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A Well-Posedness Theorem for Non-homogeneous Inviscid Fluids via a Perturbation Theorem

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1. INTRODUCTION

Continuous dependence in *strong* topology of the solution on the data (well-posedness) is a difficult but significant part of the theory of non-linear hyperbolic equations. By using the powerful abstract theory developed by Kato [12, 13], I prove in the recent paper [5] a perturbation theorem that can be successfully applied in order to show the well-posedness of a class of non-linear hyperbolic equations that appear in the applications.

Here, I show how to apply this perturbation theorem to concrete problems by considering a specific one, namely the motion of an inviscid, incompressible, non-homogeneous fluid in a bounded domain Ω. I will prove the well-posedness of this problem in Sobolev spaces $W^{m,p}(\Omega)$. The proof applies as well to unbounded domains (actually, the main difficulties are due to the boundary).

Well-posedness for the above specific problem was stated by Marsden [18] under the assumption that f vanishes identically. This author considers the $H^{s,2}$ case, and asserts that $W^{k,p}$ and $C^{k+\alpha}$ are similar. In [18] it is also claimed that the continuous dependence on the initial data holds if a fixed force term f is added to equations, provided it is divergence free and parallel to the boundary.

It is worth noting that the two approaches are completely distinct. Marsden's proof uses techniques of infinite dimensional geometry, by replacing the original problem with one of finding geodesics with respect to a weak Riemannian metric on an infinite dimensional manifold of diffeomorphisms. My proof uses Kato's perturbation theory for linear hyperbolic evolution equations (cf. [11, 12, 13]).

The Marsden (Ebin and Marsden) approach is done directly in the Lagrangian representation, where some analytical difficulties disappear because there the connective term $(v \cdot \nabla)v$ is transformed away. The pressure term becomes more complicated. In general analysts consider this

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approach to be difficult because of the heavy dose of infinite dimensional geometry used.

In Eulerian representation (used here) the continuous dependence on initial conditions is delicate. Note that in material representation the solution depends in a C^∞ way on the initial condition, while in spatial representation the dependence is only continuous (the material-spatial map is only C^0). Non-smooth external forces are easier to treat in spatial coordinates.

I point out that the perturbation theorem proved in [5] (see also Theorem 2.1 below) applies to equations of the form $a_0 D_t u + \sum_{i=1}^n a_i D_i u + au = f$, where the $N \times N$ symmetric matrices $a_i(t, x)$ satisfy the condition $\sum_{i=1}^n \nu_i a_i = 0$ on Γ and the eigenvalues of $a_0(t, x)$ are strictly positive, *provided* $p = 2$. This result then can be applied to prove well-posedness in $W^{m,2}(\Omega)$ spaces for non-linear hyperbolic problems.

Before going on, I recall that well-posedness for inviscid *homogeneous* fluids (in domains with boundaries), is studied in [10, 9, 3, 14, 5].

2. The Perturbation Theorem

I start by introducing some notation. Let Ω be an open bounded set in $\mathbf{R}^n$, $n \geqslant 2$, that lies locally on one side of its boundary Γ, a C^{m+2} manifold. The positive integer m will be fixed in the sequel. We denote by ν the unit outward normal to Γ. We denote simply by L^p the Banach space $L^p(\Omega)$, by $| \ |_p$ its canonical norm, by W^k the Sobolev space $W^{k,p}(\Omega)$, and by $\| \ \|_k$ its canonical norm defined as in [4, 5]. This notation will also be used to denote function spaces whose elements are vector fields. For instance, both W^m and $W^m \times \cdots \times W^m$ (N times) will be denoted by the same symbol W^m, and the corresponding norms by the same symbol $\| \ \|_m$ (sometimes we will write $v_0, \rho_0 \in W^m$, even when v_0 is a vector field and ρ_0 is a scalar).

We use standard notations for functional spaces consisting of functions defined on an interval $I = [t_0, t_1]$ with values in a Banach space. If $v(t, x)$ is defined on $I \times \Omega$, $v(t)$ denotes the function $v(t, \cdot)$ defined on Ω. The norm in $L^\infty(I; W^k)$ is denoted by $\| \ \|_{I,k}$, and that in $L^q(I; W^k)$ by $\| \ \|_{q,I;k}$.

If X is a Banach space, $\mathscr{L}(X)$ denotes the Banach space of all bounded linear operators from X into X.

