HBV-34

COMM. IN PARTIAL DIFFERENTIAL EQUATIONS, 7(10), 1135-1149 (1982)

HOMOGENEOUS AND NON-HOMOGENEOUS BOUNDARY VALUE PROBLEMS FOR FIRST ORDER LINEAR HYPERBOLIC SYSTEMS ARISING IN FLUID-MECHANICS (PART I)

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ABSTRACT

We prove the existence and the uniqueness of differentiable and strong solutions for a class of non-homogeneous boundary value problems for first order linear hyperbolic systems arising from the dynamics of compressible non-viscous fluids. The method provides the existence of differentiable solutions without resorting to strong or weak solutions. A necessary and sufficient condition for the existence of solutions for the non-homogeneous problem is proved. It consists of an explicit relationship between the boundary values of u and those of the data f. Strong solutions are obtained without this supplementary assumption. See Theorems 3.1, 4.1, 4.2, 4.3 and Corollary 4.4; see also Remarks 2.1 and 2.4.

In this paper we consider equation (3.1) below. In the forthcoming part II we prove similar results for the corresponding evolution problem.

1. Introduction. The motivation for the present work was the equations of motion of a compressible non-viscous fluid in a domain with boundary. In order to simplify the exposition we consider the half space $R^3_- \equiv \{x : x_1 < 0\}$. Let $T > 0$ be fixed and put $x' = (x_2, x_3)$. The governing non-linear equation are then

$$\begin{cases} \dfrac{\partial v}{\partial t} + (v\cdot\nabla)v + \rho^{-1}\nabla p = g, \\ \dfrac{\partial p}{\partial t} + v\cdot\nabla p + \rho(\dfrac{\partial \rho}{\partial p})^{-1}\mathrm{div}\, v = 0, \end{cases}$$

$$(1.1)\qquad \begin{cases} \dfrac{\partial S}{\partial t} + v\cdot\nabla S = 0, \\ v_1(t,0,x') = 0, \\ v(0,x) = v_0(x), \\ p(0,x) = p_0(x), \\ S(0,x) = S_0(x), \end{cases}$$

where the velocity $v = (v_1,v_2,v_3)$, the pressure p, and the entropy S are unknowns in $]-T,T[\times R^3_-$. The functions $g(t,x)$, $v_0(x)$, $p_0(x)$ and $S_0(x)$ are given. Furthermore the equation of state of the medium $\rho = \rho(p,S)$ is a known function of p,S verifying $\rho > 0$ and $\partial\rho/\partial p > 0$.

The motion of compressible non-viscous fluids was studied by Ebin for small initial data [5] and by us for arbitrary initial data [2], [3], [4][(1)]. However, the linearizations used in these papers are not the simple ones which consists in studying the first order hyperbolic system (the linearization procedure decouples the variable S)

$$(1.2)\qquad \begin{cases} \dfrac{\partial v}{\partial t} + (w\cdot\nabla)v + a\nabla p = g\ , \\ \dfrac{\partial p}{\partial t} + w\cdot\nabla p + b\ \mathrm{div}\ v = \ell\ , \\ v_1(t,0,x') = 0\ , \\ v(0,x) = v_0(x)\ , \\ p(0,x) = p_0(x)\ . \end{cases}$$

Here $a(t,x)$ and $b(t,x)$ are given positive functions and the given vector field $w(t,x)$ verifies $w_1(t,0,x') = 0$.

As pointed out in Ebin's paper, the known results for linear hyperbolic systems do not seem suitable for these problems. For this reason we will study the problem of the existence of differentiable solutions for a class of boundary value problems which contains as a particular case the system (1.2) and other systems arising in fluid-mechanics. This will be done in this paper (stationary case) and in a following one (part II, evolution case). The method used here also provides a simpler proof when the problem is posed in the whole space. In this last case the statements and proofs are obtained by dropping all the assumptions concerning the boundary.

In this paper we avoid the use of weak solutions, mollifiers, and negative norms by proving directly the existence of differentiable solutions which are our main concern, in view of problem (1.1). The existence of strong solutions then follows as a consequence.

Recall that in the fundamental works of K. O. Friedrichs [7] and P. D. Lax and R. S. Phillips [8] one starts by proving the existence of weak solutions.

We prove the existence of differentiable solutions without assuming the boundary space N to be maximal non-positive and the boundary matrix to be of constant rank on the boundary. See Theorems 3.1, 4.1, 4.2, 4.3 and Corollary 4.4; see also Remarks 2.1 and 2.4.

An essential tool in our method will be the introduction of a space Z of regular functions verifying not only the assigned boundary conditions but also some "complementary boundary conditions" such that: (i) the boundary integrals in (2.10) vanish for every $u,v \in Z$; (ii) Z is dense in Y (roughly speaking: the complementary conditions loose sense in Y); (iii) there exists an homeomorphism D from Z into X for which (2.28) holds; (iv) $Lu \in Y$, $\forall u \in Z$. These conditions could be weakened, but the above form is sufficient for our purposes.

In order to simplify the exposition we treat the problem in the half space $\mathbf{R}^m_-$. However, by standard methods one can adapt the results to open regular subsets Ω of $\mathbf{R}^m$.

2. <u>Basic Lemmas</u>. Let $\mathbf{R}^m_- = \{x \in \mathbf{R}^m : x_1 < 0\}$, $\mathbf{R}^{m-1} = \{x : x_1 = 0\}$ and $x' = (x_2,\dots,x_m)$. The components of the outward normal to the boundary are then $n_J = \delta_{1J}$, $J = 1,\dots,m$.

