HBV-40

# LOCAL ENERGY INEQUALITY AND SINGULAR SET FOR WEAK SOLUTIONS OF THE BOUNDARY NON-HOMOGENEOUS NAVIER-STOKES PROBLEM

by H. Beirão da Veiga
Dipartimento di Matematica
Facolta' di Scienze
38050 - POVO (Trento) - Italy

Received May, 1985

## Introduction.

In reference [3], Caffarelli, Kohn and Nirenberg constructed a special class of global weak solutions (suitable weak solutions) for the boundary homogeneous ($b\equiv 0$) Navier-Stokes problem (0.1), whose main feature is to verify the local energy estimate (0.3). The main result in reference [3] is the proof that the singular set S of a suitable weak solution verifies $\mathcal{P}^1(S)=0$, hence $\mathcal{H}^1(S)=0$. Weaker results in the same direction were previously obtained by Sheffer [6], [7]. Here, S denotes the set of the singular points of u in $]0,T[\times\Omega$, $\mathcal{H}^1(S)$ is the one-dimensional Hausdorff measure of S in the four dimensional space $R_t \times R_x^3$, and $\mathcal{P}^1(S)$ is a measure of S, constructed by using space-time parabolic cylinders (see [3], for details).

A simplified, and quite general, construction of suitable weak solutions for the boundary homogeneous problem (0.1) was given in [2] (see also [1]). In this note we utilize the method given in [2] to show that also the non-homogeneous boundary problem (0.1) admits a suitable weak solution, i.e., a weak solution which satisfies the local energy inequality (0.3). Consequently, by using the main result in [3], one has $\mathcal{P}^1(S)=\mathcal{H}^1(S)=0$. This is stated in theorem A below.

Let us now introduce some notation. Let $\Omega$ be an open bounded subset of $R^3$, locally situated on one side of his boundary $\Gamma$, a differentiable manifold of class $C^2$. Let $n$ be the unit outward normal to $\Gamma$. Moreover, $\Omega_t \equiv \{t\} \times \Omega$, $Q_t \equiv ]0,t[ \times \Omega$, $\Sigma_t \equiv ]o,t[ \times \Gamma$. T is an arbitrary positive number.

Let $L^p \equiv L^p(\Omega,R)$ and $\mathbb{L}^p \equiv \mathbb{L}^p(\Omega, R^3)$. The norms in both spaces are denoted by $|\ |_p$. Similarly, $\|\ \|_{s,p}$ denotes the usual norm in the Sobolev spaces $W^s_p \equiv W^s_p(\Omega,R)$ and $\mathbb{W}^s_p \equiv W^s_p(\Omega, R^3)$.

For a positive integer $k$ we set $\mathbb{H}^k \equiv \mathbb{W}^k_2$. $\mathbb{H}^k_0$ denotes the closure of $C^\infty_0(\Omega, R^3)$ in $\mathbb{H}^k$, and $\mathbb{H}^{-k}$ is the dual space of $\mathbb{H}^k_0$. Moreover, $V_0$ is the set of divergence free vectors in $C^\infty_0(\Omega,R^3)$, and H and V are the closure of $V_0$ in $\mathbb{L}^2$ and $\mathbb{H}^1_0$, respectively.

For vector fields in $\Omega$ we define the norm

$$\|v\|_V = |\ |\nabla v|\ |_2 = (\int_\Omega |\nabla v|^2 dx)^{1/2},$$

where $|\nabla v|^2 = \sum_{i,j=1}^{3} (\partial v_j/\partial x_i)^2$. We adopt the notations

$$\|u\|_{q,p,T} \equiv \|u\|_{L^q(0,T;\mathbb{L}^p)}, \quad \|u\|_{q,T} \equiv \|u\|_{q,q,T}.$$

We denote different constants by the same symbol $c$. When necessary, we will write $c_0, c_1, c_2, \ldots$

