HBU - 53

Annali di Matematica pura ed applicata
(IV), Vol. CLXI (1992), pp. 153-165

Attracting Properties for One Dimensional Flows of a General Barotropic Viscous Fluid. Periodic Flows (*).

H. BEIRÃO DA VEIGA

Summary. – *We consider the motion of a barotropic compressible fluid in a one dimensional bounded region with impermeable boundary, see equation* (1.1). *Here,* $u(t, q)$ *denotes the velocity and* $v(t, q)$ *the specific volume. The quantity* $\log v(t, q)$ *measures the displacement of* $v(t, q)$ *with respect to the equilibrium* $v \equiv 1$. *For the sake of brevity we denote here different norms by the simbol* $\| \ \|$. *We show that there is a positive constant* $r_0 = r_0(\mu)$, *a small ball* $B_1(r)$ *(with radius* $R_1(r)$, $\lim_{r \to 0} R_1(r) = 0$*), and a large ball* $B(r)$ *(with radius* $R(r)$, $\lim_{r \to 0} R(r) = +\infty$*) such that the following holds, for each* $r \in [0, r_0[$. (i) *If* $\|f(t)\| < r$ *for all* $t \geqq 0$, *and if* $\|(u(0), \log v(0))\| \leqq R(r)$ *(i.e.* $(u(0), \log v(0)) \in B(r)$*) then, for sufficiently large values of* t, $\|(u(t), \log v(t))\| \leqq R_1(r)$; (ii) *The solutions starting at time* $t = 0$ *from the large ball* $B(r)$ *have all the same asymptotic behaviour (see* (1.11)*)*; (iii) *If* f *is* T*-periodic then there is a (unique)* T*-periodic solution* $(u(t), \log v(t))$ *inside the small ball* $B_1(r)$. *This periodic solution atracts all solutions which intersect the large ball* $B(r)$. *Periodic solutions had been previously studied only for very specific pressure laws, namely* $p(v) = \log v$ *and* $p(v) = v^{-1}$.

1. – Main results.

In this paper we consider the motion of a barotropic compressible fluid in a one dimensional bounded region with impermeable boundary, for a *general* pressure law. By using material Lagrangian coordinates, and after a normalization, the equations of motion are

$$(1.1) \qquad \begin{cases} v_t = u_q\,, \\ u_t = \mu(v^{-1}u_q)_q - p(v)_q + f\left(t, \displaystyle\int_0^q v(t,\xi)\,d\xi\right), \\ u(t,0) = u(t,1) = 0\,, \end{cases}$$

where $q \in \Omega \equiv]0, 1[$, and $t \geqq 0$. Here, $u(t, q)$ is the velocity and $v(t, q)$ is the specific

(*) Entrata in Redazione il 7 aprile 1989.

Indirizzo dell'A.: Univ. of Pisa, Ist. Mat. Appl. «U. Dini», Via Bonanno 25/B, Pisa.

volume. The external force $f(t, x)$ is given as a known function of the Eulerian coordinates (t, x). The real function $p(\cdot)$ has a (locally) Lipschitz continuous first derivative on $]0, +\infty[$. Moreover, $p'(s) < 0$, $\forall s \in]0, +\infty[$. Without loss of generality, we assume that $p(1) = 1$.

Let us consider the conditions

$$\min_{v \in \Omega} v > 0, \tag{1.2}$$

$$\int_0^1 v(q)\, dq = 1, \tag{1.3}$$

and

$$\exists q_1 \in [0, 1] \quad \text{such that } v(q_1) = 1. \tag{1.4}$$

For convenience, we set

$$\hat{H}^1(\Omega) \equiv \{v \in H^1(\Omega) \colon (1.2), (1.3) \text{ are satisfied}\},$$

$$K^1(\Omega) \equiv \{v \in H^1(\Omega) \colon (1.2), (1.4) \text{ are satisfied}\}.$$

Clearly, $\hat{H}^1 \subset K^1$. $H_0^1(\Omega)$ is the closure of $\mathcal{D}(\Omega)$ in $H^1(\Omega)$. In the following we denote respectively by $\| \ \|$ and $(,)$ the norm and the scalar product in $L^2(\Omega)$, by $| \ |_\infty$ the norm in $L^\infty(\Omega)$, and by $\| \ \|_k$ the norm in $H^k(\Omega)$. We denote by $C^{0,1}(\Omega)$ the space of Lipschitz functions defined on $[0, 1]$ and we set $[f]_{0,1} \equiv \sup |f(x) - f(y)|/|x - y|$, for $x, y \in [0, 1]$, $x \neq y$. For convenience we drop Ω from the above symbols. Hence $L^2 = L^2(\Omega)$, and so on. We also use the notation $\int g = \int_0^1 g(q)\, dq$.

For functions $f \in L^\infty(0, +\infty; C^{0,1})$, we define

$$\langle f \rangle_{0,1} \equiv \operatorname{ess\,sup} [f(t)]_{0,1} \quad \text{for } t \in]0, +\infty[.$$

For functions $f \in L^\infty(0, +\infty; L^\infty)$ we set (for convenience) $|||f|||_\infty \equiv (10/\mu)^{1/2} \|f\|_\infty$, where $\|f\|_\infty = \operatorname{ess\,sup} |f(t)|_\infty$ for $t \in]0, +\infty[$.