Let $L^2(\mathbf{R}^m_-)$ denote the space of all measurable real-valued (classes of) functions which are square integrable on $\mathbf{R}^m_-$ and let $H^k(\mathbf{R}^m_-)$ denote the space of all functions which belong, together with all the derivatives of order less or equal to k, to $L^2(\mathbf{R}^m_-)$. Moreover $H^k_0(\mathbf{R}^m_-)$, $k \geq 1$, denotes the subspace of $H^k(\mathbf{R}^m_-)$ of all functions vanishing(2) on the boundary $\mathbf{R}^{m-1}$ in the usual trace sense and $H^k_N(\mathbf{R}^m_-)$, $k \geq 2$, denotes the subspace of functions with vanishing normal derivative on the boundary. The space of all real bounded and continuous functions together with the derivatives of order less or equal to k will be denoted by $C^k(\mathbf{R}^m_-)$. The usual norm in this space is denoted by $\| \ \|_{C^k}$. Corresponding spaces on the boundary $\mathbf{R}^{m-1}$ will be used, in particular the fractionary Sobolev spaces $H^s(\mathbf{R}^{m-1})$ for $s = \frac{1}{2}, \frac{3}{2}$. Other notations will be clear from the context.

Finally we denote by c different constants depending at most on the integers m and n. $\mathcal{N}$ denotes the set of all positive integers.

Let now H and A^J, $J = 1,\dots,m$, be $n \times n$ - matrix valued functions defined in $\mathbf{R}^m_-$. We assume that H is diagonal with diagonal elements $h_k(x)$, $k = 1,\dots,n$, verifying

$$m_0 \equiv \inf_{\substack{x \in \mathbf{R}^m_- \\ 1 \leq k \leq n}} h_k(x) > 0 . \tag{2.1}$$

For convenience we define $h(x) = (h_1(x),\dots,h_n(x))$. Moreover we suppose that the matrices A^J, $J = 1,\dots,m$, are symmetric:

(2.2)
$$a^J_{ik}(x) = a^J_{ki}(x), \ \forall \ i,k = 1,\dots,n, \ \ \forall x \in \mathbb{R}^m_- .$$

Finally, for all index i,k,J,

(2.3)
$$h_k, a^J_{ik} \in C^1(\overline{\mathbb{R}}^m_-) .$$

Remark 2.1. The results and proofs stated in this paper can be easily adapted if the assumption "for every $x \in \mathbb{R}^m_-$, $H(x)$ is diagonal" is replaced by the more general assumption "for every $x \in \mathbb{R}^m_-$, $H(x)$ is symmetric, moreover for every $x \in \mathbb{R}^{m-1}$ it has the form

(2.4)
$$H(x) = \begin{bmatrix} H_p(x) & 0 \\ 0 & H_{n-p}(x) \end{bmatrix}$$

with $H_p(x)$ and $H_{n-p}(x)$ matrices of type $p \times p$ and $(n-p) \times (n-p)$ respectively" (definition of p in (2.13)). The assumption (2.1) is then replaced by "H is uniformly positive definite". In this more general case the scalar products $(u,v)_h$ and $((u,v))_h$ (definition below) are replaced by (Hu,v) and $(Hu,v) + \sum_{J=1}^{m} (H \frac{\partial u}{\partial x_J}, \frac{\partial v}{\partial x_J})$ respectively. Moreover the operator D in (2.27) becomes $Dv \equiv Hv - \mathrm{div}(H\nabla v)$(3) where $v = (v_1,\dots,v_n)$. Note that now equation $Dv = f$ is an elliptic system of n equations instead of n single (decoupled) elliptic equations. Thanks to the boundary assumption (2.4) the operator D is again an homeomorphism of Z onto X, moreover (2.28) holds(4). The reader easily verifies that our proofs hold again.

Now let L be the partial differential operator

(2.5)
$$Lu \equiv \sum_{J=1}^{m} H^{-1} A^J \frac{\partial u}{\partial x_J}$$

where $u = (u_1,\dots,u_n)$ is a vector function defined in $\mathbb{R}^m_-$. In equations like (2.5) in which matrices act on vectors these last are always to be considered as column vectors. Note that equations (1.2) can be written in the form

(2.6)
$$\frac{\partial u}{\partial t} + \sum_{J=1}^{4} H^{-1} A^J \frac{\partial u}{\partial x_J} = f$$

where $u = (v_1, v_2, v_3, p)$, $f = (g_1, g_2, g_3, \ell)$ and

(2.7)
$$H = \begin{bmatrix} a^{-1} & & & \\ & a^{-1} & & 0 \\ & & a^{-1} & \\ 0 & & & b^{-1} \end{bmatrix}, \qquad H^{-1}A^1 = \begin{bmatrix} w_1 & 0 & 0 & a \\ 0 & w_1 & 0 & 0 \\ 0 & 0 & w_1 & 0 \\ b & 0 & 0 & w_1 \end{bmatrix},$$

$$H^{-1}A^2 = \begin{bmatrix} w_2 & 0 & 0 & 0 \\ 0 & w_2 & 0 & a \\ 0 & 0 & w_2 & 0 \\ 0 & b & 0 & w_2 \end{bmatrix}, \qquad H^{-1}A^3 = \begin{bmatrix} w_3 & 0 & 0 & 0 \\ 0 & w_3 & 0 & 0 \\ 0 & 0 & w_3 & a \\ 0 & 0 & b & w_3 \end{bmatrix}.$$