The Navier-Stokes equations are

$$(0.1)\quad \begin{cases} w' + (w\cdot\nabla)w - \Delta w = -\nabla\pi + g & \text{in } Q_T, \\ \nabla\cdot w = 0 & \text{in } Q_T, \\ w = b & \text{on } \Sigma_T, \\ w_{|t=o} = w_o(x), & \end{cases}$$

where $w' = \partial w/\partial t$. We assume, without loss of generality, that the density and the viscosity are equal to one. The initial data $w_o(x)$, the external force field $g(t,x)$ and the boundary data $b(t,x)$ are given functions. The velocity $w(t,x)$ and the pressure $\pi(t,x)$ are the unknowns. We prove the following result:

<u>Theorem A.</u> <u>Let</u> $\frac{10}{9} < p \leq \frac{5}{4}$, $r > \frac{5}{3}$, <u>and</u>

$w_o \in H \cap \mathbb{W}_p^{2-(2/p)}$,

$g \in L^1(0,T;\mathbb{L}^2) \cap \mathbb{L}^p(Q_T)$,

$b \in B_r^{2-(1/r),\,1-(1/2r)}(\Sigma_T)$ (1), $b\cdot n = 0$ <u>on</u> $\Sigma_T$,

<u>Then there exists in</u> $Q_T$ <u>a suitable weak solution</u> $w, \pi$ <u>of the non-homogeneous initial-boundary value problem</u> (0.1) <u>such that</u>

---

(1) See [8] chap. I, for the definition.

$$(0.2)\quad \begin{cases} w \in L^2(\mathbb{H}^1) \cap C_{deb}(\mathbb{L}^2) \cap L^p(\mathbb{W}^2_p), \\ w' \in L^{4/3}(\mathbb{H}^{-1}) \cap L^p(\mathbb{L}^p), \\ \pi \in L^p(W^1_p). \end{cases}$$

Moreover, for every $t \in [0,T]$ and every non-negative real function $\phi \in C^2(Q_T)$, with $\phi=0$ near $\Sigma_T$, the following local energy estimate is satisfied

$$(0.3)\quad \int_{\Omega_t} |w|^2\phi + 2\iint_{Q_t} |\nabla w|^2\phi \le \int_{\Omega_o} |w_o|^2\phi + \iint_{Q_t} |w|^2(\phi'+\Delta\phi) +$$

$$+ \iint_{Q_t} (|w|^2 + 2\pi)\, w\bullet\nabla\phi \; + \; 2\iint_{Q_t} g\bullet w\phi.$$

Finally, if $g \in L^{\alpha}_{loc}(Q_T)$, $\alpha > 5/2$, then the singular set $S$ of $w$ in $Q_T$ verifies $\mathcal{P}^1(S)=0$, hence $\mathcal{H}^1(S)=0$.

1. Preliminaries. Let us define

$$(1.1)\quad |||a|||_{r,T} \equiv \|a\|_{L^r(0,T;\mathbb{W}^2_r)} + \|a'\|_{\mathbb{L}^r(Q_T)},$$

$$(1.2)\quad s = \frac{5r}{5-2r}\;,\quad s_1 = \frac{5r}{5-r}.$$

Note that for $r > 5/3$ one has $s > 5$ and $s_1 > 5/2$.
We start by considering the auxiliary problem

$$
(1.3)\quad \begin{cases} a' - \Delta a = -\nabla q & \text{in } Q_T, \\ \nabla \bullet a = 0 & \text{in } Q_T, \\ a = b & \text{on } \Sigma_T, \\ a_{|t=o} = b_o(x), & \end{cases}
$$

where the function $b_o \in \mathbb{W}_r^{2-(2/r)}(\Omega)$ is choosen in such a way that

$$
(1.4)\quad \begin{cases} \nabla \bullet b_o = 0 & \text{in } \Omega, \\ b_o = b & \text{on } \Gamma, \quad \text{for } t=0. \end{cases}
$$

