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PERTURBATION THEORY AND WELL-POSEDNESS IN HADAMARD'S SENSE OF HYPERBOLIC INITIAL-BOUNDARY VALUE PROBLEMS

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1. INTRODUCTION

In this paper we study first order linear hyperbolic systems

$$\begin{cases} \partial_t u + \sum_{i=1}^{n} A_i(t,x)\, \partial_i u + B(t,x)u = F & \text{in } Q_T, \\ Mu|_{\Sigma_T} = 0, \qquad u(0) = f, \end{cases} \tag{1.1}$$

and corresponding nonlinear hyperbolic systems

$$\begin{cases} \partial_t u + \sum_{i=1}^{n} A_i(u)\, \partial_i u + B(u)u = F, & \text{in } Q_T, \\ Mu|_{\Sigma_T} = 0, \qquad u(0) = f, \end{cases} \tag{1.2}$$

where u is an m-vector, A_i $(i = 1, \ldots, n)$ and B are $m \times m$ matrices, M is a given $p \times m$ matrix, $F(t, x)$ and $f(x)$ are given.

Our aim is to prove that the nonlinear problem (1.2) is well-posed in Hadamard's classical sense and also to prove a sharp perturbation theorem for the linear problem (1.1). (We call perturbation theorems those results that establish the continuous dependence, on the coefficients of the operators, of the solution of linear partial differential equations.) We start by proving the perturbation theorem 2.3 for the linear problem (1.1). Then we use this result to establish the well-posedness of problem (1.2); see theorem 2.5. The lack of these basic (and expected) results in the general theory of hyperbolic equations is certainly a main gap in the theory.

In the following we assume that A_n has maximal rank on the boundary, since this is the standard hypothesis for hyperbolic first order systems. However, the methods developed here can be applied to many other interesting problems. See [1–3]. As a matter of fact, the method followed here applies to higher order hyperbolic mixed equations and systems, under various boundary conditions. However, instead of stating the theorems in a more general form we prefer to give a detailed proof for a specific case. The interested reader can easily apply it to other problems.

Typical linear problems (1.1) that satisfy the hypothesis made here are, for instance, the symmetric hyperbolic systems studied by Friedrichs [4] and Lax and Phillips [5] and, more generally, the problems treated by Rauch and Massey [6], even for nonregular coefficients. We also point out that our method, applied to the pure Cauchy problem, gives rise to proofs much simpler than that for mixed problems. See [13].

Recently, Kato [7] has extended his abstract theory, in order to cover a class of mixed problems for which, roughly speaking, the stationary part of the differential operator verifies an ellipticity hypothesis. This class does not include, in particular, first order hyperbolic systems (1.1).

Plan of the paper

Starting from hypotheses I and II, we begin by proving a differentiability result, see proposition 2.1. This result is related to other known theorems, as, for instance, those stated by Rauch and Massey [6] and Schochet [8]. Here, as in [8], the coefficients A and B are not regular (this gives rise to nontrivial difficulties). We point out that proposition 2.1 is (more or less) well known in the literature. However, we have been compelled to give here a self-contained and very careful proof. In fact, the proofs of theorems 2.3 and 2.5 (and of proposition 2.4) are very unstable with respect to small modifications (or inaccuracies) in proposition 2.1. Under this situation, imprecise references or claims should not be accepted by the interested reader. On the other hand, if we eliminate the proof of proposition 2.1 we have to enlarge the remaining proofs (in those parts in which techniques similar to those used in the proof of proposition 2.1 are used). Moreover, the proof of proposition 2.1 presents some simplifications. In particular, we do not use previous known results for equations with regular coefficients. On the other hand, our proof does not require the very technical construction of regular auxiliary data satisfying additional compatibility conditions. Such a construction will be used only for proving the perturbation theorem 2.3. Moreover, for proving this last result, we use only auxiliary data that verify, at most, compatibility conditions up to order $k - 1$, which is the same order necessary for obtaining solutions in H^k. This is a central point here, since our coefficients are not sufficiently regular to admit compatibility conditions of order larger or equal to k. We point out that approximation by regular coefficients would heavily complicate the proofs of the perturbation and of the well-posedness theorems.

After the above differentiability theorem we prove our first main result, the perturbation theorem 2.3. The perturbation of the coefficients A and B takes place in the space $\mathcal{L}_T^2(H^k)$. We start by proving a perturbation result in $\mathcal{C}_T(H^l)$, $2 \leq l \leq k - 1$; see lemma 2.2. Then, we extend this result to the case $l = k$, see theorem 2.3, by applying lemma 2.2 to the first derivatives of the solution itself. More precisely, we apply lemma 2.2 to a suitable first order system of $n(m + 1)$ equations whose solution is a vector field consisting of the tangential and time derivatives of u, and of the normal derivatives multiplied by a suitable cut-off function that vanishes near the boundary.

By using lemma 2.2 we prove an existence result to the nonlinear problem (1.2), see proposition 2.4. The proof, that follows standard methods (see, for instance, the method followed in [9]), is presented here mainly for the reader's convenience.

Finally, by using theorem 2.3 and proposition 2.4, we prove our second main result: the nonlinear problem (1.2) is well-posed in $\mathcal{C}_T(H^k)$, $k > (n/2) + 1$, in the classical Hadamard's sense; see theorem 2.5.

In order to avoid supplementary difficulties that can be overcome by using well-known (but nontrivial) devices, we consider here the half space case (i.e. $\Omega = \mathbb{R}^n_+$, x_n is the normal direction) and we assume that M is constant. The general case, i.e. Ω an open set with a compact boundary and $M = M(t, x)$, can be reduced to the previous one by using a suitable partition of unity and local change of co-ordinates, see Ikawa [10]. We also point out that our proofs adapt easily in order to treat the system (1.2) for coefficients A and B that depend on (t, x, u).

We opt here for complete, self-contained proofs. Often, equations and formulas are written in a very explicit form which, though somewhat tedious, gives the reader a complete control of certain fundamental manipulations. We hope that the resulting additional length of the paper will be well received by readers fully interested in the subject of the paper.

2. NOTATIONS AND RESULTS

Notations

We set $\mathbb{N} = \{\text{positive integers}\}$, $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$,

$$\mathbb{R}^+ = \{\lambda \in \mathbb{R};\ \lambda > 0\}, \qquad \Omega = \mathbb{R}^n_+ = \{x \in \mathbb{R}^n\colon x_n > 0\}, \qquad Q_T = [0, T] \times \Omega,$$

$$\Gamma = \{x \in \mathbb{R}^n\colon x_n = 0\}, \qquad \Sigma_T = [0, T] \times \Gamma.$$

We denote by H^l, $l \in \mathbb{N}_0$, the space $H^l(\mathbb{R}^n_+)$ endowed with the canonical norm $\|\ \|_l$. Moreover, we set

$$\mathcal{C}_T(H^l) = \bigcap_{j=0}^{l} C^j([0, T]; H^{l-j}), \qquad \mathcal{L}^p_T(H^l) = \bigcap_{j=0}^{l} W^{j,p}(0, T; H^{l-j}), \qquad p \in [1, +\infty],$$

$$|\!|\!| u |\!|\!|^2_l = \sum_{j=0}^{l} \|\partial_t^j u\|^2_{l-j}, \qquad |\!|\!| u |\!|\!|^2_{l,T} = \operatorname*{ess\,sup}_{0 \le t \le T} |\!|\!| u(t) |\!|\!|^2_l,$$

$[u]^2_{l,T} = \int_0^T |\!|\!| u(t) |\!|\!|^2_l \, dt$. Hence, $|\!|\!| \cdot |\!|\!|_{l,T}$ and $[\cdot]_{l,T}$ are norms in $\mathcal{C}_T(H^l)$ and $\mathcal{L}^2_T(H^l)$, respectively.