A quite natural quantity in order to measure elements (u, v) in the phase set $L^2 \times \hat{H}^1$ is $\|u\|^2 + \|v\|_1^2 + |v^{-1}|_\infty^2$. It is not difficult to verify (we will return to this point later on) that the above quantity is equivalent to $\|u\|^2 + \|(\log v)_q\|^2$, in the sense that, each of the above quantities is bounded away from infinity (or from zero) if and only if the other does. However the latter is more significant here and we will state below our main results in terms of it.

In the results below $\bar{r}_0$ denotes a positive constant which depends only on the particular function $p(\cdot)$ and on μ. Moreover,

$$\mathcal{R} \colon]0, \bar{r}_0[\to]0, +\infty[\quad \text{and} \quad t^* \colon]0, \bar{r}_0[\to]0, +\infty[$$

are suitable decreasing functions such that

$$\lim_{r\to 0} \mathcal{R}(r) = +\infty, \tag{1.5}$$

and

$$\mathcal{R}_1(r)\colon\ ]0, \bar{r}_0[\to]0, +\infty[$$

is an increasing function such that

$$\lim_{r\to 0} \mathcal{R}_1(r) = 0. \tag{1.6}$$

In the next section we shall prove the following result:

THEOREM 1.1. – *Let the external force $f(t, x)$ belong to $L^\infty(0, +\infty; C^{0,1})$, and assume that $\langle f\rangle_{0,1} \leqq c_0$ where c_0 is a suitable positive constant which depends only on $p(\cdot)$ and on μ. Assume that for some $r \in]0, \bar{r}_0[$ one has*

$$|||f|||_\infty < r/2, \tag{1.7}$$

$$\|u^0\|^2 + \|(\log v^0)_q\|^2 < \mathcal{R}^2(r), \tag{1.8}$$

$$\|\bar{u}^0\|^2 + \|(\log \bar{v}^0)_q\|^2 < \mathcal{R}^2(r), \tag{1.9}$$

where the initial data (u^0, v^0) and $(\bar{u}^0, \bar{v}^0)$ belong to $L^2 \times \hat{H}^1$. Let $(u(t), v(t))$ and $(\bar{u}(t), \bar{v}(t))$ be the solutions of problem (1.1) *with initial data (u^0, v^0) and $(\bar{u}^0, \bar{v}^0)$, respectively. Then, for each $t \geqq t^*(r)$ one has*

$$\|u(t)\|^2 + \|(\log v(t))_q\|^2 + \|u_q(t)\|^2 < \mathcal{R}_1^2(r), \tag{1.10}$$

and similarly for $(\bar{u}(t), \bar{v}(t))$. Furthermore,

$$\|\bar{u}(t) - u(t)\|^2 + \|(\log \bar{v}(t))_q - (\log v(t))_q\|^2 \leqq c_1 \exp(-c_2(t - t^*(r))). \tag{1.11}$$

In order to interpret the above statement it is worth noting that for «small» external forces the ball (1.8) is «large» and the ball (1.10) is «small». In fact, as $r \to 0$, the ball (1.8) invades $L^2 \times \hat{H}^1$ while the ball (1.10) shrinks to the point $(0, 1)$.

The second part of the statement shows that the large ball (1.8) is exponentially attracted by a (asymptotically) unique flow. All the solutions starting from (or intersecting) the large ball have the same asymptotic behaviour.

The following result concerning periodic solutions will also be proved in Section 3.

THEOREM 1.2. – *Let f be a T-periodic function, $T > 0$, satisfying the hypothesis of Theorem* 1.1. *Then, there is a T-periodic solution $(u(t), v(t))$ of problem* (1.1) *whose orbit lies entirely inside the small ball* (1.10). *Each solution $(\bar{u}(t), \bar{v}(t))$ of the equation* (1.1) *that intersects the large ball* (1.8) *must converge asymptotically to the periodic solution (u, v), according to the exponential law* (1.11). *In particular,* *there are no T'-periodic solutions, $T' \geqq 0$, that intersect large sphere and that are distinct from the T-periodic solution $(u(t), v(t))$.*

For pressure laws $p(v)$ such that $k_2 v^{-1} \geqq -p'(v) \geqq k_1 v^{-1}$ or $k_2 v^{-2} \geqq p'(v) \geqq k_1 v^{-2}$ $(p'(v) < 0)$, periodic and quasi-periodic solutions were studied by SHELUKHIN in [8]. For $p(v) = kv^{-1}$ periodic solutions were studied by MATSUMURA and NISHIDA in [6], also for the piston problem. For the n-dimensional case see VALLI [10].

Our results are partially related to those in BEIRÃO DA VEIGA [1], KANEL [3], KAZHIKHOV [4], [5]. The proof of Theorem 1.2 uses Serrin's technique [7].

Related results for stationary solutions were proved in BEIRÃO DA VEIGA [2].

2. – Known results.

For the reader's convenience, in this section we present a review of those results, proved in [1], that will be used in the next section. For more details, the reader is refered to [1].

Positive constants that depend at most on the particular function $p(\cdot)$ and on μ are denoted by $c, c_0, c_1, \ldots$. The symbol c may denote arbitrary different positive constants.