One has the following result:

<u>Proposition 1.1</u> <u>Problem (1.3) has a unique solution</u> $a, \nabla q$ <u>such that</u>

$$
(1.5)\qquad a \in L^r(\mathbb{W}_r^2), \quad a' \in L^r(\mathbb{L}^r), \quad q \in L^r(\mathbb{W}_r^1).
$$

<u>In particular,</u>

$$
(1.6)\qquad a \in \mathbb{L}^s(Q_T), \quad \nabla a \in \mathbb{L}^{s_1}(Q_T),
$$

<u>and</u>

$$
(1.7)\qquad \|a\|_{s,T} + \|\nabla a\|_{s_1,T} \le c\,|||a|||_{r,T}.
$$

<u>Finally</u> $a \in C(0,T;\mathbb{L}^3)$, <u>and</u> $a$ <u>and</u> $\nabla a$ <u>are bounded on compact subsets of</u> $Q_T$.

<u>Proof.</u> Existence of a unique solution satisfying (1.5) is proved in [9]. Now, by interpolation, one shows that

$$a \in W_r^{\Theta}(0,T;\mathbb{W}_r^{2(1-\Theta)}), \quad \forall\, \Theta \in [0,1].$$

By choosing $\Theta=2/5$ and by using the embeddings $W_r^{2/5}(0,T) \hookrightarrow L^s(0,T)$, and $W_r^{6/5}(\Omega) \hookrightarrow L^s(\Omega)$, one gets the first statements (1.6), (1.7). The corresponding statements for $\nabla a$ follow by choosing $\Theta=1/5$. The continuity of $a(t)$ with values in $\mathbb{L}^3$ is obtained from the choice $\Theta=3/5$. Finally, the last statements are proven as in Serrin's paper [5]. □

As in [2], we define

$$\Lambda \equiv \{v \in C^{\infty}(\overline{Q}_T,R^3): \; v(t) \in V_o \quad, \forall t\in [0,T]\}$$

and we introduce the quantities

$$(1.8)\quad \begin{cases} A(u_o,f) \equiv [\,|u_o|_2 + \|f\|_{1,2}]\exp \|\nabla a\|^{5/2}_{5/2,T} \quad , \\ A_1^2(u_o,f) \equiv c_o[\,|u_o|_2^2 + \|f\|^2_{1,2}]\,[1+\|\nabla a\|^{5/2}_{5/2,T}\;\exp\|\nabla a\|^{5/2}_{5/2,T}] \\ B(u_o,f) \equiv c_1\,\|u_o\|_{2-(2/p),p} + c_2\|f\|_{p,T} + \\ \quad + c_3\|\nabla a\|_{q_1,T}[A(u_o,f) + A_1(u_o,f)] + c_4\|a\|_{q,T}A_1(u_o,f), \end{cases}$$

where $c_o, c_1, c_2, c_3, c_4,$ are suitable positive constants, and

$$(1.9)\qquad q = 2p/(2-p)\,, \quad q_1 = 10\,p/(10-3p).$$

One has the following result:

Theorem 1.1 Let $v \in \Lambda$, $\tilde{u}_o \in V$, $\tilde{f} \in \mathbb{L}^2(Q_T)$, and let $a$ be as in (1.5) (with $r > 5/3$) and satisfying $\nabla \bullet a = 0$. Then there exists a unique solution $u$, $\nabla p$ of problem

$$
(1.10) \quad \begin{cases} u' + (v \bullet \nabla)u - \Delta u + (a \bullet \nabla)u + (u \bullet \nabla)a = -\nabla p + \tilde{f}, & \text{in } Q_T, \\ \nabla \bullet u = 0, & \text{in } Q_T, \\ u = 0, & \text{on } \Sigma_T, \\ u_{|t=o} = \tilde{u}_o(x), \end{cases}
$$

such that

$$
(1.11) \qquad u \in L^2(\mathbb{H}^2) \cap C(V), \qquad u', \nabla p \in \mathbb{L}^2(Q_T).
$$