Let us consider the system (1.1). We set $A = (A_1, \ldots, A_n)$, $A\,\partial u = \sum_{i=1}^n A_i\,\partial_i u$, $Lu = \partial_t u + A\,\partial u + Bu$. We assume that M has rank p, that

$$A \in \mathcal{L}^\infty_T(H^k), \qquad B \in \mathcal{L}^\infty_T(H^{k-1}), \tag{2.1}$$

and that there is a positive constant μ such that

$$|\det A_n| > \mu, \qquad \text{on } \Sigma_T. \tag{2.2}$$

The positive integer k, fixed once and for all, satisfies $k > 1 + (n/2)$. Sometimes we will assume that

$$B \in \mathcal{L}^\infty_T(H^k). \tag{2.3}$$

In hypotheses (2.1) and (2.3) it would be sufficient to consider $\mathcal{L}^p$ instead of $\mathcal{L}^\infty$, for some $p > 1$ (we believe that $p = 1$ would be sufficient but we did not verify it in detail). A similar remark holds for F.

We denote by $f^{(j)}$, $j \ge 1$, the function (defined on $\mathbb{R}^+_n$) that is formally obtained from equation (1.1) by solving it for $(\partial_t^j u)(0)$. By definition, $f^{(0)} = f$. The functions $f^{(j)}$ depend only on f, F, A and B. For instance, if $B = 0$, one has $f^{(1)} = F(0) - A(0)\,\partial f$, $f^{(2)} = \partial_t F(0) - A\,\partial f^{(1)} - (\partial_t A)(0)\,\partial f$, and so on. We set

$$|\!|\!| f |\!|\!|^2_l = \sum_{j=0}^{l} \|f^{(j)}\|^2_{l-j}.$$

Recall that the compatibility condition of order $j \geq 0$ for the system (1.1) is $Mf^{(j)} = 0$ on Γ. Compatibility conditions are relations on Γ between f, F, A, B and its derivatives, at time $t = 0$.

In the sequel we often deal with positive constants λ that, in fact, depend (increasingly) on various characteristic constants of the problem. In order to be more precise, we use the following notation. We denote by λ generic functions on $(\mathbb{R}^+)^l$ to $\mathbb{R}^+$ that are increasing as functions of each single variable. These functions will be called "functions of type λ". Since we are not particularly interested in the explicit form of these functions, we will denote different functions by the same symbol (even in the same formula). For instance, we are allowed to write $\lambda\lambda_1 = \lambda_1$ if $\lambda = \lambda(z_1)$ and if $\lambda_1 = \lambda_1(z_1, z_2)$, or $\lambda + \lambda = \lambda$ and so on. The symbols ϕ, ψ, ϕ', ψ', $\hat{\phi}$, $\hat{\psi}$, and $\bar{\psi}$ denote functions of type λ that depend only on the independent variables indicated below (except if a different set of independent variables is explicitly mentioned).

$$
\begin{cases} \phi = \phi(\mu^{-1}, |||A|||_{k-1,T} |||B(0)|||_{k-2}), \\ \psi = \psi(\mu^{-1}, |||A|||_{k,T}, |||B|||_{k-1,T}), \end{cases}
$$
$$
\begin{cases} \phi' = \phi'(\mu^{-1}, |||A'|||_{k-1,T}, |||B'(0)|||_{k-2}), \\ \psi' = \psi'(\mu^{-1}, |||A'|||_{k,T}, |||B'|||_{k-1,T}), \end{cases} \tag{2.4}
$$
$$
\begin{cases} \hat{\phi} = \hat{\phi}(\mu^{-1}, |||A; A'|||_{k-1,T}, |||B(0); B'(0)|||_{k-2}), \\ \hat{\psi} = \hat{\psi}(\mu^{-1}, |||A; A'|||_{k,T}, |||B; B'|||_{k-1,T}) \end{cases}
$$

and

$$
\bar{\psi} = \bar{\psi}(\mu^{-1}, |||A; A'|||_{k,T}, |||B; B'|||_{k,T}, \|f; f'\|_k, |||F(0); F'(0)|||_{k-1}, [F; F']_{k,T}).
$$

Here A_i', $i = 1, \ldots, n$, and B' denote a second set of matrices that will be used in Section 4. For convenience, we use the notation $\|u; v\|$ instead of $\|u\|, \|v\|$.

Hypotheses. As mentioned in the Introduction, our main hypotheses on the system (1.1) are uniqueness in $\mathcal{C}_T(H^1)$ and existence in $\mathcal{C}_T(H^2)$. Such results are well known for a large class of systems of type (1.1) or (1.2). Let us describe these two hypotheses. Note that we will not consider dependence on the matrix M which we will take to be fixed once and for all. On the contrary, $A = (A_1, \ldots, A_n)$ belongs to any class of coefficients, provided that their elements satisfy the hypotheses (2.1) and (2.2), for the same μ, plus hypotheses I and II below.

Hypothesis I. Assume in equation (1.1) that $B = 0$, $f = 0$, $F \in \mathcal{L}_T^2(H^1)$, $F(0) = 0$. Then there is a positive constant c (that may depend on T, F, and A) such that if $u \in \mathcal{C}_T(H^1)$ is a solution of problem (1.1), then

$$
|||u(t)|||_1^2 \leq c([F]_{1,t}^2 + [u]_{1,t}^2), \qquad \forall\, t \in [0, T].
$$

Consequences

Consider block square matrices $\mathcal{A}_i = \mathrm{diag}(A_i, \ldots, A_i)$, where the A_i are repeated q times, $q \in \mathbb{N}$. Let $U = (u^1, \ldots, u^q)$ be a $q \times m$-vector field, each u^j being an m-vector field, and similarly for $\mathfrak{F} = (F^1, \ldots, F^q)$. Consider the system

$$
\partial_t U + \sum_{i=1}^{n} \mathcal{A}_i \partial_i U = \mathfrak{F} \text{ in } Q_T, \qquad MU = 0 \text{ on } \Sigma_T, \qquad U(0) = 0, \tag{2.5}
$$

where $MU = 0$ means $Mu^j = 0$, $j = 1, \dots, q$, moreover $\mathfrak{F} \in \mathcal{L}_T^2(H^1)$, $\mathfrak{F}(0) = 0$. Since the system (2.5) consists of q decoupled systems, each one satisfying hypothesis I, (2.5) also satisfies this hypothesis. In particular, if $\mathfrak{B}$ is a square matrix of type $(q \times m) \times (q \times m)$ that belongs to the class $\mathcal{L}_T^\infty(H^{k-1})$, the system

$$\partial_t U + \sum_{i=1}^n \mathfrak{A}_i \partial_i U + \mathfrak{B} U = 0 \text{ in } Q_T, \qquad MU = 0 \text{ on } \Sigma_T, \qquad U(0) = 0, \tag{2.5'}$$

has only the zero solution, in the class $\mathcal{C}_T(H^1)$. This follows easily by setting $\mathfrak{F} = -\mathfrak{B}U$ on the above inequality and by using Gronwall's lemma.

Hypothesis II. Consider the system (1.1) for $B \equiv 0$. There are functions (of type λ) $\alpha_2 = \alpha_2(\mu^{-1}, |||A|||_{k-1,T})$, $\beta_2 = \beta_2(\mu^{-1}, |||A|||_{k,T})$, and $\gamma_2 = \gamma_2(\mu^{-1}, |||A|||_{k,T})$, such that if the pair $(f, F) \in H^2 \times \mathcal{L}_T^2(H^2)$ satisfies the compatibility conditions up to order 1 for the above system, there is a solution $u \in \mathcal{C}_T(H^2)$ of that system such that, for each $t \in [0, T]$,

$$|||u(t)|||_2^2 \le \alpha_2(|||f|||_2^2 + |||F(0)|||_1^2) + \beta_2[F]_{2,t}^2 + \gamma_2[u]_{2,t}^2. \tag{2.6}$$

The linear problem

Here, we consider the linear problem (1.1).