In this section, we recall some results proved in BEIRÃO DA VEIGA [1]. In order to state these results it is necessary to introduce some auxiliar functions and some notations. First of all it is worth noting that a natural quantity to measure elements $(u, v) \in L^2 \times \tilde{H}^1$ is $\|u\|^2 + \|v_q\|^2 + |v^{-1}|_\infty^2$, which is equivalent to $\|u\|^2 + \|v\|^2 + \|v_q\|^2 + |v|_\infty^2 + |v^{-1}|_\infty$ in the sense that both remain bounded away from zero together. However, there are equivalent (in the above sense) quantities that appear more convenient than the above ones in order to study our problems. In this regard, and for the reader's convenience, let us describe the following facts. By using well known devices ([1], [4], [5]) one shows that for each $v \in K^1$ one has:

$$\exp(-\|(\log v)_q\|) \leqq |v|_\infty \wedge |v^{-1}|_\infty \leqslant |v|_\infty \vee |v^{-1}|_\infty \leqq \exp(\|(\log v)_q\|);$$

$$\|(\log v)_q\| \exp(-\|(\log v)_q\|) \leqq \|v_q\| \wedge \|(v^{-1})_q\| \leqslant \|v_q\| \vee \|(v^{-1})_q\| \leqq \|(\log v)_q\| \exp(\|(\log v)_q\|);$$

$$\|v_q\|(1 + \|v_q\|)^{-1} \leqq \|v_q\| |v|_\infty^{-1} \leqq \|(\log v)_q\| \leqq \|v_q\| |v^{-1}|_\infty \leqq \|v_q\|(1 + \|(v^{-1})_q\|).$$

Hence, for elements $(u, v) \in L^2 \times K^1$, the quantity $\|u\|^2 + \|(\log v)_q\|^2$ is equivalent to $\|u\|^2 + \|v\|^2 + \|v_q\|^2 + \|(v^{-1})_q\|^2 + |v|_\infty^2 + |v^{-1}|_\infty^2$. In this section we will use the equivalent quantities $\psi^2[u, v]$ and $\psi^2[u, v] + \phi^2[v]$, where

$$\psi^2[u, v] \equiv (4/\mu)\|u\|^2 - 2(u, (\log v)_q) + \mu\|(\log v)_q\|^2 \tag{2.1}$$

and $\phi^2[v]$ will be defined below. Note that

$$\psi^2[u, v] = (3/\mu)\|u\|^2 + \|\mu(\log v)_q - u\|^2, \tag{2.2}$$

and that

$$\frac{1}{2}\left(\frac{4}{\mu}\|u\|^2+\mu\|(\log v)_q\|^2\right)\leqq\psi^2[u,v]\leqq\frac{3}{2}\left(\frac{4}{\mu}\|u\|^2+\mu\|(\log v)_q\|^2\right), \tag{2.3}$$

for $(u,v)\in L^2\times K^1$. Moreover,

$$\exp\left(-(2/\mu)^{1/2}\psi[u,v]\right)\leqq v(q)\leqq\exp\left((2/\mu)^{1/2}\psi[u,v]\right)\qquad\forall q\in\Omega, \tag{2.4}$$

for all $(u,v)\in L^2\times K^1$.

Defined (see [3], [5]) a real function

$$\pi\in C^1(]0,+\infty[)\quad\text{by}\quad\pi'(s)=p(s),\ \forall s>0,\ \pi(1)=1.$$

Note that $s-\pi(s)>0$, if $s\neq 1$. Set

$$\phi^2[v]=(8/\mu)\int v-\pi(v),\qquad\forall v\in K^1. \tag{2.5}$$

The quantity $\psi^2[u,v]+\phi^2[v]$ is equivalent to $\psi^2[u,v]$ since there is a real, strictly increasing function $\theta\in C([0,+\infty[)$, such that $\theta(1)=0$, and that (see [1], equation (7.8). Here $\theta(\cdot)$ is the function $(8/\mu)M^2(\cdot)$ in reference [1])

$$\phi^2[v]\leqq\theta\left(\exp\left((2/\mu)^{1/2}\psi[u,v]\right)\right). \tag{2.6}$$

Another main point is the following estimate (see [1], (7.2)):

$$\frac{1}{2}\frac{d}{dt}[\psi^2(t)+\phi^2(t)]+3\int v^{-1}\mu_q^2+\int -vp'(v)(\log v)_q^2\leqq|||f|||_\infty\psi(t),\qquad\forall t\geqq 0, \tag{2.7}$$

if $\psi(t)\equiv\psi[u(t),v(t)]$, $\phi(t)=\phi[v(t)]$ and $(u(t),v(t))$ is a solution of the system (1.1).

Now we recall some properties of the functions $\rho(r)$, $R(r)$, $\rho_1(r)$, $R_1(r)$ (see [1], section 6) which will be used to estimate the radius of significant balls in the (u,v) space. The main point is the behaviour of these functions. For precise definitions and proof we refer to [1], section 6 and 7.

