ELLIPTIC BOUNDARY VALUE PROBLEMS IN SPACES OF CONTINUOUS FUNCTIONS

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*To almost forty years friendship, on the occasion of the
60th birthdays of Alberto and Paolo*

Abstract. In these notes we consider second order linear elliptic boundary value problems in the framework of different spaces on continuous functions. We appeal to a general formulation which contains some interesting particular cases as, for instance, a new class of functional spaces, called here Hölog spaces and denoted by the symbol $C_\alpha^{0,\lambda}(\overline{\Omega})$, $0 \leq \lambda < 1$, and $\alpha \in \mathbb{R}$. One has the following inclusions

$$C^{0,\lambda+\epsilon}(\overline{\Omega}) \subset C_\alpha^{0,\lambda}(\overline{\Omega}) \subset C^{0,\lambda}(\overline{\Omega}) \subset C_{-\alpha}^{0,\lambda}(\overline{\Omega}) \subset C^{0,\lambda-\epsilon}(\overline{\Omega}),$$

for $\alpha > 0$ ($\varepsilon > 0$ arbitrarily small). Roughly speaking, for each fixed λ, the family $C_\alpha^{0,\lambda}(\overline{\Omega})$ is a refinement of the single Hölder classical space $C^{0,\lambda}(\overline{\Omega}) = C_0^{0,\lambda}(\overline{\Omega})$. On the other hand, for $\lambda = 0$ and $\alpha > 0$, $C_\alpha^{0,0}(\overline{\Omega}) = D^{0,\alpha}(\overline{\Omega})$ is a Log space. The more interesting feature is that, as for classical Hölder (and Sobolev) spaces, *full regularity* occurs. namely, for each $\lambda > 0$ and arbitrary real α, $\nabla^2 u$ and f enjoy the same $C_\alpha^{0,\lambda}(\overline{\Omega})$ regularity. All the above setup is presented as part of a more general picture.

1. **Introduction and main results.** To fix ideas, simply consider the Poisson equation $-\Delta u = f$ under the homogeneous boundary condition $u = 0$. It is well known that $f \in C(\overline{\Omega})$ does not guarantee $\nabla^2 u \in C(\overline{\Omega})$. This led us to look for "minimal assumptions" on f which guarantees continuity of the second order derivatives of u. By assuming that f belongs to a suitable functional space $C_*(\overline{\Omega})$, characterized by a Dini's continuity condition, continuity of $\nabla^2 u$ up to the boundary follows, but without any further interesting additional property, see theorem 2.1 below. Roughly speaking, we say here that $\nabla^2 u$ "totally forgets" its $C_*(\overline{\Omega})$ origin. On the contrary, a full regularity result holds for data in Hölder spaces $C^{0,\lambda}(\overline{\Omega})$, $0 < \lambda < 1$, since f and $\nabla^2 u$ have precisely the same regularity. In this situation we say that $\nabla^2 u$ "fully remembers" its $C^{0,\lambda}(\overline{\Omega})$ origin.

The above considerations led us to look for situations in which $\nabla^2 u$ "partially remembers" its origin. We have considered a family of functional spaces, called below Log spaces and denoted by the symbol $D^{0,\alpha}(\overline{\Omega})$, $0 < \alpha < +\infty$, and have shown that if $f \in D^{0,\alpha}(\overline{\Omega})$, for some $1 < \alpha < +\infty$, then $\nabla^2 u \in D^{0,\alpha-1}(\overline{\Omega})$.

This regularity result is optimal in the sense that $\nabla^2 u \in D^{0,\beta}(\overline{\Omega})$, for $\beta > \alpha - 1$, is false in general. Actually, optimality is proved in a sharper form, quite significant when, as for Log spaces, full regularity does not occurs.

In these notes we set distinct situations in a unique framework by considering a more general family of data spaces $D_{\overline{\omega}}(\overline{\Omega})$ satisfying the inclusions $C^{0,1}(\overline{\Omega}) \subset D_{\overline{\omega}}(\overline{\Omega}) \subset C(\overline{\Omega})$. Hölder spaces and Log spaces turn out to be particular cases. Furthermore, we introduce a new family of functional spaces, called here Hölog spaces and denoted by the symbol $C_\alpha^{0,\lambda}(\overline{\Omega})$, for which $\nabla^2 u$ and f enjoy the same regularity (*full regularity*) if $\lambda > 0$. For fixed λ, the family $C_\alpha^{0,\lambda}(\overline{\Omega})$, is a refinement of the single Hölder classical space $C^{0,\lambda}(\overline{\Omega}) = C_0^{0,\lambda}(\overline{\Omega})$. For $\lambda = 0$, $C_\alpha^{0,0}(\overline{\Omega}) = D^{0,\alpha}(\overline{\Omega})$ is a Log space. Proofs will be shown in a forthcoming paper.

