COMM. IN PARTIAL DIFFERENTIAL EQUATIONS, 8(4), 407-432 (1983)

HOMOGENEOUS AND NON-HOMOGENEOUS BOUNDARY VALUE PROBLEMS FOR FIRST ORDER LINEAR HYPERBOLIC SYSTEMS ARISING IN FLUID MECHANICS (PART II)

H. Beirão da Veiga

Dipartimento di Matematica - Libera Università di Trento
38050 Povo (Trento), Italy

ABSTRACT

We prove the existence and the uniqueness of differentiable and strong solutions for a class of boundary value problems for first order linear hyperbolic systems arising from the dynamics of compressible non-viscous fluids. In particular necessary and sufficient conditions for the existence of solutions for the non-homogeneous problem are studied; strong solutions are obtained without this supplementary condition. See Theorems 3.2, 3.9, 4.1, 4.2 and Corollary 4.3; see also the discussion after Theorem 4.1. In particular we don't assume the boundary space to be maximal non-positive and the boundary matrix to be of constant rank on the boundary.

In this paper we prove directly the existence of differentiable solutions without resort to weak or strong solutions. An essential tool will be the introduction of a space Z of regular functions verifying not only the assigned boundary conditions but also some suitable complementary boundary conditions; see also the introduction of Part I of this work [1].

0360-5302/83/0804-0407$3.50/0

1. Notations and results.

In this paper we study problem (1.1) with boundary conditions (1.2) or (1.3), the evolution counterpart of the stationary problem (I.3.1) studied in Part I[(1)]. This problem arises from the study of the nonlinear equations of the motion of compressible non-viscous fluids. A discussion on this subject was done in Part I, which will be assumed familiar to the reader. We start by recalling some notations.

Let $T>0$ and put $I = [-T,T]$, $\mathbb{R}^m_- = \{x\in\mathbb{R}^m : x_1 < 0\}$, $x' = (x_2,\dots,x_m)$, $\mathbb{R}^{m-1} = \{x : x_1 = 0\}$. We denote by $L^2(\mathbb{R}^m_-)$ the Hilbert space of real square integrable functions in $\mathbb{R}^m_-$ and by $H^k(\mathbb{R}^m_-)$ the Sobolev space of functions belonging to $L^2(\mathbb{R}^m_-)$ together with all the derivatives of order less than or equal to k. Moreover $H^k_0(\mathbb{R}^m_-)$, $k\geq 1$, denotes the subspace of functions vanishing (only the function, not the derivatives) on the boundary $\mathbb{R}^{m-1}$ and $H^k_N(\mathbb{R}^m_-)$, $k\geq 2$, the subspace of functions with vanishing normal derivative on the boundary. $H^s(\mathbb{R}^{m-1})$, $s = 1/2, 3/2$, denotes the usual Sobolev spaces of fractional order on the boundary $\mathbb{R}^{m-1}$. The space of all real functions which are bounded and continuous together with their derivatives up to order k will be denoted by $C^k(\mathbb{R}^m_-)$, and the usual norm by $\| \|_{C^k}$. Let n and p be fixed integers, $0\leq p\leq n$. We define $X = [L^2(\mathbb{R}^m_-)]^n$, $\mathcal{Y} = [H^1(\mathbb{R}^m_-)]^n$, $\mathcal{Z} = [H^2(\mathbb{R}^m_-)]^n$, $Y = [H^1_0(\mathbb{R}^m_-)]^p \times [H^1(\mathbb{R}^m_-)]^{n-p}$, $Z = [H^2_0(\mathbb{R}^m_-)]^p \times [H^2_N(\mathbb{R}^m_-)]^{n-p}$. In section 4 we also utilize the trace spaces $\mathcal{H}^s_p \equiv [H^s(\mathbb{R}^{m-1})]^p$.

The canonical scalar products and norms in X and $\mathcal{Y}$ will be denoted by (u,v), $((u,v))$, $|u|$ and $\|u\|$. Given a fixed vector field $h = (h_1,\dots,h_n)$ in $I\times\mathbb{R}^m_-$ we also utilize (see Part I) the families of weighted scalar products $(u,v)_{h(t)}$ and $((u,v))_{h(t)}$ and corresponding norms in X and Y, namely $|u|_{h(t)}$ and $\|u\|_{h(t)}$; here $h(t)$ stands for $h(t,\cdot)$. For

(1) I.3.1. means equation (3.1) in part I, and so on; a corresponding notation is also used for the statements.

convenience we will write $(u,v)_t$ instead of $(u,v)_{h(t)}$ and so on. Hence

$$(u,v)_t \equiv \sum_{k=1}^{n} \int_{\mathbf{R}^m_-} u_k v_k h_k(t)dx, \quad ((u,v))_t \equiv (u,v)_t + (\nabla u, \nabla v)_t$$

where $(\nabla u, \nabla v)_t \equiv \sum_{J=1}^{m} (\frac{\partial u}{\partial x_J}, \frac{\partial v}{\partial x_j})_t$. To point out that X is endowed with the norm $|\cdot|_t$ we sometimes write X_t. A similar notation holds for Y_t. We note from now on that (under the choice of $h(t,x)$ made in the sequel) the norms $\|\cdot\|_t$ and $\|\cdot\|$ are equivalent in Y, uniformly respect to $t \in I$. The same holds for $|\cdot|_t$ and $|\cdot|$ in X (weighted norms are also utilized in Kato's paper [5]). We also introduce the following forms:

$$(u,v)_{h'(t)} \equiv \sum_{k=1}^{n} \int_{\mathbf{R}^m_-} u_k v_k h_k'(t)dx,$$

$$((u,v))_{h'(t)} \equiv (u,v)_{h'(t)} + \sum_{J=1}^{m} (\frac{\partial u}{\partial x_J}, \frac{\partial v}{\partial x_j})_{h'(t)}.$$

Moreover we put

$$\|h'(t)\|_{C^0} \equiv \max_{1 \leq k \leq n} \|h_k'(t)\|_{C^0(\mathbf{R}^{m-1})};$$

here $h'(t)$ stands for the time derivative.

Let now X be a general Banach space. We will utilize classical notations for Banach spaces consisting of X-vector valued measurable functions defined on I. We don't repeat the definitions of $L^q(I;X)$, $1 \leq q \leq +\infty$. $AC(I;X)$ means X-valued absolutely continuous functions on I. The meaning of $C^k(I;X)$ is clear. Moreover $L[X;X_1]$ denotes the Banach space of the linear continuous operators from X into X_1.

Finally we define $N = \{u \in \mathbf{R}^n : u_1 = \cdots = u_p = 0\}$, $N^\perp = \{u \in \mathbf{R}^n : u_{p+1} = \cdots = u_n = 0\}$, $P_N u = (u_{p+1}, \cdots, u_n)$, $P_N^\perp u = (u_1, \cdots, u_p)$.

Let now $u_0(x)$ and $f(t,x)$ be given n-dimensional vector fields defined in $\mathbf{R}^m_-$ and in $I \times \mathbf{R}^m_-$ respectively. We want to study the mixed initial boundary-value problem

$$(1.1) \qquad \begin{cases} u' + (L(t) + B(t))u = f & \text{in} \quad I \times \mathbf{R}^m_-, \\ u(0,x) = u_0(x) & \text{in} \quad \mathbf{R}^m_-, \end{cases}$$

with homogeneous boundary conditions

$$(1.2) \qquad u_k(t,0,x') = 0, \quad k = 1, \cdots, p, \qquad \text{for} \quad (t,x') \in I \times \mathbf{R}^{m-1},$$

or with non-homogeneous boundary conditions

$$(1.3) \qquad u_k(t,0,x') = w_k(t,x'), \quad k = 1,\cdots,p, \quad \text{for} \quad (t,x') \in I \times \mathbf{R}^{m-1} .$$

In (1.1) the unknown u is an n-dimensional vector field defined in $I \times \mathbf{R}^m_-$, moreover(2)

$$(1.4) \qquad Lu \equiv \sum_{J=1}^{m} H^{-1} A^J \frac{\partial u}{\partial x_J}$$

where H and A^J, $J = 1,\cdots,m$, are $n \times n$ matrix valued functions defined in $I \times \mathbf{R}^m_-$. We assume that H is diagonal (this condition is not essential; see Remark 1.2) with diagonal $h = (h_1,\cdots,h_n)$ and that

$$(1.5) \qquad m_0(t) \equiv \inf_{\substack{x \in \mathbf{R}^m_- \\ 1 \leq k \leq n}} h_k(t,x) > 0, \qquad \forall\, t \in I.$$