F: $[0,+\infty[\to]0,+\infty[$ is a continuous function, such that

$$F(0)=0,\qquad F(y)>0\quad\text{if}\quad y>0,\ F(+\infty)\equiv\lim_{y\to+\infty}F(y)=0.$$

R: $]0,r_0[\to]y_0,+\infty[$ and ρ: $]0,r_0[\to]r_0,+\infty[$ are strictly decreasing functions such that $\rho(r)<R(r)$, $\forall r\in]0,r_0[$. Moreover,

$$\lim_{r\to 0}\rho(r)=\lim_{r\to 0}R(r)=+\infty. \tag{2.8}$$

R_1: $]0,r_0[\to]0,\overline{y}_0[$ and ρ_1: $]0,r_0[\to]0,y_0[$ are strictly increasing functions such that $\rho_1(r)<R_1(r)$, $\forall r\in]0,r_0[$. Moreover,

$$\lim_{r\to 0}\rho_1(r)=\lim_{r\to 0}R_1(r)=0. \tag{2.9}$$

Finally, for all $r \in]0, r_0[$ and for all $(u, v) \in L^2 \times \hat{H}^1$, one has (*)

$$\begin{cases} \psi^2[u,v] < \rho^2(r) \Rightarrow \psi^2[u,v] + \phi^2[v] < R^2(r), \\ \psi^2[u,v] < \rho_1^2(r) \Rightarrow \psi^2[u,v] + \phi^2[v] < R_1^2(r). \end{cases} \tag{2.10}$$

The above constants r_0, y_0, $\bar{y}_0$ and the functions F, ρ, R, ρ_1, R_1 depend only on μ and $p(\cdot)$. Let the initial data (u^0, v^0) belong to $L^2 \times \hat{H}^1$ and the external force $f(t, x)$ belong to $L^\infty(Q_\infty)$, where $Q_\infty = \Omega \times]0, +\infty[$, and denote by $(u(t), v(t))$ the global weak solution of the problem (1.1) constructed as in [1], section 4. Then

$$u \in C([0, +\infty[; L^2) \cap L^2_{\mathrm{loc}}([0, +\infty[; H_0^1), \qquad v \in C([0, +\infty[; \hat{H}_1).$$

If, in addition, $f \in L^1_{\mathrm{loc}}([0, +\infty[; C^{0,1})$ then the solution (u, v) is unique, moreover *strong* well-posedness holds. More precisely, if (u_n^0, v_n^0) and $(u_n(t), v_n(t))$ are sequences as above and if (u_n^0, v_n^0) converges to (u^0, v^0) in $L^2 \times H^1$ as $n \to +\infty$, then $(u_n(t), v_n(t))$ converges to $(u(t), v(t))$ in $L^2 \times H^1$, the convergence being uniform on bounded intervals $[0, T]$ (moreover, $u_n \to u$ in $L^2(0, T; H_0^1)$, $\forall T > 0$). See [1], section 5.

From Theorems 7.2, 7.3, 7.4 of [1], in particular, the following result follows. We point out that there are not smallness assumptions on $f(t)$, as t goes to ∞.

THEOREM 2.1. – *Let* $(u^0, v^0) \in L^2 \times \hat{H}^1$, $f \in L^\infty(Q_\infty)$, *and let* $r < r_0$.

(i) *If*

$$|||f|||_\infty < r \tag{2.11}$$

and if

$$\psi^2[u^0, v^0] + \phi^2[v^0] < R^2(r) \tag{2.12}$$

(*in particular, if* $\psi^2[u^0, v^0] < \rho^2(r)$) *then*

$$\psi^2[u(t), v(t)] + \phi^2[v(t)] < R^2(r), \qquad \forall t \geqq 0. \tag{2.13}$$

(ii) *If* (2.11) *holds and if*

$$\psi^2[u^0, v^0] + \phi^2[v^0] < R_1^2(r) \tag{2.14}$$

(*in particular if* $\psi^2[u^0, v^0] < \rho_1^2(r)$), *then*

$$\psi^2[u(t), v(t)] + \phi^2[v(t)] < R_1^2(r), \qquad \forall t \geqq 0. \tag{2.15}$$

(iii) *If*

$$|||f|||_\infty < r/2 \tag{2.16}$$

(*) Since $R^2(r) = \rho^2(r) + \theta(\exp[(2/\mu)^{1/2} \rho(r)])$; see [1], eq. (6.7).

and if (2.12) *holds, then* (2.15) *holds for all* $t \geqq T^*$ *where*

$$T^* \equiv \frac{\psi^2[u^0, v^0] + \phi^2[v^0]}{r\rho_1(r)} < \frac{R^2(r)}{r\rho_1(r)}. \tag{2.17}$$

3. – Proofs.

We start by proving the following

LEMMA 3.1. – *Here* $\bar{r}_0 \in]0, r_0]$ *and* c_2 *are positive constants which depend only on* μ *and* $p(\cdot)$. *Let* $(u(t), v(t))$ *be the solution of* (1.1) *with the initial data* $(u^*, v^*) \in H_0^1 \times \times \hat{H}^1$. *Assume that for some* $r \in]0, \bar{r}_0[$ f *satisfies* (2.11), $\psi^2[u^*, v^*] + \phi^2[v^*] < R_1^2(r)$, *and* $\|u_q^*\|^2 < a(R_1^2(r) + r^2)$ *where* $a \geqq c_2$. *Then*

$$\|u_q(t)\|^2 < a(R_1^2(r) + r^2), \qquad \forall t \geqq 0.$$

PROOF. – For convenience, we will sometimes use the notation $\psi^2(t) = \psi^2[u(t), v(t)]$, $\phi^2(t) = \phi^2[v(t)]$. Let $r < r_0$. By theorem 2.1 (ii) one has $\psi^2(t) + \phi^2(t) < R_1^2(r) \leqq \bar{y}_0^2$ and by (2.4) one gets $c^{-1} \leqq v(t, q) \leqq c$, for a suitable c. By multiplying equation $(1.1)_2$ by u_{qq} and integrating in Ω one has