The symbol c denotes positive constants that will depend at most on Ω, n, N, p, m. In Sections 3 and 4, these constants may also depend on q. The symbol $\mathbb{N}$ denotes the set of all positive integers.

Let $I = [-T, T]$, $T > 0$, and let $v = (v_1, ..., v_n)$ be a vector field defined on $I \times \bar{\Omega}$, such that

$$v \in L^\infty(I; W^m) \cap C(I; W^{m-1}), \tag{2.1}$$

and that

$$v \cdot \nu = 0 \qquad \text{on } \Gamma, \text{ for each } t \in I, \tag{2.2}$$

where $p \in]1, +\infty[$, and $m > 1 + (n/p)$. Let $u = (u_1, ..., u_N) \in W^2$, $N \geq 1$, and define (for each $t \in I$) the differential operator

$$A(t)\, u(x) = (v(t, x) \cdot \nabla)\, u(x), \tag{2.3}$$

with domain $D(t) = \{u \in W^2 : (v(t) \cdot \nabla)\, u \in W^2\}$. In [5] (see also [4]) I show that the family of unbounded operators $A(t)$, $t \in I$, generates an evolution operator $U(t, s)$ (cf. [11, 12, 19]), $t, s \in I$, in W^2. In particular, $U(t, s) \in \mathscr{L}(W^2)$, $U(s, s) = \text{Identity}$, $U(t, r)\ U(r, s) = U(t, s)$, $\forall t, r, s \in I$. Moreover $U(t, s)$ is strongly continuous on $I \times I$; i.e., the map $U(\cdot, \cdot) u_0$ is continuous on $I \times I$ with values in W^2, for each fixed $u_0 \in W^2$. The evolution operator $U(t, s)$ generated by the family $\{A(t)\}$, $t \in I$, is characterized by the following property. For each fixed $s \in I$, and for each $u_0 \in W^2$, the function $u(t) = U(t, s) u_0$ is the solution of the Cauchy problem $D_t u + A(t)\, u(t) = 0$ on I, $u(s) = u_0$.

If $f \in L^1(I; W^2)$, the solution of the Cauchy problem $D_t u + (v(t) \cdot \nabla) u = f(t)$, $t \in I$, $u(0) = u_0$, is given by

$$u(t) = U(t, 0) u_0 + \int_0^t U(t, s)\, f(s)\, ds. \tag{2.4}$$

Let us recall the part of the perturbation theorem proved in [5] that will be used here. Assume that $\{v_k\}$, $k \in \mathbb{N}$, is a sequence of coefficients that verify assumptions (2.1) and (2.2), and such that

the quantities $\|v_k\|_{I,m}$ are uniformly bounded with respect to k, (2.5)

and that

$$\lim_{k \to +\infty} v_k = v \qquad \text{in} \quad C(I; W^{m-1}). \tag{2.6}$$

By using as coefficients the functions v_k instead of v, we define (in the obvious way) operators $A_k(t)$, domains $D_k(t)$, and evolution operators $U_k(t, s)$. One has the following result [5, Theorem 4.2].

THEOREM 2.1. *Let* $p \in]1, +\infty[$, *and let* $m \in \mathbb{N}$ *be such that* $m > \max\{2, 1 + (n/p)\}$. *Under the above assumptions on the coefficients* v *and* v_k, *one has*

$$\lim_{k \to +\infty} U_k(t, s) = U(t, s) \tag{2.7}$$

strongly in W^2, uniformly on $I \times I$. Moreover

$$|||U_k(t, s)||| \leqslant e^{\theta_k T}, \qquad \forall t, s \in I, \tag{2.8}$$

where $\theta_k = c(\Omega, n, N, p, m) \|v_k\|_{I,m}$, *for a suitable constant* c, *and* $|||\ |||$ *denotes the norm in* $\mathscr{L}(W^2)$.