PROPOSITION 2.1. Assume that the hypotheses (2.1), (2.2), I and II hold and let $(f, F) \in H^l \times \mathcal{L}_T^2(H^l)$, $2 \le l \le k$, satisfy the compatibility conditions up to order $l - 1$. Then, there is a (unique) solution $u \in \mathcal{C}_T(H^l)$ of problem (1.1). Moreover, for each $t \in [0, T]$,

$$|||u(t)|||_l^2 \le \alpha_l'(|||f|||_l^2 + |||F(0)|||_{l-1}^2) + \beta_l'[F]_{l,t}^2 + \gamma_l'[u]_{l,t}^2. \tag{2.7}$$

In particular

$$|||u(t)|||_l^2 \le e^{t\psi}\{\phi(||f||_l^2 + |||F(0)|||_{l-1}^2) + \psi[F]_{l,t}^2\}. \tag{2.8}$$

Above, the functions α_l' are of type ϕ, β_l' and γ_l' of type ψ, see (2.4), with the following exception. If $l = k$, the dependence on $|||B(0)|||_{k-2}$ and on $|||B|||_{k-1,t}$ should be replaced, respectively, by dependence on $|||B(0)|||_{k-1}$ and on $|||B|||_{k,t}$. The same modification must be made for ϕ and ψ in equation (2.8).

Now, we state the perturbation theorems. We will distinguish between the cases $l \le k - 1$ and $l = k$. We note that, for $l \le k - 1$, the method developed here works also if the coefficients A and B belong to suitable functional spaces involving less than k (but more regular) derivatives. In this case, the perturbation of the coefficients should be done in these functional spaces.

Consider a second system of type (1.1), namely

$$\begin{cases} \partial_t u' + \sum_{i=1}^n A_i'(t, x)\, \partial_i u' + B'(t, x)u' = F' & \text{in } Q_T, \\ Mu'|_{\Sigma_T} = 0 \qquad u'(0) = f'. \end{cases} \tag{2.9}$$

We prove the following perturbation results.

LEMMA 2.2. Assume that the coefficients A, B and A', B' satisfy the hypotheses (2.1) and (2.2), and that the hypotheses I and II hold for A and for A'. Let $2 \leq l \leq k-1$. If $l = k-1$, assume that B satisfies (2.3). Let the pairs f, F and f', F' belong to $H_l \times \mathcal{L}_T^2(H^l)$ and satisfy the compatibility conditions up to order $l-1$ for the systems (1.1) and (2.9), respectively. Denote by u and u' the solutions of these two systems, which belong to $\mathcal{C}_T(H^l)$.

Given $\varepsilon \in]0, 1]$ there is a positive integer $\Lambda(\varepsilon)$ (that depends only on ε and T, on the particular coefficients A and B, on μ, and on the particular data f, F, but not on A', B', f', F') such that, for each $t \in [0, T]$,

$$\begin{aligned} &|||(u-u')(t)|||_l^2 \\ &\quad \leq \hat{\phi}\, e^{\hat{\psi}t}\{\|f-f'\|_l^2 + |||(F-F')(0)|||_{l-1}^2 + \Lambda_0|||(A-A')(0)|||_{k-1}^2 + \Lambda_0|||(B-B')(0)|||_{k-2}^2\} \\ &\qquad + \hat{\psi}\, e^{\hat{\psi}t}\{\varepsilon + [F-F']_{l,t}^2 + \Lambda_0[B-B']_{k-1,t}^2 + \Lambda(\varepsilon)[A-A']_{k-1,t}^2\}. \end{aligned} \tag{2.10}$$

In particular, consider a sequence of problems (2.9) each one satisfying the above assumptions. Assume, in addition, that $|||A'|||_{k,T}$ and $|||B'|||_{k-1,T}$ are uniformly bounded; that $A' \to A$ in $\mathcal{L}_T^2(H^{k-1})$ and $B' \to B$ in $\mathcal{L}_T^2(H^{k-1})$; that $f' \to f$ in H^l and $F' \to F$ in $\mathcal{L}_T^2(H^l)$; then $u' \to u$ in $\mathcal{C}_T(H^l)$.

The quantity Λ_0 satisfies

$$\Lambda_0 \leq e^{\psi t}[\phi(\|f\|_l^2 + |||F(0)|||_{l-1}^2) + \psi[F]_{l,t}^2]. \tag{2.11}$$

Now, we state the perturbation theorem for $l = k$.

THEOREM 2.3. Assume that the coefficients A, B and A', B' satisfy the conditions (2.1)–(2.3) and the hypotheses I and II. Let R be a constant such that $|||A'|||_{k,T} \leq R$. Moreover, assume that the pairs f, F and f', F' belong to $H^k \times \mathcal{L}_T^2(H^k)$ and satisfy the compatibility conditions up to order $k-1$ for the systems (1.1) and (2.9), respectively. Denote by $u \in \mathcal{C}_T(H^k)$ and $u' \in \mathcal{C}_T(H^k)$ the solutions of these systems. Then, given $\varepsilon \in]0, 1]$, there is a positive $\tilde{\Lambda}(\varepsilon)$ (that depends on the same quantities on which depends the $\Lambda(\varepsilon)$ in lemma 2.2 and also on R†), such that for each $t \in [0, T]$, one has

$$\begin{aligned} |||(u-u')(t)|||_k^2 \leq \bar{\psi}\exp(\bar{\psi}\, e^{t\bar{\psi}})\{&\varepsilon + \|f-f'\|_k^2 + |||(F-F')(0)|||_{k-1}^2 + |||(A-A')(0)|||_{k-1}^2 \\ &+ |||(B-B')(0)|||_{k-1}^2 + [F-F']_{k,t}^2 + [A-A']_{k,t}^2 + [B-B']_{k,t}^2 \\ &+ \tilde{\Lambda}(\varepsilon)[A-A']_{k-1,t}^2\}. \end{aligned} \tag{2.12}$$

In particular, consider a sequence of problems (2.9) each of them satisfying the above assumptions. Assume, in addition, that $|||A'|||_{k,T}$ and $|||B'|||_{k,T}$ are uniformly bounded; that $A' \to A$, $B' \to B$, and $F' \to F$ in $\mathcal{L}_T^2(H^k)$; that $f' \to f$ in H^k; then $u' \to u$ in $\mathcal{C}_T(H^k)$.

The nonlinear problem

Now, we consider problem (1.2). The hypothesis are the following:

(N.1) $A(\cdot)$ and $B(\cdot)$ are defined and of class C^k over the whole space $\mathbb{R}^m$;‡
(N.2) $A_n(v)$ is nonsingular for each $v \in N$, where N is the kernel in $\mathbb{R}^m$ of the matrix M;

† Dependence on R can be dropped.
‡ See the remark at the end of this section.

(N.3) for each fixed $v \in \mathcal{L}_T^\infty(H^k)$ such that $Mv|_{\Sigma_T} = 0$ the system

$$\partial_t u + \sum_{i=1}^n A_i(v)\,\partial_i u = F \text{ in } Q_T, \qquad Mu|_{\Sigma_T} = 0, \qquad u(0) = f, \tag{2.13}$$

satisfies the hypotheses I and II.

In assumption (N.3), the meaning of hypothesis I is clear. Let us clarify the meaning of hypothesis II. For each $v \in \mathcal{L}_T^\infty(H^k)$ such that $Mv|_{\Sigma_T} = 0$, define

$$\mu(v) = \inf_{(t,x) \in \Sigma_T} |\det A_n(v(t,x))|. \tag{2.14}$$

Clearly $\mu(v) > 0$, by assumption (N.2). Hypothesis II means that if $(f, F) \in H^2 \times \mathcal{L}_T^2(H^2)$ satisfy the compatibility conditions up to order 1 for the system (2.13), then there is a solution $u \in \mathcal{C}_T(H^2)$ of this system satisfying

$$|\!|\!| u(t) |\!|\!|_2^2 \le \phi(|\!|\!| f |\!|\!|_2^2 + |\!|\!| F(0) |\!|\!|_1^2) + \psi_1 [F]_{2,T}^2 + \psi_2 [u]_{2,T}^2, \tag{2.15}$$

where $\phi = \phi(\mu(v)^{-1}, |\!|\!| A(v) |\!|\!|_{k-1,T})$ and $\psi_1 = \psi_1(\mu(v)^{-1}, |\!|\!| A(v) |\!|\!|_{k,T})$, $i = 1, 2$, are functions of type λ.

