimensional vector field, $N \geq 1$, defined in $Q_T \equiv (0, T) \times \Omega$, where $\Omega \subset \mathbb{R}^n$, $n \geq 3$, is a regular, bounded open set, which we assume is of class C^2.

We recall that scalar multiplication of both sides of (1.1) by u, followed by classical manipulations, will lead to the well-known a priori estimate

$$\begin{aligned} &\| u \|^2_{L^\infty(0, T; L^2(\Omega))} + \| u \|^p_{L^p(0, T; W^{1, p}(\Omega))} \\ &\qquad \leq c \big(\| u_0 \|_2^2 + \| f \|^{p'}_{L^{p'}(0, T; W^{-1, p'}(\Omega))} \big). \end{aligned} \tag{1.2}$$

2010 *Mathematics Subject Classification.* Primary 35B65, 35J57.

Key words and phrases. Initial-boundary value problems, p-Laplacian parabolic singular systems, regularity up to the boundary.

So we assume in the sequel that

$$u_0 \in L^2(\Omega) \quad \text{and} \quad f \in L^{p'}(0, T; W^{-1,p'}(\Omega)), \tag{1.3}$$

where p' is the exponent conjugate to p. Further, by appealing to the first equation (1.1) together with (1.3), we get

$$\partial_t u \in L^{p'}(0, T; W^{-1,p'}(\Omega)). \tag{1.4}$$

We call u a weak solution of problem (1.1) if

$$u \in L^\infty(0, T; L^2(\Omega)) \cap L^p(0, T; W_0^{1,p}(\Omega)),$$

if $\partial_t u$ satisfies (1.4), if $u(0) = u_0$, and if (for almost all $t \in (0, T)$)

$$\langle \partial_t u, w \rangle + \langle (\mu + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u, w \rangle = \langle f, w \rangle, \quad \forall w \in W_0^{1,p}(\Omega).$$

Here $\langle \cdot, \cdot \rangle$ denotes the natural duality pairings.

For the existence and uniqueness ($p \geq \frac{2n}{n+2}$) of weak solutions under our hypotheses, we refer the reader to [**21**, Theorem 6.2.1] (see also [**30**, Chap. II, Theorem 1.1], where $p \geq 2$). In reference [**21**] it is assumed that $f = 0$. However, as remarked in [**19**], the proof continuous to hold without this condition. For instance, the above assumptions on f are sufficient.

In the following a main point is that the external force f is assumed to be square-integrable in Q_T. This low integrability prevents boundedness of $\nabla u(t, x)$ (which holds, for instance, if $f \in L^q(Q_T)$, $q > n+2$). So, the ellipticity coefficient $(\mu + |\nabla u|^2)^{\frac{p-2}{2}}$ does not remain bounded away from zero. Actually, we also assume that f satisfies (1.3). However, this does not invalidate the substance of the above remark (in particular for values of p near 2).

Our aim is to prove that, if $u(x, t)$ denotes the solution of problem (1.1), then

$$f \in L^2(0, T; L^2(\Omega)) \quad \Longrightarrow \quad u \in L^{2(p-1)}(0, T; W^{2,q}(\Omega)), \tag{1.5}$$

where q is given by (1.7) below.

As a rule, we are generous in assumptions in order to avoid complicating remarks. So, we assume that

$$\begin{cases} \frac{2n}{n+2} < p \leq 2, & \text{if } n > 3, \\ \frac{5}{4} < p \leq 2, & \text{if } n = 3. \end{cases} \tag{1.6}$$

In particular, the inclusion $L^2(\Omega) \subset W^{-1,p'}(\Omega)$ holds. We define

$$q = q(p) = \frac{2n(p-1)}{n - 2(2-p)}. \tag{1.7}$$

The central role of this exponent will be clear in the following. Our assumptions on p imply that $q \in (1, 2]$, and also that the immersion $W^{2,q}(\Omega) \subset W^{1,p}(\Omega)$ is compact.