We suppose that the matrices A^J are symmetric

$$(1.6) \qquad a^J_{ik}(t,x) = a^J_{ki}(t,x), \quad \forall i,k = 1,\cdots,n, \ \forall (t,x) \in I \times \mathbf{R}^m_- .$$

Furthermore B is an $n \times n$ matrix valued function (not necessarily symmetric) defined in $I \times \mathbf{R}^m_-$. We assume that for all indices i,k,J(3)

$$(1.7) \qquad h_k, a^J_{ik}, b_{ik} \in L^1(I;\, C^1(\mathbf{R}^m_-)),$$

$$(1.8) \qquad \frac{dh_k}{dt}, \frac{da^J_{ik}}{dt}, \frac{db_{ik}}{dt} \in L^1(I;\, C^0(\mathbf{R}^m_-)).$$

From (1.7), (1.8) it follows that the functions in (1.7) belong to $C^0(I;C^0(\mathbf{R}^m_-))$. Hence $m_0 \in C^0(I;\mathbf{R})$ is positive and bounded away from zero. For convenience we use the notation $\|A^J(t)\|_{C^\ell} \equiv \sup_{i,k} \|a^J_{ik}(t)\|_{C^\ell(\mathbf{R}^m_-)}$, $1 \leq i,\, k \leq n$; similar notations will be used for other matrices and also for norms on the boundary. Moreover $\|A(t)\|_{C^\ell} \equiv \sup_{1 \leq J \leq m} \|A^J(t)\|_{C^\ell}$.

We now give the assumptions for the matrices on the boundary. First of all note that the boundary matrix is the restriction of A^1 to $\mathbf{R}^{m-1}$. We assume that A^1 verifies the condition (I.2.14) or equivalently that

---

(2) Matrix action is understood as multiplication from the left; in this case vectors are understood as column vectors.

(3) In (1.7), (1.8) one can replace $C^1(\mathbf{R}^m_-)$ by the space of all Lipschitz continuous functions $W^{1,\infty}(\mathbf{R}^m_-)$ and $C^0(\mathbf{R}^m_-)$ by the space of all bounded and measurable functions $L^\infty(\mathbf{R}^m_-)$.

$$A^1 = \left[\begin{array}{c|c} 0 & M \\ \hline M^T & 0 \end{array}\right], \quad \forall\ (t,x) \in I \times \mathbf{R}^{m-1}, \tag{1.9}$$

where M is a $p \times (n-p)$ matrix valued function defined in $I \times \mathbf{R}^{m-1}$ and M^T is it transpose; instead of (1.9) we can assume that the boundary matrix A^1 has the more general form indicated in Remark I.2.4,(i). In this case all the results and proofs hold again if one replaces $[-T,T]$ by $[0,T]$.

On the other hand we assume that the matrices A^J, $J = 2,\cdots,m$, verify (I.2.15) or equivalently

$$A^J = \left[\begin{array}{c|c} R^J & 0 \\ \hline 0 & S^J \end{array}\right], \quad \forall(t,x) \in I \times \mathbf{R}^{m-1}, \tag{1.10}$$

where R^J and S^J are $p \times p$ and $(n-p) \times (n-p)$ matrix valued functions defined on $I \times \mathbf{R}^{m-1}$. Finally the matrix B verifies (I.2.23) i.e.

$$B = \left[\begin{array}{c|c} B_1 & 0 \\ \hline B_2 & B_3 \end{array}\right], \quad \forall\ (t,x) \in I \times \mathbf{R}^{m-1}, \tag{1.11}$$

where B_1, B_2 and B_3 are matrix valued functions defined on the boundary, of types $p \times p$, $(n-p) \times p$ and $(n-p) \times (n-p)$ respectively. As pointed out in part I we don't assume the boundary matrix A^1 to be of constant rank on the boundary. Even when this condition is utilized (Corollary 4.3) the rank is not assumed constant near the boundary. Moreover we don't assume the operator L to be formally dissipative and the boundary space to be maximal non-positive.

As pointed out in Part I we avoid the use of mollifiers, negative norms and distributions by proving directly the existence of differentiable solutions instead of starting by the existence of weak solutions; see for instance the fundamental papers of K. O. Friedrichs [3] and P. D. Lax - R. S. Phillips [6].

<u>Remark 1.1.</u> As for Part I, all the results and proofs hold again if the problem is posed in the whole space $\mathbf{R}^m$; for this easier case it suffices to drop in statements and proofs all assumptions and arguments concerning the boundary. We get in this way a new and simpler proof for that particular case.

Concerning our results, we prove that if $u^0 \in Y$ and $f \in L^1(I;Y)$ then the problem (1.1), (1.2) has a unique differentiable solution $u \in C^0(I;Y) \cap AC(I;X)$; see Theorem 3.2. Clearly if f is more regular one easily gets from Equation (1.1) more regularity for $u'(t)$.

Furthermore if $f \in L^1(I;X)$, $u^0 \in X$ and $w = (w_1,\cdots,w_p) \in L^1(I;H_p^{1/2})$ then there exists a (unique) strong solution u of problem (1.1), (1.3), without any additional assumption on the operators; see Theorems 3.9 and 4.1. The corresponding result for differentiable solutions is not true in general; however we prove that a compatibility condition for w and the boundary values of f suffices to guarantee differentiable solutions. This condition is also necessary, at least in order to get smooth solutions; see Theorem 4.2 and the corresponding discussion. Note that our compatibility condition has to hold for any time and is independent of the classical condition (4.7).

The compatibility condition is always verified if $w = 0$ and $f \in L^1(I;Y)$, and in this sense the statement of Theorem 3.2 follows from that of Theorem 4.2. We also prove that the compatibility condition is always verified if rank $M(t,x) = p$ on $I \times \mathbf{R}^{m-1}$; hence in this case a (unique) differentiable solution exists for all pairs f,w (see Corollary 4.3).

In the sequel we give a direct proof of Theorem 3.2 which is the evolution counterpart of the proof given in Part I for Theorem I.3.1; we show directly that the function $u(t)$ constructed below is an X-valued absolutely continuous function, without resort to approximation of the time dependent operator $L(t) + B(t)$ by piecewise constant (respect to t) elements; for another proof, using T. Kato's results and part I, see Remark 1.3.

Remark 1.2. As pointed out in Remark I.2.1 the method used in these papers still works under the following conditions on the matrix $H(t,x)$:

(i) $H(t,x)$ is symmetric and uniformly positive definite in $I \times \mathbf{R}^m_-$ (instead of diagonal plus condition (1.5));

(ii) on the boundary $I \times \mathbf{R}^{m-1}$ the matrix $H(t,x)$ has the form

$$H = \begin{bmatrix} H_p & 0 \\ 0 & H_{n-p} \end{bmatrix} \tag{1.12}$$

where $H_p(t,x')$ and $H_{n-p}(t,x')$ are matrices of type $p \times p$ and $(n-p) \times (n-p)$ respectively.(4)

In this case the scalar products $(u,v)_t$ and $((u,v))_t$ are defined by

$$(u,v)_t \equiv (H(t)u,v), \quad \forall\, u,v \in X,$$

and

$$((u,v))_t \equiv (H(t)u,v) + \sum_{J=1}^{m} (H(t) \frac{\partial u}{\partial x_J}, \frac{\partial v}{\partial x_J}), \quad \forall\, u,v \in \mathcal{Y}.$$

The operator $D(t)$ (see Section 2) acting on Z becomes $D(t)v \equiv H(t)v - \operatorname{div}(H(t)\nabla v)$, or more explicitely

$$(D(t)v)_k \equiv \sum_{i=1}^{n} h_{ki}(t)v - \sum_{J=1}^{m} \frac{\partial}{\partial x_J} (\sum_{i=1}^{n} h_{ki}(t) \frac{\partial v_i}{\partial x_J}), \quad k = 1,\cdots,n,$$

where $(D(t)v)_k$ is the k^{th} component of $D(t)v$. Now equation $D(t)v = f$ is an elliptic system of n equations instead of n decoupled elliptic equations. Due to the boundary assumption (1.12) the operator $D(t)$ is again an homeomorphism from Z onto X (for almost all $t \in I$) and (2.8) holds again.

Remark 1.3. Existence theorems for abstract evolution equations (and applications) have been given by T. Kato in a sequence of well known papers (see [4], [5] and references; see also [2]). By using Friedrich's classical method one shows (see [5]) that Kato's hypothesis hold for the pure Cauchy problem for equation (1.1) posed in the whole space $\mathbf{R}^m$. However for mixed initial-boundary value problems like (1.1), (1.2) this was still an open problem; nevertheless from Theorem I.3.1 it easily follows that $L(t) + B(t)$ is $M - \mu_0$ stable in X and in Y, for suitable constants M and μ_0; by combining this result with Kato's theorems one gets immediately a statement similar to Theorem 3.2. We show the above claim in the appendix.