PROPOSITION 2.4. Assume that (N.1)–(N.3) hold and let the pair $(f, F) \in H^k \times \mathcal{L}_{T_0}^2(H^k)$ satisfy the compatibility conditions up to order $k - 1$ for problem (1.2).† Then, there is a positive T such that this problem has a unique solution $u \in \mathcal{C}_T(H^k)$. A lower bound for T is given by inequalities of the form

$$T\lambda_1(\|f\|_k, |\!|\!| F(0) |\!|\!|_{k-1}) \le 1 \qquad \text{and} \qquad [F]_{k,T}\lambda_2(\|f\|_k, |\!|\!| F(0) |\!|\!|_{k-1}) \le 1, \tag{2.16}$$

where λ_1 and λ_2 are suitable functions of type λ. Moreover, $|\!|\!| u |\!|\!|_{k,T} \le \lambda_3(\|f\|_k, |\!|\!| F(0) |\!|\!|_{k-1})$, for a suitable λ_3.

Finally we state our result on Hadamard's well-posedness for system (1.2). Consider a sequence of problems

$$\begin{cases} \partial_t u_\nu + \sum_{i=1}^n A_i(u_\nu)\,\partial_i u_\nu + B(u_\nu)u_\nu = F_\nu & \text{in } Q_T, \\ Mu_\nu|_{\Sigma_T} = 0, \qquad u_\nu(0) = f_\nu, \end{cases} \tag{2.17$_\nu$}$$

where, for each $\nu \in \mathbb{N}$, the pair $(f_\nu, F_\nu) \in H^k \times \mathcal{L}_{T_0}^2(H^k)$ satisfy the compatibility conditions up to order $k - 1$ for system $(2.17)_\nu$. Let f, F be as in proposition 2.4 and let $u \in \mathcal{C}_{T_0}(H^k)$ be a solution of problem (1.2) in Q_{T_0}, for some $T_0 > 0$.‡ One has the following well-posedness result.

THEOREM 2.5. Assume that $A(\cdot)$, $B(\cdot)$, and M satisfy hypotheses (N.1)–(N.3) and let f, F, u, f_ν, and F_ν be as described above. Assume that

$$\lim_{\nu\to\infty} f_\nu = f \text{ in } H^k, \qquad \lim_{\nu\to\infty} F_\nu = F \text{ in } \mathcal{L}_{T_0}^2(H^k). \tag{2.18}$$

† Recall that the compatibility conditions are independent of eventual solutions u of (1.2).
‡ T_0 is arbitrarily large, provided that the solution u exist on $[0, T_0]$.

Then, for sufficiently large values of ν, there is a solution $u_\nu \in \mathcal{C}_{T_0}(H^k)$ of $(2.17)_\nu$. Moreover,

$$\lim_{\nu\to\infty} u_\nu = u \qquad \text{in } \mathcal{C}_{T_0}(H^k). \tag{2.19}$$

The above result still holds if in equations $(2.17)_\nu$ we replace $A(u_\nu)$ and $B(u_\nu)$ by $A^\nu(u_\nu)$ and $B^\nu(u_\nu)$, respectively, provided that $A^\nu(\cdot)$ and $B^\nu(\cdot)$ converge, respectively, to $A(\cdot)$ and $B(\cdot)$ with respect to the C^k-norm, on compact subsets of $\mathbb{R}^m$.

Remark. If $A(\cdot)$ and $B(\cdot)$ are defined on an open subet $\mathcal{O}$ of $\mathbb{R}^m$, the results and proofs remain essentially unchanged, provided that in hypotheses (N.1)–(N.3) the range of v is contained in $\mathcal{O}$ and that the set $\{f(x): x \in \mathbb{R}^n_+\}$, contained in $\mathcal{O}$, has a positive distance to $\partial\mathcal{O}$. Moreover, we can also consider, without difficulty, the case in which A and B depend on (t, x, u).

3. PROOF OF PROPOSITION 2.1

The proof of proposition 2.1, completed by induction, follows from lemmas 3.1 and 3.2 below.

LEMMA 3.1. Let A satisfy (2.1) and let l be an integer such that $2 \le l \le k$. Assume that for $B \equiv 0$ and for every pair $(f, F) \in H^l \times \mathcal{L}^2(H^l_T)$ that satisfies the compatibility conditions up to order $l - 1$ the system (1.1) has a solution $u \in \mathcal{C}_T(H^l)$ such that

$$|||u(t)|||_l^2 \le \alpha_l(|||f|||_l^2 + |||F(0)|||_{l-1}^2) + \beta_l[F]_{l,t}^2 + \gamma_l[u]_{l,t}^2, \qquad \forall\, t \in [0, T], \tag{3.1}$$

where $\alpha_l = \alpha_l(\mu^{-1}, |||A|||_{k-1,T})$, $\beta_l = \beta_l(\mu^{-1}, |||A|||_{k,T})$, and $\gamma_l = \gamma_l(\mu^{-1}, |||A|||_{k,T})$ are functions of type λ.

Then, for each B satisfying (2.1) (and also (2.3), if $l = k$) and for each pair $(f, F) \in H^l \times \mathcal{L}^2_T(H^l)$ satisfying the compatibility conditions up to order $l - 1$ for system (1.1), there is a solution $u \in \mathcal{C}(H^l_T)$ of this system. Moreover, (2.7) holds.

Remark. Since lemma 3.1 holds for each fixed positive integer m, it holds (in particular) if m is replaced by $(n + 1) \times m$, i.e. it holds for systems of $(n + 1) \times m$ equations

$$\begin{cases} \partial_t U + \sum_{i=1}^n \mathcal{A}_i \partial_i U + \mathcal{B} U = \mathcal{F}, \\ MU|_{\Sigma_T} = 0, \qquad U(0) = \mathfrak{f}, \end{cases} \tag{3.2}$$

where $\mathcal{A}_i = \operatorname{diag}(A_i, \ldots, A_i)$, with the A_is repeated $n + 1$ times, $\mathcal{B}$ is a $[(n + 1)m] \times [(n + 1)m]$ square matrix, and so on.

LEMMA 3.2. Assume that A satisfies (2.1) and (2.2), and that hypothesis I holds. Let $2 \le l \le k - 1$ and assume that for each $\mathcal{B}$ satisfying (2.1) and for each pair $(\mathfrak{f}, \mathcal{F}) \in H^l \times \mathcal{L}^2_T(H^l)$ satisfying the compatibility conditions up to order $l - 1$ for the system (3.2), there is a solution $U \in \mathcal{C}_T(H^l)$ of this system satisfying the inequality (2.7), in which u, f, F, B and A should be replaced, respectively, by U, $\mathfrak{f}$, $\mathcal{F}$, $\mathcal{B}$ and $\mathcal{A}$.

Then, if the pair $(f, F) \in H^{l+1} \times \mathcal{L}^2_T(H^{l+1})$ satisfies the compatibility conditions up to order l for the system (1.1) with $B \equiv 0$, there is a solution $u \in \mathcal{C}_T(H^l)$ to this last system which satisfies the estimate (3.1) for the value $l + 1$.