Next, we recall the well-known inequality

$$\|D^2 v\|_q \leq C_0(q) \|\Delta v\|_q \tag{1.8}$$

for $v \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$. Moreover, there is a constant K, independent of q, such that

$$C_0(q) \leq K q, \tag{1.9}$$

at least for $q > \frac{2n}{n+2}$ (see [**32**]).

Our main result is the following; note that $\mu = 0$ is allowed. As in [**10**], Ω is assumed to be of class C^2.

THEOREM 1.1. *Let be* $\mu \geq 0$. *Assume that* p *satisfies* (1.6), *define* q *by* (1.7), *and assume that*

$$(2-p)\, C_0(q) < 1\,, \tag{1.10}$$

where $C_0(q)$ *is defined by* (1.8). *Let* $u_0 \in W_0^{1,\,p}(\Omega)$, *and let*

$$f \in L^2(0,\, T;\, L^2(\Omega)) \tag{1.11}$$

for some $T \in]\,0,\, +\infty\,]$. *Define* $\mathcal{C} = \mathcal{C}(u_0,\, f,\, \mu,\, T)$ *by*

$$\mathcal{C}^2 = \mu^{\frac{p}{2}}\, |\,\Omega\,| + \|\,\nabla\, u_0\,\|_p^p + \frac{2}{p}\, \|\,f\,\|^2_{L^2(0,\,T;\,L^2(\Omega))}\,. \tag{1.12}$$

Let u *be the weak solution of problem* (1.1). *Then* $u \in L^{2(p-1)}(0,\, T;\, W^{2,\,q}(\Omega))$. *Moreover, the estimate*

$$\|\,u\,\|^{2(p-1)}_{L^{2(p-1)}(\,0,\,T;\,W^{2,\,q}(\Omega))} \leq C\,(\,T^{2-p}\,\mathcal{C}^{2\,(p-1)} + \mathcal{C}^2) \tag{1.13}$$

holds. Furthermore, the estimates (3.13), (3.14), *and* (3.15) *are satisfied.*

Note that if $n = 3$ and $p > \frac{3}{2}$, then

$$u \in L^{2\,(p-1\,)}(0,\; T\,;\, C^{\,0,\,\alpha}(\overline{\Omega}))\,,$$

where $\alpha = \frac{p-\frac{3}{2}}{p-1}$.

In Section 3, the extension of the above result (in particular the regularity result $L^{2(p-1)}(0,\, T; W^{2,\,q}(\Omega))$) to systems of the "same family" will be proposed.

REMARK 1.1. Due to (1.9), condition (1.10) holds if

$$(2-\,p)\, q < \frac{1}{K}\,. \tag{1.14}$$

Furthermore, by appealing to the definition of q, we show that (1.14) holds if

$$2 - \frac{n}{2nK+2} < p \leq 2\,, \tag{1.15}$$

where the constant $K = K(\Omega)$ is independent of p and q. Hence, (1.15) is a sufficient condition on p to guarantee the results claimed in Theorem 1.1.

It is worth noting that, for $p = 2$, the above estimates turn into the classical "heat equation" estimates. For instance, the main estimate (3.17) leads to

$$\|\,u\,\|^2_{L^2(\,0,\,T;\,W^{2,\,2}(\Omega))} \leq C\,\big(\,\|\,\nabla\,u_0\,\|_2^2 + \|\,f\,\|^2_{L^2(0,\,T;\,L^2(\Omega))}\big)\,. \tag{1.16}$$

We may also consider the nonhomogeneous evolution boundary value problem, by appealing to the extension of the results of reference [**10**] to the above problem, which is shown in [**6**]. Other possible extensions are to consider slip boundary conditions, and also to replace, in the first equation (1.1), the gradient $\nabla\, u$ by the symmetric gradient $\nabla\, u + \nabla^T\, u$. In the elliptic case these extensions are considered in the forthcoming paper [**8**].