Let us return to the general case (2.5). Let us define $\tilde{A}^J = H^{-1}A^J$ hence $\tilde{a}^J_{ik} = h_i^{-1}a^J_{ik}$. For convenience we will use the notations $\|A^J\|_{C^\ell} \equiv \max_{i,k}\|a^J_{ik}\|_{C^\ell}$, $\|A\|_{C^\ell} \equiv \max_J\|A^J\|_{C^\ell}$ and similarly for the other matrices used in this paper. By definition $\|h\|_{C^\ell} \equiv \max_k\|h_k\|_{C^\ell} = \|H\|_{C^\ell}$. Let be

$$X \equiv [L^2(\mathbf{R}^m_-)]^n, \quad Y \equiv [H^1(\mathbf{R}^m_-)]^n, \quad Z \equiv [H^2(\mathbf{R}^m_-)]^n$$

and define the scalar products in X

$$(u,v) \equiv \sum_{k=1}^n \int_{\mathbf{R}^m_-} u_k v_k dx, \quad (u,v)_h \equiv \sum_{k=1}^n \int_{\mathbf{R}^m_-} u_k v_k h_k dx$$

and also the corresponding norms $|\cdot|$ and $|\cdot|_h$ (which are equivalent) and the scalar products in Y

$$((u,v)) \equiv (u,v) + (\nabla u, \nabla v), \qquad ((u,v))_h \equiv (u,v)_h + (\nabla u, \nabla v)_h$$

and corresponding (equivalent) norms $\|\cdot\|$ and $\|\cdot\|_h$. By definition $(\nabla u, \nabla v)_h \equiv \sum_{J=1}^m \left(\frac{\partial u}{\partial x_J}, \frac{\partial v}{\partial x_J}\right)_h$. To point out that X is endowed with the norm $|\cdot|_h$ we sometimes write X_h instead of X. A similar remark holds for Y.

For the reader's convenience we state the following two lemmas (which proofs are classical):

<u>Lemma 2.2.</u> <u>For any pair</u> $u,v \in Y$ <u>the following identity holds:</u>

$$(2.8) \qquad (Lu,v)_h = -(u,Lv)_h + \alpha(u,v) + \int_{\mathbf{R}^{m-1}} v\cdot A^1 u\, dx'$$

<u>where</u> α <u>is the continuous and symmetric bilinear form on</u> X <u>defined by</u>

$$(2.9) \qquad \alpha(u,v) \equiv -\sum_{k,i=1}^n \int_{\mathbf{R}^m_-} \left(\sum_{J=1}^m \frac{\partial a^J_{ki}}{\partial x_J}\right) u_i v_k dx.$$

<u>Lemma 2.3.</u> <u>For any pair</u> $u,v \in Z$ <u>the following identity holds:</u>

$$(2.10) \qquad ((Lu,v))_h = -((u,Lv))_h + \beta(u,v) + \sum_{\ell=1}^m \int_{\mathbf{R}^{m-1}} \frac{\partial v}{\partial x_\ell}\cdot A^1 \frac{\partial u}{\partial x_\ell} dx' + \int_{\mathbf{R}^{m-1}} v\cdot A^1 u dx',$$

<u>where</u> $\beta(u,v)$ <u>is the bilinear continuous and symmetric form on</u> Y

(2.11) $$\beta(u,v) \equiv \bar{\beta}(u,v) + \bar{\beta}(v,u) + \sum_{i=1}^{m} \alpha\left(\frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_i}\right) + \alpha(u,v)$$

and

(2.12) $$\bar{\beta}(u,v) \equiv \sum_{\ell,J=1}^{m} \left(\frac{\partial \tilde{A}^J}{\partial x_\ell} \frac{\partial u}{\partial x_J}, \frac{\partial v}{\partial x_\ell}\right)_h .$$

Proof. Clearly

$$((Lu,v))_h = (Lu,v)_h + \sum_{\ell=1}^{m} \left(\sum_{J=1}^{m} \frac{\partial \tilde{A}^J}{\partial x_\ell} \frac{\partial u}{\partial x_J}, \frac{\partial v}{\partial x_\ell}\right)_h + \sum_{\ell=1}^{m} \left(L \frac{\partial u}{\partial x_\ell}, \frac{\partial v}{\partial x_\ell}\right)_h .$$

On the other hand (2.8) yields

$$\left(L \frac{\partial u}{\partial x_\ell}, \frac{\partial v}{\partial x_\ell}\right)_h + \left(\frac{\partial u}{\partial x_\ell}, L \frac{\partial v}{\partial x_\ell}\right)_h = \alpha\left(\frac{\partial u}{\partial x_\ell}, \frac{\partial v}{\partial x_\ell}\right) + \int_{R^{m-1}} \frac{\partial v}{\partial x_\ell} \cdot A^1 \frac{\partial u}{\partial x_\ell} dx' .$$

By adding the first equation with that obtained by switching u and v, and by using the last equation above, one easily gets (2.10). □

Now let p be an integer, $0 \leq p \leq n$ and let us define $N = \{u \in R^n : u_1 = \cdots = u_p = 0\}$, $N^\perp = \{u \in R^n : u_{p+1} = \cdots = u_n = 0\}$, $P_N u = (u_{p+1},\ldots,u_n)$, $P_N^\perp u = (u_1,\ldots,u_p)$. We assume that the boundary conditions are given by $P_N^\perp u = 0$ on the boundary or more explicitly by

(2.13) $$u_k(0,x') = 0, \quad k = 1,\ldots,p, \quad \text{for } x' \in R^{m-1} .$$

We also assume that the boundary matrix $A^1 = \sum_{J=1}^{m} n_J A^J$ verifies for each $x' \in R^{m-1}$ the following assumptions (the reader is also referred to the papers [7] and [8]; see in particular the sections 5 and 8 of [7]):

(2.14) $$\begin{cases} A^1(N) \subset N^\perp , \\ A^1(N^\perp) \subset N . \end{cases}$$

Moreover we assume that for each $x \in R^{m-1}$

(2.15) $$A^J(N) \subset N, \quad J = 2,\ldots,m .$$

With the boundary conditions in canonical form (2.13) assumptions (2.14) and (2.15) are respectively equivalent to the following ones: The matrices A^J take on the boundary the form

(2.16) $$A^1 = \begin{bmatrix} 0 & M \\ M^T & 0 \end{bmatrix}, \qquad A^J = \begin{bmatrix} R^J & 0 \\ 0 & S^J \end{bmatrix}, \quad J = 2,\ldots,m ,$$

where $M(x')$, $R^J(x')$ and $S^J(x')$ are $p \times (n-p)$, $p \times p$ and $(n-p) \times (n-p)$ matrices respectively. M^T denotes the transpose of M. By preceding assumptions R^J and S^J are symmetric. The ranks of these matrices are free.

