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On the Stationary Motion of Granulated Media.

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1. Introduction and main results.

Let Ω be an open, bounded domain in $\mathbb{R}^3$, locally situated on one side of its boundary Γ, a differentiable manifold of class C^2. In this paper we consider the following system of equations

$$(1)\qquad \begin{cases} -\nu\Delta u + (u\cdot\nabla)u - \eta\omega\times u = -\nabla p + f, & \text{in } \Omega, \\ \nabla\cdot u = 0, & \text{in } \Omega, \\ (u\cdot\nabla)\omega + F(p)\omega = g, & \text{in } \Omega, \\ u = 0, & \text{on } \Gamma, \end{cases}$$

which describes the stationary motion of a granulated medium with constant density. For more information, we refer the reader to Antocec and Leluch [3], Leluch and Nenashev [5], Antocev, Kazhykov and Monachov [2], Lukaszewicz [8], and to the bibliography quoted in these references. Here, the vector fields $u = (u_1, u_2, u_3)$ and $\omega = (\omega_1, \omega_2, \omega_3)$ denote the velocity and the angular velocity of rotation of the particles, respectively. The scalar p denotes the pressure. The quantities $u(x)$, $\omega(x)$ and $p(x)$ are the unknowns, in problem (1). The positive constants η, ν are the Magnus and viscosity coefficients. The given vector fields $f = (f_1, f_2, f_3)$ and $g = (g_1, g_2, g_3)$ denote the exterior mass forces and the density of momentum of the forces, respect-

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ively. The function $F = F(p)$ describes the friction between the particles.

The time-dependent motion of granulated media was studied by several authors (see [8], for references). On the contrary (as far as we know), the only existence theorem available in the literature for the stationary motion was proved by Lukaszewicz [8]. In [8], the author proved the existence of weak solutions for problem (1), under the assumption $F(\xi) \geqslant m > 0$, for $\xi \in \mathbb{R}$, where m is a positive constant.

In the sequel we will assume that $F(\xi)$, $\xi \in \mathbb{R}$, is a real continuous function, for which there exist two constants $m > 0$ and $p_0 \in R$ such that

$$F(\xi) \geqslant m\,, \quad \text{if } \xi \geqslant p_0\,. \tag{2}$$

This condition is more general then those described in [2], § 5, eq. (5.2), and includes in particular the physicaly important case

$$F(\xi) = n + k\xi\,, \tag{3}$$

where $n > 0$ is a schift cohesion constant and $k > 0$ is the friction constant (see [2], § 1, section 6, eq. (1.45)).

Under assumption (2), we succed in proving the existence of a solution u, ω, p such that $p(x) \geqslant p_0$, $\forall x \in \Omega$. The lower bound $F(p(x)) \geqslant m$, $\forall x \in \Omega$, follows then as a consequence. More precisely, we will prove the following result:

THEOREM A. *Let $f \in \mathbb{L}^q(\Omega)$, $q > 3$, $g \in \mathbb{L}^\infty(\Omega)$, and let F be a real continuous function verifying* (2). *Fix a constant a, such that $a \geqslant p_0$. Then, there exists a solution u, ω, p of problem* (1) *such that*

$$\min_{x \in \overline{\Omega}} p(x) = a\,. \tag{4}$$

Moreover, $u \in \mathbb{W}^2_q(\Omega)$, $p \in W^1_q(\Omega)$, $\omega \in \mathbb{L}^\infty(\Omega)$, and the estimates (8), (9), (23) *hold.*

In theorem A, equations $(1)_1$ and $(1)_2$ are verified almost everywhere in Ω, and $(1)_4$ is verified for all $x \in \Gamma$. Equation $(1)_3$ is verified in the following weak sense:

$$-\int_\Omega [(u \cdot \nabla)\varphi] \cdot \omega \, dx + \int_\Omega F(p)\omega \cdot \varphi \, dx = \int_\Omega g \cdot \varphi \, dx\,, \quad \forall \varphi \in W^1_2(\Omega)\,. \tag{5}$$

This definition is meaningful, since

$$-\int_\Omega [(u\cdot\nabla)\varphi]\cdot\omega\,dx = \int_\Omega [(u\cdot\nabla)\omega]\cdot\varphi\,dx\,, \qquad \forall\varphi\in \mathbf{W}_2^1(\Omega)\,,$$

if (say) $\omega\in \mathbf{W}_2^1(\Omega)$.