In references [**15**], [**16**] (see [**13**, chapters IX, X]) local Hölder continuity in $(0,\, T) \times \Omega$ of the space gradient of local weak solutions is proved. Regularity results, *up to the boundary*, are stated in [**13**, chapter X] (see the Theorems 1.1 and 1.2 therein). In particular, in Theorem 1.2, Hölder continuity up to the parabolic boundary (where $u = 0$) of the spatial gradient of weak solutions u is proved. However, regularity results, up to the boundary, for second order space derivatives in $L^q(\Omega)$ spaces and for solutions to the parabolic singular case, seem not known in the literature. Actually, the two types of estimates are not comparable. It is worth noting that in the elliptic case (see [**10**]) the $W^{2,\,q}(\Omega)$ estimates imply $C^{1,\,\alpha}(\overline{\Omega})$ regularity, since $q > n$ is admissible.

A very interesting related subject concerns linear elliptic and parabolic equations with discontinuous coefficients (in particular by appealing to BMO and VMO functional spaces). See, for instance, [**20**], [**23**], [**24**], [**25**], and references therein. Results in this direction may be useful to try to extend the regularity results for second order space derivatives without assumptions like (1.10).

Another classical related subject (here $N = 1$) are Harnack's inequalities. See references and results in the recent monograph [**14**]. The first parabolic versions go back to Hadamard [**22**] (see also [**31**]).

For related results we also refer, in addition to the above references, to the monographs [**26**], [**30**], and to references [**1**], [**2**], [**3**], [**4**], [**5**], [**11**], [**12**], [**13**], [**17**], [**18**], [**27**], [**28**], [**29**].

The results stated here were announced in [**7**].

2. An auxiliary result concerning the stationary problem

By $L^p(\Omega)$ and $W^{m,p}(\Omega)$, m a nonnegative integer, and $p \in (1, +\infty)$, we denote the usual Lebesgue and Sobolev spaces with the standard norms $\|\cdot\|_p$ and $\|\cdot\|_{m,p}$. We set $\|\cdot\| = \|\cdot\|_2$. We denote by $W^{1,p}_0(\Omega)$ the closure of $C^\infty_0(\Omega)$ in $W^{1,p}(\Omega)$, and by $W^{-1,p'}(\Omega)$, $p' = p/(p-1)$, the strong dual of $W^{1,p}_0(\Omega)$ with norm $\|\cdot\|_{-1,p'}$. We set $\partial_t\, u = \frac{\partial\, u}{\partial\, t}$.

The symbols c, c_0, c_1, etc., denote positive constants that may depend on μ; by capital letters, C, we denote positive constants independent of $\mu \geq 0$ (for convenience, assume the range of μ bounded from above). The same symbol c or C may denote different constants, even in the same equation.

The proof of Theorem 1.1 shown below strongly appeals to the following regularity result.

Proposition 2.1. *Let the hypothesis assumed in Theorem* 1.1 *concerning* Ω, p, q, *and* μ *hold, and assume that* $f \in L^2(\Omega)$. *Further, let* u *be the unique weak solution to the stationary problem*

$$\begin{cases} -\nabla \cdot \big((\mu + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u \big) = f \quad \textit{in } \Omega\,, \\ u = 0 \quad \textit{on } \partial\Omega\,. \end{cases} \tag{2.1}$$

Then u *belongs to* $W^{2,q}(\Omega)$. *Moreover, the following estimate holds:*

$$\|u\|_{2,q} \leq C \left(\|f\|_q + \|f\|_2^{\frac{1}{p-1}} \right). \tag{2.2}$$

This result is essentially a particular case of a more general result, namely, Theorem 2.1 in reference [**10**] (see also [**9**]). So, for the proof, we refer the reader to this last paper. However a small (actually obvious) adaptation is needed. For completeness, we discuss this point in the Appendix.