Moreover,

$$
(1.12) \qquad \|u\|_{\infty,2} \leq A(\tilde{u}_o, \tilde{f}),
$$

$$
(1.13) \qquad \|\nabla u\|^2_{2,T} \leq A_1(\tilde{u}_o, \tilde{f}).
$$

Proof. Existence and uniqueness are shown in [9] (theorem 15) or in [10] (theorem 4.2). Let us prove the estimates (1.12) and (1.13). By multiplying $(1.10)_1$ by $u$ and by integrating over $\Omega$, one gets

$$
(1.14) \qquad \frac{1}{2}\frac{d}{dt}|u|_2^2 + \frac{1}{2}|\nabla u|_2^2 \leq |\tilde{f}|_2 |u|_2 + c\, |\nabla a|_{5/2}^{5/2} |u|_2^2,
$$

where integrations by parts, Hölder's inequality and Sobolev's embeddings theorems have been utilized. Hence, $(d/dt)\,|u|_2 \le |\tilde{f}|_2 +$ $+ c\,|\nabla a|^{5/2}_{5/2}\,|u|_2$, from which (1.12) follows.

By using again (1.14) one easily obtains, for every $t \in [0,T]$,

$$\frac{1}{2}\,\|u\|^2_{\infty,2,T} + \frac{1}{2}\,\|\nabla u\|^2_{2,T} \le |\tilde{u}_0|^2_2 + 2\,\|\tilde{f}\|^2_{1,2,T} +$$

$$+ \frac{1}{2}\,\|u\|^2_{\infty,2,T} + 2c\,\|u\|^2_{\infty,2,T}\,\|\nabla a\|^{5/2}_{5/2,T}\,.$$

This last inequality, togheter with (1.12), yields (1.13). Note that, from (1.7), one has

(1.15) $$\|\nabla a\|_{5/2,T} \le c\,|||\,a\,|||_{5/3,T}\,.$$

□

To conclude this section we prove the following result

Theorem 1.2 Let $v$, $a$, $\tilde{u}_0$ and $\tilde{f}$ be as defined in theorem 1.1, and let $1 < p \le 5/4$. Then the solution $u$, $p$ of problem (1.10) verifies the following estimate

(1.16) $$|||u|||_{p,T} + \|\nabla p\|_{p,T} \le B(\tilde{u}_0, \tilde{f}) + c_5\,(\,\|v\|_{\infty,2,T} + \|\nabla v\|_{2,T})\,\|\nabla u\|_{2,T}.$$

Proof. From Hölder's inequality one gets

$$\|(v\bullet\nabla)u\|_{p,T} \leq \|v\|_{q,T}\, \|\nabla u\|_{2,T}.$$

Moreover,

$$\|v\|_{q,T} \leq \|v\|_{\infty,2,T}^{1-(2/q)}\, \|v\|_{2,6,T}^{2/q} \leq c\, (\|v\|_{\infty,2,T} + \|\nabla v\|_{2,T}).$$

Consequently,

$$\|(v\bullet\nabla)u\|_{p,T} \leq c\, (\|v\|_{\infty,2,T} + \|\nabla v\|_{2,T})\, \|\nabla u\|_{2,T}. \tag{1.17}$$

Similarly, $\|(u\bullet\nabla)a\|_{p,T} \leq \|u\|_{10/3,T}\, \|\nabla a\|_{q_1,T}$, which implies

$$\|(u\bullet\nabla)a\|_{p,T} \leq c\, \|\nabla a\|_{q_1,T}\, (\|u\|_{\infty,2,T} + \|\nabla u\|_{2,T}). \tag{1.18}$$

Moreover,

$$\|(a\bullet\nabla)u\|_{p,T} \leq \|a\|_{q,T}\, \|\nabla u\|_{2,T}. \tag{1.19}$$

Estimate (1.16) follows from [9] (theorem 15) and from (1.17), (1.18), (1.19). Note that, since $r>5/3$ and $p \leq 5/4$, one has $1/r \leq (1/p)-(1/10)$. Thus, proposition 1.1 implies that $a \in \mathbb{L}^q(Q_T)$ and $\nabla a \in L^{q_1}(Q_T)$.