Another interesting research field is the extension of theorem 2.1 to data spaces larger then $C_*(\overline{\Omega})$. In fact, there may be other significant functional spaces, possibly larger then $C_*(\overline{\Omega})$, satisfying the required properties. An attempt in this direction was done in the preparation's manuscript to reference [1], where a functional space $B_*(\overline{\Omega})$ was defined and studied. For some information and results, see section 7.

2. **Some preliminaries.** In the following Ω is an open, bounded, connected set in $\mathbb{R}^n$, locally situated on one side of its boundary Γ. The boundary Γ is of class $C^{2,\lambda}$, for some $\lambda > 0$. By $C(\overline{\Omega})$ we denote the Banach space of all real continuous functions f defined in $\overline{\Omega}$. The "sup" norm is denoted by $\| f \|$. We also appeal to the classical spaces $C^k(\overline{\Omega})$ endowed with their usual norms $\| u \|_k$, and to the Hölder spaces $C^{0,\lambda}(\overline{\Omega})$, endowed with the standard semi-norms and norms. $C^{0,1}(\overline{\Omega})$, is sometimes denoted by $Lip(\overline{\Omega})$, the space of Lipschitz continuous functions in $\overline{\Omega}$.

Symbols c and C denote generic positive constants. We may use the same symbol to denote different constants.

In these notes we consider linear elliptic boundary value problems with data and solutions belonging to suitable spaces of continuous functions, which have the main role here. For simplicity, consider the very basic case of constant coefficients, second order, elliptic operators

$$\boldsymbol{L} = \sum_1^n a_{ij}\partial_i\,\partial_j\,, \tag{1}$$

under the homogeneous Dirichlet boundary condition

$$\begin{cases} \boldsymbol{L}\,u = f \quad \text{in} \quad \Omega\,, \\ u = 0 \quad \text{on} \quad \Gamma\,. \end{cases} \tag{2}$$

The main lines of the proofs apply to more general situations, at the cost of additional technicalities.

The starting point of these notes was reference [1], where the main goal was to look for minimal assumptions on the data which guarantee classical solutions to the $2 - D$ Euler equations in a bounded domain. For a brief, clear, exposition on the links between the Euler equations and the problems treated in these notes, the reader is invited to have a look at reference [3]. The study of the above problem led to consider a Banach space of Dini's type, denoted by the symbol $C_*(\overline{\Omega})$. Let us recall here definition and some properties of $C_*(\overline{\Omega})$ (see [1] and, for complete proofs, [2]).

We set

$$I(x;\,r) = \{\, y : |y - x| \leq r \,\}\,, \quad \Omega(x;\,r) = \Omega \cap I(x;\,r)\,,$$

and define, for arbitrary $f \in C(\overline{\Omega})$, and $r > 0$,

$$\omega_f(r) \equiv \sup_{x,\, y \in \Omega\, ;\, 0 < |x - y| \leq r} |f(x) - f(y)|. \tag{3}$$

Further we define the the semi-norm

$$[f]_* = [f]_{*,\, R} \equiv \int_0^R \omega_f(r)\, \frac{dr}{r} \tag{4}$$

and the functional space

$$C_*(\overline{\Omega}) \equiv \{\, f \in C(\overline{\Omega}) : [f]_* < \infty \,\}. \tag{5}$$

A norm is introduced in $C_*(\overline{\Omega})$ by setting $\|f\|_* = [f]_* + \|f\|$. Norms defined for two distinct values of R are equivalent. Clearly, $C^{0,\lambda}(\overline{\Omega}) \subset C_*(\overline{\Omega}) \subset C(\overline{\Omega})$.

The following are some of the main properties of this space: $C_*(\overline{\Omega})$ is a Banach space; The embedding $C_*(\overline{\Omega}) \subset C(\overline{\Omega})$ is compact; The set $C^\infty(\overline{\Omega})$ is dense in $C_*(\overline{\Omega})$.

The following result holds (see Theorem 4.5, in [1]).