3. An Application

Let m and p be as in Theorem 2.1. In this section we apply Theorem 2.1 to prove the well-posedness of the equations describing the motion of a non-homogeneous, inviscid, incompressible fluid, namely

$$\begin{aligned} &D_t v + (v \cdot \nabla) v = -\frac{1}{\rho} \nabla \pi + f, \\ &D_t \rho + v \cdot \nabla \rho = 0, \\ &\operatorname{div} v = 0 \qquad \text{in} \quad I \times \Omega, \\ &v \cdot \nu = 0 \qquad \text{on} \quad I \times \Gamma, \\ &v(0) = v_0, \qquad \rho(0) = \rho_0 \end{aligned} \tag{3.1}$$

(see Sédov [23, Chap. IV, Sect. 1, p. 164]). For a mathematical study of system (3.1) see Marsden [18], Beirão da Veiga and Valli [7, 8], and Valli and Zajackowski [22]. Let us give some references also on papers concerning the motion of *incompressible*, non-homogeneous fluids in the case in which *viscosity terms* are added to Eq. (3.1). Inviscid, non-homogeneous fluids in the presence of diffusion (say, a continuous medium consisting of two components, for example, water and a dissolved salt) are studied by Beirão da Veiga, Serapioni, and Valli [6]. Viscous, non-homogeneous fluids are studied by Kazhikhov [15], Ladyzhenskaya and Solonnikov [16], and Simon [20]. Viscous, non-homogeneous fluids in the presence of diffusion are studied by Beirão da Veiga [2] and Secchi [21]. Let us return to the proof of the well-posedness of system (3.1). For convenience we will restrict the time variable to the non-negative real axis. Hence, $I = [0, T]$, $T > 0$. We assume that

$$v_0 \in W^m, \qquad \operatorname{div} v_0 = 0 \text{ in } \Omega,\ v_0 \cdot \nu = 0 \text{ on } \Gamma, \tag{3.2}$$

$$\rho_0 \in W^m, \qquad a \leqslant \rho_0(x) \leqslant b,\ \forall x \in \Omega, \tag{3.3}$$

where a and b are positive constants, and that

$$f \in L^q(I; W^m), \qquad \text{where} \quad q > 1. \tag{3.4}$$

It is well known that the system (3.1) has a unique solution $(v, \rho, \nabla\pi)$ on $I^* = [0, T^*]$, if $T^* > 0$ is sufficiently small. Here we are interested in solutions in Sobolev spaces W^m. We refer the reader to H. Beirão da Veiga and A. Valli [8] ($n = 2, 3$; $p = 2$. The method applies as well if $p \neq 2$) and to A. Valli and W. M. Zajaczkowski [22]. An alternative proof of the existence theorem for local solutions in Sobolev spaces can be done by adapting to non-homogeneous fluids the proof given for homogeneous fluids in [4, Sect. 5].

The local existence result asserts that there is a positive T^* and a (unique) solution $v, \rho \in L^\infty(I^*; W^m)$, $\nabla\pi \in L^q(I^*; W^m)$ of the system (3.1) on $I^* = [0, T^*]$. Moreover $a \leqslant \rho(t, x) \leqslant b$ on $I^* \times \Omega$. Actually, a lower bound for T^* depends only on the norms of the data v_0, ρ_0, f, and decreases if these norms increase. More precisely, there exists a positive real function $\tau(M, M')$, defined for positive values of M and M', which is non-increasing as a function of each of the variables separately, and such that if

$$\|v_0\|_m < M, \qquad \|\rho_0\|_m < M, \qquad \|f\|_{1,I;m} < M', \tag{3.5}$$

then the solution (v, ρ) exists (at least) on $I_\tau = [0, \tau]$.

Moreover, there exists a real positive function $L(M, M')$, which is non-decreasing as a function of each of the variables separately, and such that $\|v\|_{I_\tau, m} < L(M, M')$, $\|\rho\|_{I_\tau, m} < L(M\ M')$, if (3.5) holds.

Note that by using Eqs. (3.1) and (3.10) (Eq. (3.10) follows from Eq. (3.1)), one easily verifies that the norms $\|\nabla\pi\|_{q, I_\tau; m}$, $\|D_t v\|_{q, I_\tau; m-1}$, $\|D_t \rho\|_{I_\tau, m-1}$ are bounded by quantities that depend (non-decreasingly) on the above norms of the data, and on $\|v\|_{I_\tau, m}$ and $\|\rho\|_{I_\tau, m}$. Hence, we can assume, by eventually changing the bound L, that the above norms of $\nabla\pi$, $D_t v$, $D_t \rho$, are bounded by $L(M, M')$. Here, we do not take into account the dependence on $a\ b$, Ω, n, p, m, q, since all these parameters are fixed.