□

## 2. Proof of theorem A.

In this section we assume the reader familiar with reference [2]. Let $w_o$ and $g$ be as in theorem A, and define

(2.1) $u_o \equiv w_o - b_o$, $f = g - (a \bullet \nabla) a$.

Consider sequences $u_o^{(n)} \in V$, $f_n \in \mathbb{L}^2(Q_T)$ such that

$$|u_o^{(n)} - u_o|_2 < 1/n, \quad \|u_o^{(n)} - u_o\|_{2-(2/p),p} < 1/n,$$

$$\|f_n - f\|_{1,2,T} < 1/n, \quad \|f_n - f\|_{p,T} < 1/n.$$

The following result holds:

Theorem 2.1. Let $u_o^{(n)}$ and $f_n$ be defined as above. Then there exist $V_n \in \Lambda$ and $u_n, p_n$ such that

$$(2.2) \begin{cases} u'_n + (v_n \bullet \nabla) u_n - \Delta u_n + (a \bullet \nabla) u_n + (u_n \bullet \nabla) a + \nabla p_n = f_n & \text{in } Q_T, \\ \nabla \bullet u_n = 0 & \text{in } Q_T, \\ u_n = 0 & \text{on } \Sigma_T, \\ (u_n)_{|t=o} = u_o^{(n)} . \end{cases}$$

Moreover

(2.3) $\|u_n - v_n\|_{2,T} < \frac{1}{n}$, and

$$
(2.4)\quad \begin{cases}
\|u_n\|_{\infty,2,T} \le A(u_o + \frac{1}{n}, f + \frac{1}{n}), \\
\|\nabla u_n\|^2_{2,T} \le A_1^2(u_o + \frac{1}{n}, f + \frac{1}{n}), \\
|||u'_n|||_{p,T} + \|\nabla p\|_{p,T} \le B(u_o + \frac{1}{n}, f + \frac{1}{n}) + \\
\qquad + c_6 A_1^2(u_o + \frac{1}{n}, f + \frac{1}{n}).
\end{cases}
$$

Estimates (2.4) hold also for $\|v_n\|_{\infty,2,T}$, $\|\nabla v_n\|^2_{2,T}$ and $\|v'_n\|_{p,T}$.

The proof of the above theorem is given in reference [2] (theorem 2.1). On the right hand sides of (2.4) we assume that the terms 1/n are added to the norms of $u_o$ and f appearing on the definitions (1.8), and not directly to the functions $u_o$ and f.

From the estimates (2.4) and (2.3) it follows the existence of subsequences $u_\nu$, $v_\nu$, $p_\nu$ and functions u, p, such that

$$
(2.5)\quad \begin{cases}
u_\nu \to u \quad \text{weakly in } L^2(V), \text{ weakly in } L^p(\mathbb{W}^2_p), \text{ and weak-* in } L^\infty(H). \\
p_\nu \to p \quad \text{weakly in } L^p(\mathbb{W}^1_p), \\
v_\nu \to v \quad \text{weakly in } L^2(V) \text{ and weak-* in } L^\infty(H).
\end{cases}
$$

Moreover, a well known compactness theorem (J.L. Lions, [4], chap. 1, theorem 5.1) guarantees that we can further select subsequences (still denoted by the same index ν) verifying

(2.6) $u_\nu \to u,\ v_\nu \to v,$ strongly in $\mathbb{L}^2(Q_T)$; strongly in $\mathbb{L}^2(\Omega)$, for almost all $t \in ]0, T[$; and a.e. in $Q_T$.