3. Proof of Theorem 1.1

Besides proving Theorem 1.1, we also want to show that the main devices in our argument easily apply to other systems of equations of the form

$$\begin{cases} \partial_t u - \nabla \cdot S(\nabla u)0 = f(t, x) \text{ in } (0, T) \times \Omega, \\ u = 0 \text{ on } (0, T) \times \partial\Omega, \\ u(0) = u_0 \text{ in } \Omega, \end{cases} \tag{3.1}$$

where $S(\cdot)$ is given by

$$S(\nabla u) := A(|\nabla u|^2)\nabla u, \tag{3.2}$$

and $A(y)$, $y \geq 0$, satisfies suitable p-growth conditions. This will be discussed briefly in the next section. In this section we treat system (1.1) by assuming that

$$A(y) = (\mu + y)^{\frac{p-2}{2}}. \tag{3.3}$$

We define

$$G(y) := \int A(y)\, dy, \tag{3.4}$$

for $y \geq 0$. Hence, in this section,

$$G(y) = \frac{2}{p}(\mu + y)^{\frac{p}{2}}. \tag{3.5}$$

Proof of Theorem 1.1. By integration by parts we show that

$$\begin{aligned} -\int_\Omega (\partial_t u) \cdot \nabla \cdot \big(A(|\nabla u|^2)\nabla u\big)\, dx = \frac{1}{2}\int_\Omega A(|\nabla u|^2)\, \partial_t |\nabla u|^2\, dx \\ - \int_{\partial\Omega} A(|\nabla u|^2)(\partial_t u) \cdot (\partial_n u)\, dS. \end{aligned} \tag{3.6}$$

So, by scalar multiplication of both sides of the first equation (3.1) by $-\nabla \cdot \big(A(|\nabla u|^2)\nabla u\big)$, and by taking into account that the boundary integral vanishes, one gets

$$\begin{aligned} \frac{1}{2}\int_\Omega A(|\nabla u|^2)\, \partial_t |\nabla u|^2\, dx + \int_\Omega |\nabla \cdot S(\nabla u)|^2\, dx \\ = -\int_\Omega f \cdot (\nabla \cdot S(\nabla u))\, dx. \end{aligned} \tag{3.7}$$

By appealing to (3.5), we show from (3.6) that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_\Omega G(|\nabla u|^2)\, dx + \int_\Omega |\nabla \cdot S(\nabla u)|^2\, dx \\ = -\int_\Omega f \cdot (\nabla \cdot S(\nabla u))\, dx. \end{aligned} \tag{3.8}$$

Hence,

$$\frac{d}{dt}\int_\Omega G(|\nabla u|^2)\, dx + \int_\Omega \big|\nabla \cdot \big(A(|\nabla u|^2)\nabla u\big)\big|^2\, dx \leq \int_\Omega |f|^2\, dx. \tag{3.9}$$

By integration with respect to t, one gets

$$\int_\Omega G(|\nabla u(t)|^2)dx + \int_0^t \| \nabla \cdot S(\nabla u(s)) \|_2^2 \, ds \leq \int_\Omega G(|\nabla u_0|^2)dx + \int_0^t \| f(s) \|_2^2 \, ds \,. \tag{3.10}$$

Further, by appealing to the inequality

$$(\mu + |\nabla u|^2)^{\frac{p}{2}} \leq \mu^{\frac{p}{2}} + |\nabla u|^p \,, \tag{3.11}$$

we obtain (for $0 < t \leq T$)

$$\| \nabla u(t) \|_p^p + \frac{2}{p} \int_0^t \| \nabla \cdot S(\nabla u(s)) \|_2^2 \, ds \leq \mathcal{C}^2 \,, \tag{3.12}$$

where $\mathcal{C}^2$ is defined by (1.12). We have used the inequality $(a+b)^\alpha \leq a^\alpha + b^\alpha$, for nonnegative a and b, and $0 < \alpha < 1$. From (3.10), standard manipulations, prove that

$$\| \nabla u \|^p_{L^\infty(0,T;\,L^p(\Omega))} \leq \mathcal{C}^2 \tag{3.13}$$

and that

$$\| \nabla \cdot \big((\mu + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u \big) \|^2_{L^2(0,T;\,L^2(\Omega))} \leq \mathcal{C}^2 \,. \tag{3.14}$$