$$\frac{1}{2}\frac{d}{dt}\|u_q\|^2 + c\|u_{qq}\|^2 \leqq c\|(\log v)_q\| \, \|u_q\|^{1/2} \|u_{qq}\|^{3/2} + c(\|(\log v)_q\| + \||f\||_\infty)\|u_{qq}\|,$$

since $|u_q|_\infty \leqq \sqrt{2}\|u_q\|^{1/2}\|u_{qq}\|^{1/2}$. Recall that $\int u_q = 0$. By Young's inequality it readily follows that

$$\frac{1}{2}\frac{d}{dt}\|u_q\|^2 + c_3\|u_q\|^2 \leqq c_4 R_1^4(r)\|u_q\|^2 + c[R_1^2(r) + r^2].$$

Note that $\|u_q\| \leqq (1/\sqrt{2})\|u_{qq}\|$. By fixing $\bar{r}_0 \in]0, r_0]$ so that $c_4 R_1^4(r) \leqq c_3/2$, and by taking into account that $R_1(\cdot)$ is an increasing function, it follows that

$$\frac{d}{dt}\|u_q\|^2 + c_3\|u_q\|^2 \leqq c(R_1^2(r) + r^2) \tag{3.1}$$

for each $r \in]0, \bar{r}_0[$. In particular, $D_t\|u_q\|^2 \leqq 0$ if $\|u_q\|^2 \geqq c_2(R_1^2(r) + r^2)$, where $c_2 = = c/c_3$. ■

LEMMA 3.2. – *Let* $r \in]0, \bar{r}_0[$ *and assume that* (2.16) *and* (2.12) *hold, where* $(u^0, v^0) \in \in L^2 \times \hat{H}^1$. *Then, there is a positive constant* $\bar{c}$ *such that*

$$\|u_q(t)\|^2 < \bar{c}(R_1^2(r) + r^2), \qquad \forall t \geqq 1 + T^*, \tag{3.2}$$

where T^* *is defined by the equation* (2.17).

PROOF. – By Theorem 2.1 (iii) one has

$$\psi^2(t) + \phi^2(t) < R_1^2(r), \qquad \forall t \geqq T^*. \tag{3.3}$$

Hence, (2.7) and (2.4) yield

$$\frac{1}{2}\frac{d}{dt}[\psi^2(t) + \phi^2(t)] + c\|u_q\|^2 \leqq c\|\|f\|\|_\infty^2 .$$

By integrating with respect to t on $[T^*, T^*+1]$ and using (3.3) it follows that

$$\int_{T^*}^{T^*+1} \|u_q(\tau)\|^2 \, d\tau < \bar{c}(R_1^2(r) + r),$$

where $\bar{c}$ is a suitable positive constant. We assume that $\bar{c} \geqq c_2$. Consequently, $\|u_q(t_0)\|^2 < \bar{c}(R_1^2(r) + r^2)$, for some $t_0 \in [T^*, T^*+1]$. Lemma 2.2 shows that (3.2) is satisfied. ∎

In the following we consider two solutions (u, v), $(\bar{u}, \bar{v})$ of the problem (1.1) with initial data (u^0, v^0), $(\bar{u}^0, \bar{v}^0) \in L^2 \times \widehat{H}^1$ satisfying

$$\begin{cases} \psi^2[u^0, v^0] + \phi^2[v^0] < R^2(r), \\ \psi^2[\bar{u}^0, \bar{v}^0] + \phi^2[\bar{v}^0] < R^2(r), \end{cases} \tag{3.4}$$

for some $r \in]0, \bar{r}_0[$ such that $\|\|f\|\|_\infty < r$. Set, for convenience,

$$w = \bar{u} - u, \qquad \tau = \bar{v} - v, \qquad l = (\log \bar{v})_q - (\log v)_q . \tag{3.5}$$

From Theorem 1.1 (i) and from (2.4) it follows that there is a constant $N \in [1, +\infty[$ such that

$$N^{-1} \leqq \bar{v}(t, q) \wedge v(t, q) \leqslant \bar{v}(t, q) \vee v(t, q) \leqslant N, \qquad \forall (t, q) \in Q_\infty . \tag{3.6}$$

For each $N \in [1, +\infty[$ define

$$L(N) \equiv \sup_{N^{-1} \leqq s, s' \leqq N} \left| \frac{p'(s) - p(s)}{s - s'} \right| .$$

Set $k = -p'(1)$. It readly follows from equation (1.1) that

$$\begin{cases} \tau_t = w_q, \\ w_t = \mu(\bar{v}^{-1}\bar{u}_q - v^{-1}u_q)_q - (p(\bar{v}) - p(v))_q + f[\bar{v}] - f[v], \end{cases} \tag{3.7}$$

where $f[v](t, q) = f\left(t, \int_0^q v(t, \xi)\, d\xi\right)$, etc. By multiplying $(3.7)_2$ by w and integrating with respect to q, one gets

$$\frac{1}{2}\frac{d}{dt}\int w^2 + \mu \int \bar{v}^{-1} w_q^2 = \mu \int \frac{\tau}{v\bar{v}} u_q w_q + \int [p(\bar{v}) - p(v)]\, w_q + \int (f[\bar{v}] - f[v])\, w . \tag{3.8}$$