On the other hand, from the embedding $L^\infty(\mathbb{L}^2) \cap L^2(\mathbb{L}^6) \subsetneq L^4(\mathbb{L}^3)$ it follows that the sequences $(v_\nu)_i\,(u_\nu)_j$, $i,j=1,2,3$, are bounded in $L^2(\mathbb{L}^{3/2})$. Arguing as in [4] p. 76, one has

(2.7) $(v_\nu)_i\,(u_\nu)_j \to u_i\,u_j$, weakly in $L^2(\mathbb{L}^{3/2})$.

In particular, $(v_\nu \bullet \nabla)u_\nu \to (u\bullet\nabla)u$ weakly in $L^2(\mathbb{W}_2^{-2})$.

Finally, by recalling that $a \in \mathbb{L}^5(Q_T)$, the first statement (2.5) implies

$$\iint_{Q_T} [(a\bullet\nabla)u_\nu]\bullet\psi \to \iint_{Q_T} [(a\bullet\nabla)u]\bullet\psi\ ,\quad \forall\psi\in \mathbb{L}^{10/3}(Q_T).$$

Hence,

(2.8) $(a\bullet\nabla)u_\nu \to (a\bullet\nabla)u$, weakly in $\mathbb{L}^{10/7}(Q_T)$.

Similarly, since $\nabla a \in \mathbb{L}^{5/2}(Q_T)$, the weak convergence of $u_\nu$ to $u$ in $L^2(\mathbb{L}^6)$ implies

(2.9) $(u_\nu\bullet\nabla)a \to (u\bullet\nabla)a$, weakly in $L^{10/9}(\mathbb{L}^{3/2})$.

The statements (2.5) to (2.9) allows us to pass to the limit in (2.2), as $\nu \to +\infty$, to yield

$$(2.10)\quad \begin{cases} u' + (u\bullet\nabla)u - \Delta u + (a\bullet\nabla)u + (u\bullet\nabla)a + \nabla p = f, & \text{in } Q_T, \\ \nabla\bullet u = 0 & \text{in } Q_T, \\ u = 0 & \text{on } \Sigma_T, \\ u_{|t=o} = u_o(x). \end{cases}$$

By defining $w$ and $\pi$ as

$$(2.11)\qquad w = u+a, \quad \pi \equiv p+q,$$

and by using (2.10), (1.3) and (2.1), one shows that $w,\pi$ is a solution of problem (0.1).

Now we prove the local energy estimate (0.3). By setting

$$(2.12)\quad \begin{cases} w_n \equiv u_n + a, & \pi_n \equiv p_n + q, \\ g_n \equiv f_n + (a\bullet\nabla)a, & w_o^{(n)} \equiv u_o^{(n)} + b_o, \end{cases}$$

equation (2.2) can be written equivalently in the form

$$(2.13)\quad \begin{cases} w'_n - \Delta w_n + [(v_n+a)\bullet\nabla]\, w_n + [(u_n - v_n)\bullet\nabla]\, a + \nabla\pi_n = g_n, & \text{in } Q_T, \\ \nabla\bullet w_n = 0 & \text{in } Q_T, \\ w_n = b & \text{on } \Sigma_T, \\ (w_n)_{|t=o} = w_o^{(n)}. \end{cases}$$