Furthermore, the estimate

$$\| \partial_t u \|^2_{L^2(0,T;\,L^2(\Omega))} \leq 2\,\mathcal{C}^2 \tag{3.15}$$

holds since

$$\partial_t u = \nabla \cdot S(\nabla u) + f(t,x) \,.$$

Finally, we prove the main estimate (1.13) by applying to the stationary equation

$$\nabla \cdot S(\nabla u) = \partial_t u - f(t,x) \,,$$

for a.a. $t \in (0,T)$, the regularity result stated in Proposition 2.1. So, by appealing to (2.2), it follows that

$$\| u(t) \|_{2,q}^{2(p-1)} \leq C \big(\| \partial_t u - f \|_q^{2(p-1)} + \| \partial_t u - f \|_2^2 \big) \,, \tag{3.16}$$

for a.a. $t \in (0,T)$. Further, by integration in $(0,T)$, we show that

$$\begin{aligned} &\| u \|^{2(p-1)}_{L^{2(p-1)}(0,T;\,W^{2,q}(\Omega))} \\ &\leq C \Big(\| f \|^{2(p-1)}_{L^{2(p-1)}(0,T;\,L^2(\Omega))} + \| \partial_t u \|^{2(p-1)}_{L^{2(p-1)}(0,T;\,L^2(\Omega))} \\ &\qquad + \| f \|^2_{L^2(0,T;\,L^2(\Omega))} + \| \partial_t u \|^2_{L^2(0,T;\,L^2(\Omega))} \Big) \,, \end{aligned} \tag{3.17}$$

where, for brevity, we have appealed to $q \leq 2$. Finally, by appealing to (1.12), (3.15), and the Hölder inequality, straightforward calculations show that the right-hand side of (3.17) is bounded by $C\,(T^{2-p}\,\mathcal{C}^{2\,(p-1)} + \mathcal{C}^2)$. Hence, the main estimate (1.13) holds. This completes the proof of Theorem 1.1.

Note that from (3.8) it follows that the quantity

$$\int_\Omega G(|\nabla u(t)|^2)\,dx$$

is decreasing with respect to time, if $f = 0$. Hence the norm $\| \nabla u(t) \|_p$ decreases with time.

Professor S. Antontsev kindly informed the author that the device leading to equation (3.9) was previously used by him. We refer, for instance, to the argument around equation (6.4) in reference [**2**]. We are grateful to Professor Antontsev for leading our attention to his wide and deep work on the subject.

4. A more general setting

The extension of the results stated in Theorem 1.1 to systems of equations of the form (3.1), where $S(\cdot)$ is given by (3.2), looks not difficult, provided that a corresponding extension of Proposition 2.1 to the related stationary system

$$\begin{cases} -\nabla\cdot\left(A(\,|\nabla u|^2\,)\,\nabla u\right) = f \quad \text{in } \Omega\,, \\ u = 0 \quad \text{on } \partial\Omega\,, \end{cases} \tag{4.1}$$

holds. This last possibility, claimed in reference [**10**], seems not difficult to prove. Roughly speaking, if the "stationary" extension holds, then the argument followed in the previous section applies to the evolution problem (3.1). Clearly, suitable p-growth conditions on $A(y^2)$ are necessary. Concerning the calculations developed in the previous section, the existence of positive constants c_0 and c_1 such that

$$c_0\, y^p - c_1 \leq G(y^2) \leq \widetilde{c}_0\, y^p + \widetilde{c}_1\,, \tag{4.2}$$

for $y \geq 0$, would be sufficient. For instance, if $(\,\mu + |\,\nabla u|^2\,)^{\frac{p-2}{2}}$ is replaced by $(\,\mu + |\,\nabla u|\,)^{p-2}$, the estimates (4.2) hold by setting

$$c_0 = \frac{1}{p}\,,\quad c_1 = \frac{2^p\,\mu^2}{p\,(p-1)}\,,\quad \widetilde{c}_0 = \widetilde{c}_1 = \frac{2^p}{p}\,.$$

For more detail we refer to [**7**].