By setting $f(t) = 0$, for $t > T$, we assume (without loss of generality) that $I = [0, +\infty[$. Let now (v_0^k, ρ_0^k), $k \in \mathbb{N}$, be a sequence of data satisfying the assumptions (3.2), (3.3), and (3.4), for each $k \in \mathbb{N}$. Moreover, assume that

$$\lim_{k \to +\infty} v_0^k = v_0, \qquad \lim_{k \to +\infty} \rho_0^k = \rho_0, \qquad \text{in } W^m, \tag{3.6}$$

and that

$$\lim_{k \to +\infty} f_k = f, \qquad \text{in} \quad L^q(I; W^m). \tag{3.7}$$

For each $k \in \mathbb{N}$ denote by $v_k, \rho_k, \nabla\pi_k$, the solution of Eq. $(3.1)_k$. We denote by $(3.1)_k$ Eq (3.1) if v_0, ρ_0, f are replaced by v_0^k, ρ_0^k, f_k, respectively. Note that the solutions v_k, ρ_k are defined on all I_τ, for sufficiently large values of k, since for these values of k the data v_0^k, ρ_0^k, f_k verify the assumptions (3.5). One has the following well-posedness theorem.

THEOREM 3.1. *Let* v_0, ρ_0, f, *and also* v_0^k, ρ_0^k, f_k, $k \in \mathbb{N}$, *verify the assumptions* (3.2), (3.3), *and* (3.4). *Assume that* (3.6) *and* (3.7) *hold. Let* $\tau = \tau(M, M')$, *where* M *and* M' *satisfy* (3.5). *Then*

$$\lim_{k \to +\infty} v_k = v, \qquad \lim_{k \to +\infty} \rho_k = \rho, \qquad in \quad C(I_\tau; W^m). \tag{3.8}$$

The following result is a trivial consequence of (3.8) together with Eqs. (3.10), $(3.1)_1$, $(3.1)_2$.

COROLLARY 3.2. *Under the hypothesis described in Theorem* 3.1, *one has* $\nabla\pi_k \to \nabla\pi$ *in* $L^q(I_\tau; W^m)$, $D_t v_k \to D_t v$ *in* $L^q(I_\tau; W^{m-1})$, *and* $D_t \rho_k \to D_t \rho$ *in* $C(I_\tau; W^{m-1})$. *If* $f_k \to f$ *in* $C(I; W^m)$ *then* $\nabla\pi_k \to \nabla\pi$ *in* $C(I_\tau; W^m)$; *and* $D_t v_k \to D_t v$ *in* $C(I_\tau; W^{m-1})$.

In order to prove Theorem 3.1, we state the following auxiliary result.

LEMMA 3.3. *Under the hypothesis of Theorem* 3.1, *one has*

$$\lim_{k \to +\infty} v_k = v, \qquad in \quad C(I_\tau; W^{m-1}). \tag{3.9}$$

Proof. By applying the divergence operator to both sides of Eq. $(3.1)_1$, and by taking the scalar product of both sides of $(3.1)_1$ with ν, it readily follows that

$$\begin{aligned} \operatorname{div}\left(\frac{1}{\rho}\nabla\pi\right) &= -\sum_{i,j=1}^{n} (D_i v_j)(D_j v_i) + \operatorname{div} f, && \text{in } \Omega, \\ \frac{\partial\pi}{\partial\nu} &= \rho \sum_{i,j=1}^{n} (D_i \nu_j) v_i v_j + \rho f, && \text{on } \Gamma, \end{aligned} \tag{3.10}$$

for each $t \in I_\tau$. Similar equations (denoted by $(3.10)_k$) hold for v_k, ρ_k, $\nabla\pi_k$, f_k, instead of v, ρ, $\nabla\pi$, f. Since the functions v_k are uniformly bounded on $L^\infty(I_\tau; W^m)$ and on $W^{1,q}(I_\tau; W^{m-1})$, and since this last space is continuously embedded on $C^{1-(1/q)}(I_\tau; W^{m-1})$, it follows (by Ascoli and Arzelà's theorem) that the sequence $\{v_k\}$ is relatively compact in $C(I_\tau; W^{m-1})$. Similarly, the sequence $\{\rho_k\}$ is relatively compact in $C(I_\tau; W^{m-1})$. Hence, from the elliptic boundary value problem $(3.10)_k$ it readily follows that the sequence $\{\nabla\pi_k\}$ is relatively compact in $L^q(I_\tau; W^{m-1})$. The desired result follows, since limits of convergent subsequences of the sequence $(v_k, \rho_k, \nabla\pi_k)$ are solutions of (3.1), and the solution of (3.1) is unique. ∎