Remarks. If F is defined by (3), assumption (2) holds by setting $m=n$, $p_0=0$. By choosing $a=0$, condition (4) coincides with condition (5.5), in reference [2], § 5.

We also note that condition (4) can easily be replaced by other conditions on the pressure term, as for instance a condition on the mean-value of p in Ω.

I am indebted to Grzegorz Lukaszewicz for kindly providing me with a copy of reference [2], during his stay in my University (in fact, my interest on problem (1) originated from a preprint of his paper [8]).

2. Notations.

$L^p, |\ |_p$ 1) usual $L^p(\Omega)$ space ($1\leq p\leq+\infty$), and usual norm in L^p. For convenience we set $|\ |_\infty=\|\ \|$.

$C^0, \|\ \|$ 2) usual $C^0(\bar\Omega)$ space, of real continuous functions in $\bar\Omega$, with the uniform convergence norm $\|\ \|$. More generaly, $\|\ \|$ will denote the norm in L^∞.

$C^{0,\alpha}, \|\ \|_{0,\alpha}$ 3) Space of α-Hölder continuous real functions in $\bar\Omega$, $0<\alpha<1$, with the natural norm $\|\ \|_{0,\alpha}$.

$W_p^k, \|\ \|_{k,p}$ 4) Sobolev space $W_p^k(\Omega)$, $1\leq p<+\infty$, k positive integer, and usual norm in that space.

$\overset{\circ}{W}{}_p^k$ 5) Closure of $C_0^\infty(\Omega)$ in $W_p^k(\Omega)$.

The norm in W_p^k is

$$\|h\|_{k,p}=\left(\sum_{l=0}^{k}\sum_{|\alpha|=l}|D^\alpha h|_p^p\right)^{1/p}.$$

Similarly, for vector fields $u=(u_1,u_2,u_3)$ in Ω, we define the spaces $\mathbf{L}^p$, $\mathbf{W}_p^k$, $\overset{\circ}{\mathbf{W}}{}_p^k$, $\mathbf{C}^0$, $\mathbf{C}^{0,\alpha}$, and so on. Norms will be denoted by the same symbol in both the scalar and vector cases.

For vector functions we also define

$$|\nabla u|^2 = \sum_{i,j=1}^{3} \left(\frac{\partial u_j}{\partial x_i}\right)^2, \qquad \|u\|_V = |\nabla u|_2 = \left(\int_\Omega |\nabla u|^2\, dx\right)^{\frac{1}{2}},$$

and

$$(v\cdot\nabla)\, u = \sum_{i=1}^{3} v_i \frac{\partial u}{\partial x_i}\,.$$

Moreover,

$$V = \{v \in \mathring{\mathbf{W}}_2^1\colon \nabla\cdot v = 0 \text{ in } \Omega\}\,,$$

and V' is the dual space of V, with the canonical norm $\|\ \|_{V'}$.

We denote positive constants depending at most on Ω, q, ν, η, and m, by $c, c_0, c_1, \ldots$. For convenience, we denote different constants by the same symbol c. Otherwise, we will write $c_0, c_1, c_2, \ldots$.

Proof of theorem A. In the sequel, $q > 3$ is fixed. Set

$$(6)\qquad \begin{cases} R_0 = |f|_q + \|f\|_{V'}^2 + \|g\|\,\|f\|_{V'}\,, \\ R_1 = |f|_q + \|g\|\,\|f\|_{V'} + R_0^2\,, \end{cases}$$

and define

$$\mathbf{K} = \left\{v \in V \cap \mathbf{C}^0\colon \|v\|_V \leqslant \frac{1}{\nu}\|f\|_{V'},\ |v|_q \leqslant c_0 R_0,\ \|v\| \leqslant c_1 R_1\right\},$$

$$\mathbf{Q} = \left\{\alpha \in \mathbf{C}^0\colon \|\alpha\| \leqslant \frac{1}{m}\|g\|\right\}.$$

The constants c_0 and c_1 will be defined in the proof of theorem 1. One has the following result:

Theorem 1. *Let f and g be as in theorem A, and let $\alpha \in \mathbf{Q}$, $v \in \mathbf{K}$. Then, the problem*

$$(7)\qquad \begin{cases} -\nu\Delta u + (v\cdot\nabla)\, u - \eta\alpha\times u + \nabla p = f & \text{in } \Omega\,, \\ \nabla\cdot u = 0 & \text{in } \Omega\,, \\ u = 0 & \text{on } \Gamma\,, \end{cases}$$

has a unique solution $u \in \mathbb{K} \cap \mathrm{W}^2_q$, $p \in W^1_q$, *such that* (4) *holds. Moreover,*

$$\|u\|_{2,q} + |\nabla p|_q \leqslant c[|f|_q + \|g\| R_0 + R_1^2] \,. \tag{8}$$

In particular,

$$a \leqslant p(x) \leqslant a + c[|f|_q + \|g\| R_0 + R_1^2]\,, \tag{9}$$

for every $x \in \bar{\Omega}$.

PROOF. – The existence and the uniqueness of a solution $u \in V$ of problem (7) follows as for the Stokes linearized stationary problem, by using Galerkin's method [7]. Since $\omega \times u \cdot u = 0$, one has

$$\|u\|_V \leqslant \frac{1}{\nu} \|f\|_{V'} \,. \tag{10}$$

By taking in account that $|\ |_6 \leqslant c \|\ \|_V$, one has

$$|(v \cdot \nabla) u|_{\frac{3}{2}} \leqslant c |v|_6 \|u\|_V \leqslant c \|f\|^2_{V'} \,. \tag{11}$$

On the other hand,

$$|\alpha \times u|_6 \leqslant \|\alpha\| \, |u|_6 \leqslant c \|g\| \, \|f\|_{V'} \,. \tag{12}$$

From (7), (11), (12), and from well known regularity results for the linear Stokes problem [4], one gets

$$\|u\|_{2,\frac{3}{2}} + |\nabla p|_{\frac{3}{2}} \leqslant c R_0 \,. \tag{13}$$

Since $W^2_{\frac{3}{2}} \hookrightarrow W^1_3 \hookrightarrow L^q$, there exists a positive constant c_0 such that

$$|u|_q \leqslant c_0 R_0 \,. \tag{14}$$

Let us define

$$\frac{1}{r} = \frac{1}{q} + \frac{1}{3} \,. \tag{15}$$

By again using regularity theorem for the linear Stokes problem, one obtains

$$\|u\|_{2,r} + |\nabla p|_r \leqslant c|f|_q + c\|g\| \, \|f\|_{V'} + c R_0^2 \leqslant c R_1 \,,$$

since $|(v\cdot\nabla)u|_r \leqslant cR_0^2$. By utilizing now the embeddings $\mathbf{W}_r^2 \hookrightarrow \mathbf{W}_q^1 \hookrightarrow \mathbf{C}^0$, one deduces that there exists a positive constant c_1 for which

$$\|u\| \leqslant c_1 R_1 . \tag{16}$$

From (10), (14), (16) it follows that $u \in \mathbf{K}$. Furthermore, $|(v\cdot\nabla)u|_q \leqslant \leqslant c\|v\||\nabla u|_q \leqslant cR_1^2$, and $|\alpha \times u|_q \leqslant \|\alpha\||u|_q \leqslant c\|g\|R_0$. Hence, a last regularization of u, p in equation (7), yields (8).

Let us denote by $\overline{p}(x)$ the particular solution $p(x)$ + constant, for which

$$\int_\Omega \overline{p}(x)\,dx = 0 . \tag{17}$$

Let $\alpha = 1-(3/q)$. Since $W_q^1 \hookrightarrow C^{0,\alpha}$, one deduce from (8) that $\overline{p} \in C^{0,\alpha}$, moreover

$$\|\overline{p}\| \leqslant c\big[|f|_q + \|g\|R_0 + R_1^2\big] . \tag{18}$$

Finally, by defining

$$p(x) = \overline{p}(x) - \min_{x\in\overline{\Omega}} \overline{p}(x) + a , \tag{19}$$

one shows that $p \in C^{0,\alpha}$ and that (4), (9) hold. □

THEOREM 2. *The map $(v,\alpha) \to (u,p)$, defined in theorem 1, is continuous on $\mathbf{K}\times\mathbf{Q}$, with respect to the uniform topologies.*

PROOF. – Let (v_n,α_n) and (v,α) be elements of $\mathbf{K}\times\mathbf{Q}$ and let u_n, p_n and u, p be the corresponding solutions, constructed in theorem 1. Assume that $\|v_n - v\| \to 0$, and $\|\alpha_n - \alpha\| \to 0$, as $n \to +\infty$.