5. Appendix

In this section we give some specifications concerning the proof of Proposition 2.1 as a corollary of Theorem 2.1 in reference [**10**] (to which the interested reader is necessarily referred). It is worth noting that the points raised in the sequel are minor points in the context of the complete proof. But they are just those for which a small adaptation is necessary (actually, the reader may easily do this task by himself). To avoid repetition of large parts of the original proof presented in [**10**], we assume that the reader, while reading the original proof, decided to take advantage of the remarks made below.

In [**10**] the authors proved the following result (see [**10**], Theorem 2.1).

Theorem 5.1. *Let $p \in (1,2]$, and let $q \geq 2$, $q \neq n$, be given. Assume that $(2-p)\,C_0(q) < 1$, where $C_0(q)$ satisfies* (1.8)*. Further, assume that $\mu \geq 0$. Let $f \in L^{r(q)}(\Omega)$, where*

$$r(q) = \begin{cases} \dfrac{nq}{n(p-1)+q(2-p)} & \textit{if } \; q \in [\,2,\,n]\,, \\ q & \textit{if } \; q \geq n\,, \end{cases} \tag{5.1}$$

and let u be the unique weak solution of problem (2.1)*. Then u belongs to $W^{2,q}(\Omega)$. Moreover, the following estimate holds:*

$$\|u\|_{2,q} \leq C\left(\|f\|_q + \|f\|_{r(q)}^{\frac{1}{p-1}}\right)\,. \tag{5.2}$$

Our first remark is that, in reference [**10**], the authors consider the case in which the term $(\mu + |\nabla u|^2)^{\frac{p-2}{2}}$ is replaced by $(\mu + |\nabla u|)^{p-2}$. As expected, the proof applies exactly by following the same way.

The second remark is the following. Assumption (1.11), namely, $f(t) \in L^2(\Omega)$, a.e. in t, leads us to consider the above theorem for the particular value q for which $r(q) = 2$. This value of q, given by (1.7), is less then 2. This is in contrast to Theorem 5.1, where it is assumed that $q \geq 2$. However, as already claimed in [**10**], the proof given in this last reference immediately applies to a range of values which includes $q < 2$, by making a couple of small changes in the original proof (shown below just for completeness). We necessarily assume that the reader has in hand the proof presented in reference [**10**]. Notation is that used in this last reference.

Actually, the following is the unique real modification needed to adapt the proof to our value of q. In [**10**, Sec. 3, third to last row], the authors claim the convergence result

$$f\,(\mu + |\nabla v^m|)^{2-p} \to f\,(\mu + |\nabla v|)^{2-p} \tag{5.3}$$

in the $L^{\frac{q}{2}}$-norm, where v^m denotes auxiliary "technical" and not significant, suitable subsequences (see [**10**]). The above convergence result is not suitable here, since $\frac{q}{2} < 1$. However, as already remarked in [**10**], convergence in the distributional sense is sufficient here. As shown in [**10**], one has

$$|\,(\mu + |\nabla v^m|\,)^{2-p} - (\mu + |\nabla v|\,)^{2-p}\,| \leq \frac{2-p}{\mu^{p-1}}\,|\nabla v^m - \nabla v|\,.$$

In particular, it follows that (5.3) holds a.e. in Ω, since, as in [**10**], ∇v^m converges a.e. in Ω to ∇v.

On the other hand, $(\mu + |\nabla v^m|)^{2-p}$ is bounded in L^t, where $t := \frac{q*}{2-p} > \frac{n}{2}$, where $q^* = \frac{q\,n}{n-q}$. So, it follows from [**30**, Chap. 1, Lemma 1.3] that

$$(\mu + |\nabla v^m|)^{2-p} \to (\mu + |\nabla v|)^{2-p}$$

weakly in L^t. Moreover, (1.6) implies that $(\frac{n}{2})' \leq 2$. Consequently, $f \in L^{(\frac{n}{2})'}$. It readily follows that (5.3) holds in the distributional sense.