Theorem 2.1. *Let $f \in C_*(\overline{\Omega})$ and let u be the solution of problem* (2). *Then $u \in C^2(\overline{\Omega})$, moreover,*

$$\|\nabla^2 u\| \leq c\, \|f\|_*. \tag{6}$$

The regularity results proved for data in $C_*(\overline{\Omega})$, like theorem 2.1, led us to look for data spaces, between Hölder and $C_*(\overline{\Omega})$ spaces, for which solutions "remember", at least partially, their origin, see section 1. The following is a significant example of a functional space of "intermediate type", based on the well known formulae

$$\int \frac{(-\log r)^{-\alpha}}{r}\, dr = \frac{1}{\alpha - 1}\, (-\log r)^{1-\alpha}, \tag{7}$$

where $0 < \alpha < +\infty$ (for $\alpha = 1$, the right hand side should be replaced by $-\log(-\log r)$). We assume that $0 < r < 1$. Equation (7) shows that the $C_*(\overline{\Omega})$ semi-norm (4) is finite if

$$\omega_f(r) \leq C\, (-\log r)^{-\alpha}, \tag{8}$$

for some $\alpha > 1$. This led to define, for each fixed $\alpha > 0$, a semi-norm

$$[f]_\alpha \equiv \sup_{x,\, y \in \overline{\Omega}\ \ 0 < |x - y| < 1} \frac{|f(x) - f(y)|}{(-\log|x - y|)^{-\alpha}} = \sup_{r \in (0,1)} \frac{\omega_f(r)}{(-\log r)^{-\alpha}}, \tag{9}$$

and a related functional space $D^{0,\alpha}(\overline{\Omega})$, as follows.

Definition 2.2. For each real positive α, we set

$$D^{0,\alpha}(\overline{\Omega}) \equiv \{\, f \in C(\overline{\Omega}) : [f]_\alpha < \infty \,\}. \tag{10}$$

A norm is introduced in $D^{0,\alpha}(\overline{\Omega})$ by setting $\|f\|_\alpha \equiv [f]_\alpha + \|f\|$.

Note that we have merely replaced in the definition of Hölder spaces the quantity

$$\frac{1}{|x - y|} \quad \text{by} \quad \log \frac{1}{|x - y|},$$

and allow α to be arbitrarily large. This similitude led us to have called these spaces, in reference [5], H-log spaces. Below we call these spaces simply Log spaces. Spaces $D^{0,\alpha}(\overline{\Omega})$ are Banach spaces. Furthermore, the (compact) embeddings

$$C^{0,\lambda}(\overline{\Omega}) \subset D^{0,\alpha}(\overline{\Omega}) \subset C_*(\overline{\Omega}) \subset D^{0,\beta}(\overline{\Omega}) \subset C(\overline{\Omega}) \tag{11}$$

hold, for $0 < \beta < 1 < \alpha$, and $0 < \lambda \leq 1$.

In reference [5] we claimed, and left the proof to the reader, that $C^\infty(\overline{\Omega})$ is dense in $D^{0,\alpha}(\overline{\Omega})$. Actually, as shown below in theorem 4, this result *is false*.

In reference [5] we proved the following result.

Theorem 2.3. *Let* $f \in D^{0,\alpha}(\overline{\Omega})$ *for some* $\alpha > 1$, *and let* u *be the solution of problem* (2). *Then* $\nabla^2 u \in D^{0,\alpha-1}(\overline{\Omega})$, *moreover*

$$\| \nabla^2 u \|_{\alpha-1} \leq C \, \| f \|_\alpha \, . \tag{12}$$

The above result is optimal. If $\beta > \alpha-1$, *then* $\nabla^2 u \in D^{0,\beta}(\overline{\Omega})$ *is false in general.*

Concerning the *optimality* claimed above it is worth noting that it is not confined to the particular family of spaces under consideration, but is something stronger. Let us illustrate this distinction. Let $\alpha > 1$ be given, and let u be the solution of problem (2), where $f \in D^{0,\alpha}(\overline{\Omega})$. The theorem claims that $\nabla^2 u \in D^{0,\alpha-1}(\overline{\Omega})$. Optimality *restricted* to the Log spaces framework means that, given $\beta > \alpha - 1$, there is at least a data f as above for which $\nabla^2 u$ does not belong to $D^{0,\beta}(\overline{\Omega})$. This situation does not exclude that (for instance, and to fix ideas) for all $f \in D^{0,\alpha}(\overline{\Omega})$ the oscillations $\omega(r)$ of $\nabla^2 u$ satisfy the estimate

$$\omega(r) \leq C_f \, (-\log r)^{-(\alpha-1)} \left(\log^2 \frac{1}{r} \right)^{-1} \tag{13}$$

since

$$(-\log r)^{-\beta} = o\Big((-\log r)^{-(\alpha-1)} \left(\log^2 \frac{1}{r} \right)^{-1} \Big),$$

for all $\beta > \alpha - 1$.