Proof of proposition 2.1 (by assuming lemmas 3.1 and 3.2). We start by proving the thesis of the theorem when $B = 0$. The proof is done by induction. If $l = 2$, and if $(f, F) \in H^2 \times \mathcal{L}_T^2(H^2)$ satisfy the c.c. up to order 1, the solution $u \in \mathcal{C}_T(H^2)$ exists and satisfies (3.1), by hypothesis II. Assume now that $l \in [2, k-1]$ and that the solution of system (1.1) with $B = 0$ exists and satisfies (3.1) provided that $(f, F) \in H^l \times \mathcal{L}_T^2(H^l)$ satisfies the c.c. up to order $l - 1$. Then this last result also holds for systems (3.2) with $\mathcal{B} = 0$, since these systems consist of the union of $n + 1$ systems of type (1.1) with $B = 0$. Then, lemma 3.1 applies to system (3.2) (see the remark after lemma 3.1) and shows that the hypotheses of lemma 3.2 hold. Hence, if $B \equiv 0$ and if $(f, F) \in H^{l+1} \times \mathcal{L}_T^2(H^{l+1})$, satisfy the c.c. up to order l, the solution of system (1.1) exists, belongs to $\mathcal{C}_T(H^{l+1})$, and satisfies (3.1) for the value $l + 1$. This proves proposition 2.1 for $B = 0$. The result for $B \neq 0$ follows from the result for $B = 0$, together with lemma 3.1.

Finally, (2.7) and equation (B.5) in Appendix B show that

$$|||u(t)|||_l^2 \leq \phi(\|f\|_l^2 + |||F(0)|||_{l-1}^2) + \psi[F]_{l,t}^2 + \psi \int_0^t |||u(s)|||_l^2 \, ds.$$

An application of Gronwall's lemma yields (2.8). ■

Proof of lemma 3.1. In order to fix the ideas, assume that $2 \leq l < k$. Let $B \in \mathcal{L}_T^\infty(H^{k-1})$ be given and assume that the pair $(f, F) \in H^l \times \mathcal{L}_T^2(H^l)$ satisfy the compatibility conditions up to order $l - 1$ for system (1.1). Consider the functions $f^{(j)}$, $j = 0, \ldots, l - 1$, defined in correspondence to the system (1.1); for the definition of $f^{(j)}$ see Section 2. Lemma B.1 in Appendix B shows that $f^{(j)} \in H^{l-j}$. Consider the equations

$$\partial_t^j w(0) = f^{(j)}, \qquad j = 0, \ldots, l - 1, \tag{3.3}$$

and define, for $\tau \in]0, T]$,

$$\mathbb{K} = \{w \in \mathcal{L}_\tau^2(H^l) : [w]_{l,\tau} \leq c_0 |||f|||_l, \text{ and (3.3) holds}\}.$$

Since $\mathcal{L}_T^2(H^l) = H^l(Q_\tau)$, the set $\mathbb{K}$ is nonempty, for a suitable choice of the constant c_0 (independent of τ). For each $w \in \mathbb{K}$ consider the problem

$$Lu = F - Bw \text{ in } [0, \tau], \qquad Mu|_{\Sigma_\tau} = 0, \qquad u(0) = f. \tag{3.4}$$

From (3.3) it follows that the pair $(f, F - Bw)$ satisfy the c.c. up to order $l - 1$. Denote by $u = \Lambda(w)$ the solution of (3.4). Equation (3.1), with F replaced by $F - Bw$, together with equations (A.4) and (A.4′) in Appendix A, show that

$$\begin{aligned} |||u(t)|||_l^2 \leq{} & \alpha_l(|||f|||_l^2 + |||F(0)|||_{l-1}^2 + c|||B(0)|||_{k-2}^2 |||f|||_l^2) \\ & + \beta_l([F]_{l,T}^2 + c|||B|||_{k-1,T}^2 [w]_{l,\tau}^2) + \gamma_l [u]_{l,\tau}^2. \end{aligned} \tag{3.5}$$

It readily follows that $[u]_{l,\tau}^2 \leq c_0$ if τ is sufficiently small. Hence, $\Lambda(\mathbb{K}) \subset \mathbb{K}$. Finally, if $u = \Lambda(w)$ and $u' = \Lambda(w')$, one has $L(u - u') = B(w - w')$ in Q_τ, $M(u - u') = 0$ on Σ_τ, and $(u - u')(0) = 0$. Hence, by applying (3.5) to $u - u'$, one shows that $[u - u']_{l,\tau}^2 \leq c\tau[w - w']_{l,\tau}^2$. It follows that Λ is a contraction, for τ small enough. The fixed point $u = \Lambda(u) \in \mathbb{K}$ is a solution of problem (1.1) in $[0, \tau]$. This solution belongs to $\mathcal{C}_\tau(H^l)$, since $\Lambda(\mathbb{K}) \subset \mathcal{C}_\tau(H^l)$. By setting

$w = u$ in equation (3.5) one shows that

$$|||u(t)|||_l^2 \le \alpha_l(1 + c|||B(0)|||_{k-2}^2)(|||f|||_l^2 + |||F(0)|||_{l-1}^2) + \beta_l[F]_{l,t}^2$$
$$+ (\gamma_l + c\beta_l|||B|||_{k-1,T}^2)[u]_{l,t}^2, \tag{3.6}$$

which proves (2.7) and also furnishes explicit expressions for α_l', β_l', γ_l' in terms of α_l, β_l, γ_l.

By using (3.6) and Gronwall's lemma one gets an a priori estimate for $|||u(t)|||_l^2$ on $[0, T]$. A continuation argument shows that the solution u of (1.1) exists in $[0, T]$. Note that the above value of τ is bounded from below if $|||f|||_l$ is bounded from above. ■

Proof of lemma 3.2. By assumption, $B \equiv 0$. Nevertheless, some formulas will be established without assuming this condition since, later on (in Section 4), they will be used for $B \not\equiv 0$.

Suppose that the pair $(f, F) \in H^{l+1} \times \mathcal{L}_T^2(H^{l+1})$ satisfies the compatibility conditions up to order l, $2 \le l \le k - 1$, for system (1.1). By assumption, this system admits a solution $u \in \mathcal{C}_T(H^l)$. Since $l \ge 2$, we are allowed to take first derivatives of both sides of equation (1.1) with respect to the variables x_j, $j = 1, \dots, n - 1$, and t. This yields

$$\begin{cases} L(\partial_j u) + \sum_{i=1}^{n} (\partial_j A_i)\partial_i u = \partial_j F - (\partial_j B)u & \text{in } Q_T, \\ M(\partial_j u)|_{\Sigma_T} = 0, \qquad (\partial_j u)(0) = \partial_j f, \end{cases} \tag{3.7$_j$}$$

for $j = 1, \dots, n - 1$, and

$$\begin{cases} L(\partial_t u) + \sum_{i=1}^{n} (\partial_t A_i)\,\partial_i u = \partial_t F - (\partial_t B)u & \text{in } Q_T, \\ M(\partial_t u)|_{\Sigma_T} = 0, \qquad (\partial_t u)(0) = f^{(1)}. \end{cases} \tag{3.8}$$

Let us consider the normal direction x_n. We start by arguing as in [11]. Since $A_n \in \mathcal{C}_T(H^{k-1})$ and $H^{k-1} \hookrightarrow C^{0,\alpha}$ for some $\alpha(k, n) > 0$, one has

$$|\det A_n(t, x', x_n) - \det A_n(t, x', 0)| \le c_0\|A_n\|_{k-1,T}^m x_n^\alpha. \tag{3.9}$$

We do not use the injection $H^k \hookrightarrow C^{0,1}$ since we want to have the H^{k-1} norm (instead of the H^k norm) on the right-hand side of (3.9). See [11]. By using equation (2.2) it follows that

$$|\det A_n| > \mu/2 \qquad \text{if } 0 < x_n < r, \tag{3.10}$$

where $r = (\mu/(2c_0))^{1/\alpha}\|A\|_{k-1,T}^{-m/\alpha}$. In particular

$$r^{-1} = \phi(\mu^{-1}, \|A\|_{k-1,T}). \tag{3.11}$$

Now, we fix a function $\Theta \in C^\infty(\mathbb{R}^+; \mathbb{R}^+)$ such that $\Theta(x_n) = 0$ if $0 < x_n < 1/2$, $\Theta(x_n) = 1$ if $x_n > 1$, and we set $\vartheta(x_n) = \Theta(x_n/r)$. From (3.11) it follows that

$$|\vartheta^{(j)}(x_n)| \le cr^{-j} \le \phi(\mu^{-1}, \|A\|_{k-1,T}), \tag{3.12}$$

for each $j \in \mathbb{N}_0$. Now, by differentiating with respect to x_n both sides of (1.1), we get equation $(3.7)_j$ for $j = n$. It readily follows that

$$L(\partial_n(\vartheta u)) + \sum_{i=1}^{n} (\vartheta(\partial_n A_i) - \vartheta' A_i)(\partial_i u) - \vartheta' A_n(\partial_n u) - \vartheta' \partial_t u$$
$$= \vartheta \partial_n F + (\vartheta'' A_n - \vartheta(\partial_n B) + \vartheta' B)u. \tag{3.13}$$