We end with two remarks:

i) The set

$$\mathbb{K} = \{v \in W^{2,\,q}(\Omega) \,:\, \|\Delta v\|_q \leq R\,,\, v = 0 \text{ on } \partial\Omega\}\,,$$

introduced in [**10**, Section 3], is still contained in $W_0^{1,\,p}(\Omega)$, since $p > \frac{2\,n}{n+2}$.

ii) As at the very beginning of [**10**, Section 4], here one also has $p < q^*$. So, as in [**10**], we show that the solutions u^μ of problem (1.1) for $\mu > 0$ converge in $W^{1,p}(\Omega)$ to the solution u of (1.1) for $\mu = 0$.

References

[1] S. Antontsev and S. Shmarev, *Elliptic equations and systems with nonstandard growth conditions: Existence, uniqueness and localization properties for solutions*, Nonlinear Analysis, **65** (2006), 728–761. MR2232679 (2007f:35072)

[2] S. Antontsev and S. Shmarev, *Anisotropic parabolic equations with variable nonlinearity*, Publ. Mat., **53** (2009), 355–399. MR2543856 (2010m:35258)

[3] S. Antontsev and S. Shmarev, *Vanishing solutions of anisotropic parabolic equations with variable nonlinearity*, J. Math. Anal. Appl., **361** (2010), 371-391. MR2568702 (2010i:35196)

[4] S. Antontsev and S. Shmarev, *Blow-up of solutions to parabolic equations with nonstandard growth conditions*, J. Comp. Appl. Math., **234** (2010), 2633-2645. MR2652114 (2011g:35205)

[5] D. E. Apushkinskaya, A. I. Nazarov, *The elliptic Dirichlet problem in weighted spaces*, J. Math. Sci., **123** (2004), no. 6, 4527–4538. MR1923543 (2003e:35073)

[6] H. Beirão da Veiga, *On the global regularity for singular p-systems under non-homogeneous Dirichlet boundary conditions*, J. Math. Anal. Appl., **398** (2013), 527–533. doi: 10.1016/j.jmaa.2012.08.058. MR2990077

[7] H. Beirão da Veiga, *Singular parabolic p-Laplacian systems under non-smooth external forces. Regularity up to the boundary*, Online arXiv:1206.1808v1 [math.AP], 8 Jun 2012.

[8] H. Beirão da Veiga, *On the singular p-Laplacian system under Navier slip type boundary conditions. The gradient-symmetric case*, to appear.

[9] H. Beirão da Veiga and F. Crispo, *On the global regularity for nonlinear systems of the p-Laplacian type*, to appear in Discrete and Continuous Dynamical Systems—Series S., Online arXiv:1008.3262v1 [math.AP], 19 Aug 2010. MR3039691

[10] H. Beirão da Veiga and F. Crispo, *On the global $W^{2,\,q}$ regularity for nonlinear $N-$systems of the p-Laplacian type in n space variables*, Nonlinear Analysis-TMA , **75** (2012), 4346–4354. DOI: 10.1016/j.na.2012.03.021 MR2921994

[11] Y. Z. Chen, *Hölder continuity of the gradients of solutions of non-linear degenerate parabolic systems*, Acta Math. Sinica, **2** (1986), 309–331. MR900477 (89m:35036)

[12] H. Choe, *Hölder continuity of solutions of certain degenerate parabolic systems*, Non-linear Anal. **8** (1992), 235–243. MR1148287 (92m:35147)

[13] E. DiBenedetto, *Degenerate parabolic equations*, Springer, Berlin, 1993. MR1230384 (94h:35130)

[14] E. DiBenedetto, U. Gianazza and V. Vespri, *Harnack's inequality for degenerate and singular parabolic equations*, Springer Monographs in Math., Springer, New York, 2012. MR2865434

[15] E. DiBenedetto and A. Friedman, *Regularity of solutions of non-linear degenerate parabolic systems*, J. Reine Angew. Math., **349**, (1984), 83–128. MR743967 (85j:35089)