Our optimality's proof, reported below in section 6, avoid the above possibility. This fact is significant in all cases in which full regularity is not reached, as in the above example. In fact, full regularity implies the above sharp optimality.

In section 6 we prove the *sharp* optimality result, as an opportunity to show a proof in these notes.

3. **The spaces** $D_{\overline{\omega}}(\overline{\Omega})$**.** Our next aim has been to extend the theorem 2.3 to more general data spaces, denoted here by the symbol $D_{\overline{\omega}}(\overline{\Omega})$. These functional spaces satisfy the inclusions

$$Lip(\overline{\Omega}) \subset D_{\overline{\omega}}(\overline{\Omega}) \subset C(\overline{\Omega})\,.$$

The basic results proved for data in $D^{0,\alpha}(\overline{\Omega})$ and in $C^{0,\lambda}(\overline{\Omega})$, are now a particular case. Clearly, specific proofs in particular cases could be more stringent (dependence of constants, for instance).

We start by defining these spaces and showing of their main properties. Consider real, continuous, non-decreasing functions $\overline{\omega}(r)$, defined for $0 \leq r < R$. Furthermore, $\overline{\omega}(0) = 0$, and $\overline{\omega}(0) > 0$ for $r > 0$. We call these functions *oscillation functions*.

We set

$$[f]_{\overline{\omega}} = \sup_{0 < r < R} \frac{\omega_f(r)}{\overline{\omega}(r)}\,. \tag{14}$$

Hence,

$$\omega_f(r) \leq [f]_{\overline{\omega}} \overline{\omega}(r)\,, \quad \forall\, 0 < r < R\,. \tag{15}$$

Further, we define the linear space

$$D_{\overline{\omega}}(\overline{\Omega}) = \{\, f \in C(\overline{\Omega}) : [f]_{\overline{\omega}} < \infty \,\}\,. \tag{16}$$

One easily shows that $[f]_{\overline{\omega}}$ is a semi-norm $D_{\overline{\omega}}(\overline{\Omega})$. We define a norm by setting

$$\|f\|_{\overline{\omega}} = [f]_{\overline{\omega}} + \|f\|. \tag{17}$$

Two norms, with distinct values of the parameter R, are equivalent.

Next we establish some useful properties of the above functional spaces.

Proposition 1. *If*

$$0 < k_0 \le \frac{\overline{\omega}(r)}{\overline{\omega}_0(r)} \le k_1 < +\infty, \tag{18}$$

for r in some neighborhood of the origin, then $D_{\overline{\omega}}(\overline{\Omega}) = D_{\overline{\omega}_0}(\overline{\Omega})$, with equivalent norms.

Proposition 2. *$D_{\overline{\omega}}(\overline{\Omega})$ is a Banach space.*

Proposition 3. *Assume that*

$$\lim_{r \to 0} \frac{\overline{\omega}(r)}{\overline{\omega}_1(r)} = 0. \tag{19}$$

Then the embedding

$$D_{\overline{\omega}}(\overline{\Omega}) \subset D_{\overline{\omega}_1}(\overline{\Omega}),$$

is compact.

Also note that, by Ascoli-Arzela's Theorem, the embedding

$$D_{\overline{\omega}}(\overline{\Omega}) \subset C(\overline{\Omega})$$

is compact.

Proposition 4. *Assume that $\overline{\omega}(r)$ is concave near the origin, and that $D_{\overline{\omega}}(\overline{\Omega})$ does not coincide with the space of constant functions. Then $C^1(\overline{\Omega})$ is not dense in $D_{\overline{\omega}}(\overline{\Omega})$.*

4. **Spaces $D_{\overline{\omega}}(\overline{\Omega})$ and regularity.** We start by putting each oscillation function $\overline{\omega}(r)$, satisfying the assumption

$$\int_0^R \overline{\omega}(r) \frac{dr}{r} \le C_R \tag{20}$$

for some constant C_R, in correspondence with a unique, related oscillation function $\widehat{\omega}(r)$. Hence, to a functional space $D_{\overline{\omega}}(\overline{\Omega})$ there corresponds a well defined functional space $D_{\widehat{\omega}}(\overline{\Omega})$. Note that assumption (20) is equivalent to the inclusion $D_{\overline{\omega}}(\overline{\Omega}) \subset C_*(\overline{\Omega})$.