Note that if $p = 0$ or if $p = n$ the conditions (2.15) disappear and condition (2.14) becomes $A^1 \equiv 0$ on the boundary; in case $p = 0$ it suffices that A^1 was negative semi-definite; see Remark 2.4. Note also that in the particular case (1.2) one has $p = 1$ and the matrices (2.7) verify the assumptions (2.16).

Finally, we describe the assumptions on the lower order term Bu in equation (3.1). We assume that B is an $n \times n$ matrix valued function defined in $\mathbb{R}^m_-$ and verifying

$$b_{ik} \in C^1(\mathbb{R}^m_-), \qquad i,k = 1,\dots,n . \tag{2.17}$$

Moreover we assume that for each x on the boundary one has

$$B(N) \subset N . \tag{2.18}$$

This last condition means that on the boundary

$$B = \begin{bmatrix} B_1 & 0 \\ B_2 & B_3 \end{bmatrix} \tag{2.19}$$

where $B_1(x')$, $B_2(x')$ and $B_3(x')$ are matrices of types $p \times p$, $(n-p) \times p$ and $(n-p) \times (n-p)$ respectively. Clearly $B \in \mathcal{L}[Y;Y]$; see definitions below.

We now define

$$\lambda_0 \equiv \max\Big\{\sup_{u \in X} \frac{|(Bu,u)_h + \frac{1}{2}\alpha(u,u)|}{|u|_h^2}, \ \sup_{u \in Y} \frac{|((Bu,u))_h + \frac{1}{2}\beta(u,u)|}{\|u\|_h^2}\Big\} . \tag{2.20}$$

From our assumptions on the coefficients it follows the existence of a constant c such that

$$\lambda_0 \leq \tilde{\lambda}_0 \equiv c\{\frac{1}{m_0}\|A\|_{C^1} + \frac{1}{m_0^2}\|A\|_{C^0}\|h\|_{C^1} + \|B\|_{C^1}\} . \tag{2.21}$$

Moreover we define the following spaces:

$$Y \equiv \{u \in \mathcal{Y} : u \in N \text{ on the boundary}\} = [H_0^1(\mathbb{R}^m_-)]^p \times [H^1(\mathbb{R}^m_-)]^{n-p} ,$$

and

$$Z \equiv \{u \in \mathcal{Z} : u \in N \text{ and } \frac{\partial u}{\partial x_1} \in N^\perp \text{ on the boundary}\} = [H_0^2(\mathbb{R}^m_-)]^p \times [H_N^2(\mathbb{R}^m_-)]^{n-p} .$$

Note that Z is dense in Y and Y is dense in X.

<u>Remark 2.4.</u> Assume that, instead of $(2.16)_1$, the boundary matrix has the more general form

$$A_1 = \begin{bmatrix} 0 & M \\ M^T & Q \end{bmatrix}$$

where Q is a symmetric and negative semi-definite matrix; this is equivalent to replace condition $(2.14)_1$ by condition

$$u \cdot A^1 u \leq 0, \quad \forall u \in N, \quad \forall x' \in \mathbb{R}^{m-1} . \tag{2.22}$$

Under this weaker assumption the existence results proved in our paper (including those of Section 4) holds again provided that one assume that $\lambda > \lambda_0$ instead of $|\lambda| > \lambda_0$. The proofs remain unchanged in the essential features.

(ii) Energy estimates are obtained under weaker assumptions than those needed to get existence. The a priori bound $|u|_h \leq (|\lambda| - \lambda_0)^{-1}|\lambda u + Lu + Bu|_h$, $\forall u \in Y$, holds without assumptions $(2.14)_2$, (2.15) and (2.19); moreover assumption $(2.14)_1$ can be replaced by (2.22) if we take in account only the values λ for which $\lambda > \lambda_0$. Analogously the a priori bound $\|u\|_h \leq (|\lambda| - \lambda_0)^{-1}\|\lambda u + Lu + Bu\|_h$, $\forall u \in Z$, holds without the assumptions (2.15) and (2.19); moreover if we take in account only values $\lambda > \lambda_0$, the assumptions $(2.14)_1$ and $(2.14)_2$ can be replaced respectively by (2.22) and by $u \cdot A^1 u \leq 0$, $\forall u \in N^{\perp}$, $\forall x' \in \mathbb{R}^{m-1}$. The remaining assumptions $(2.14)_2$, (2.15) and (2.19) are utilized only to get $L(Z) \subset Y$ and $B(Y) \subset Y$; it seems clear that these assumptions could be weakened if one only wants to prove existence for strong solutions.

Lemma 2.5. Assume that $(2.14)_1$ holds. Then

$$(Lu,v)_h = -(u,Lv)_h + \alpha(u,v), \quad \forall u,v \in Y . \tag{2.23}$$

In particular

$$|(Lu + Bu,u)_h| \leq \lambda_0 |u|_h^2, \quad \forall u \in Y . \tag{2.24}$$

In fact, under the hypothesis of the lemma the last term in equation (2.8) vanishes.