2. Some basic estimates.

In this paper c denotes any constant which depends at most on the integers m and n; $\mathbf{N}$ is the set of all positive integers.

---

(4) The regularity assumptions remain (1.7), (1.8).

For convenience we denote by E the subset of I consisting on all points t where at least one of the coefficients in equation (1.7) is not in $C^1(\mathbf{R}^m_-)$; note that these coefficients are continuous functions in $I \times \mathbf{R}^m_-$. Clearly E has zero Lebesgue measure.

From the above definitions and assumptions it easily follows that

$$(2.1)\qquad \begin{cases} \|L(t)\|_{\mathcal{L}[\mathcal{Y};X]} \leq c\|\tilde{A}(t)\|_{C^0}, \\ \\ \|L(t)\|_{\mathcal{L}[Z;\mathcal{Y}]} \leq c\|\tilde{A}(t)\|_{C^1} \end{cases}$$

where $\tilde{A}^J \equiv H^{-1}A^J$, $J = 1,\cdots,m$, moreover

$$(2.2)\qquad \begin{cases} \|B(t)\|_{\mathcal{L}[X;X]} \leq c\|B(t)\|_{C^0}, \\ \\ \|B(t)\|_{\mathcal{L}[\mathcal{Y};\mathcal{Y}]} \leq c\|B(t)\|_{C^1}, \end{cases}$$

where in the left hand sides $\|B(t)\|$ denotes the norm of the linear operator $B(t)$ and in the right hand sides the notation introduced in Section 1 for matrix-valued functions is used. Clearly the above estimates concerning C^1 norms hold for $t \in I/E$ and those concerning C^0 norms for all $t \in I$.

We also define as in (I.2.9) and (I.2.11) the bilinear continuous and symmetric forms $\alpha_t(u,v)$ and $\beta_t(u,v)$ on X and Y respectively; these forms are now time depending. Moreover we define $\lambda_0(t)$ as in (I.2.20)

$$\lambda_0(t) \equiv \max\left\{\sup_{u\in X} \frac{|(B(t)u,u)_t + \frac{1}{2}\alpha_t(u,u)|}{|u|_t^2},\ \sup_{u\in Y} \frac{|((B(t)u,u))_t + \frac{1}{2}\beta_t(u,u)|}{\|u\|_t^2}\right\}.$$

One easily verifies that there exists a constant c such that

$$\lambda_0(t) \leq \tilde{\lambda}_0(t) \equiv c\left\{\frac{1}{m_0(t)}\|A(t)\|_{C^1} + \frac{1}{m_0(t)^2}\|A(t)\|_{C^0}\|h(t)\|_{C^1} + \|B(t)\|_{C^1}\right\}.$$

We recall the following result (see Lemmas I.2.5 and I.2.6):

<u>Lemma 2.1</u> <u>Assume that</u> (1.9) <u>holds.</u> <u>Then for each fixed</u> $t \in I/E$ <u>one has</u>

$$(2.3)\qquad (Lu,v)_t + (u,Lv)_t = \alpha_t(u,v), \quad \forall\, u,v \in Y,$$

$$(2.4)\qquad ((Lu,v))_t + ((u,Lv))_t = \beta_t(u,v), \quad \forall\, u,v \in Z.$$

<u>In particular</u>

$$(2.5)\qquad |(Lu + Bu,u)_t| \leq \tilde{\lambda}_0(t)|u|_t^2, \quad \forall u \in Y,$$

(2.6) $$|((Lu + Bu,u))_t| \leq \tilde{\lambda}_0(t)\|u\|_t^2, \qquad \forall u \in Z.$$

Lemma 2.2 Assume that (1.9), (1.10) and (1.11) hold. Then for each $t \in I/E$ one has

(2.7) $$L(t) \in \mathcal{L}[Z;Y], \quad B(t) \in \mathcal{L}[Y;Y].$$

Hence estimates (2.1) and (2.2) hold with $\mathcal{Y}$ and $\mathcal{Z}$ replaced by Y and Z respectively.

Finally we need some results on the operator D introduced in (I.2.27). By definition $D_k(t)\phi \equiv h_k\phi - \operatorname{div}(h_k\nabla\phi)$; here $\phi(x)$ is a real function defined in $\mathbf{R}^m_-$ and the operators divergence and gradient concern only the x variable. The time variable can be viewed as a parameter. Hence

$$D(t) \equiv \begin{bmatrix} D_1(t) & & 0 \\ & \ddots & \\ 0 & & D_n(t) \end{bmatrix}$$

is a differential operator acting on n-dimensional vector fields in $\mathbf{R}^m_-$; this operator is well defined for each $t \in I/E$.

Lemma 2.3 For each $t \in I/E$ the operator $D(t)$ is a homeomorphism from Z onto X. Moreover for each pair $t,s \in I/E$ one has

(2.8) $$((u,v))_t = (u,D(t)v), \qquad \forall\, u \in Y, \quad \forall\, v \in Z,$$

(2.9) $$\|D(t)\|_{\mathcal{L}[Z;X]} \leq c\|h(t)\|_{C^1},$$

(2.10) $$\|D(t)^{-1}\|_{\mathcal{L}[X;Z]} \leq \frac{c}{m_0(t)}\Big[1 + \frac{\|h(t)\|_{C^1}}{m_0(t)}\Big],$$

(2.11) $$\|D(t)D(s)^{-1}\|_{\mathcal{L}[X;X]} \leq \Big\{\frac{\|h(t)\|_{C^0}}{m_0(s)}\Big[1 + \frac{\|h(s)\|_{C^1}}{m_0(s)}\Big] + \frac{\|h(t)\|_{C^1}}{m_0(s)}\Big\}.$$

Proof. Equation (2.8) was proved in Lemma I.2.9. The proof of (2.9) is obvious. Let now $\psi \in X$ and let $z \in Z$ be the solution of $D(t)z = \psi$. By the definitions [5]

[5] For convenience we drop the index k and the parameter t.

(2.12) $$hz - \operatorname{div}(h\nabla z) = \psi,$$

consequently $(1-\Delta)z = h^{-1}(\psi + \nabla h \cdot \nabla z)$. Well known results for the Dirichlet and the Neumann boundary value problems for the operator $1 - \Delta$ yield (see [7])

(2.13) $$\|z\|_Z \leq \frac{c}{m_0}(|\psi| + \|h\|_{C^1}\|z\|).$$

Now by multiplying both sides of (2.12) by z and by integrating on R^m_- one easily gets $\|z\| \leq m_0^{-1}|\psi|$. Using this estimate in Equation (2.13) one proves (2.10).

The above estimate

(2.14) $$\|z\| \leq m_0(t)^{-1}|D(t)z|, \quad \forall z \in Z, \ \forall t \in I/E,$$

will also be useful in Section 3.

Finally let $t,s \notin E$, $\psi \in X$ and $z = D(s)^{-1}\psi$. By the definitions one easily gets $z - \Delta z = h(s)^{-1}(\psi + \nabla h(s) \cdot \nabla z)$. Hence $D(t)z = h(t)h(s)^{-1}(\psi + \nabla h(s) \cdot \nabla z) - \nabla h(t) \cdot \nabla z$, consequently

(2.15) $$|D(t)z| \leq \frac{\|h(t)\|_{C^0}}{m_0(s)} [|\psi| + \|h(s)\|_{C^1} + \|z\|] + \|h(t)\|_{C^1}\|z\|.$$

By using now (2.14) at the point s one easily gets (2.11). □

For the reader's convenience we also state the following result:

__Lemma 2.4.__ __Let__ u __be an__ X__-valued measurable function on__ I __and let__ α __and__ β __be non-negative integrable functions on__ I. __Assume that for all__ $t \in I$ __and for almost all__ $s \in I$ __one has__

(2.16) $$|u(t) - u(s)| \leq |\int_s^t \alpha(\tau)d\tau| + |t-s|\beta(s).$$

__Then__ $u \in AC(I;X)$.

__Proof.__ Let E be the exceptional set of points s and let $t_0 \in]t,s[$, $t_0 \notin E$. Then $|u(t) - u(s)| \leq (|u(t) - u(t_0)| + |u(t_0) - u(s)|$ and by the hypothesis $|u(t) - u(s)| \leq \int_t^{t_0} \alpha(\tau)d\tau + (t_0-t)\beta(t_0) + \int_{t_0}^s \alpha(\tau)d\tau + (s - t_0)\beta(t_0)$, hence

$$|u(t)-u(s)| \leq \int_t^s \alpha(\tau)d\tau + (s - t)\beta(t_0).$$

Integrating with respect to t_0 over $[t,s]$ one easily gets $|u(t) - u(s)| \leq \int_t^s [\alpha(\tau) + \beta(\tau)]d\tau$. □