Since $|p(\overline{v}) - p(v) + k\tau| \leqq L(|v-1| + |\tau|)|\tau|$ $\forall v, \overline{v} \in [N^{-1}, N]$, and $\int \tau w_q = (1/2) D_t \int \tau^2$, it follows that

$$(3.9) \qquad \frac{1}{2}\frac{d}{dt}(\|w\|^2 + k\|\tau\|^2) + \mu \int \overline{v}^{-1} w_q^2 \leqq$$

$$\leqq \mu N^2 \|u_q\| \, |\tau|_\infty \|w_q\| + L|\tau|_\infty^2 \|w_q\| + L|\tau|_\infty |v-1|_\infty \|w_q\| + \int (f[\overline{v}] - f[v])\, w \, .$$

Note that (cf. [1], (5.3))

$$(3.10) \qquad \begin{cases} |\tau|_\infty \leqq N\|l\|, \\ |v-1|_\infty \leqq N\|(\log v)_q\| . \end{cases}$$

Since $(\log v)_t = v^{-1} u_q$, the equation $(3.7)_2$ may be written as

$$(3.11) \qquad w_t = \mu l_t - (p'(\overline{v})\overline{v}_q - p'(v)\, v_q) + f[\overline{v}] - f[v] \, .$$

On the other hand $-p'(\overline{v})\overline{v}_q + p'(v)\, v_q = -\overline{v}p'(\overline{v})\, l + (vp'(v) - \overline{v}p'(\overline{v}))(\log v)_q$. Then, by multiplying (3.11) by l and integrating on Ω with respect to q, one gets

$$(3.12) \qquad \frac{\mu}{2}\frac{d}{dt}\|l\|^2 + \alpha(N)\|l\|^2 - \int w_t \, l \leqq L_0(N) \int |(\log v)_q| \, |\tau| \, |l| + \int (f[v] - f[\overline{v}])\, l \, ,$$

where by definition

$$\alpha(N) \equiv \min_{N^{-1} \leqq s \leqq N} - sp'(s) \, ,$$

$$L_0(N) \equiv \sup_{N^{-1} \leqq s < s' \leqq N} \left| \frac{sp'(s) - s'p'(s')}{s' - s} \right| .$$

Since

$$\int w_t \, l = \frac{d}{dt}\int wl + \int \overline{v}^{-1} w_q^2 - \int \frac{\tau}{v\overline{v}} u_q w_q \, ,$$

one obtains

$$(3.13) \qquad \frac{\mu}{2}\frac{d}{dt}\|l\|^2 + \alpha(N)\|l\|^2 - \frac{d}{dt}\int wl \leqq$$

$$\leqq \overline{v}^{-1} w_q^2 + \int \frac{|u_q|}{v\overline{v}} |\tau| \, |w_q| + L_0 \int |(\log v)_q| \, |\tau| \, |l| + \int (f[v] - f[\overline{v}])\, l \, .$$

Now we multiply the equation (3.9) by $4/\mu$ and we add side by side to the equation

(3.13). By using (3.10) it readily follows that

$$(3.14)\qquad \frac{1}{2}\frac{d}{dt}\left(\tilde{\psi}^2(t)+\frac{4k}{\mu}\|\tau\|^2\right)+3N^{-1}\|w_q\|^2+\alpha(N)\|l\|^2\leqq 5N^3\|u_q\|\,\|l\|\,\|w_q\|+$$

$$+4LN^2\mu^{-1}\|l\|^2\|w_q\|+4LN^2\mu^{-1}\|(\log v)_q\|\,\|l\|\,\|w_q\|+L_0N\|(\log v)_q\|\,\|l\|^2+$$

$$+|f[\bar{v}]-f[v]|_\infty(4\mu^{-1}\|w\|+\|l\|),$$

where, for convenience (see (2.1)),

$$\tilde{\psi}^2(t)\equiv\psi^2[w(t),(\bar{v}/v)(t)]=4\mu^{-1}\|w(t)\|^2-2(w(t),l(t))+\mu\|l(t)\|^2.$$

Note that (cf. (2.3))

$$(3.15)\qquad \frac{1}{2}(4\mu^{-1}\|w\|^2+\mu\|l\|^2)\leqq\tilde{\psi}^2(t)\leqq\frac{3}{2}(4\mu^{-1}\|w\|^2+\|l\|^2).$$

For convenience, we denote by C positive constants which depend at most on μ, $p(\cdot)$, and N, and which are nondecreasing function of N for $N\in[1,+\infty[$. By using this notation and Cauchy-Schwarz inequality, (3.14) yields

$$(3.16)\qquad \frac{1}{2}\frac{d}{dt}(\tilde{\psi}^2(t)+4k\mu^{-1}\|\tau\|^2)+C^{-1}\|w_q\|^2+C^{-1}\|l\|^2\leqq$$

$$\leqq C(\|u_q\|^2+\|l\|^2+\|(\log v)_q\|^2+\|(\log v)_q\|)\|l\|^2+cN[f]_{0,1}(\|w\|+\|l\|)\|l\|.$$