Assume now both hypotheses (2.14) and let $u,v \in Z$. If $\ell \neq 1$ the tangential derivatives $\frac{\partial u}{\partial x_\ell}$ and $\frac{\partial v}{\partial x_\ell}$ belong to N and from $(2.14)_1$ it follows that the corresponding integrals in equation (2.10) vanish. This also holds for $\ell = 1$ as one shows by using $(2.14)_2$ and by recalling that the normal derivatives $\frac{\partial u}{\partial x_1}$ and $\frac{\partial v}{\partial x_1}$ belong to $N^{\perp}$. Hence one gets the following result:

Lemma 2.6. Assume that (2.4) holds. Then

$$((Lu,v))_h = -((u,Lv))_h + \beta(u,v), \quad \forall u,v \in Z . \tag{2.25}$$

In particular

$$|((Lu + Bu,u))_h| \leq \lambda_0 \|u\|_h^2, \quad \forall u \in Z . \tag{2.26}$$

Lemma 2.7. The operator L with domain Y is preclosed in X. Moreover its closure $\bar{L}$ verifies (2.23) for each pair $u,v \in D(\bar{L})$.

The proof follows easily from (2.23).

Lemma 2.8. Assume that $(2.14)_2$ and (2.15) hold. Then $L \in L[Z;Y]$, i.e. L is a bounded linear operator from Z into Y.

Proof. Clearly $L \in L[Z;\mathcal{Y}]$. Let $u \in Z$. Then $\frac{\partial u}{\partial x_1} \in N^{\perp}$ on the boundary hence $A^1(\frac{\partial u}{\partial x_1}) \in N$. On the other hand for $J \neq 1$ one has $\frac{\partial u}{\partial x_J} \in N$ hence $A^J(\frac{\partial u}{\partial x_J}) \in N$. Consequently $Lu \in N$ on the boundary. □

Define now the operators $D_k g \equiv h_k g - \mathrm{div}(h_k \nabla g)$ and put

(2.27)
$$D = \begin{bmatrix} D_1 & & 0 \\ & \ddots & \\ 0 & & D_n \end{bmatrix} .$$

Lemma 2.9. The operator D is an homeomorphism from Z onto X. Moreover

(2.28)
$$((u,v))_h = (u,Dv), \quad \forall u \in Y, \quad \forall v \in Z .$$

Proof. The first statement follows from well known results for the Dirichlet and Neumann boundary value problems for second order linear elliptic equations; the reader is referred to the classical paper of L. Nirenberg [9]. Equation (2.28) follows by integration by parts. □

3. Classical Solutions for $f \in Y$. Strong Solutions. The assumptions in this section are (2.1), (2.2), (2.3), (2.14), (2.15), (2.17) and (2.18). Recall also the definitions (2.5) and (2.20). The boundary conditions are given by (2.13). Under these assumptions we prove the following theorem of existence and unicity for differentiable and strong solutions of equation (3.1)[5]:

Thereom 3.1. Let the above conditions hold and let $\lambda \in \mathbb{R}$ be such that $|\lambda| > \lambda_0$. Then (i) for each $f \in Y$ the equation

(3.1)
$$\lambda u + Lu + Bu = f$$

has a unique differentiable solution $u \in Y$. Moreover

(3.2)
$$\|u\|_h \leq \frac{1}{|\lambda| - \lambda_0} \|f\|_h .$$

(ii) For each $f \in X$ the equation

(3.3)
$$\lambda u + \bar{L}u + Bu = f$$

has a unique solution $u \in D(\bar{L})$. Moreover

(3.4)
$$|u|_h \leq \frac{1}{|\lambda| - \lambda_0} |f|_h .$$

Proof. We give two different approximations. By Galerkin's method and by elliptic regularization.

1^{st} method. Let $\{a_s\}$, $s \in N$, be a base for Z and put

(3.5)
$$u^{(\ell)} = \sum_{s=1}^{\ell} c_s^{(\ell)} a_s, \quad \forall \ell \in N .$$

Select the real numbers $c_s^{(\ell)}$ as the solutions of the linear non-homogeneous system

(3.6)
$$((\lambda u^{(\ell)}, a_r))_h + (((L + B)u^{(\ell)}, a_r))_h = ((f, a_r)), \quad \forall r \leq \ell .$$

For each $\ell \in N$ this problem is uniquely solvable. By using (2.26) and (2.20) it easily follows that

(3.7) $$\|u^{(\ell)}\|_h \le \frac{1}{|\lambda| - \lambda_0} \|f\|_h .$$

This gives the unicity for the linear homogeneous system hence the existence for the non-homogeneous system (3.6). From (3.7) and from the weak compactness of the spheres in Hilbert spaces it follows the existence of a subsequence $u^{(\nu)}$ and an element $u \in Y$ such that $u^{(\nu)} \longrightarrow u$ weakly in Y. Since $L \in L[Y;X]$ one has $Lu^{(\nu)} \longrightarrow Lu$ weakly in X. Using now (3.6) and (2.28) one gets $(\lambda u^{(\nu)} + (L + B)u^{(\nu)} - f, Da_r) = 0$, $r = 1,\dots,\ell$, and by passing to the limit

(3.8) $$(\lambda u + (L + B)u - f, Da_r) = 0, \quad \forall r \in N .$$

Since $\{Da_r\}$ is a base for X equation (3.1) holds. Estimate (3.2) follows from (3.7). Finally let $u \in Y$ be an arbitrary solution of equation (3.1). By multiplying scalarly in X_h the equation by u and by using (2.23) one gets (3.4). Hence the solution is unique (in particular one gets the convergence of all the sequences $u^{(\ell)}$ and $Lu^{(\ell)}$ in the proof given above).