3. The homogeneous problem. In the following the expressions "differentiable solution" and "strong solution" will be used in the following senses: let $f \in L^1(I;X)$ and $u^0 \in X$. A function u is a differentiable solution of problem (1.1), (1.2) if: (i) $u \in C^0(I;Y) \cap AC(I;X)$; (ii) u solves equation (1.1). Recall that $u'(t)$ exists and is in $L^1(I;X)$. A function u is a strong solution of problem (1.1), (1.2) if: (i) $u \in C^0(I;X)$; (ii) for every pair of sequences $f^{(\ell)} \in L^1(I;Y)$, $u_0^{(\ell)} \in Y$ such that $f^{(\ell)} \to f$ in $L^1(I;X)$ and $u_0^{(\ell)} \to u_0$ in X there exists a classical solution $u^{(\ell)}$ of problem (1.1), (1.2) with data $f^{(\ell)}$, $u_0^{(\ell)}$ such that $u^{(\ell)} \to u$ in $C^0(I;X)$. We will use indifferently the notations $\frac{du}{dt}$, u' and $d_t u$. We begin with the uniqueness result:

Proposition 3.1. Let $f \in L^1(I;X)$, $u_0 \in X$ and let $u \in L^1(I;X)$ be a solution of (1.1), (1.2) in the following sense: there exist sequences $f^{(\ell)} \in L^1(I;X)$ and $u_0^{(\ell)} \in X$ such that $f^{(\ell)} \to f$ in $L^1(I;X)$, $u_0^{(\ell)} \to u_0$ in X and the problem $d_t u^{(\ell)}(t) + (L+B)u^{(\ell)}(t) = f^{(\ell)}(t)$, $u^{(\ell)}(0) = u_0^{(\ell)}$ has a solution $u^{(\ell)} \in L^1(I;Y) \cap AC(I;X)$ such that $u^{(\ell)} \to u$ in $L^1(I;X)$.

Then u is the unique solution (in the sense described above) of problem (1.1), (1.2).

Proof: Let u and v be two solutions and let $f^{(\ell)}$, $u_0^{(\ell)}$, $u^{(\ell)}$ and $g^{(\ell)}$, $v_0^{(\ell)}$, $v^{(\ell)}$ be as in the above statement. By putting $w^{(\ell)} = u^{(\ell)} - v^{(\ell)}$, by multiplying scalarly in X_t both sides of $d_t w^{(\ell)} + (L+B)w^{(\ell)} = f^{(\ell)} - g^{(\ell)}$ by $w^{(\ell)}$, and by using (2.5) one easily gets

$$\tfrac{1}{2}\frac{d}{dt}\,|w^{(\ell)}|_t^2 \leq \tilde{\lambda}_0(t)|w^{(\ell)}|_t^2 + |f^{(\ell)} - g^{(\ell)}|_t |w^{(\ell)}|_t + \tfrac{1}{2}\|h'(t)\|_{C^0}|w^{(\ell)}|_t^2 .$$

By using Gronwall's lemma and by passing to the limit when $\ell \to +\infty$, the proposition follows. □

We now prove the existence of a differentiable solution.

Theorem 3.2. *Assume that the conditions described in section* 1 *concerning the operators* L *and* B *hold. Let* $u_0 \in Y$ *and* $f \in L^1(I;Y)$ *be given. Then there exists a (unique) differentiable solution* $u \in C^0(I;Y) \cap AC(I;X)$ *of problem* (1.1), (1.2). *Moreover for each* $t \in I$

$$(3.1)\quad \|u(t)\|_t \leq e^{\left|\int_0^t [\tilde{\lambda}_0(\tau) + \frac{1}{2} m_0(\tau)^{-1} \|h'(\tau)\|_{C^0}] d\tau\right|} \cdot \{\|u_0\|_0 + \|f\|_{L^1(0,t;Y_t)}\},$$

hence in particular

$$(3.2)\quad \|u(t)\| \leq \left[\frac{\|h(0)\|_{C^0}}{m_0(t)}\right]^{1/2} e^{\left|\int_0^t [\tilde{\lambda}_0(\tau) + m_0(\tau)^{-1} \|h'(\tau)\|_{C^0}] d\tau\right|} \cdot \{\|u_0\| + \|f\|_{L^1(0,t;Y)}\}.$$

Finally if $f \in L^1(I;Y) \cap L^q(I;X)$, $1 \leq q \leq +\infty$, *then* $u \in W^{1,q}(I;X)$; *if* $f \in L^1(I;Y) \cap C^0(I;X)$ *then* $u \in C^1(I;X)$. *The corresponding estimates are obvious.*

For convenience some lemmas will be stated during the proof. Denote by $\{a_s\}$, $s \in \mathbf{N}$ a base of Z and put

$$(3.3)\qquad u^{(\ell)}(t) = \sum_{s=1}^{\ell} c_s^{(\ell)}(t)\, a_s\,, \quad \ell \in \mathbf{N}\,.$$

Select the real functions $c_s^{(\ell)}(t)$, $s = 1,\cdots,\ell$, as the solutions of the linear non-homogeneous system of ℓ ordinary differential equations $((d_t u^{(\ell)}(t), a_r))_t + (((L+B)u^{(\ell)}(t), a_r))_t = ((f, a_r))_t$, $r = 1,\cdots,\ell$, with initial data $((u^{(\ell)}(0), a_r))_0 = ((u_0, a_r))_0$ or equivalently

$$(3.4)\quad \begin{cases} \sum_{s=1}^{\ell} ((a_s, a_r))_t \frac{d}{dt} c_s^{(\ell)}(t) + \sum_{s=1}^{\ell} (((L+B)a_s, a_r))_t c_s^{(\ell)}(t) = ((f, a_r)), \\ \sum_{s=1}^{\ell} ((a_s, a_r))_0 c_s^{(\ell)}(0) = ((u_0, a_s))_0, \qquad r = 1,\cdots,\ell. \end{cases}$$

The coefficients $((a_s, a_r))_t$ belong to $C^0(I;\mathbf{R})$, the corresponding matrix is invertible and $(((L+B)a_s, a_r)) \in L^1(I;\mathbf{R})$. Consequently problem (2.5) is uniquely solvable in I and the solution $(c_1^{(\ell)}(t),\cdots,c_\ell^{(\ell)}(t))$ is an $\mathbf{R}^\ell$-valued absolutely continuous function in I. By multiplying $(3.4)_1$ by

$c_r^{(\ell)}(t)$ and by adding in r one gets

$$((\frac{d}{dt}u^{(\ell)},u^{(\ell)}))_t + (((L+B)u^{(\ell)},u^{(\ell)}))_t = ((f,u^{(\ell)}))_t \tag{3.5}$$

for each $\ell \in \mathbf{N}$. On the other hand

$$\tfrac{1}{2}\frac{d}{dt}\|u^{(\ell)}(t)\|_t^2 = ((\frac{d}{dt}u^{(\ell)}(t),u^{(\ell)}(t)))_t + \tfrac{1}{2}((u^{(\ell)}(t),u^{(\ell)}(t)))_{h'(t)}. \tag{3.6}$$

Moreover

$$|((u,u))_{h'(t)}| \leq m_0(t)^{-1}\|h'(t)\|_{C^0}\|u\|_t^2, \quad \forall\, u \in Y. \tag{3.7}$$

From (3,5), (3.6), (3.7) and (2.6) one finally gets

$$|\tfrac{1}{2}\frac{d}{dt}\|u^{(\ell)}(t)\|_t^2| \leq \{\tfrac{1}{2}m_0(t)^{-1}\|h'(t)\|_{C^0} + \tilde{\lambda}_0(t)\}\|u^{(\ell)}(t)\|_t^2 + \tag{3.8}$$

$$+ \|f(t)\|_t\|u^{(\ell)}(t)\|_t.$$

Hence by comparison theorems

$$\|u^{(\ell)}(t)\|_t \leq e^{|\int_0^t [\tilde{\lambda}_0(\tau) + \frac{1}{2}\frac{\|h'(\tau)\|_{C^0}}{m_0(\tau)}]d\tau|}\{\|u_0\|_0 + \|f\|_{L^1(0,t;Y_t)}\} \tag{3.9}$$

for every $t \in I$; note that $\|u^{(\ell)}(0)\|_0 \leq \|u_0\|_0$. Let L be the linear operator defined by $(Lu)(t) = L(t)u(t)$ a.e. in I. By (2.1), it follows that L is a bounded operator from $L^\infty(I;Y)$ into $L^\infty(I;X)$. Hence (3.9) implies the existence of a subsequence $u^{(\nu)}$ and of elements $u \in L^\infty(I;Y)$ and $w \in L^\infty(I;X)$ such that

$$\begin{cases} u^{(\nu)} \to u & \text{weak-* in } L^\infty(I;Y), \\ (L+B)u^{(\nu)} \to w & \text{weak-* in } L^\infty(I;X). \end{cases} \tag{3.10}$$

We utilize above a well known result on functional analysis; see for instance [8], Chap. V, Section 1, Theorem 10. Recall also that $L^1(I;X')' = L^\infty(I;X)$, where X' is the dual of the Hilbert space X. We remark that if we replace L^1 by L^2 in the assumptions (1.7), (1.8) concerning $h(t)$ and $h'(t)$ then in (3.10) it suffices to use the weak convergence in L^2.