Define $\widehat{\omega}(r)$ by setting

$$\widehat{\omega}(r) = \int_0^r \overline{\omega}(s) \frac{ds}{s} \tag{21}$$

for $0 < r \le R$, and $\widehat{\omega}(0) = 0$. Obviously, $\widehat{\omega}$ satisfies all the properties described in section 3 for generic oscillation functions. In particular, Banach spaces

$$D_{\widehat{\omega}}(\overline{\Omega}) = \{\, f \in C(\overline{\Omega}) : [f]_{\widehat{\omega}} < \infty \,\} \tag{22}$$

turn out to be well defined.

We extend Theorem 2.3 to data in $D_{\overline{\omega}}(\overline{\Omega})$ spaces. For clearness, and for the reader's convenience, we impose simple conditions to the oscillation functions $\overline{\omega}(r)$, which hold in the more interesting cases. Here we do not discuss more general assumptions.

Consider the linear elliptic boundary value problem (2). We have excluded, in advance, data spaces whose elements are characterized by boundedness or continuity of f, since these singular cases have been largely investigated in the past. Hence we imposed the limitation

$$Lip(\overline{\Omega}) \subset D_{\overline{\omega}}(\overline{\Omega}) \subset C(\overline{\Omega}) \tag{23}$$

to the data spaces $D_{\overline{\omega}}(\overline{\Omega})$. Exclusion of $Lip(\overline{\Omega})$ means that $\overline{\omega}(r)$ does not verify $\overline{\omega}(r) \le c\,r$, for any positive constant c. Hence $\limsup(\overline{\omega}(r)/\,r\,) = +\infty$, as $r \to 0$. We simplify, by assuming that

$$\lim_{r \to 0} \frac{\overline{\omega}(r)}{r} = +\infty. \tag{24}$$

In particular the graph of $\overline{\omega}(r)$ is tangent to the vertical axis, at the origin (as for Hölder and Log spaces). This picture also shows that *concavity* of the graph is a quite natural assumption. Concavity implies that left and right derivatives are well defined, for $r > 0$. By also taking into account that $\overline{\omega}(r)$ is non-decreasing, we realize that pointwise differentiability of $\overline{\omega}(r)$, for $r > 0$, is not a particularly restrictive assumption. This claim is reinforced by the equivalence result for norms, under condition (18), which allows regularization of oscillation functions $\overline{\omega}(r)$, staying inside the same original functional space $D_{\overline{\omega}}(\overline{\Omega})$. Summarizing, *differentiability*, for $r > 0$, and *concavity*, both in a neighborhood of the origin, are natural assumptions here. In the sequel "differentiability" and "concavity" have this localized meaning.

Furthermore, if $\overline{\omega}(r)$ is concave, not flat, and differentiable for $r > 0$, then necessarily

$$\frac{\overline{\omega}(r)}{r\,\overline{\omega}'(r)} > 1, \tag{25}$$

for all $r > 0$. This led us to the condition

$$\lim_{r \to 0} \frac{\overline{\omega}(r)}{r\,\overline{\omega}'(r)} = C_1 > 1, \tag{26}$$

where $C_1 = +\infty$ is admissible. Furthermore, "limit" could be replaced by "lower limit". The significance of assumption (26) is reinforced by the particular situation in Lipschitz, Hölder, and Log cases in which the limit exists and is given by, respectively, 1, $\frac{1}{\lambda}$, and $+\infty$. As expected, the Lipschitz case stays outside the admissible range. Note that, basically, the larger is the space, the larger is the limit.

On the other hand, since we look for classical solutions, we have to impose assumption (20). Note that, due to a possible loss of regularity, it could happen that a "regularity space" $D_{\widehat{\omega}}(\overline{\Omega})$, necessarily contained in $C(\overline{\Omega})$, is not contained in $C_*(\overline{\Omega})$.