On the other hand, one has $\partial_n u = \partial_n(\vartheta u) + (1 - \vartheta)\partial_n u - \vartheta' u$. Moreover, equation $(1.1)_1$ shows that

$$(1 - \vartheta)\partial_n u = (1 - \vartheta)A_n^{-1}\left[F - \partial_t u - \sum_{j=1}^{n-1} A_j \partial_j u - Bu\right],$$

where, by definition, the right-hand side is equal to zero when $1 - \vartheta = 0$, even if A_n^{-1} does not exist. Hence,

$$\partial_n u = \partial_n(\vartheta u) - (1 - \vartheta) \sum_{j=1}^{n-1} A_n^{-1} A_j \partial_j u - (1 - \vartheta)A_n^{-1} \partial_t u$$
$$- (1 - \vartheta)A_n^{-1}Bu + (1 - \vartheta)A_n^{-1}F - \vartheta' u. \tag{3.14}$$

Finally, by replacing in equations (3.7), (3.8), and (3.13) the derivative $\partial_n u$ by the right-hand side of equation (3.14), we obtain the system $(3.15)_j$ below, $j = 1, \ldots, n + 1$, which is a system of type (3.2) for the $n \times (m + 1)$-dimensional vector field $U = (\partial_1 u, \ldots, \partial_{n-1} u, \partial_n(\vartheta u), \partial_t u)$. Let us describe this system in detail.

For $j = 1, \ldots, n - 1$, one has

$$\begin{cases} \partial_t(\partial_j u) + \sum_{i=1}^{n} A_i \partial_i(\partial_j u) + \sum_{r=1}^{n-1} B_{jr}(\partial_r u) + B_{jn} \partial_n(\vartheta u) + B_{j,n+1}(\partial_t u) F_j \quad \text{in } Q_T, \\ M(\partial_j u)|_{\Sigma_T} = 0; \qquad (\partial_j u)(0) = \partial_j f, \end{cases} \tag{3.15$_j$}$$

where

$$B_{jr} = (\partial_j A_r) - (1 - \vartheta)(\partial_j A_n)A_n^{-1}A_r + \delta_{jr}B, \qquad \text{for } 1 \le r \le n - 1,$$
$$B_{jn} = \partial_j A_n, \qquad B_{j,n+1} = -(1 - \vartheta)(\partial_j A_n)A_n^{-1},$$

and

$$F_j = \partial_j F - (1 - \vartheta)(\partial_j A_n)A_n^{-1}F + \vartheta'(\partial_j A_n)u - (\partial_j B)u + (1 - \vartheta)(\partial_j A_n)A_n^{-1}Bu.$$

For $j = n + 1$ one has

$$\begin{cases} \partial_t(\partial_t u) + \sum_{i=1}^{n} A_i \partial_i(\partial_t u) + \sum_{r=1}^{n-1} B_{n+1,r}(\partial_r u) + B_{n+1,n} \partial_n(\vartheta u) \\ \qquad + B_{n+1,n+1}(\partial_t u) = F_{n+1} \qquad \text{in } Q_T, \\ M(\partial_t u)|_{\Sigma_T} = 0; \qquad (\partial_t u)(0) = f^{(1)}, \end{cases} \tag{3.15$_{n+1}$}$$

where

$$B_{n+1,r} = \partial_t A_r - (1 - \theta)(\partial_t A_n)A_n^{-1}A_r, \qquad B_{n+1,n+1} = B - (1 - \vartheta)(\partial_t A_n)A_n^{-1},$$
$$B_{n+1,n} = \partial_t A_n,$$

and

$$F_{n+1} = \partial_t F - (1 - \vartheta)(\partial_t A_n)A_n^{-1}F + \vartheta'(\partial_t A_n)u - (\partial_t B)u + (1 - \vartheta)(\partial_t A_n)A_n^{-1}Bu.$$

Recall that $f^{(1)} = F(0) - \sum_{i=1}^{n} A_i(0)\,\partial_i f - B(0)f$. Finally, for $j = n$, one has

$$\begin{cases} \partial_t\,\partial_n(\vartheta u) + \sum_{i=1}^{n} A_i\,\partial_i\,\partial_n(\vartheta u) + \sum_{r=1}^{n-1} B_{nr}(\partial_r u) + B_{nn}\,\partial_n(\vartheta u) \\ \qquad + B_{n,n+1}(\partial_t u) = F_n \quad \text{in } Q_T, \\ M\,\partial_n(\vartheta u)|_{\Sigma_T} = 0; \qquad \partial_n(\vartheta u)(0) = \partial_n(\vartheta f), \end{cases} \tag{3.15$_n$}$$

where

$$B_{nr} = \vartheta(\partial_n A_r) + 2(1 - \vartheta)\vartheta' A_r - \vartheta(1 - \vartheta)(\partial_n A_n)A_n^{-1}A_r - \vartheta' A_r, \qquad \text{for } r = 1, \ldots, n-1,$$

$$B_{nn} = -2\vartheta' A_n + \vartheta\,\partial_n A_n, \qquad B_{n,n+1} = (1 - 2\vartheta)\vartheta' - \vartheta(1 - \vartheta)(\partial_n A_n)A_n^{-1},$$

and

$$\begin{aligned} F_n = {} & \vartheta\,\partial_n F + [2(1 - \vartheta)\vartheta' - \vartheta(1 - \vartheta)(\partial_n A_n)A_n^{-1}]F \\ & + [-2(1 - \vartheta)\vartheta' B + \vartheta(1 - \vartheta)(\partial_n A_n)A_n^{-1}B - \vartheta'^2 A_n + \vartheta\vartheta'(\partial_n A_n) + \vartheta' B - \vartheta\,\partial_n B]u. \end{aligned}$$

Equations $(3.15)_j$, $j = 1, \ldots, n+1$ (abbreviate to (3.15)), form a system of type (3.2), where $\mathcal{B}$ is the block matrix $[B_{rs}]$, $r, s = 1, \ldots, n+1$, and $\mathcal{F}$ and $\mathfrak{f}$ are the $m \times (n+1)$-vector fields $(F_1, \ldots, F_{n+1})$ and $(\partial_1 f, \ldots, \partial_{n-1} f, \partial_n(\vartheta f), f^{(1)})$, respectively.

Compatibility conditions for system (3.15) hold up to the order $l - 1$ since they hold, for original system (1.1), up to the order l. In fact, in order to get (3.5) from (1.1), we performed only the following two operations. To take first derivatives of both sides of (1.1) and to use equations $(1.1)_1$ and (3.14), which involve u and its first derivatives. More precisely, the compatibility conditions for equations $(3.15)_j$, $j = 1, \ldots, n-1$, follow by differentiation, with respect to the tangential direction x_j, of the compatibility conditions of the same order for equation (1.1). The compatibility conditions for $(3.15)_n$ hold trivially since, near the boundary, ϑ and its derivatives vanish identically. The compatibility conditions of orders up to l for equation (1.1) yield compatibility conditions of orders up to $l - 1$ for equation $(3.15)_{n+1}$.

Finally, by taking into account the concrete expression of $\mathfrak{f}$, $\mathcal{F}$, and $\mathcal{B}$, it readily follows that $\mathfrak{f} \in H^l$, $\mathcal{F} \in \mathcal{L}_T^2(H^l)$, and $\mathcal{B} \in \mathcal{L}_T^\infty(H^{k-1})$.