By taking the difference (side by side) of the equations

$$\begin{cases} -\nu\Delta u + (v\cdot\nabla)u - \nabla p = f + \eta\alpha\times u , \\ -\nu\Delta u_n + (v_n\cdot\nabla)u_n - \nabla p_n = f + \eta\alpha_n\times u_n , \end{cases} \tag{20}$$

by multiplying both sides of the equation just obtained by $u - u_n$, and by integrating over Ω, one easily shows that

$$\nu\|u-u_n\|_V \leqslant c(\|v-v_n\| + \|\alpha-\alpha_n\|)\|u\|_V .$$

Hence, $u_n \to u$ in V. In particular,

$$f + \eta\alpha_n \times u_n - (v_n \cdot \nabla) u_n \to f + \eta\alpha \times u - (v \cdot \nabla) u$$

in L^2, as $n \to +\infty$. From (20) and from the estimates in L^2 norm for the linear Stokes problem it follows that $u_n \to u$ in $\mathbf{W}_2^2$ and $\nabla p_n \to \nabla p$ in $\mathbf{L}^2$. In particular, $u_n \to u$ uniformly in $\bar{\Omega}$. On the other hand, by recalling definition (17), one has $\bar{p}_n \to \bar{p}$ in W_2^1. Moreover, the sequence $\bar{p}_n$ is uniformly bounded in $C^{0,\alpha}$, $\alpha = 1 - (3/q)$, by estimate (8). In particular, by using Ascoli-Arzela's theorem, $\|\bar{p}_n - \bar{p}\| \to 0$, and $\min p_n(x) \to \min p(x)$, as $n \to +\infty$. From definition (19), it follows that $\|p_n - p\| \to 0$. □

We consider now the auxiliary problem (see also [8])

$$(21) \qquad \begin{cases} -\varepsilon \Delta\omega + (u \cdot \nabla)\omega + F_0(x)\omega = g & \text{in } \Omega, \\ \omega = 0 & \text{on } \Gamma, \end{cases}$$

where $\varepsilon > 0$ in fixed, and $F_0(x)$ is a real function defined on $\bar{\Omega}$. We assume that

$$(22) \qquad u \in \mathbf{C}^1, \quad F_0 \in C^0, \quad \min_{x \in \bar{\Omega}} F_0(x) \geqslant m,$$

where by definition $\mathbf{C}^1 = \mathbf{C}^1(\bar{\Omega})$. Here and in the sequel attention will not be given to the minimal assumptions under which the auxiliary results hold.

The uniform bound stated below is crucial in order to prove theorem A.

Theorem 3. *Let $g \in L^\infty$, and u and F_0 be as in (22). Then, there exists a unique solution $u \in \mathbf{W}_q^2$ of problem (21). Moreover,*

$$(23) \qquad \|\omega\| \leqslant \frac{1}{m} \|g\|.$$

The map $(u, F_0) \to \omega$ is bounded and continuous from $\mathbf{C}^1 \times \mathbf{C}$ into $\mathbf{W}_q^2$, with respect ot the canonical topologies in these functional spaces.

PROOF. Existence, uniqueness, and continuous dependence of the solution ω in the space W_q^2 (for arbitrary finite q), follow from well known results; see for instance [6].

Let us prove estimate (23). For convenience we define $g(x) = 0$, $F_0(x) = m$, for $x \in \mathbb{R}^3 \setminus \Omega$.

Denote by J_δ, $\delta > 0$, the Friedrichs mollification operator (see for instance [1]), and set $g_\delta = J_\delta g$, $F_\delta = J_\delta F_0$. Then $g_\delta \in C^\infty(\bar{\Omega})$, $\|g\|_\delta \leqslant \|g\|$, and $\|g_\delta - g\| \to 0$ as $\delta \to 0$. Similarly, $F_\delta \in C^\infty(\bar{\Omega})$, $\|F_\delta\| \leqslant \|F_0\|$, $\|F_\delta - F_0\| \to 0$ as $\delta \to 0$, and

$$\min_{x \in \bar{\Omega}} F_\delta(x) \geqslant m . \tag{24}$$

Let now ω_δ be the solution of problem (21) with F_0 and g replaced by F_δ and g_δ, respectively. Since $\omega_\delta \to \omega$ in W_q^2, it follows in particular that $\|\omega_\delta - \omega\| \to 0$ as $\delta \to 0$. Hence, if (23) holds for every pair ω_δ, g_δ, $\delta > 0$, it holds also for the pair ω, g. Let us then prove (23) for ω_δ, g_δ. Note that ω_δ are regular functions (for instance, $\omega_\delta \in C^{2,\beta}(\Omega) \cap C^{1,\beta}(\bar{\Omega})$, for $\beta > 0$). For convenience we denote in the sequel the functions $\omega_\delta, g_\delta, F_\delta$, by ω, g, F_0, respectively.