Clearly, we must have $D_{\overline{\omega}}(\overline{\Omega}) \subset D_{\widehat{\omega}}(\overline{\Omega})$. By appealing to a de l'Hôpital's rule one shows that

$$\lim_{r \to 0} \frac{\widehat{\omega}(r)}{\overline{\omega}(r)} = \lim_{r \to 0} \frac{\overline{\omega}(r)}{r\,\overline{\omega}'(r)} = C_1. \tag{27}$$

Note that if $0 < C_1 < \infty$ proposition 1 shows equivalence of norms.

The following result holds.

Theorem 4.1. *Assume that the oscillation function $\overline{\omega}$ is concave and differentiable, and satisfies assumptions* (20), (24), *and* (26). *Further, let $f \in D_{\overline{\omega}}(\overline{\Omega})$*

and let u be the solution of problem (2). Then $\nabla^2 u \in D_{\widehat{\omega}}(\overline{\Omega})$, where $\widehat{\omega}(r)$ is defined by (21). The estimate

$$\| \nabla^2 u \|_{\widehat{\omega}} \leq C \| f \|_{\overline{\omega}} \tag{28}$$

holds, for some positive constant C. If in equation (26) the constant C_1 is finite then full regularity holds, namely $D_{\widehat{\omega}}(\overline{\Omega}) = D_{\overline{\omega}}(\overline{\Omega})$.

The above regularity result is optimal, in the sharp sense (see below).

The above theorem holds under more general assumptions. The proof of theorem 4.1 follows that developed in Hölder spaces in [6], part II, section 5.

For previous related results we refer to [7] and [10]. The author is grateful to Piero Marcati who, after an exposition of our results, found the above related references.

Concerning other references, not related to our regularity results but merely to Log spaces (mostly for $n = 1$, or $\alpha = 1$), the author is grateful to Francesca Crispo for calling our attention to the treatise [8], to which the reader is referred. In particular, as claimed in the introduction of this volume, the space $D^{0,1}(\overline{\Omega})$ was considered in reference [11]. See also definition 2.2 in reference [8].

5. **Hölog spaces $C_{\lambda,\alpha}(\overline{\Omega})$ and full regularity.** Assume that, for some $\lambda > 0$,

$$\widehat{\omega}(r) = \lambda \overline{\omega}(r) \tag{29}$$

in a neighborhood of the origin. Then there is a $k > 0$, such that

$$\overline{\omega}(r) = k r^\lambda. \tag{30}$$

This fact could suggest that Hölder spaces are the unique full regularity class inside our framework. However, *full regularity* is also enjoyed by other spaces. The following is a quite challenging example. Consider oscillation functions of the form

$$\overline{\omega}(r) \equiv \omega_{\alpha,\lambda}(r) = r^\lambda (-\log r)^{-\alpha}, \quad r < 1, \tag{31}$$

where $0 < \lambda < 1$ and $\alpha \in \mathbb{R}$. For $\lambda = 0$ and $\alpha > 0$ we re-obtain $D^{0,\alpha}(\overline{\Omega})$, and for $\alpha = 0$ and $\lambda > 0$ we re-obtain $C^{0,\lambda}(\overline{\Omega})$. The compact inclusions

$$C_{\lambda,\alpha}(\overline{\Omega}) \subset C^{0\lambda}(\overline{\Omega}) \subset C_{\lambda,-\alpha}(\overline{\Omega})$$

hold, where here $\alpha > 0$.

We set

$$[f]_{\lambda,\alpha} \equiv [f]_{\overline{\omega}}, \quad \text{and} \quad \| f \|_{\lambda,\alpha} \equiv \| f \|_{\overline{\omega}},$$

where $\overline{\omega}(r)$ is given by (31). The following result follows from theorem 4.1.

Theorem 5.1. *Let $f \in C_{\lambda,\alpha}(\overline{\Omega})$ for some $\lambda \in (0,1)$ and some $\alpha \in \mathbb{R}$. Let u be the solution of problem (2). Then $\nabla^2 u \in C_{\lambda,\alpha}(\overline{\Omega})$. Moreover*

$$\| \nabla^2 u \|_{\lambda,\alpha} \leq C \| f \|_{\lambda,\alpha}. \tag{32}$$

6. **On the optimality of the regularity result.** In this section we discuss and prove the sharp optimality of the regularity result, claimed in theorem 4.1.