Note that $|f[\bar{v}]-f[v]|_\infty\leqq[f]_{0,1}|\tau|_\infty\leqq N[f]_{0,1}\|l\|$. ∎

Assume again that (3.4) holds and assume that (1.7) is satisfied. Theorem 2.1 (iii) shows that

$$(3.17)\qquad \begin{cases}\psi^2[u(t),v(t)]+\phi^2[v(t)]<R_1^2(r)\\ \psi^2[\bar{u}(t),\bar{v}(t)]+\phi^2[\bar{v}(t)]<R_1^2(r)\qquad \forall t\geqq T^*.\end{cases}$$

Since $R_1^2(r)\leqq R_1^2(\bar{r}_0)$, it follows that the left hand sides in equations (3.17) are bounded by $\bar{y}_0^2$, for $t\geqq T^*$. Hence, by equation (2.14), there is a suitable $N\in[1,+\infty[$, that depends only on μ and $p(\cdot)$, such that (3.6) holds if $t\geqq T^*$. Consequently, for $t\geqq T^*$, the equation (3.14) holds even if the contants C are replaced by constants c. In particular, the term $cN[f]_{0,1}(\|w\|+\|l\|)\|l\|$ can be dropped provided that

$$(3.18)\qquad \langle f\rangle_{0,1}\leqq c_0,$$

for a suitable c_0 (note that $\|w\|\leqq\|w_q\|$). Finally, by using Lemma 3.2, and also (3.17), (2.3), one gets

$$(3.19)\qquad \|u_q\|^2+\|l\|^2+\|(\log v)_q\|^2+\|(\log v)_q\|\leqq c(R_1^2(r)+r^2+R_1(r)),\qquad \forall t\geqq T^*+1,$$

for a suitable c. By taking into account the above remarks, one gets from (3.16)

$$(3.20)\qquad \frac{1}{2}\frac{d}{dt}(\widehat{\psi}(t)+4k\mu^{-1}\|\tau\|^2)+c\|w_q\|^2+c_5\|l\|^2\leqq$$

$$\leqq c_6(R_1^2(r)+r^2+R_1(r))\|l\|^2,\qquad \forall t\geqq T^*+1,$$

under the additional hypothesis (3.18). Fix now a value $\tilde{r}_0\in]0,\bar{r}_0]$ so that $c_6[R_1^2(\tilde{r}_0)+\tilde{r}_0^2+R_1(\tilde{r}_0)]\leqq c_5/2$. The value $\tilde{r}_0$ depends only on μ and $p(\cdot)$. For convenience we continue to use the notation $\bar{r}_0$ instead of $\tilde{r}_0$. Since $R_1(r)$ is an increasing function, for each $r\in]0,\bar{r}_0[$ one has

$$(3.21)\qquad \frac{1}{2}\frac{d}{dt}(\widehat{\psi}(t)+4k\mu^{-1}\|\tau\|^2)+c(\widehat{\psi}(t)+4k\mu^{-1}\|\tau\|^2)<0,$$

for each $t\geqq T^*+1$, for a suitable constant c. We have also used (3.15) and $(3.10)_1$. Consequently,

$$\widehat{\psi}^2(t)+4\mu k^{-1}\|\tau\|^2\leqq[\widehat{\psi}^2(T^*+1)+4\mu k^{-1}\|\tau(T^*+1)\|^2]\exp[-c(t-T^*-1)].$$

Using again (3.15) and $(3.10)_1$ it readily follows that

$$(3.22)\qquad \|\overline{u}(t)-u(t)\|^2+\|(\log\overline{v}(t))_q-(\log v(t))_q\|^2\leqq$$

$$\leqq c\exp(-c(t-T^*-1))\{\|\overline{u}(T^*+1)-u(T^*+1)\|^2+$$

$$+\|(\log\overline{v}(T^*+1))_q-(\log v(T^*+1))_q\|^2\}.$$

The term $\{\ldots\}$ on the right hand side of (3.22) can be estimated in terms of the initial data (see also [1], eq. (5.8)). For convenience we follow here a more crude way. This term is bounded by $c\widehat{\psi}(T^*+1)$, hence by $cR_1^2(r)\leqq c\overline{y}_0^2$. In particular

$$(3.23)\qquad \|\overline{u}(t)-u(t)\|^2+\|(\log\overline{v}(t))_q-(\log v(t))_q\|^2\leqq c_1\exp(-c_2(t-T^*+1)).$$

The above arguments prove the following theorem:

THEOREM 3.2. – *There are positive constants $\bar{r}_0$, c_0, c_1, c_2, which depend only on μ and $p(\cdot)$, such that the following result holds. Let (u^0,v^0), $(\overline{u}^0,\overline{v}^0)\in L^2\times\widehat{H}^1$ satisfy* (3.4), *and let f satisfy* (1.7), (3.18), *for some $r\in]0,\bar{r}_0[$. Let (u,v) and $(\overline{u},\overline{v})$ be the solution of the equation* (1.1) *with initial data (u^0,v^0) and $(\overline{u}^0,\overline{v}^0)$ respectively. Then, for $t\geqq T^*+1$ (see* (2.17)) *both solutions satisfy* (3.17) *and* (3.2). *Further,* (3.22) *and* (3.23) *are satisfied.*

Theorem 1.1 is an immediate consequence of Theorem 3.3. Note that (2.12) holds if $\psi^2[u^0,v^0]<\rho^2(r)$ (see Theorem 2.1 (i)) and recall (2.3).