We prove now the second part of the theorem. Let $f \in X$ and consider a sequence $f^{(\ell)} \in Y$, $f^{(\ell)} \to f$ in X. Let $u^{(\ell)} \in Y$ be the solutions of

(3.9) $$\lambda u^{(\ell)} + (L + B)u^{(\ell)} = f^{(\ell)} .$$ (6)

By taking the scalar product in X_h of the difference of the ℓ-th and the k-th equations with $u^{(\ell)} - u^{(k)}$, one gets $|u^{(\ell)} - u^{(k)}|_h \le (|\lambda| - \lambda_0)^{-1}|f^{(\ell)} - f^{(k)}|_h$. It follows that $u^{(\ell)} \to u \in X$ strongly in X. Hence $Bu^{(\ell)} \to Bu$ and from (3.9) $LU^{(\ell)} \to f - \lambda u - Bu$ strongly in X. This means that $u \in D(\bar{L})$ and that (3.3) holds. Let now $u \in D(\bar{L})$ be an arbitrary solution of (3.3). Then from Lemma 2.7 one easily gets the estimate (3.4) hence the unicity.

<u>2^{th} method</u>. Let $\varepsilon > 0$ be a parameter and look for $u_\varepsilon \in Z$ such that

(3.10) $$\lambda((u^\varepsilon, v))_h + (((L + B)u^\varepsilon, v))_h \pm \varepsilon((u^\varepsilon, v))_Z = ((f, v))_h, \quad \forall v \in Z .$$

We take $+\varepsilon$ if $\lambda > \lambda_0$ and $-\varepsilon$ if $\lambda < -\lambda_0$. The left hand side of the above equation is a bilinear continuous and coercive form over Z, consequently by a classical result of P. D. Lax and A. N. Milgram there exists a unique solution u^ε of problem (3.10). By taking $v = u^\varepsilon$ it follows that $(|\lambda| - \lambda_0)\|u^\varepsilon\|_h^2 + \varepsilon\|u^\varepsilon\|_Z^2 \le \|f\|_h \|u^\varepsilon\|_h$. Hence

(3.11) $$\|u^\varepsilon\|_h^2 \le \frac{1}{|\lambda| - \lambda_0} \|f\|_h ,$$

and from the first estimate above it follows that $\varepsilon\|u^\varepsilon\|_Z^2$ is bounded by a constant independent of ε. Thus there exist a subsequence u^ε and an element $u \in Y$ such that

$$(3.12)\qquad \begin{cases} u^\varepsilon \rightharpoonup u \quad \text{weakly in } Y, \\ \varepsilon \|u^\varepsilon\|_Z \to 0, \end{cases}$$

when $\varepsilon \to 0$. In particular $(L + B)u^\varepsilon \rightharpoonup (L + B)u$ weakly in Y. By using now (2.28) we write (3.10) in the form $(\lambda u^\varepsilon + (L + B)u^\varepsilon - f, Dv) = \mp \varepsilon((u^\varepsilon, v))_Z$, $\forall v \in Z$. By passing to the limit when $\varepsilon \to 0$ it follows that $\lambda u + (L + B)u - f = 0$. The remainder of the proof is as in the 1st method above. □

4. <u>The Non-Homogeneous Problem</u>. For convenience define

$$H_p^s \equiv [H^s(\mathbf{R}^{m-1})]^p$$

for $s = \frac{1}{2}$ or $\frac{3}{2}$ and $p \in N$. In this section we consider non-homogeneous boundary condition $P_N^\perp u = w$ or more explicitly

$$(4.1)\qquad u_k(0,x') = w_k(x'), \quad k = 1,\dots,p, \quad \text{for } x' \in \mathbf{R}^{m-1}.$$

Let $f \in X$ and $w \in H_p^{1/2}$. We said that $u \in X$ is a <u>strong solution</u> of problem (3.1), (4.1) if there exist sequences $u^{(\ell)} \in Y$, $f^{(\ell)} \in X$, $\ell \in N$, such that $\lambda u^{(\ell)} + Lu^{(\ell)} + Bu^{(\ell)} = f^{(\ell)}$, $P_N^\perp u^{(\ell)} = w$ on the boundary (in the usual trace sense) and $u^{(\ell)} \to u$, $f^{(\ell)} \to f$ strongly in X.

<u>Theorem 4.1</u>. <u>Let</u> $f \in X$ <u>and</u> $w \in H_p^{1/2}$ <u>be given, and let</u> $|\lambda| > \lambda_0$. <u>Then there exists a unique strong solution</u> u <u>of problem</u> (3.1), (4.1). <u>Moreover</u>

$$(4.2)\qquad |u|_h \le \frac{\|h\|_{C^0}^{1/2}}{|\lambda| - \lambda_0} \{|f| + c(2|\lambda| - \lambda_0 + \|\tilde{A}\|_{C^0} + \|B\|_{C^0})\|w\|_{H_p^{1/2}}\}.$$

<u>Proof</u>. Consider a linear continuous operator $w \to \bar{w}$ from $H_p^{1/2}$ into Y such that $P_N^\perp \bar{w} = w$ on the boundary $\mathbf{R}^{m-1}$ (in the trace sense) and $P_N \bar{w} \equiv 0$ in $\mathbf{R}_-^m$. By carrying out the change of variables $u = \bar{w} + v$ and by using Theorem 3.1 the result follows easily. □