On the other hand from $(3.4)_1$ and (2.8) it easily follows for each $r \leq \nu$ and each $t \in I$, that

$$((u^{(\nu)}(t),a_r))_t = ((u^{(\nu)}(0),a_r))_0 + \int_0^t ((u^{(\nu)}(\tau),a_r))_{h'(\tau)}d\tau - \tag{3.11}$$

$$- \int_0^t ((L+B)u^{(\nu)}(\tau), D(\tau)a_r)d\tau + \int_0^t ((f(\tau),a_r))_\tau d\tau .$$

By passing to the limit in (3.11) one gets

(3.12) $$\lim_{\nu\to+\infty} ((u^{(\nu)}(t),a_r))_t = ((u_0,a_r))_0 + \int_0^t ((u(\tau),a_r))_{h'(\tau)}d\tau -$$

$$- \int_0^t ((w(\tau),D(\tau)a_r))d\tau + \int_0^t ((f(\tau),a_r))_\tau d\tau,$$

for each $r \in \mathbf{N}$ and each $t \in I$. Recall that $D(t)a_r \in L^1(I;X)$ thanks to (2.9) and (1.7). From (3.12) and (3.9) it follows that

(3.13) $$u^{(\nu)}(t) \to u(t) \quad \text{weakly in } Y, \quad \text{for every} \quad t \in I.$$

Hence by (2.1) $(L(t) + B(t))u^{(\nu)}(t) \to (L(t) + B(t))u(t)$ weakly in X, for each $t \in I$, consequently $w = (L+B)u$.

Since the left hand side of (3.12) is equal to $((u(t),a_r))_t$ a density argument in Z shows that

(3.14) $$((u(t),z))_t = ((u_0,z))_0 + \int_0^t ((u(\tau),z))_{h'(\tau)}d\tau -$$

$$- \int_0^t ((L+B)u(\tau), D(\tau)z)d\tau + \int_0^t ((f(\tau),z))_\tau d\tau, \quad \forall z \in Z.$$

Note that $u(0) = u_0$ by (3.13), $(3.4)_2$; moreover (3.13) and (3.9) yields (3.1).

Furthermore an easy computation shows that

(3.15) $$((v,z))_t = ((v,z))_s + \int_s^t ((v,z))_{h'(\tau)}d\tau, \quad \forall v,z \in Y,$$

for every $t,s \in I$, consequently (3.14) yields the following statement:

Lemma 3.3. *Let u be the function obtained above. Then for every pair $s,t \in I$ one has*

(3.16) $$((u(t),z))_s - ((u(s),z))_s = \int_s^t ((u(\tau),z))_{h'(\tau)}d\tau -$$

$$- \int_s^t ((u(t),z))_{h'(\tau)}d\tau - \int_s^t ((L+B)u(\tau),D(\tau)z)d\tau + \int_s^t ((f(\tau),z))_\tau d\tau,$$

for every $z \in Z$.

Corollary 3.4. *The function u is a Y-valued weakly continuous function in I*$^{(6)}$.

Proof. From (3.16) one has $\lim_{t\to s}((u(t),z))_s = ((u(s),z))_s$, $\forall z \in Z$. Moreover Z is dense in Y_s and $u(t)$ is uniformly bounded in Y_s. Hence when $t \to s$ one has $u(t) \rightharpoonup u(s)$ weakly in Y_s hence in Y.

Proposition 3.5. The function u is an X-valued absolutely continuous function in I, i.e.

(3.17) $$u \in AC(I;X).$$

Proof. In the sequel we denote by K_0 different constants which will not be specified. Since $u \in L^\infty(I;Y)$ one easily gets, by using (2.14), that for each $s \in I/E$:

(3.18) $$\left|\int_s^t ((u(\tau),z))_{h'(\tau)}d\tau\right| \leq K_0\left|\int_s^t \|h'(\tau)\|_{C^0}d\tau\right| \; |D(s)z|,$$

(3.19) $$\left|\int_s^t ((u(t),z))_{h'(\tau)}d\tau\right| \leq K_0\left|\int_s^t \|h'(\tau)\|_{C^0}d\tau\right| \; |D(s)z|,$$

(3.20) $$\left|\int_s^t((f(\tau),z))_\tau d\tau\right| \leq K_0\left|\int_s^t \|f(\tau)\|d\tau\right| \; |D(s)z|.$$

Moreover

$$\left|\int_s^t ((L+B)u(\tau),D(\tau)z)d\tau\right| \leq \|(L+B)u\|_{L^\infty(I;X)} \left|\int_s^t \|D(\tau)D(s)^{-1}\|_{\mathcal{L}[X;X]}\right| \; |D(s)z|.$$

Hence by preceding estimates and (2.11)

(3.21) $$\left|\int_s^t((L+B)u(\tau),D(\tau)z)d\tau\right| \leq \{K_0(t-s)[1 + \|h(s)\|_{C^1}] + K_0\left|\int_s^t\|h(\tau)\|_{C^1}d\tau\right|\} \quad |D(s)z|,$$

for each $s \in I/E$. From (3.16) and from the above estimates it follows that

(3.22) $$|((u(t)-u(s),z))_s| \leq \{\left|\int_s^t \alpha(\tau)d\tau\right| + |t-s|\beta(s)\} \; |D(s)z|, \; \forall t \in I, \; \forall s \in I/E,$$

where α and β are real non-negative integrable functions in I. On the other hand a classical result on functional analysis yields

$$|u(t)-u(s)| = \sup_{z\in Z} \frac{|(u(t)-u(s),D(s)z)|}{|D(s)z|}, \quad \forall s \in I/E,$$

(6) Actually u is strongly continuous; see Lemma 3.7 below.

since $D(s)Z = X$. Since $((u(t)-u(s),z))_s = (u(t)-u(s),D(s)z)$ one gets $|u(t) - u(s)| \leq |\int_s^t \alpha(\tau)d\tau| + |t-s|\beta(s)$; by Lemma 2.4 the statement follows.

□

<u>Lemma 3.6.</u> <u>The function</u> u <u>verifies equation</u> $(1.1)_1$ <u>almost everywhere.</u>

<u>Proof.</u> By (3.17) the derivative $u'(t)$ exists and is in $L^1(I;X)$. Hence it suffices to verify that there exists a subset $E_1 \subset I$ of zero Lebesgue measure such that for each $z \in Z$

$$\lim_{t \to s} \left(\frac{u(t)-u(s)}{t-s}, D(s)z\right) = ((L+B)u(s)-f(s),D(s)z), \quad \forall\, t \in I/E_1 . \tag{3.23}$$

Let us verify (3.23). From the definitions one has

$$\frac{1}{t-s}\int_s^t ((u(t),z))_{h'(\tau)}d\tau = (u(t), \frac{1}{t-s}\int_s^t h'(\tau)d\tau : z) \tag{3.24}$$

$$+ \sum_{J=1}^{m} (\frac{\partial}{\partial x_J} u(t), \frac{1}{t-s}\int_s^t h'(\tau)d\tau : \frac{\partial z}{\partial x_J}),$$

where for convenience $a:z$ is the vector with components $(a:z)_k = a_k v_k$, $k = 1,\cdots,n$. On the other hand $\frac{\partial}{\partial x_J} u(t)$ is X-valued weakly continuous in I since $u(t)$ is Y-valued weakly continuous and $\frac{\partial}{\partial x_J} \in L[Y;X]$. Moreover by (1.8) and by a well known generalization of a Lebesgue's theorem there exists a subset $E_1 \subset I$ of zero measure such that

$$\lim_{t \to s} \frac{1}{t-s}\int_s^t h'(\tau)d\tau = h'(s), \quad \forall\, s \in I/E_1,$$

where the limit is in the $C^0(\mathbf{R}^m_-)$ norm. Hence by passing to the limit in (3.24) one gets

$$\lim_{t \to s} \frac{1}{t-s}\int_s^t ((u(t),z))_{h'(\tau)}d\tau = ((u(s),z))_{h'(s)}, \quad \forall\, s \in I/E_1 . \tag{3.25}$$

Now by dividing both sides of (3.16) by $t-s$, by using $((u(t)-u(s),z))_s = (u(t)-u(s),D(s)z)$ and by passing to the limit when $t \to s$ one gets (3.23). Note that the functions in the first, third and fourth integrals on the right hand side of (3.16) are integrable on I. □

Lemma 3.7. The function u is in $C(I;Y)$.