Proof of Theorem 3.1. Let D^α be any space derivative such that $0 \leqslant |\alpha| \leqslant m-2$. By applying D^α to both sides of Eqs. $(3.1)_1$, $(3.1)_2$, $(3.1)_5$, $(3.1)_6$, one gets

$$D_t(D^\alpha v) + (v\cdot\nabla)D^\alpha v = F^\alpha[v] - D^\alpha\left(\frac{1}{\rho}\nabla\pi\right) + D^\alpha f,$$
$$D_t(D^\alpha \rho) + v\cdot\nabla D^\alpha\rho = H^\alpha[v,\rho], \qquad \text{for} \quad t\in I_\tau, \tag{3.11}$$
$$(D^\alpha v)(0) = D^\alpha v_0, \qquad (D^\alpha\rho)(0) = D^\alpha\rho_0,$$

where by definition $F^\alpha[v] = (v\cdot\nabla)D^\alpha v - D^\alpha[(v\cdot\nabla)v]$, $H^\alpha[v,\rho] = v\cdot\nabla D^\alpha\rho - D^\alpha[v\cdot\nabla\rho]$. Note that the higher order terms are of order $m-2$. The reader should note that in our approach to the well-posedness problem the boundary condition and the divergence free condition do not appear.

By replacing in Eq. (3.11) the functions v, ρ, $\nabla\pi$, v_0, ρ_0, f by v_k, ρ_k, $\nabla\pi_k$, v_0^k, ρ_0^k, f_k, respectively, we get a sequence of equations, denoted by $(3.11)_k$. By using Sobolev's embedding theorems and Hölder's inequality it readily follows that, for each $t\in I_\tau$,

$$\|F^\alpha[v] - F^\alpha[v_k]\|_2 \leqslant C\|v - v_k\|_m, \tag{3.12}$$

and that

$$\|H^\alpha[v,\rho] - H^\alpha[v_k,\rho_k]\|_2 \leqslant C(\|v-v_k\|_m + \|\rho-\rho_k\|_m), \tag{3.13}$$

since the norms $\|v_k\|_{I_\tau,m}$ and $\|\rho_k\|_{I_\tau,m}$ are uniformly bounded. Here, as well as in the sequel, the symbol C denotes any positive constant that depends only on M and M' (and on the fixed parameters described above). On the other hand one has for each $t\in I_\tau$,

$$\left\|\frac{1}{\rho}\nabla\pi - \frac{1}{\rho_k}\nabla\pi_k\right\|_m \leqslant C\|v-v_k\|_m + C(1+\|f_k\|_m)\,\|\rho-\rho_k\|_m + C\|f-f_k\|_m. \tag{3.14}$$

This last estimate is proved by applying well-known L^p-estimates to the elliptic boundary value problems (3.10), $(3.10)_k$. For the reader's convenience, the proof is sketched in the Appendix.

By setting $u=(v,\rho)$, $u_0=(v_0,\rho_0)$, $G^\alpha = (F^\alpha[v] - D^\alpha(\rho^{-1}\nabla\pi) + D^\alpha f, H^\alpha[v,\rho])$, $G_k^\alpha = (F^\alpha[v_k] - D^\alpha(\rho_k^{-1}\nabla\pi_k) + D^\alpha f_k, H^\alpha[v_k\ \rho_k])$, Eq. (3.11) can be written in the more compact form

$$D_t(D^\alpha u) + (v\cdot\nabla)D^\alpha u = G^\alpha, \qquad t\in I_\tau,$$
$$(D^\alpha u)(0) = D^\alpha u_0, \tag{3.15}$$

and Eq. $(3.11)_k$ in a similar form, denoted by $(3.15)_k$. For each fixed multi-index α, (3.15) and $(3.15)_k$ are systems of $N=n+1$ linear evolution equations in the Banach space W^2. Formula (2.4) shows that

$$(D^\alpha u)(t) = U(t,0)D^\alpha u_0 + \int_0^t U(t,s)\,G^\alpha(s)\,ds, \tag{3.16}$$