<u>For differentiable solutions a corresponding result fails</u>. We start by giving a necessary condition, later on we will prove its sufficiency. Define <u>smooth solution</u> as a differentiable solution which belongs to Z. Let now $f \in Y$ and let u be a smooth solution of equation (3.1) with the non-homogeneous boundary condition (4.1). At each point $x \in \mathbf{R}^{m-1}$ one has

$$(4.3)\qquad \begin{cases} P_N^\perp H^{-1} A^1 = H_p^{-1} M P_N, \\ P_N^\perp H^{-1} A^J = H_p^{-1} R^J P_N^\perp, \qquad J = 2,\dots,m, \\ P_N^\perp B = B_1 P_N^\perp, \end{cases}$$

where the operators act on column vectors of $\mathbf{R}^n$ and H_p^{-1} is the $p \times p$ diagonal matrix with diagonal elements $h_k^{-1}(x)$, $k = 1,\dots,p$. Hence by restricting equation (3.1) to the boundary and by applying $P_N^\perp$ to both sides one gets

$$H_p^{-1}M\left(\frac{\partial u_p}{\partial x_1}, \ldots, \frac{\partial u_n}{\partial x_1}\right) = F \tag{4.4}$$

where by definition

$$F \equiv P_N^{\perp} f - F_\lambda[w] \tag{4.5}$$

and F_λ is the operator

$$F_\lambda[w] \equiv \lambda w + \sum_{J=2}^{m} H_p^{-1} R^J \frac{\partial w}{\partial x_J} + B_1 w \tag{4.6}$$

which acts on vector fields $w(x') = (w_1(x'), \ldots, w_p(x'))$ defined on the boundary R^{m-1}. Hence

$$F(x) \in \text{range of } H_p^{-1}(x)M(x) \quad \text{a.e. on the boundary} , \tag{4.7}$$

is a necessary condition for the existence of a smooth solution. In order to reverse condition (4.7) we want to state it in terms of the functional spaces used in this paper. One has the following result:

Theorem 4.2 (necessary condition). Let the data $f \in Y$ and $w \in H_p^{3/2}$ be given and define F by equation (4.5). If there exists a smooth solution of problem (3.1), (4.1) then the equation

$$H_p^{-1}M(g_{p+1}, \ldots, g_n) = (F_1, \ldots, F_p) \tag{4.8}$$

admits at least one solution $g \in H_{n-p}^{1/2}$.

Proof. Note that $F_\lambda \in L[H_p^{3/2}; H_p^{1/2}]$. Hence $F \in H_p^{1/2}$. Let now $u \in Z$ be a solution of (3.1), (4.1). Clearly $P_N\left(\frac{\partial u}{\partial x_1}\right) = \left(\frac{\partial u_{p+1}}{\partial x_1}, \ldots, \frac{\partial u_n}{\partial x_1}\right) \in H_p^{1/2}$ on the boundary. Moreover from equation (4.4) one shows that $P_N\left(\frac{\partial u}{\partial x_1}\right)$ is a solution of equation (4.8). □

We will now prove in the next theorem that the necessary condition stated in Theorem 4.2 is a sufficient condition in order to get a differentiable solution $u \in Y$.

Theorem 4.3 (sufficient condition). Let f, w and F be as in Theorem 4.2. If equation (4.8) admits a solution $g \in H_{n-p}^{1/2}$ then problem (3.1), (4.1) is uniquely solvable in Y (hence in Y if $w \equiv 0$). Moreover the following estimate holds

$$\|u\|_h \leq \frac{c\|h\|_{C^0}^{1/2}}{|\lambda| - \lambda_0} \{[1 + K(2|\lambda| - \lambda_0 + \|\tilde{A}\|_{C^1} + \|B\|_{C^1})]\|f\| + \tag{4.9}$$
$$+ [2|\lambda| - \lambda_0 + \|\tilde{A}\|_{C^1} + \|B\|_{C^1}][1 + K(|\lambda| + \|\tilde{A}\|_{C^1} + \|B\|_{C^1})]\|w\|_{H_p^{3/2}}\}$$

where K is a real number such that (recall (4.8))

$$\|g\|_{H_{n-p}^{1/2}} \leq K\|F\|_{H_p^{1/2}} . \tag{4.10}$$

Proof. For each $\psi \in H^2(\mathbf{R}_-^m)$ denote by $\gamma_0\psi$ and $\gamma_1\psi$ the values (in the usual trace sense) of ψ and $\frac{\partial\psi}{\partial x_1}$ on the boundary. It is well known that there exist right inverses $\gamma_0^{-1} \in L[H^{3/2}(\mathbb{R}^{m-1}); H^2(\mathbb{R}_-^m)]$ and $\gamma_1^{-1} \in L[H^{1/2}(\mathbb{R}^{m-1}); H^2(\mathbb{R}_-^m)]$ of γ_0 and γ_1 respectively. Define $\tilde{u}_J = \gamma_0^{-1} w_J$, $J = 1,\dots,p$, and $\tilde{u}_J = \gamma_1^{-1} g_J$, $J = p+1,\dots,n$. Clearly $\|\tilde{u}\|_Z \le c(\|w\|_{H_p^{3/2}} + \|g\|_{H_p^{1/2}})$. Carry out the change of variables $u = v + \tilde{u}$. Equation (3.1) becomes

$$\lambda v + Lv + Bv = f_1 \equiv f - (\lambda\tilde{u} + L\tilde{u} + B\tilde{u}) \ . \tag{4.11}$$

Obviously $f_1 \in Y$. By using (2.16), (2.19) and (4.8) one easily gets $f_1 \in N$ on the boundary. Hence $f_1 \in Y$. By Theorem 3.1 equation (4.11) has a unique solution $v \in Y$, moreover

$$\|v\|_h \le \frac{1}{|\lambda| - \lambda_0} \|f_1\|_h \ . \tag{4.12}$$

This means that equation (3.1) admits a solution $u \in Y$ verifying (4.1) and verifying the estimate

$$\|u\|_h \le \|h\|_{C^0}^{1/2} \left(\|\tilde{u}\| + \frac{1}{|\lambda| - \lambda_0} \|\lambda\tilde{u} + L\tilde{u} + B\tilde{u} - f\| \right) .$$

Recalling the above estimates and recalling also that

$$\|F_\lambda[w]\|_{H_p^{1/2}} \le c(|\lambda| + \|\tilde{A}\|_{C^1} + \|B\|_{C^1}) \|w\|_{H_p^{3/2}}$$

and that $\|P_N f\|_{H_p^{1/2}} \le c\|f\|$, the estimate (4.9) follows with straightforward calculations. The uniqueness is obvious. □