Assume that $\boldsymbol{L} = \Delta$. Consider the function

$$u(x) = \psi(|x|) \widehat{\omega}(|x|) x_1 x_2 \tag{33}$$

defined in the $n-$sphere $I(0,1)$, $n \geq 2$. Furthermore, $u(0) = 0$. The function $\psi(r)$ is non-negative, indefinitely differentiable, and vanishes for $r \geq \frac{1}{2}$. Moreover, $\psi(r) = 1$ for $|x| < \frac{1}{4}$. Homogeneous boundary conditions are obviously verified. The minimal regularity of u, and of its derivatives, is that reached inside $I(0, \frac{1}{4})$.

The "singular point" is the origin. It would be more elegant summation for all indexes $i \neq j$ however the conclusion is the same. To fix ideas, assume that $n = 3$.

The point here is that, due to the term $x_1 x_2$ in (33), the second order derivative $\partial_1 \partial_2 u(x)$ leave unchanged the "bad term" $\widehat{\omega}(|x|)$. This does not occur for square derivatives $\partial_i^2 u(x)$, hence for $\Delta u(x)$.

Straightforward calculations show that in $I(0, \frac{1}{4})$ one has

$$f(x) = \Delta u(x) = \frac{x_1 x_2}{|x|^2} \left(5 \overline{\omega}(|x|) + |x| \overline{\omega}'(|x|) \right),$$

where $f(0) = 0$. Furthermore, for $x \neq 0$,

$$\partial_1 \partial_2 u(x) = \widehat{\omega}(|x|) + \left(1 - 2 \frac{x_1^2 x_2^2}{|x|^4} \right) \overline{\omega}(|x|) + \frac{x_1^2 x_2^2}{|x|^4} \left(|x| \overline{\omega}'(|x|) \right). \qquad (34)$$

The functions $f(x)$ and $\partial_1 \partial_2 u(x)$ are continuous, and vanishes for $x = 0$. In the above equations, the specific expressions of the coefficients of $\overline{\omega}(|x|)$ and $|x| \overline{\omega}'(|x|)$, are, essentially, secondary (up to some remarks). The point is that they are homogeneous of degree zero. Hence they have no effect on the minimal regularity.

It readily follows from the above expressions that $f \in D_{\overline{\omega}}(I)$, and $\partial_1 \partial_2 u \in D_{\widehat{\omega}}(I)$. Due to the explicit term $\widehat{\omega}(|x|)$, the regularity claimed for the mixed second order derivative is optimal. For instance, the presence of the term $\widehat{\omega}(|x|)$ in (34) does not allow the estimate (13), since in this example

$$\widehat{\omega}(r) = (-\log r)^{-(\alpha - 1)}.$$

To conclude, note that, in accordance to the regularity result claimed in theorem 4.1, the second and third terms in the right hand side of (34) *can not* be less regular then $\widehat{\omega}(|x|)$.

Clearly, the above argument is fruitful if, in (34), a possible elimination of the term $\widehat{\omega}(|x|)$ by means of the other two terms is excluded. This would make fruitless the counterexample. In particular this is not possible since these coefficients are positive.

Let us briefly present a more "compact" argument. Denote by $H_k(x)$, k integer, generic homogeneous functions of degree k. Recall the differentiation rules for homogeneous functions. One has

$$\partial_i \widehat{\omega}(x) = \frac{\overline{\omega}(x)}{|x|} \frac{x_i}{|x|} = H_{-1}(x) \overline{\omega}(x),$$

$$\partial_j \overline{\omega}(x) = \overline{\omega}'(x) \frac{x_j}{|x|} = H_{-2}(x) \left(|x| \overline{\omega}'(x) \right).$$

Setting now $u(x) = \widehat{\omega}(|x|) x_i x_j$, it readily follows $\partial_i u(x) = x_j \widehat{\omega}(x) + H_1(x) \overline{\omega}(x)$, and

$$\partial_j \partial_i u(x) = \delta_{ij} \widehat{\omega}(x) + H_0(x) \left(\overline{\omega}(x) + |x| \overline{\omega}(x) \right). \qquad (35)$$

Hence

$$\Delta u(x) = H_0(x) \left(\overline{\omega}(x) + |x| \overline{\omega}(x) \right).$$

However, if $i \neq j$, the term $\widehat{\omega}(x)$ is still present in the right hand side of (35).