and that

$$(D^\alpha u_k)(t) = U_k(t, 0) D^\alpha u_0^k + \int_0^t U_k(t, s)\, G_k^\alpha(s)\, ds, \qquad (3.16)_k$$

for $t \in I_\tau$.[1] By subtracting side by side Eq. $(3.16)_k$ from the Eq. (3.16), one easily verifies that

$$\begin{aligned}\|D^\alpha u(t) - D^\alpha u_k(t)\|_2 &\\ \leqslant \|(U(t,0) - U_k(t,0)) D^\alpha u_0\|_2 &+ |||U_k(t,0)|||\ \|D^\alpha u_0 - D^\alpha u_0^k\|_2 \\ &+ \int_0^t \|(U(t,s) - U_k(t,s))\, G^\alpha(s)\|_2\, ds \\ &+ \int_0^t |||U_k(t,s)|||\ \|G^\alpha(s) - G_k^\alpha(s)\|_2\, ds. \end{aligned} \qquad (3.17)$$

The estimates (3.12), (3.13), (3.14) show that $\|G^\alpha(s) - G_k^\alpha(s)\|_2$ is bounded by the right hand side of Eq. (3.14), for suitable values of the constants C. On the other hand, (2.8) shows that $|||U_k(t,s)||| \leqslant C$. Hence, for fixed $\sigma \in]0, \tau]$, one has

$$\begin{aligned}\|D^\alpha u(t) - D^\alpha u_k(t)\|_2 &\\ \leqslant \|(U(t,0) - U_k(t,0)) D^\alpha u_0\|_2 &+ C\|u_0 - u_0^k\|_m \\ &+ \int_0^\sigma \|(U(t,s) - U_k(t,s))\, G^\alpha(s)\|_2\, ds + C(\sigma + \sigma^{1-1/q})\, \|u - u_k\|_{I_\sigma, m} \\ &+ C\int_0^\sigma \|f(s) - f_k(s)\|_m\, ds, \end{aligned} \qquad (3.18)$$

for every $t \in [0, \sigma]$. By adding side by side these inequalities for all multi-index α satisfying $0 \leqslant |\alpha| \leqslant m-2$, and by taking the supremum of both sides as t runs over I_σ, one gets

$$\begin{aligned}\|u - u_k\|_{I_\sigma, m} \leqslant \sum_\alpha \sup_{t\in I_\sigma} \|(U(t,0) - U_k(t,0)) D^\alpha u_0\|_2 &+ C\|u_0 - u_0^k\|_m \\ + c\sum_\alpha \int_0^\sigma \sup_{t \in I_\sigma} &\|(U(t,s) - U_k(t,s))\, G^\alpha(s)\|_2\, ds \\ + C_1(\sigma + \sigma^{1-(1/q)})\, &\|u - u_k\|_{I_\sigma, m} + C\int_0^\sigma \|f(s) - f_k(s)\|_m\, ds. \end{aligned} \qquad (3.19)$$

[1] Equations (3.16) and $(3.16)_k$ prove, in particular, that $u, u_k \in C(I_\tau; W^m)$.

Let σ be the solution of the equation $C_1(\sigma+\sigma^{1-(1/q)})=\frac{1}{2}$ if this solution is less than $\tau(M, M')$. Otherwise, we set $\sigma=\tau(M, M')$. For this choice of σ, we are allowed to drop the forth term on the right hand side of (3.19), by multiplying the left hand side by $\frac{1}{2}$. It readily follows, by using in particular the dominated convergence theorem, that $u_k \to u$ in $C([0, \sigma]; W^m)$. By applying this result successively to the intervals $[j\sigma, (j+1)\sigma] \cap [0, \tau]$, one shows that $u_k \to u$ in $C([0, \tau]; W^m)$. ∎

Let us explicitely state the following consequence of the above result.