Consider now the matrix $M(x)$. We say that $\operatorname{rank} M(x) = p$ <u>uniformly</u> for $x \in \mathbb{R}^{m-1}$ if the sum of the squares of the determinants of order p contained in $M(x)$ is bounded below by a positive constant independent of x, i.e. if

$$\sum_{\alpha\in I} M_\alpha^2 \ge d^2 > 0, \qquad \forall x \in \mathbb{R}^{m-1} \ . \tag{4.13}$$

<u>Corollary 4.4. Assume that</u>(7)

$$\operatorname{rank} M(x) = p, \ \underline{\text{uniformly on}} \ \mathbb{R}^{m-1} \ . \tag{4.14}$$

<u>Then to each pair</u> $(f,w) \in Y \times H_p^{3/2}$ <u>corresponds a unique differentiable solution</u> $u \in Y$ <u>of problem</u> (3.1), (4.1). <u>Moreover</u> u <u>verifies</u> (4.9) <u>with a value</u> K <u>such that</u>

$$\begin{cases} K \le \dfrac{c}{d^2} \|A\|_{C^1} \|h\|_{C^1}, & \text{if } p = 1 , \\[2ex] K \le \dfrac{c}{d^2} \|A\|_{C^0}^{2p-3} \|A\|_{C^1}^2 \|h\|_{C^1}, & \text{if } p > 1 , \end{cases} \tag{4.15}$$

<u>where the norms concern the boundary spaces</u> $C^\ell(\mathbb{R}^{m-1})$, $\ell = 0,1$.

Proof. Let I be the set of all p-tuples of integers $\alpha = (\alpha_1,\ldots,\alpha_p)$ such that $1 \leq \alpha_1 < \cdots < \alpha_p \leq n-p$. Denote by A_J, $J = 1,\ldots,n-p$ the J-column of the matrix M, by M_α, $\alpha = (\alpha_1,\ldots,\alpha_p) \in I$, the value of the determinant whose columns are $A_{\alpha_1},\ldots,A_{\alpha_p}$ and by $M_\alpha^J(F)$ the value of the determinant obtained from M_α by replacing the J-column A_{α_J} by $H_p(F_1,\ldots,F_p) = (h_1F_1,\ldots,h_pF_p)$. Let finally $g_\alpha \equiv (g_1,\ldots,g_{n-p})$ be the vector column such that $g_{\alpha_i} = M_\alpha^i(F)$ for $i = 1,\ldots,p$ and $g_k \equiv 0$ for $k \notin \{\alpha_1,\ldots,\alpha_p\}$. From the definitions one has $Mg_\alpha = \sum_{i=1}^p A_{\alpha_i} g_{\alpha_i}$ and arguing as in the proof of Crammer's rule one shows that $Mg_\alpha = M_\alpha H_p(F_1,\ldots,F_p)$. The vector

$$g = \Big(\sum_{\alpha\in I} M_\alpha^2\Big)^{-1} \sum_{\alpha \in I} M_\alpha g_\alpha \tag{4.16}$$

verifies equation (4.8) i.e. the vector $g(x)$ is a solution of (4.8), for $x \in \mathbf{R}^{m-1}$. Obviously $g \in H_{n-p}^{1/2}$. Finally estimate (4.15) follows from (4.16) with straightforward calculations. Recall that $\|\xi\eta\|_{H^{1/2}(\mathbf{R}^{m-1})} \leq c\|\xi\|_{C^1(\mathbf{R}^{m-1})}\|\eta\|_{H^{1/2}(\mathbf{R}^{m-1})}$. □

Remark 4.5. Condition (4.7) determines the linear subspace of data (f,w) for which a differentiable solution u of problem (3.1), (4.1) exists.

For simplicity assume the homogeneous boundary condition $w \equiv 0$. By neglecting w the above linear subspace becomes(8)

$$Y_1 = \{f \in Y : P_n^\perp f(x) \in \text{ range of } H_p^{-1}(x)M(x) \text{ a.e. on the boundary}\} .$$

If we want to solve (3.1) for every $y \in Y$ (this means $Y_1 = Y$) condition (4.7) becomes rank $M(x) \equiv p$ on the boundary. In the other extreme case, namely rank $M(x) \equiv 0$ on the boundary, condition (4.7) says that (smooth) solutions can not exist if $f \notin Y$. In intermediate cases for which rank $M(x) \equiv q$, $0 < q < p$, explicitly necessary and sufficient conditions could be obtained from equation (4.4) and from Theorem 4.3. This was specified only for $q = p$ (Corollary 4.4) because this is the situation in problem (1.2) where $M = (0 \;\; 0 \;\; 1)$, consequently $q = p = 1$.

Footnotes

1. Later, in a paper independent from ours, Agemi extended the approach of Ebin to arbitrary initial data (see [1]).
2. Only the function. Not necessarily the derivatives.
3. This means that $(Dv)_k \equiv \sum_{i=1}^n h_{ki}v_i - \sum_{j=1}^m \frac{\partial}{\partial x_j}\big(\sum_{i=1}^n h_{ki}\frac{\partial v_i}{\partial x_j}\big)$, for $k = 1,\ldots,n$.
4. Clearly with $((u,v))_h$ replaced by the scalar product indicated above.
5. See also Remark 2.4, (i).

6. Actually one has $(u^{(\ell)}, Lu^{(\ell)}) \in Y \times Y$, $\forall \ell \in \mathbb{N}$.

7. Note that condition (4.14) can't be verified if $2p > n$.

8. Note by the way that $Y \subset Y_1 \subset \mathcal{Y}$.

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Received November 1981