Clearly, $w_n \to w$ and $\pi_n \to \pi$, in the same topologies in wich $u_n \to u$ and $p_n \to p$, respectively. By multiplying both sides of equation (2.13) by $\phi w_\nu$, by integrating over $Q_t$, and by using suitable integrations by parts, one has

$$(2.14)\qquad \frac{1}{2}\int_{\Omega}|w_{\nu}|^{2}\phi+\iint_{Q_t}|\nabla w_{\nu}|^{2}\phi=$$

$$=\frac{1}{2}\int_{\Omega_o}|w_o^{(\nu)}|^{2}\phi+\frac{1}{2}\iint_{Q_t}|w_{\nu}|^{2}(\phi'+\Delta\phi)+$$

$$+\frac{1}{2}\iint_{Q_t}|w_{\nu}|^{2}(v_{\nu}+a)\bullet\nabla\phi+$$

$$+\frac{1}{2}\iint_{Q_t}\{[(u_{\nu}-v_{\nu})\bullet\nabla]a\}\bullet w_{\nu}\phi+$$

$$+\iint_{Q_t}\pi_{\nu}w_{\nu}\bullet\nabla\phi+\iint_{Q_t}g_{\nu}\bullet w_{\nu}\phi.$$

By passing to the limit in (2.14) as $\nu\to+\infty$, one gets (0.3) (see reference [2], for details). The terms with $w_{\nu}$ are treated now as the corresponding terms with $u_{\nu}$ in reference [2]. Moreover, (2.6) shows that $u_{\nu}-v_{\nu}\to 0$ and $w_{\nu}\to w$, a.e. in $Q_T$. On the other hand, since $\mathbb{L}^{10/3}(Q_T)\subsetneq L^{\infty}(\mathbb{L}^2)\cap L^2(\mathbb{L}^6)$, the functions $(u_{\nu}-v_{\nu})_i(w_{\nu})_j$ are uniformely bounded in $L^{5/3}(Q_T)$. Consequently (see [4], chap. I, lemma 1.3) it follows that $(u_{\nu}-v_{\nu})_i(w_{\nu})_j\to 0$ weakly in $\mathbb{L}^{10/3}(Q_T)$, as $\nu\to+\infty$. Hence, recalling that $\nabla a\in L^{5/2}(Q_T)$, one gets

$$\lim_{\nu\to+\infty}\iint_{Q_t}\{[(u_{\nu}-v_{\nu})\bullet\nabla]a\}\bullet w_{\nu}\phi=0.$$

Finally, the last statement in theorem A follows by using (0.3), togheter with theorem B, in [3]; the conditions $g\in\mathbb{L}^2(Q_T)$, $\nabla\bullet g=0$ and $p=5/4$ are not strictly necessary.

## REFERENCES.


[1] H. Beirão da Veiga, "On the suitable weak solutions to the Navier-Stokes equations in the whole space", J. Math. Pures Appl., in the press.

[2] H. Beirão da Veiga, "On the construction of suitable weak solutions to the Navier-Stokes equations via a general approximation theorem", J. Math. Pures Appl.", in the press.

[3] L. Caffarelli, R. Kohn and L. Nirenberg, "Partial regularity of suitable weak solutions of the Navier-Stokes equations", Comm. Pure Appl. Math., 35 (1982), 771-831.

[4] J. L. Lions, "Quelques méthodes de résolution des problèmes aux limites non-linéaires", Dunod, Paris 1969.

[5] J. Serrin, "On the interior regularity of weak solutions of the Navier-Stokes equations", Arch. Rat. Mech. Anal., 9 (1962), 187-195.

[6] V. Scheffer, "Hausdorff measure and the Navier-Stokes equation", Comm. Math. Phys., 55 (1977), 97-112.

[7] V. Scheffer, "The Navier-Stokes equation on a bounded domain", Comm. Math. Phys., 73 (1980), 1-42.

[8] V.A. Solonnikov, "A priori estimates for second order parabolic equations", Amer. Math. Soc. Transl., 65 (1967), 51-137.

[9] V.A. Solonnikov, "Estimates of the solutions of a nonstationary linearized system of Navier-Stokes equations", Amer. Math. Soc. Transl., 75 (1968), 1-116.

[10] V.A. Solonnikov, "Estimates for the solutions of nonstationary Navier-Stokes equations", J. of Soviet Math., 8 (1977), 467-529.