[16] E. DiBenedetto and A. Friedman, *Hölder estimates for non-linear degenerate parabolic systems*, J. Reine Angew. Math., **357**, (1985), 1–22. MR783531 (87f:35134a)

[17] E. DiBenedetto, Y.C. Kwong and V. Vespri, *Local space analycity of solutions of certain singular parabolic equations*, Indiana Univ. Math. J., **40**, (1991), 741–765. MR1119195 (93b:35014)

[18] E. DiBenedetto and Y. C. Kwong, *Intrinsic Harnack estimates and extinction profile for certain singular parabolic equations*, Trans. AMS., **330**, (1992), 783–811. MR1076615 (92f:35033)

[19] L. Diening, C. Ebmeyer, M. Růžička, *Optimal convergence for the implicit space-time discretization of parabolic systems with p-structure*, SIAM J. Numer. Anal., **45**, (2007), 457–472. MR2300281 (2008b:65118)

[20] H. Dong and D. Kim, *On the L_p-solvability of higher order parabolic and elliptic systems with BMO coefficients*, Arch. Rat. Mech. Anal., **199** (2011), 889-941. MR2771670 (2012h:35152)

[21] H. Gajewsky, K. Gröger and K. Zacharias, *Nichtlineare Operatorgleichungen und Operatordifferentialgleichungen*, Akademie-Verlag, Berlin, 1974. MR0636412 (58:30524a)

[22] J. Hadamard, *Extension à l'équation de la chaleur d'un theoreme de A. Harnack*, Rend. Circ. Mat. Palermo, **23** (1954), 337–346. MR0068713 (16:930a)

[23] D. Kim and N. V. Krylov, *Parabolic equations with measurable coefficients*, Potential Anal., **26**, (2007), 345-361. MR2300337 (2008f:35161)

[24] V. Kozlov and A. Nazarov, *The Dirichlet problem for non-divergence parabolic equations with discontinuous in time coefficients*, Math. Nachr., **282**, (2009), 1220–1241. MR2561181 (2011b:35189)

[25] N. V. Krylov, *Parabolic and elliptic equations with VMO coefficients*, Comm. in P.D.E., **32**, (2007), 453–475. MR2304157 (2008a:35125)

[26] O. A. Ladyzhenskaya, V. A. Solonnikov and N. N. Ural'tseva, *Linear and quasilinear equations of parabolic type*. Trans. Math. Mono., **23** AMS, Providence, RI 1968. MR0241822 (39:3159b)

[27] G. M. Lieberman, *A new regularity estimate for solutions of singular parabolic equations*, Discrete and Continuous Dynamical Systems—Supplement volume, 2005, 605–610. MR2192719 (2006i:35159)

[28] G. M. Lieberman, *The first initial-boundary value problem for quasilinear second order parabolic equations*, Ann. Sc. Norm. Sup. Pisa **13** (1986), 347–387. MR881097 (88e:35108)

[29] G.M. Lieberman, *Boundary and initial regularity for solutions of degenerate value parabolic equations*, Nonlinear Anal. TMA **20** (1993), 551–570. MR1207530 (94e:35041)

[30] J.-L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod; Gauthier-Villars, Paris, 1969. MR0259693 (41:4326)

[31] B. Pini, *Sulla soluzione generalizzata di Wiener per il primo problema di valori di contorno nel caso parabolico*, Rend. Sem. Mat, Univ. Padova, **23** (1954), 422–434. MR0065794 (16:485c)

[32] V. I. Yudovic, *Some estimates connected with integral operators and with solutions of elliptic equations*, Dokl. Akad. Nauk SSSR, **138**(1961), 805–808; English translation in Soviet Math. Doklady **2** (1961), 746–749. MR0140822 (25:4236)

Dipartimento di Matematica, Università di Pisa, Largo Bruno Pontecorvo 5, 56127 Pisa, Italy

E-mail address: bveiga@dma.unipi.it