THEOREM 3.4. *Let v_0 ρ_0, and f, verify the assumptions* (3.2), (3.3), *and* (3.4), *and let $[0, T[$ be the maximal interval of existence of a solution v, ρ of problem* (3.1) *in the space $L^\infty_{\text{loc}}([0, T[; W^m)$.*[2] *Then, to each $\varepsilon>0$ and to each $T_0 \in]0, T[$ there corresponds a positive $\delta=\delta(\varepsilon, T_0)$, such that the following result holds. If data v_0', ρ_0', f' verify assumptions* (3.2), (3.3), (3.4), *and belong to the neighborhood*

$$\|v_0'-v_0\|_m<\delta, \qquad \|\rho_0'-\rho_0\|_m<\delta, \qquad \|f-f'\|_{q,I_0;m}<\delta, \tag{3.20}$$

then the solution v', ρ' of problem (1.3) *with data v_0', ρ_0', f' exists on $I_0=[0, T_0]$ and belongs to $C(I_0; W^m)$. Furthermore,*

$$\|v'(t)-v(t)\|_m<\varepsilon, \qquad \|\rho'(t)-\rho(t)\|_m<\varepsilon, \tag{3.21}$$

uniformly on $[0, T_0]$.

COROLLARY 3.5. *Under the hypothesis of Theorem* 3.4, *one also has*

$$\|\nabla\pi-\nabla\pi'\|_{q,I_0;m}<\varepsilon, \quad \|D_t v-D_t v'\|_{q,I_0;m-1}<\varepsilon, \quad \|D_t\rho-D_t\rho'\|_{I_0,m-1}<\varepsilon. \tag{3.22}$$

Moreover, if f, $f_k \in C(I_0; W^k)$, and $f_k \to f$ in this functional space, then (3.22) *holds by replacing $L^q(I_0)$ by $C(I_0)$.*

Proof of Theorem 3.4. Let M, M' be positive real numbers such that $\|v\|_{I_0,m}<M$, $\|\rho\|_{I_0,m}<M$, and $\|f\|_{q,I_0;m}<M'$. Set $\tau=\tau(M, M')$, and let $[t_0, t_0+\tau]$ be an arbitrary subinterval of $[0, T_0]$, of length τ. Since $\|v(t_0)\|_m<M$, $\|\rho(t_0)\|_m<M$, and $\|f\|_{q,[t_0,t_0+\tau];m}<M'$, Theorem 3.1 shows that to each $\varepsilon>0$ there corresponds a $\delta>0$ such that if $(v'(t_0), \rho'(t_0))$ belongs to the δ-neighborhood of $(v(t_0), \rho(t_0))$ then $(v'(t), \rho'(t))$ belongs to the ε-neighborhood of $(v(t), \rho(t))$, for all $t \in [t_0, t_0+\tau]$. By applying this result successively on intervals $[j\tau, (j+1)\tau] \cap [0, T_0]$, $j=0, 1, ..., J$, where $(T_0/\tau)-1 \leqslant J<T_0/\tau$, we prove (3.21). ∎

[2] This means that $v, \rho \in L^\infty(0, T_0, W^m)$, for all $T_0<T$. Note that, from the previous results, it follows that $v, \rho \in C([0, T[; W^m)$.

4. Appendix

Here we prove the estimate (3.14). Since $t \in I_\tau$ is fixed, we drop it from the notation. From Eqs. (3.10) and $(3.10)_k$ it readily follows that (for convenience, the ith component of v_k is denoted by v_i^k)

$$\begin{aligned}\Delta(\pi-\pi_k) = {} & \rho^{-1}\nabla\rho\cdot\nabla(\pi-\pi_k)+\rho^{-1}\nabla(\rho-\rho_k)\cdot\nabla\pi_k \\ & +[(\rho_k-\rho)/\rho\rho_k]\,\nabla\rho_k\cdot\nabla\pi_k \\ & -\rho\sum_{i,j}[(D_iv_j)(D_jv_i)-(D_iv_j^k)(D_jv_i^k)] \\ & -(\rho-\rho_k)\sum_{i,j}(D_iv_j^k)(D_jv_i^k) \\ & +\rho\,\operatorname{div}(f-f_k)+(\rho-\rho_k)\operatorname{div} f_k, \qquad \text{in } \Omega, \\ \partial(\pi-\pi_k)/\partial\nu = {} & \rho\sum_{i,j}(D_i\nu_j)(v_iv_j-v_i^kv_j^k) \\ & +(\rho-\rho_k)\sum_{i,j}(D_i\nu_j)v_i^kv_j^k \\ & +\rho(f-f_k)\cdot\nu+(\rho-\rho_k)f_k\cdot\nu, \qquad \text{on } \Gamma.\end{aligned} \tag{4.1}$$