The above facts, together with the hypothesis in lemma 3.2, show that system (3.15) admits a solution $U \in \mathcal{C}_T(H^l)$ satisfying (2.7), i.e. satisfying

$$|||U(t)|||_l^2 \le \alpha_l'(|||\mathfrak{f}|||_l^2 + |||\mathcal{F}(0)|||_{l-1}^2) + \beta_l'[\mathcal{F}]_{l,t}^2 + \gamma_l'[U]_{l,t}^2, \tag{3.16}$$

where $\alpha_l' = \phi(\mu^{-1}, |||A|||_{k-1,T}, |||\mathcal{B}(0)|||_{k-2})$, $\beta_l' = \psi(\mu^{-1}, |||A|||_{k,T}, |||\mathcal{B}|||_{k-1,T})$, and $\gamma_l' = \psi(\mu^{-1}, |||A|||_{k,T}, |||\mathcal{B}|||_{k-1,T})$ are functions of type λ.

On the other hand, $U \equiv (\partial_1 u, \ldots, \partial_{n-1} u, \partial_n(\vartheta u), \partial_t u)$ is a solution of (3.15). Since this solution belongs to the space $\mathcal{C}_T(H^{l-1}) \hookrightarrow \mathcal{C}_T(H^1)$, hypothesis I guarantees that the two solutions must coincide. Hence, $u \in \mathcal{C}_T(H^{l+1})$.

Finally, by taking into account the definitions of the functions $\mathfrak{f}$, $\mathcal{F}$, $\mathcal{B}$, and U (note that $\mathcal{B}$ appears in the expressions of the coefficients α_l', β_l', γ_l', in equation (3.16)), by proving suitable estimates for the norms of these functions, and by using these estimates in equation

(3.16), we get (3.1). Let us show it. By applying corollary A.2 (see Appendix A) and by taking into account equation (3.12) it readily follows that (recall that $B \equiv 0$)

$$|||\mathfrak{g}|||_l \le \phi(\mu^{-1}, |||A|||_{k-1,T})|||f|||_{l+1} + |||F(0)|||_l,$$

$$|||\mathfrak{F}(0)|||_{l-1} \le \phi(\mu^{-1}, |||A|||_{k-1,T})(|||f|||_l + |||F(0)|||_l),$$

$$|||\mathfrak{F}(s)|||_l \le \psi(\mu^{-1}, |||A|||_{k,T})(|||u(s)|||_l + |||F(s)|||_{l+1}),$$

$$|||\mathfrak{B}(0)|||_{k-2} \le \phi(\mu^{-1}, |||A|||_{k-1,T}),$$

$$|||\mathfrak{B}|||_{k-1,T} \le \psi(\mu^{-1}, |||A|||_{k,T}).$$

The two last estimates show that, in equation (3.16),

$$\alpha_l' = \phi(\mu^{-1}, |||A|||_{k-1,T}), \qquad \beta_l' = \psi(\mu^{-1}, |||A|||_{k,T}), \qquad \gamma_l' = \psi(\mu^{-1}, |||A|||_{k,T}).$$

Finally, from (3.14), we get

$$|||\partial_n u(t)|||_l^2 \le \phi(\mu^{-1}, |||A|||_{k-1,T})(|||U(t)|||_l^2 + |||F(t)|||_l^2 + |||u(t)|||_l^2).$$

From (3.16), together with the above estimates and the definition of U, it readily follows that

$$|||u(t)|||_{l+1}^2 \le \phi(|||f|||_{l+1}^2 + |||F(0)|||_l^2) + \psi([F]_{l+1,t}^2 + [u]_{l+1,t}^2) + \phi(|||F(t)|||_l^2 + |||u(t)|||_l^2), \quad (3.17)$$

where $\phi = \phi(\mu^{-1}, |||A|||_{k-1,T})$ and $\psi = \psi(\mu^{-1}, |||A|||_{k,T})$. Since

$$|||u(t)|||_l^2 \le |||u(0)|||_l^2 + [u]_{l+1,t}^2,$$

and analogously for $|||F(t)|||_l^2$, the equation (3.1) holds with l replaced by $l + 1$. ■

4. THE PERTURBATION THEOREM

We start by stating a technical result that will be of help in this section.

PROPOSITION 4.1. Assume that (2.1) and (2.2) hold and let $(f, F) \in H^l \times \mathcal{L}_T^2(H^l)$ for some integer l, $2 \le l \le k - 1$. Assume that the pair f, F satisfies the compatibility conditions up to order $l - 1$ for the system (1.1). Then, given $\varepsilon > 0$ there are functions $f_\varepsilon \in H^{l+1}$ and $F_\varepsilon \in \mathcal{L}_T^2(H^{l+1})$ that satisfy the compatibility conditions up to order l. Moreover,

$$\|f - f_\varepsilon\|_l^2 \le \varepsilon, \qquad [F - F_\varepsilon]_{l,T}^2 \le \varepsilon, \qquad |||(F - F_\varepsilon)(0)|||_{l-1}^2 \le \varepsilon.$$

The proof, a modification of that of lemma 3.3 in [6], will be given in Appendix B.

In the sequel, together with the system (1.1), namely,

$$Lu = F \text{ in } Q_T, \qquad Mu = 0 \text{ on } \Sigma_T, \qquad u(0) = f, \tag{4.1}$$

we also consider system (2.9), namely

$$L'u' = F' \text{ in } Q_T, \qquad Mu' = 0 \text{ on } \Sigma_T, \qquad u(0) = f', \tag{4.1'}$$

where the meaning of L' is clear (see (2.9)).

Proof of lemma 2.2. The reader should recall the notations introduced in Section 2, concerning the symbols $\phi, \ldots, \bar{\Psi}$. Let f_ε and F_ε be as in proposition 4.1, and consider the problem

$$Lu_\varepsilon = F_\varepsilon \text{ in } Q_T, \qquad Mu_\varepsilon = 0 \text{ on } \Sigma_T, \qquad u_\varepsilon(0) = f_\varepsilon. \tag{4.2}$$

From (2.8), we get

$$|||u_\varepsilon(t)|||_l^2 \leq \Lambda_0 + \psi\, e^{\psi t}\varepsilon, \qquad |||u_\varepsilon(t)|||_{l+1}^2 \leq \Lambda(\varepsilon), \tag{4.3}$$

where

$$\Lambda(\varepsilon) = e^{t\psi}\{\phi(\|f_\varepsilon\|_{l+1}^2 + |||F_\varepsilon(0)|||_l^2) + \psi[F_\varepsilon]_{l+1,t}^2\}.$$

If $l = k - 1$, replace, in the expressions of ϕ and ψ, $|||B(0)|||_{k-2}$ and $|||B|||_{k-1,T}$ by, respectively, $|||B(0)|||_{k-1}$ and $|||B|||_{k,T}$. Note that, given ε, T, A, B, f, and F, we can fix f_ε and F_ε. Hence, $\Lambda(\varepsilon)$ has just the dependence claimed in lemma 2.2.