It is easy to verify the identity

$$\Delta\omega^2 = 2\Delta\omega\cdot\omega + 2\sum_{i,j=1}^{3}\left(\frac{\partial\omega_j}{\partial x_i}\right)^2, \tag{25}$$

where by definition $\omega^2 = \omega\cdot\omega = |\omega|^2$. Hence, by taking the scalar product of both sides of equation $(21)_1$, with $\omega(x)$, one obtains

$$-\frac{1}{2}\varepsilon\Delta\omega^2 + \sum_{i,j=1}^{3}\left(\frac{\partial\omega_j}{\partial x_i}\right)^2 + \frac{1}{2}u\cdot\nabla\omega^2 + F\omega^2 = g\cdot\omega .$$

Consequently,

$$-\tfrac{1}{2}\varepsilon\Delta\omega^2 + \tfrac{1}{2}u\cdot\nabla\omega^2 + F\omega^2 \leqslant |g|\,|\omega| . \tag{26}$$

If ω^2 vanishes identically in Ω, then (23) is obvious. Otherwise, let $x_0 \in \Omega$ be a point of maximum for ω^2 in $\bar{\Omega}$. From (26) together with (24) it follows that

$$m\omega^2(x_0) \leqslant F(x_0)\omega^2(x_0) \leqslant |g(x_0)|\,|\omega(x_0)| ,$$

hence $|\omega(x_0)| \leqslant m^{-1}|g(x_0)|$. This yields (23). □

THEOREM 4. *Let f, g, F and the constant a be defined as in the statement of theorem A, and let $\varepsilon > 0$. Then, there exists a solution $u_\varepsilon \in \in \mathbb{K} \cap \mathbb{W}^2_q$, $p_\varepsilon \in W^1_q$, $\omega_\varepsilon \in \mathbb{Q} \cap \mathbb{W}^2_q$ of problem*

$$(27)\qquad \begin{cases} -\nu\Delta u_\varepsilon + (u_\varepsilon\cdot\nabla)\,u_\varepsilon - \eta\omega_\varepsilon \times u_\varepsilon = f - \nabla p_\varepsilon\,, & \text{in } \Omega\,,\\ -\varepsilon\Delta\omega_\varepsilon + (u_\varepsilon\cdot\nabla)\,\omega_\varepsilon + F(p_\varepsilon)\,\omega_\varepsilon = g\,, & \text{in } \Omega\,,\\ \nabla\cdot u_\varepsilon = 0\,, & \text{in } \Omega\,,\\ u_\varepsilon = \omega_\varepsilon = 0\,, & \text{on } \Gamma\,, \end{cases}$$

where p_ε verifies (4). *Moreover,* (8) *and* (9) *hold.*

PROOF. Let $v \in \mathbb{K}$, $\alpha \in \mathbb{Q}$, and consider the (unique) solution u, p of problem (7) constructed in theorem 1. As shown in that theorem, $u \in \mathbb{K}$ and the estimates (8) and (9) hold.

Now we define $F_0(x) = F(p(x))$, and we consider the solution ω of problem (21) constructed in theorem 3. Clearly, $\omega \in \mathbb{Q} \cap \mathbb{W}^2_q$. In order to prove theorem 4, it suffices to show that the map $\Phi(v, \alpha) = = (u, \omega)$, from $\mathbb{K}\times\mathbb{Q}$ into $\mathbb{K}\times\mathbb{Q}$, has a fixed point. This will be done by using Shauder's fixed point theorem. The continuity of Φ with respect to the uniform topologies follows from the results stated before. Infact, the continuity of the map $(v, \alpha) \to (u, p)$ was proved theorem 2. Let us prove that the map $(u, p) \to \omega$ in continuous. If $p_n \to p$ uniformly in $\bar{\Omega}$ then $F(p_n(x)) \to F(p(x))$ uniformly in $\bar{\Omega}$, since p_n and p verify (9) and $F(\xi)$ is a continuous function in $\mathbb{R}$. On the other hand, $u_n \to u$ in the $\mathbb{C}^1(\bar{\Omega})$ norm, since the sequence u_n is bounded in $\mathbb{W}^2_p$ (by (8)) and the embedding $\mathbb{W}^2_p \hookrightarrow \mathbb{C}^1$ is compact. The last statement in theorem 3 shows that $\omega_n \to \omega$ in $\mathbb{W}^2_q \hookrightarrow \mathbb{C}^0$.