Well-known L^p-estimates for solutions of elliptic boundary value problems (cf. [1] and references) show that $\|\nabla(\pi-\pi_k)\|_m \leqslant c(\|\Delta(\pi-\pi_k)\|_{m-1}+\|\partial(\pi-\pi_k)/\partial\nu\|_{m-(1/p),\Gamma})$. Hence, by using (4.1) together with Sobolev's embedding theorems and Hölder inequalities, and by recalling that $\|v_k\|_{I_\tau,m}\leqslant C$ and that $\|\rho_k\|_{I_\tau,m}\leqslant C$, it readily follows that

$$\begin{aligned}\|\nabla\pi-\nabla\pi_k\|_m \leqslant {} & C_1\|\nabla\pi-\nabla\pi_k\|_{m-1}+C\|v-v_k\|_m \\ & +C(1+\|\nabla\pi_k\|_{m-1}+\|f_k\|_{m-1})\,\|\rho-\rho_k\|_m+C\|f-f_k\|_m.\end{aligned} \tag{4.2}$$

On the other hand, the norm of the coefficient $\rho^{-1}\nabla\rho$ is bounded in $C^{0,\alpha}$ by a constant C, since $W^{m-1}\subsetneq C^{0,\alpha}$ for a suitable $\alpha>0$. Hence, by well-known L^p-estimates for elliptic operators, one has

$$\begin{aligned}|\nabla(\pi-\pi_k)|_p \leqslant c(& \|\Delta(\pi-\pi_k)-\rho^{-1}\nabla\rho\cdot\nabla(\pi-\pi_k)\|_{m-1} \\ & +\|\partial(\pi-\pi_k)/\partial\nu\|_{m-(1/p),\Gamma}).\end{aligned}$$ [3]

Hence

$$\begin{aligned}|\nabla(\pi-\pi_k)|_p \leqslant {} & C\|v-v_k\|_m+C(1+\|\nabla\pi_k\|_{m-1}+\|f_k\|_{m-1}) \\ & \times\|\rho-\rho_k\|_m+C\|f-f_k\|_m.\end{aligned} \tag{4.3}$$

[3] This very rough estimate is sufficient for our purposes here. See the following remark.

Remark. We could claim directly for $\|\nabla(\pi-\pi_k)\|_m$ the estimate stated above for $|\nabla(\pi-\pi_k)|_p$. In fact, the regularity of the coefficient $\rho^{-1}\cdot\nabla\rho$ in the elliptic operator $\Delta-\rho^{-1}\nabla\rho\cdot\nabla$, namely $\rho^{-1}\cdot\nabla\rho\in W^{m-1}$, is sufficient for that purpose. However, for the reader's convenience, we utilize only L^p-estimates for elliptic operators with coefficients in C^l spaces (rather than with coefficients in suitable Sobolev spaces) since this is the case usually treated in the current literature.

Now, by a well-known lemma of J. L. Lions [17], to each $\varepsilon>0$ there corresponds a $\lambda(\varepsilon)>0$ such that $\|\nabla(\pi-\pi_k)\|_{m-1}\leqslant\varepsilon\|\nabla(\pi-\pi_k)\|_m+\lambda(\varepsilon)|\nabla(\pi-\pi_k)|_p$. By setting $\varepsilon=C_1/2$, and by using (4.2) and (4.3), we show that $\|\nabla\pi-\nabla\pi_k\|_m\leqslant C\|v-v_k\|_m+C(1+\|\nabla\pi_k\|_{m-1}+\|f_k\|_{m-1})\|\rho-\rho_k\|_m+C\|f-f_k\|$. Consequently,

$$\|\nabla\pi-\nabla\pi_k\|_m\leqslant C\|v-v_k\|+C(1+\|f_k\|_m)\|\rho-\rho_k\|_m+C\|f-f_k\|_m, \qquad (4.4)$$

since $\|\nabla\pi_k\|_m\leqslant C(1+\|f_k\|_m)$. This last estimate can be proved by applying the above arguments, this time to the elliptic problem $(3.10)_k$ (written in the form that corresponds to (4.1)). From (4.4), together with the estimates $\|\rho\|_{I_\tau,m}\leqslant C$, $\|\rho_k\|_{I_\tau,m}\leqslant C$, (3.14) readily follows.

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