Proof. From (3.1) one gets $\limsup_{t\to 0}\|u(t)\|_t \leq \|u(0)\|_0$ and by (3.26) below it follows that $\limsup_{t\to 0}\|u(t)\|_0 \leq \|u(0)\|_0$; hence Corollary 3.4 yields $u(t) \to u(0)$ strongly in Y_0, hence in Y. Now from the uniqueness of the solution it follows the strong continuity of $u(t)$ in every point $t \in I$. []

The last statement in Theorem 3.2 follows directly from equation (1.1), as well as the corresponding estimates. Hence it remains only to show (3.2). This estimate follows easily from (3.1) and from the estimates

$\|v\|^2 \leq m_0(t)^{-1}\|v\|_t^2$, $\|v\|_0^2 \leq \|h(0)\|_{C^0}\|v\|^2$ and

$$\|v\|_t^2 \leq e^{\left|\int_s^t m_0(\tau)^{-1}\|h'(\tau)\|_{C^0}d\tau\right|}\|v\|_s^2, \quad \forall\, v \in Y. \tag{3.26}$$

To verify (3.26) use the estimate

$$\left|\frac{d}{dt}\|v\|_t^2\right| \leq \frac{\|h'(t)\|_{C^0}}{m_0(t)}\|v\|_t^2 .$$

Note by the way that (3.26) also holds with $\|v\|_t$ and $\|v\|_s$ replaced by $|v|_t$ and $|v|_s$ respectively. Theorem 3.2 is proved. □

Remark 3.8. If h is not time dependent a shorter proof of Theorem 3.2 is obtained by showing that (notation of Part I)

$$((u(t),a_r))_h = ((u_0,a_r))_h + \int_0^t (f(\tau)-(L+B)u(\tau),Da_r)d\tau, \quad \forall\, r \in \mathbf{N};$$

from this equation one easily sees that u is a solution because it verifies the integral equation

$$u(t) = u_0 + \int_0^t [f(\tau) - (L+B)u(\tau)]d\tau, \quad \forall\, t \in I.$$

The existence result for this particular case can also be utilized to get the existence for the general case; to this end one extends it successively to the following cases: (i) $h(t)$ is a piecewise constant function (respect to t) with values in $[C^1(\mathbf{R}^m_-)]^n$; (ii) $h(t) \in C^0(I;[C^1(\mathbf{R}^m_-)]^n)$; (iii) $h(t) \in L^1(I;[C^1(\mathbf{R}^m_-)]^n)$.

We now prove the existence of strong solutions for the homogeneous boundary value problem.

Theorem 3.9. *Let $f \in L^1(I;X)$ and $u_0 \in X$. Then there exists a unique strong solution u of problem (1.1), (1.2). Moreover u satisfies the estimate*

$$(3.27) \qquad |u(t)|_t \leq e^{|\int_0^t [\tilde{\lambda}_0(\tau) + \frac{1}{2} m_0(\tau)^{-1} \|h'(\tau)\|_{C^0}] d\tau|} \cdot \{|u_0|_0 + \|f\|_{L^1(0,t;X_t)}\},$$

and also the estimate (3.2) with $\|u(t)\|$, $\|u_0\|$ and $\|f\|_{L^1(0,t;Y)}$ replaced by $|u(t)|$, $|u_0|$ and $\|f\|_{L^1(0,t;X)}$ respectively.

Proof. Let $u_0^{(\ell)} \in Y$ and $f^{(\ell)} \in L^1(I;Y)$, $\ell \in \mathbf{N}$, be such that $u_0^{(\ell)} \to u_0$ strongly in X and $f^{(\ell)} \to f$ strongly in $L^1(I;X)$. Let $u^{(\ell)} \in C^0(I;Y) \cap AC(I;X)$ be the solution of $d_t u^{(\ell)} + (L+B)u^{(\ell)} = f^{(\ell)}$, $u^{(\ell)}(0) = u_0^{(\ell)}$. By multiplying both sides of the last equation (scalarly in X_t) by $u^{(\ell)}$ one gets $(d_t u^{(\ell)}(t), u^{(\ell)}(t))_t + ((L+B)u^{(\ell)}(t), u^{(\ell)}(t))_t = (f^{(\ell)}(t), u^{(\ell)}(t))_t$. On the other hand

$$\frac{1}{2}\frac{d}{dt}|u^{(\ell)}(t)|_t^2 = (\frac{du^{(\ell)}}{dt}, u^{(\ell)})_t + \frac{1}{2}(u^{(\ell)}, u^{(\ell)})^2_{h'(t)},$$

moreover $|(u,u)_{h'(t)}| \leq m_0(t)^{-1}\|h'(t)\|_{C^0}|u|_t^2$, $\forall\, u \in Y$. Hence, as for (3.9), one easily gets the estimate (3.27) for the approximating solutions $u^{(\ell)}(t)$. By applying this last estimate to $u^{(\ell)}(t) - u^{(k)}(t)$ one shows that $u^{(\ell)}$ is a Cauchy sequence in $C^0(I;X)$. Hence there exists $u \in C^0(I;X)$ such that $u_n \to u$ in $C^0(I;X)$. Clearly (3.27) holds for $u(t)$. Finally the last statement in Theorem 3.9 follows from (3.27) by using formulae (3.26) with $\|v\|_t$ and $\|v\|_s$ replaced by $|v|_t$ and $|v|_s$ respectively. □

4. The non-homogeneous problem. We recall some usual notations and results. If $\psi \in H^1(\mathbf{R}^m_-)$ we denote by $\gamma_0\psi$ the trace of ψ on $\mathbf{R}^{m-1}$. If $\psi \in H^2(\mathbf{R}^m_-)$ we denote by $\gamma_1\psi$ the trace of the normal derivative of ψ on $\mathbf{R}^{m-1}$. It is well known that $\gamma_0 \in L[H^1(\mathbf{R}^m_-);\ H^{1/2}(\mathbf{R}^{m-1})]$, $\gamma_0 \in L[H^2(\mathbf{R}^m_-);\ H^{3/2}(\mathbf{R}^{m-1})]$, $\gamma_1 \in L[H^2(\mathbf{R}^m_-);\ H^{1/2}(\mathbf{R}^{m-1})]$. It is also well known that there exists (non unique) linear continuous right inverses γ_0^{-1} and γ_1^{-1} in the above spaces.

In the sequel, boundary values are always to be understood in the trace sense.

We start by studying the existence of strong solutions for the non-homogeneous problem (1.1), (1.3). Assume that $w = (w_1,\cdots,w_p) \in L^1(I;H_p^{1/2})$ is the trace on the boundary of a function $\bar{w} = (\bar{w}_1,\cdots,\bar{w}_p)$ such that

$$(4.1)\qquad \begin{cases} \bar{w} \in L^1(I;[H^1(R^m_-)]^p), \\ \bar{w}' \in L^1(I;[L^2(\mathbf{R}^m_-)]^p). \end{cases}$$

We also denote by $\bar{w}$ the n-dimensional vector field $\bar{w} = (\bar{w}_1,\cdots,\bar{w}_p,0,\cdots,0)$; clearly $\bar{w} \in L^1(I;\mathcal{Y})$ and $\bar{w}' \in L^1(I;X)$. Note that from (4.1) it follows that $\bar{w} \in C^0(I;X)$, consequently $\bar{w}(0) \in X$. By carrying out the change of variables $u = v + \bar{w}$ in (1.1), (1.3) our problem is equivalent to proving the existence of a strong solution v of the homogeneous boundary value problem $v' + (L+B)v = f - [\bar{w}' + (L+B)\bar{w}]$ with initial condition $v(0) = u_0 - \bar{w}(0)$. Since $f - [\bar{w}' + (L+B)\bar{w}] \in L^1(I;X)$ and $u_0 - \bar{w}(0) \in X$ this problem has a unique solution, due to Theorem 3.9. Consequently the non-homogeneous problem (1.1), (1.3) has a strong solution $u = v + \bar{w}$ in the above sense. Clearly this solution satisfies the following property:

(S) There exist sequences $f^{(\ell)} \in L^1(I;X)$, $u_0^{(\ell)} \in X$ and $u^{(\ell)} \in L^1(I;\mathcal{Y}) \cap AC(I;X)$ such that $P_N^{\perp} u^{(\ell)} = w$ on $I \times R^{m-1}$, $d_t u^{(\ell)} + (L+B)u^{(\ell)} = f^{(\ell)}$, $u^{(\ell)}(0) = u_0^{(\ell)}$, $u_0^{(\ell)} \to u_0$ in X, $f^{(\ell)} \to f$ in $L^1(I;X)$ and $u^{(\ell)} \to u$ in $C^1(I;X)$.