Finally $\Phi(\mathbb{K}\times\mathbb{Q})$ is a bounded set in $\mathbb{W}^2_q \times \mathbb{W}^2_q$, hence it is relatively compact in $\mathbb{C}^0\times\mathbb{C}^0$. □

PROOF OF THEOREM A. From theorem 4, it easily follows that there exists a *subsequence* $(u_\varepsilon, p_\varepsilon, \omega_\varepsilon)$, solution of (27), and functions $u \in \mathbb{K}\times\mathbb{W}^2_q$, $p \in W^1_q$, $\omega \in \mathbb{L}^\infty$, such that (8), (9), (23) hold, and

$$(28)\qquad \begin{cases} u_\varepsilon \to u & \text{weakly in } \mathbb{W}^2_q\,,\\ u_\varepsilon \to u & \text{in } \mathbb{C}^1\,,\\ p_\varepsilon \to p & \text{weakly in } W^1_q\,,\\ p_\varepsilon \to p & \text{in } \mathbb{C}^0\,,\\ \omega_\varepsilon \to \omega & \text{weak-* in } \mathbb{L}^\infty\,. \end{cases}$$

This is proved by using well known compactness theorems. Note that from the uniform estimate $\|\omega_\varepsilon\| \leqslant m^{-1}\|g\|$, it follows that there exists a subsequence ω_ε verifying $(28)_5$, i.e. verifying

$$\lim_{\varepsilon\to 0}\int_\Omega \omega_\varepsilon\cdot\psi\,dx = \int_\Omega \omega\cdot\psi\,dx\,, \qquad \forall\psi\in L^1\,.$$

From (27) and (28) it follows that u, p, ω is a solution of equations $(1)_1$ and $(1)_2$. Moreover $u(x) = 0$, for every $x\in\Gamma$. In order to acomplish the proof of theorem A, we will prove that equation (5) holds. We multiply both sides of equation $(27)_2$ by ω_ε and we integrate over Ω. This gives

$$\varepsilon\|\omega_\varepsilon\|_V^2 + m|\omega_\varepsilon|_2^2 \leqslant |g|_2\,|\omega_\varepsilon|_2 \leqslant \frac{1}{m}\,|g|_2^2\,.$$

Hence $\sqrt{\varepsilon}\|\omega_\varepsilon\|_V \leqslant \sqrt{1/m}\,|g|_2$. In particular

$$\lim_{\varepsilon\to 0}\varepsilon\|\omega_\varepsilon\|_V = 0\,. \tag{29}$$

Multiply $(27)_2$ by $\varphi\in \mathbb{W}_2^1$, and integrate over Ω. By doing some integrations by parts, one has

$$\varepsilon\sum_{i,j=1}^{3}\int_\Omega \frac{\partial(\omega_\varepsilon)_j}{\partial x_i}\frac{\partial\varphi_j}{\partial x_i}\,dx - \int_\Omega (u_\varepsilon\cdot\nabla)\varphi\cdot\omega_\varepsilon\,dx + \int_\Omega F(p_\varepsilon)\,\omega_\varepsilon\cdot\varphi\,dx = \int g\cdot\varphi\,dx\,. \tag{30}$$

By taking into account (28), (29), and by passing to the limit in (30) as $\varepsilon\to 0$, equation (5) follows. Note that

$$\lim_{\varepsilon\to 0} F(p_\varepsilon(x)) = F(p(x))\,,$$

uniformly in $\bar{\Omega}$. □

Uniqueness. By assuming, for convenience, that $F(\xi)$ is locally lipschitz continuous, it is not difficult to verify that $\nabla\omega\in L^1$ is the main additional assumption in order to prove the uniqueness of the solution, for *small data* f and g.

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