7. **On elliptic problems with more general data. Uniform boundedness of** $\nabla^2 u$ **..** In the context of [1], the Theorem 2.1 was marginal. So the proof, written in a still existing manuscript, remained unpublished. Actually, at that time, we have proved the above result for more general elliptic boundary value problems. The proof depends only on the behavior of the related Green's functions. Recently, by following the same ideas, we have shown the following result for the Stokes system (see the Theorem 1.1 in [2]):

Theorem 7.1. *For every* $\boldsymbol{f} \in \boldsymbol{C}_*(\overline{\Omega})$ *the solution* $(\boldsymbol{u}, p)$ *to the Stokes system*

$$\begin{cases} -\Delta\,\boldsymbol{u} + \nabla\,p = \boldsymbol{f} & in \quad \Omega\,, \\ \nabla \cdot \boldsymbol{u} = 0 & in \quad \Omega\,, \\ \boldsymbol{u} = 0 & on \quad \Gamma \end{cases} \tag{36}$$

belongs to $\boldsymbol{C}^2(\overline{\Omega}) \times C^1(\overline{\Omega})$. *Moreover, there is a constant* c_0, *depending only on* Ω, *such that the estimate*

$$\|\,\boldsymbol{u}\,\|_2 + \|\nabla\,p\,\| \leq c_0 \,\|\,\boldsymbol{f}\,\|_*\,, \quad \forall\,\boldsymbol{f} \in \boldsymbol{C}_*(\overline{\Omega})\,, \tag{37}$$

holds.

The proof of the above theorem, as that of theorem 2.1, is quite different from that of theorem 4.1. Both are based on estimates for Green's functions like those shown in the classical treatise of Olga Ladyzhenskaya, see [9]. They may also be found in Solonnikov paper [12]. In the manuscript quoted above we also tried to extend the result claimed in Theorem 2.1 to data belonging to functional spaces *larger* then $C_*(\overline{\Omega})$. Together with $C_*(\overline{\Omega})$, we have considered a functional space $B_*(\overline{\Omega})$ obtained by commuting *integral* and *sup* operators in the right hand side of definition

$$[\,f\,]_* = \int_0^R \sup_{x \in \overline{\Omega}} \omega_f(x;\,r)\,\frac{dr}{r}\,, \tag{38}$$

see (4). For each $f \in C(\overline{\Omega})$, we defined the semi-norm

$$\langle\,f\,\rangle_* = \sup_{x \in \overline{\Omega}} \int_0^R \omega_f(x;\,r)\,\frac{dr}{r}\,, \tag{39}$$

and the related functional space

$$B_*(\overline{\Omega}) \equiv \{\,f \in C(\overline{\Omega}) : \langle\,f\,\rangle_* < +\infty\,\} \tag{40}$$

endowed with the norm $\|f\|^* \equiv \langle\,f\,\rangle_* + \|f\|$. $B_*(\overline{\Omega})$ is a Banach space. We have shown that the inclusion $C_*(\overline{\Omega}) \subset B_*(\overline{\Omega})$ is proper, by constructing strongly oscillating functions which belong to $B_*(\overline{\Omega})$ but not to $C_*(\overline{\Omega})$. This construction was recently published in reference [3], Proposition 1.7.1. Furthermore, we have shown that Theorem 2.1, and similar, holds in a weaker form for data $f \in B_*(\overline{\Omega})$. We have proved that the first order derivatives of the solution u are Lipschitz continuous in $\overline{\Omega}$. Furthermore, the estimate

$$\|\,\nabla^2\,u\,\|_{L^\infty(\Omega)} \leq c_0 \,\|\|f\,\|^* \tag{41}$$

holds. The proof is published in reference [3], actually for data in a functional space $D_*(\overline{\Omega})$ containing $B_*(\overline{\Omega})$. See Theorem 1.3.1 in [3]. A similar extension holds for the Stokes problem, as shown in reference [4], Theorem 6.1, where we have proved that if $\boldsymbol{f} \in \boldsymbol{D}_*(\overline{\Omega})$, then the solution $(\boldsymbol{u}, p)$ of problem (36) satisfies the estimate

$$\|\,\boldsymbol{u}\,\|_{1,\,1} + \|\,p\,\|_{0,\,1} \leq C \,\|\|\,\boldsymbol{f}\,\|\|_*\,. \tag{42}$$

So $\nabla^2 \boldsymbol{u}$, $\nabla p \in L^\infty(\Omega)$. Furthermore, we have shown that if smooth functions are dense in $B_*(\overline{\Omega})$ then full regularity occurs, namely (37) holds for all $\boldsymbol{f} \in B_*(\overline{\Omega})$.

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