Now we prove Theorem 1.2. We argue as Serrin in reference [7]. It is sufficient to show the existence of periodic solution $(u(t),v(t))$, since, if an «initial data» $(\overline{u}^0,\overline{v}^0)$ belongs to the ball (1.8), then the corresponding solution $(\overline{u}(t),\overline{v}(t))$ satisfies (3.23), hence converges asymptotically to $(u(t),v(t))$.

Let $(u^0, v^0) \in L^2 \times \hat{H}^1$ be an initial data satisfying (1.8) and set

$$(u_n(t), v_n(t)) = (u(nT + t), v(nT + t)), \quad \forall t \geqq 0,$$

for each $n \in N$. Since f is T-periodic, each pair (u_n, v_n) is a solution of (1.1) with the initial data $(u_n(0), v_n(0)) = (u(nT), v(nT))$. Hence (3.23) shows that (assume $m > n$)

$$\|u(nT) - u(mT)\|^2 + \|(\log v(nT))_q - (\log v(mT)_q\|^2 =$$

$$= \|u(nT) - u_{m-n}(nT)\|^2 + \|(\log v(nT))_q - (\log v_{m-n}(nT))_q\|^2 \leqq$$

$$\leqq c_1 \exp(-c_2(nT - T^* - 1)),$$

for $n \geqq (1 + T^*)/T$ (it is also clear that the trajectories of $(u_n(t), v_n(t))$ lie inside the ball (1.10)). The above inequality proves that $(u(nT), (\log v(nT))_q)$ is a Cauchy sequence in $L^2 \times L^2$, as $n \to +\infty$. Let (u_0^*, v_0^*) be the limit in $L^2 \times \hat{H}^1$ of the sequence $(u(nT), v(nT))$, and denote by $(u^*(t), v^*(t))$ the solution of (1.1) with the initial data (u_0^*, v_0^*). From the continuous dependence of the solution on the initial data (see [1], corollary 5.4) it follows that $(u^*(T), v^*(T)) = \lim_{n \to +\infty} (u((n+1)T), v((n+1)T)) = (u_0^*, v_0^*)$ in $L^2 \times H^1$. Hence $(u^*(t), v^*(t))$ is T-periodic. The last point of the proof can also be done using (3.22). In fact, for $nT > 1 + T^*$ the trajectories $(u_n(t), v_n(t))$, $t \geqq 0$, and the point (u_0^*, v_0^*) lie inside the ball (1.10). Hence the estimate (3.22) can be applied to the solutions (u_n, v_n) and (u^*, v^*) with $T^* + 1$ replaced by 0. In fact, the assumption $t > T^* + 1$ (in order to get (3.22)) was needed only the guarantee that (3.19) holds. In the present case this smallness assumption holds for each $t \geqq 0$, since for large values of n all the picture lies inside the ball (1.10). Consequently

$$\|u^*(T) - u((n+1)T)\|^2 + \|(\log v^*(T))_q - (\log v((n+1)T))_q\|^2 \leqq$$

$$\leqq c \exp(-cT)[\|u_0^* - u(nT)\|^2 + \|(\log v_0^*)_q - (\log v(nT))_q\|^2],$$

which tends to zero as $n \to +\infty$.

REFERENCES

[1] H. BEIRÃO DA VEIGA, *Long time behavior for one dimensional motion of a general barotropic viscous fluid*, Arch. Rat. Mech. Analysis, **108** (1989), pp. 141-160.

[2] H. BEIRÃO DA VEIGA, *The stability of one dimensional stationary flows of compressible viscous fluids*, Ann. Institut H. Poicaré, Analyse Non-linéaire, **7**(1990), pp. 259-268.

[3] YA. I. KANEL, *On a model system of equations of one dimensional motion*, translation from russian in Diff. Equations, 4 (1968), pp. 374-380.

[4] A. V. KAZHIKHOV, *Correctness «in the large» of initial-boundary value problems for model system of equations of a viscous gas*, Din. Sploshnoi Sredy, **21** (1975), pp. 18-47.

[5] A. V. KAZHIKHOV, *Stabilization of solutions of an initial-boundary-value problem: for the equations of motion of a barotropic viscous fluid*, translation from russian in Diff. Equations, **15** (1979), pp. 463-467.
[6] A. MATSUMURA - T. NISHIDA, *Periodic solutions of a viscous gas equation*, Lecture Notes in Num. Anal., **10** (1989) pp. 49-82.
[7] J. SERRIN, *A note on the existence of periodic solutions of the Navier-Stokes equations*, Arch. Rat. Mech. Analysis, **3** (1959), pp. 120-122.
[8] V. V. SHELUKHIN, *Periodic flow of viscous gas*, Din. Sploshnoi Sredy, **42** (1979), pp. 80-102 (in russian).
[9] V. A. SOLONNIKOV - V. KAZHIKHOV, *Existence theorems for the equations of motion of a compressible viscous fluid*, Ann. Rev. Fluid Mech., **13** (1981), pp. 79-95.
[10] A. VALLI, *Periodic and stationary solutions for compressible Navier-Stokes equations via a stability method*, Ann. Sc. Norm. Sup. Pisa, (1984), pp. 607-647.