Note that from Proposition 3.1 it follows that a strong solution of problem (1.1), (1.3) in the sense (S) is unique. Hence we proved the following result:

Theorem 4.1. Let $u_0 \in X$, $f \in L^1(I;X)$ and $w \in L^1(I;H_p^{1/2})$. Moreover assume that w verifies the regularity assumption (4.1). Then there exists a unique strong solution u of the non-homogeneous boundary value problem (1.1), (1.3). Moreover $u \in C^0(I;X)$.

We leave to the reader the deduction of the estimate of $|u(t)|$ in terms of the data.

We are looking now for differentiable solutions for the non-homogeneous problem. For the sake of simplicity we assume from now on that for each index i,k,J,

$$(4.2) \qquad h_k\ ,\ a_{ik}^J \in L^\infty(I;C^1(\overline{R}_-^m)).$$

We start by remarking that a statement similar to Theorem 4.1 fails for differentiable solutions. In fact besides the usual compatibility condition (4.7) between u_0 and w also a compatibility condition between w and the boundary values of f is needed (for any time t). For, assume that $u_0 \in \mathcal{Y}$, $f \in L^2(I;\mathcal{Y})$ and $w \in L^2(I;H_p^{3/2})$ with $w' \in L^2(I;H_p^{1/2})$. Let u be a differentiable solution of problem (1.1), (1.3) and assume u smooth in the following sense: $u \in L^2(I;Z)$ with $u' \in L^2(I;\mathcal{Y})$. Define

$$(4.3) \qquad \mathcal{F}[w] \equiv w' + \sum_{J=2}^{m} H_p^{-1} R^J \frac{\partial w}{\partial x_J} + B_1 w.$$

$\mathcal{F}$ is a bounded linear operator from $\mathcal{A} \equiv \{w : w \in L^2(I;H_p^{3/2}),\ w' \in L^2(I;H_p^{1/2})\}$ into $L^2(I;H_p^{1/2})$. By restricting equation (1.1) to the boundary, by applying $P_N^\perp$ to both sides and by using (I.4.3) one gets

$$(4.4) \qquad H_p^{-1} M\left(\frac{\partial u_{p+1}}{\partial x_1}, \dots, \frac{\partial u_n}{\partial x_1}\right) = F[(f,w)],$$

where by definition[7]

$$(4.5) \qquad F[(f,w)] \equiv P_N^\perp f - \mathcal{F}[w].$$

---

[7] Note by the way that the map $(f,w) \to F[(f,w)]$ is linear-continuous from $L^2(I;\mathcal{Y}) \times \mathcal{A}$ to $L^2(I;H_p^{1/2})$.

Hence if there exists a smooth solution u *of problem* (1.1), (1.3) *the equation*

$$H_p^{-1} M(g_{p+1}, \cdots, g_n) = F[(f,w)], \tag{4.6}$$

admits at least one solution $g \in L^2(I; H_p^{1/2})$. Conversely, one has

Theorem 4.2. Let $u_0 \in \mathcal{Y}$ *and* $f \in L^2(I; \mathcal{Y})$ *verify*

$$P_N^{\perp} u_0 = w(0) \quad \text{on } \mathbf{R}^{m-1}, \tag{4.7}$$

and let $w \in \mathcal{A}$, *i.e.*

$$w \in L^2(I; H_p^{3/2}) \ \text{with} \ w' \in L^2(I; H_p^{1/2}). \tag{4.8}$$

Assume that (4.6) *has at least one solution* $g \in L^2(I; H_p^{1/2})$ *satisfying the additional regularity assumption: there exists* $\bar{g}$ *such that*

$$g = \frac{\partial \bar{g}}{\partial x_1} \tag{4.9}$$

with

$$\bar{g} \in L^2(I; [H^2(\mathbf{R}_-^m)]^{n-p}), \quad \bar{g}' \in L^2(I; [H^1(\mathbf{R}_-^m)]^{n-p})^{(8)}. \tag{4.10}$$

Then problem (1.1), (1.3) *admits a (unique) differentiable solution* $u \in C^0(I; \mathcal{Y}) \cap AC(I; X)$, $u' \in L^2(I; X)$ *and additionally*

$$\|u(t)\| \leq \Big\{1 + \Big[\frac{\|h(0)\|_{C^0}}{m_0(t)}\Big]^{1/2} e^{\left|\int_0^t [\tilde{\lambda}_0(\tau) + m_0(\tau)^{-1} \|h'(\tau)\|_{C^0}] d\tau\right|}\Big\} \cdot \tag{4.11}$$

$$\cdot \{\|u_0\| + \|\tilde{u}(0)\| + t^{1/2} [\|f\|_{L^2(0,t;\mathcal{Y})} + \|\tilde{u}'\|_{L^2(0,t;\mathcal{Y})} +$$

$$+ c(\|\tilde{A}\|_{L^\infty(0,t;C^1)} + \|B\|_{L^\infty(0,t;C^1)}) \|\tilde{u}\|_{L^2(0,t;Z)}]\}.$$

The function $\tilde{u}$ *depends only on the data (see below) and its norms can be estimated from those of the data.*

(8) Note that for every $g \in L^2(I; H_p^{1/2})$ there exists $\bar{g}$ such that (4.9) and $(4.10)_1$ hold. Hence the additional regularity assumption is only $(4.10)_2$.

Proof. Define $\tilde{u}$ by $\tilde{u}_J \equiv \gamma_0^{-1} w_J$ if $J = 1,\cdots,p$, and $\tilde{u}_J \equiv \bar{g}_J$ if $J = p+1,\cdots,n$. Clearly $P_N^{\perp}\tilde{u} = w$ on the boundary, moreover

$$(4.12)\quad \begin{cases} \|\tilde{u}\|_{L^2(I;Z)} + \|\tilde{u}'\|_{L^2(I;\mathcal{Y})} \leq c\Big[\|w\|_A + \|\bar{g}\|_{L^2(I;[H^2]^{n-p})} + \|\bar{g}'\|_{L^2(I;[H^1]^{n-p})}\Big], \\ \\ \|\tilde{u}(0)\| \leq c(\|\bar{g}(0)\|_{[H^1]^{n-p}} + \|u_0\|). \end{cases}$$

By carrying out the change of variables $u = v + \tilde{u}$ our problem can be written as $v' + (L+B)v = f - [\tilde{u}' + (L+B)\tilde{u}]$, with the initial condition $v(0) = u_0 - \tilde{u}(0)$ and the boundary condition $P_N^{\perp}v = 0$. By using (I.4.3), (4.9) and the definition of $\tilde{u}$ one easily gets

$$P_N^{\perp}[f - (\tilde{u}' + (L+B)\tilde{u})] = P_N f - [H_p^{-1} M(g_{p+1},\cdots,g_n) +$$
$$+ \sum_{J=2}^{m} H_p^{-1} R^J \frac{\partial w}{\partial x_j} + B_1 w + \frac{\partial w}{\partial t}] \quad \text{on } \mathbf{R}^{m-1},$$

for almost all $t \in I$. Recalling definitions (4.3), (4.5) and Equation (4.6) one gets $P_N^{\perp}[f - (\tilde{u}' + (L+B)\tilde{u})] = 0$. Hence $f - [\tilde{u}' + (L+B)\tilde{u}] \in L^2(I;Y)$ and by Theorem 3.2 there exists a (unique) differentiable solution v for the above homogeneous problem, $v \in C^0(I;Y) \cap AC(I;X)$ with $v' \in L^2(I;X)$. Clearly $u = \tilde{u} + v$ is a differentiable solution of the non-homogeneous problems (1.1), (1.3). Since $\|\tilde{u}(t)\| \leq \|\tilde{u}(0)\| + t^{1/2}\|\tilde{u}'\|_{L^2(0,t;\mathcal{Y})}$ and $\|v(t)\|$ verifies (3.2) with $\|v(0)\| \leq \|u_0\| + \|\tilde{u}(0)\|$ and with

$$(4.13)\quad \|f - [\tilde{u}' + (L+B)\tilde{u}]\|_{L^2(I;Y)} \leq \|f\|_{L^2(I;\mathcal{Y})} + \|\tilde{u}'\|_{L^2(I;\mathcal{Y})} +$$
$$+ [\|\tilde{A}\|_{L^\infty(I;C^1)} + \|B\|_{L^\infty(I;C^1)}]\|\tilde{u}\|_{L^2(I;Z)},$$

one easily gets (4.10). □

Consider now the matrix $M(t,x)$. We say that rank $M=p$ uniformly on $I \times \mathbf{R}^{m-1}$ if the sum of the squares of the determinants of order p contained in $M(t,x)$ is bounded below by a positive constant d^2 independent of $(t,x) \in I \times \mathbf{R}^{m-1}$. We prove the following result:

Corollary 4.3. *Let* $u_0 \in \mathcal{Y}$, $f \in L^2(I;\mathcal{Y})$ *with* $f' \in L^2(I;X)$ *and*

$$w \in [H^{3/2}(I \times \mathbf{R}^{m-1})]^p. \tag{4.14}$$

Assume that (4.7) *holds and that*

$$\text{rank } M(t,x) = p, \quad \text{uniformly on} \quad I \times \mathbf{R}^{m-1}. \tag{4.15}$$

Then the non-homogeneous initial-boundary value problem (1.1), (1.3) *has a (unique) differentiable solution* $u \in C^0(I;\mathcal{Y}) \cap AC(I;X)$ *with* $u' \in L^2(I;X)$. *An estimate for* $\|u(t)\|$ *follows from* (4.10) *and from the explicit construction below.*

Proof. Condition (4.14) means that w is the trace on $I \times \mathbf{R}^{m-1}$ of a function $\tilde{w}$ verifying

$$\begin{cases} \tilde{w} \in L^2(I;[H^2(\mathbf{R}^m_-)]^p), \\ \tilde{w}' \in L^2(I;[H^1(\mathbf{R}^m_-)]^p), \\ \tilde{w}'' \in L^2(I;[L^2(\mathbf{R}^m_-)]^p). \end{cases} \tag{4.16}$$

Consider for each $(t,x) \in I \times \mathbf{R}^m_-$ the equation

$$H_p^{-1} M(\tilde{g}_{p+1},\dots,\tilde{g}_n) = P_N^{\perp} f - \left(\frac{\partial \tilde{w}}{\partial t} + \sum_{J=2}^{m} H_p^{-1} R^J \frac{\partial \tilde{w}}{\partial x_J} + B_1 \tilde{w}\right), \tag{4.17}$$

where the unknown is the vector field $\tilde{g} = (\tilde{g}_{p+1},\dots,\tilde{g}_n)$ defined in $I \times \mathbf{R}^m_-$. In Equation (4.17) the matrix M is extended from the boundary to the interior by shifting it in the negative x_1 direction. In particular property (4.15) holds uniformly on $I \times \mathbf{R}^m_-$ (9). To solve system (4.17) in $I \times \mathbf{R}^m_-$ we argue as in the proof of Corollary I.4.4 (see also part I for notations); one easily shows that the right hand side $\tilde{F}$ of (4.17) verifies $H_p\tilde{F} \in L^2(I;[H^1(\mathbf{R}^m_-)]^p)$, $(H_p\tilde{F})' \in L^2(I;[L^2(\mathbf{R}^m_-)]^p$. Hence by using the explicit formula (I.4.16) one gets $\tilde{g} \in L^2(I;[H^1(\mathbf{R}^m_-)]^{n-p})$, $\tilde{g}' \in L^2(I;[L^2(\mathbf{R}^m_-)]^{n-p})$ or equivalently $\tilde{g} \in [H^1(I \times \mathbf{R}^m_-)]^{n-p}$.

Clearly $g = \gamma_0\tilde{g}$ verifies equation (4.17) on the boundary, moreover the right hand side of (4.17) coincides on the boundary with the right hand side

(9) One could also let the coefficients have their original values in the interior and argue in a neighbourhood of the boundary.

of (4.6); hence $g = \gamma_0 \tilde{g}$ verifies equation (4.6) and the result follows from Theorem 4.2. Note that the existence of $\bar{g}$ verifying (4.9), (4.10) follows from $g \in [H^{1/2}(I \times R^{m-1})]^{n-p}$. □

APPENDIX

5. This section concerns Remark 1.3. For convenience we take in account the time interval $[0,T]$; since Theorem I.3.1, Part I holds for $|\lambda| > \lambda_0$ a similar proof is valid for the backward problem in time.

Let $D(T) \subset X$ denote the domain of $\bar{L}(t)$, the closure in X of the operator $L(t)$: $Y \to X$. Clearly $Y \subset D(t)$. In the sequel we denote by $L(t)$ both the closure $\bar{L}(t)$ and the restriction of $L(t)$ to the subspace $\{u \in Y: L(t)u \in Y\}$. Let us put for brevity $\tilde{L}(t) \equiv L(t) + B(t)$.

For convenience we assume in the sequel that

(5.2) $$h_k,\ a^J_{ik},\ b_{ik} \in L^\infty(0,T;C^1(\mathbf{R}^m_-)).$$

It follows that

(5.3) $$\mu_0 \equiv \sup_{t\in[0,T]} \lambda_0(t) \leq c\{\frac{1}{m_0}\|A\|_{L^\infty(0,T;C^1)} + \frac{1}{m_0^2}\|A\|_{L^\infty(0,T;C^0)}\|h\|_{L^\infty(0,T;C^1)} + \|B\|_{L^\infty(0,T;C^1)}\},$$

where

$$m_0 \equiv \inf_{t\in[0,T]} m_0(t).$$

Define

$$M^2 \equiv m_0^{-1}\|h\|_{L^\infty(0,T;C^0)}\, e^{m_0^{-1}\|h'\|_{L^1(0,T;C^0)}}.$$

We shall prove that the family $\tilde{L}(t)$, $t \in [0,T]$, is (M,μ_0)-stable in Y and in X. Denote by $R(\lambda,t)$ the resolvent operator of $\tilde{L}(t)$ and let $0 \leq t_k < \cdots < t_1 \leq T$ ($k \in \mathbf{N}$, arbitrary) and $\lambda > \mu_0$. Then for every $u \in Y$ one has

$$K^2 \equiv \|\prod_{i=1}^k R(\lambda,t_i)u\|_Y^2 \leq \frac{1}{m_0}\|\prod_{i=1}^k R(\lambda,t_i)u\|^2_{Y_{t_1}}$$

and by using (I.3.2)

$$K^2 \leq \frac{1}{m_0}(\lambda-\mu_0)^{-2}\ \|\prod_{i=2}^k R(\lambda,t_i)u\|^2_{Y_{t_1}};$$

hence by (3.26)

$$K^2 \leq \frac{1}{m_0} (\lambda-\mu_0)^{-2} e^{\int_{t_2}^{t_1} m_0(\tau)^{-1} \| h'(\tau) \|_{C^0} d\tau} \| \prod_{i=2}^{k} R(\lambda,t_i) u \|^2_{Y_{t_2}} .$$

By repeating this argument one gets

$$K^2 \leq \frac{1}{m_0} (\lambda-\mu_0)^{-2k} e^{\int_{t_k}^{t_1} m_0(\tau)^{-1} \| h'(\tau) \|_{C^0} d\tau} \| u \|^2_{Y_{t_k}} \leq \frac{M^2}{(\lambda-\mu_0)^{2k}} \| u \|^2$$

hence

$$\| \prod_{i=1}^{k} R(\lambda,t_i) \|_{L[Y\to Y]} \leq \frac{M}{(\lambda-\mu_0)^k} .$$

The (M,μ_0)-stability in X is proved analogously. □

REFERENCES

1. H. Beirão da Veiga, "Homogeneous and non-homogeneous boundary value problems for first order linear hyperbolic systems arising in fluid-mechanics (Part I)", Comm. P. D. E. (to appear).

2. G. Da Prato - M. Iannelli, "On a method for studying abstract evolution equations in the hyperbolic case", Comm. P.D.E., 1 (1976), 585-608.

3. K. O. Friedrichs, "Symmetric positive linear differential equations", Comm. Pure Appl. Math. 11 (1958), 333-418.

4. T. Kato, "Quasi-linear equations of evolution, with applications to partial differential equations", Spectral theory and differential equations, Lecture Notes in Mathematics 448, Springer (1975), 25-70.

5. T. Kato, "The Cauchy problem for quasi-linear symmetric hyperbolic systems", Arch. Rat. Mech. Anal., 58 (1975), 181-205.

6. P. D. Lax and R. S. Phillips, "Local boundary conditions for dissipative symmetric linear differential operators", Comm. Pure Appl. Math. 13 (1960), 427-455.

7. L. Nirenberg, "On elliptic partial differential equations", Ann. Sc. Norm. Sup. Pisa 13 (1959), 115-162.

8. K. Yosida, "Functional Analysis" Springer (1974), Berlin-Heidelberg-New York.

Received July 1982