By taking the difference, side by side, between equations (4.1′) and (4.2), we show that

$$\begin{cases} L'(u' - u_\varepsilon) = \sum_i (A_i - A_i')\,\partial_i u_\varepsilon + (B - B')u_\varepsilon + F' - F_\varepsilon & \text{in } Q_T, \\ M(u' - u_\varepsilon) = 0 \text{ on } \Sigma_T; \qquad (u' - u_\varepsilon)(0) = f' - f_\varepsilon. \end{cases} \tag{4.4}$$

By applying (2.8) to the above solution $u' - u_\varepsilon$, and by taking (4.3) into account, we show that

$$\begin{aligned} |||(u' - u_\varepsilon)(t)|||_l^2 \leq \phi' e^{\psi' t}\{&\|f' - f_\varepsilon\|_l^2 + |||(F' - F_\varepsilon)(0)|||_{l-1}^2 \\ &+ (\Lambda_0 + \psi\, e^{\psi t}\varepsilon)(|||(A' - A)(0)|||_{k-1}^2 + |||(B' - B)(0)|||_{k-2}^2) \\ &+ \psi'([F' - F_\varepsilon]_{l,t}^2 + \Lambda(\varepsilon)[A' - A]_{k-1,t}^2 + (\Lambda_0 + \psi\, e^{\psi t}\varepsilon)[B' - B]_{k-1,t}^2)\}. \end{aligned} \tag{4.5}$$

By using the estimate in proposition 4.1 one easily gets

$$|||(u' - u_\varepsilon)(t)|||_l^2 \leq \text{right-hand side of (2.10)}. \tag{4.6}$$

Note that $[B' - B]_{k-1,t}^2 \leq \hat{\psi}\exp(t\hat{\psi})$. A similar, but much simpler argument shows that

$$|||(u - u_\varepsilon)(t)|||_l^2 \leq e^{\psi t}\{\phi(\|f - f_\varepsilon\|_l^2 + |||(F - F_\varepsilon)(0)|||_{l-1}^2) + \psi[F - F_\varepsilon]_{l,t}^2\}.$$

Hence, $|||(u - u_\varepsilon)(t)|||_l^2 \leq \psi\, e^{\psi t}\varepsilon$. This inequality together with (4.6) shows that (2.10) holds. ■

Remark. Let $l = k - 1$ and assume that $f \in H^k$ and $F \in \mathfrak{L}_T^2(H^k)$ satisfy the compatibility conditions up to order $k - 1$ for the system (4.1). Then, by arguing for $u' - u$ as above for $u' - u_\varepsilon$, one shows that (compare to (4.5))

$$\begin{aligned} |||(u' - u)(t)|||_{k-1}^2 \leq \phi' e^{\psi' t}\{&\|f' - f\|_{k-1}^2 + |||(F' - F)(0)|||_{k-2}^2 \\ &+ \Lambda_0(|||(A' - A)(0)|||_{k-1}^2 + |||(B' - B)(0)|||_{k-2}^2) \\ &+ \psi'([F' - F]_{k-1,t}^2 + |||u|||_{k,t}^2[A' - A]_{k-1,t}^2 + \Lambda_0[B' - B]_{k-1,t}^2)\}. \end{aligned} \tag{4.7}$$

Proof of theorem 2.3. By arguing for (4.1′) as in Section 3 for (4.1), we get systems $(3.15')_j$, $j = 1, \ldots, n + 1$, which can be obtained by the reader by replacing everywhere† in

† Also in the expressions of the B_{jr}, of the F_j, and of $f^{(1)}$.

$(3.15)_j$ the elements A, B, f, F, and u by, respectively, A', B', f', F', and u'. We denote by (3.15) [resp. (3.15′)] the union of the systems $(3.15)_j$ [resp. $(3.15')_j$], for $j = 1, \ldots, n+1$. For convenience, we write these systems in the abbreviate form

$$\begin{cases} \partial_t U + \sum_{i=1}^{n} \mathcal{A}_i \partial_i U = \mathcal{F} - \mathcal{B} U & \text{in } Q_T, \\ MU = 0 \text{ on } \Sigma_T, \qquad U(0) = \mathfrak{f}, \end{cases} \tag{4.8}$$

and

$$\begin{cases} \partial_t U' + \sum_{i=1}^{n} \mathcal{A}'_i U' = \mathcal{F}' - \mathcal{B}' U' & \text{in } Q_T, \\ MU' = 0 \text{ on } \Sigma_T, \qquad U'(0) = \mathfrak{f}'. \end{cases} \tag{4.8′}$$

Hence, $\mathcal{A}_i$ is the matrix $(A_i, \ldots, A_i)$ (A_i repeated $n+1$ times), $\mathcal{B}$ is the block matrix $[B_{rs}]$, $r, s = 1, \ldots, n+1$, U is the $m \times (n+1)$ vector field $(\partial_1 u, \ldots, \partial_{n-1} u, \partial_n(\vartheta u), \partial_t u)$, $\mathcal{F}$ is the $m \times (n+1)$ vector field $(F_1, \ldots, F_{n+1})$, and $\mathfrak{f} = (\partial_1 f, \ldots, \partial_{n-1} f, \partial_n(\vartheta f), \partial_t f)$; see Section 3. Similar definitions hold for $\mathcal{A}'_i$, $i = 1, \ldots, n$, $\mathcal{B}'$, $\mathcal{F}'$ and $\mathfrak{f}'$.

System (4.8), a system of $m \times (n+1)$ equations, satisfies hypothesis I (see "consequences", after the statement of hypothesis I). It also satisfies hypothesis II since the system

$$\partial_t U + \sum_{i=1}^{n} \mathcal{A}_i \partial_i U = \mathcal{F} \text{ in } Q_T, \qquad MU = 0 \text{ on } \Sigma_T, \qquad U(0) = \mathfrak{f}, \tag{4.9}$$

(where the pair $\mathfrak{f}, \mathcal{F}$ satisfy the regularity and the compatibility conditions required in hypothesis II) is the union of $n+1$ systems which satisfy that same hypothesis. Note that the constants $\alpha_2, \beta_2, \gamma_2$ do not change "type" since $|||\mathcal{A}|||_{k,T} \cong |||A|||_{k,T}$ and $|\det \mathcal{A}_n| = |\det A_n|^{n+1}$. The same arguments show that (4.8′) satisfies hypotheses I and II. By applying to the couple of systems (4.8), (4.8′), the equation (2.10) in theorem 2.2 (for the value $l = k-1$) we get

$$\begin{aligned} |||(U - U')(t)|||_{k-1}^2 \le \hat{\Phi}\, e^{\hat{\Psi} t}\{ & \|\mathfrak{f} - \mathfrak{f}'\|_{k-1}^2 + |||(\mathcal{F} - \mathcal{F}')(0)|||_{k-2}^2 + |||(\mathcal{B}U - \mathcal{B}'U')(0)|||_{k-2}^2 \\ & + \tilde{\Lambda}_0 |||(\mathcal{A} - \mathcal{A}')(0)|||_{k-1}^2 \} \\ & + \hat{\Psi}\, e^{\hat{\Psi} t}\{\varepsilon + [\mathcal{F} - \mathcal{F}']_{k-1,t}^2 + [\mathcal{B}U - \mathcal{B}'U']_{k-1,t}^2 + \tilde{\Lambda}(\varepsilon)[\mathcal{A} - \mathcal{A}']_{k-1,t}^2\} \end{aligned}$$

where $\tilde{\Lambda}_0$, given by

$$\tilde{\Lambda}_0 = e^{\Psi t}\{\Phi(\|\mathfrak{f}\|_{k-1}^2 + |||\mathcal{F}(0)|||_{k-2}^2 + |||\mathcal{B}U(0)|||_{k-2}^2) + \Psi[\mathcal{F}]_{k-1,T}^2 + \Psi[\mathcal{B}U]_{k-1,T}^2\}$$

satisfies $\tilde{\Lambda}_0 \le \bar{\Psi}\, e^{\Psi t}$ and $\tilde{\Lambda}(\varepsilon)$ depends only on the particular ε, T, $\mathcal{A}$, $\mathcal{B}$, $\mathfrak{f}$, and $\mathcal{F}$. Hence,

$$|||(U - U')(t)|||_{k-1}^2 \le \bar{\Psi}\, e^{\hat{\Psi} t}\{\cdots\} + \bar{\Psi}\, e^{\hat{\Psi} t}[U - U']_{k-1,t}^2$$

where the term $\{\cdots\}$ is the term inside double brackets in equation (4.10) below. By using Gronwall's lemma it readily follows that

$$\begin{aligned} |||(U - U')(t)|||_{k-1}^2 \le \bar{\Psi} \exp(\bar{\Psi}\, e^{\hat{\Psi} t})\{ & \varepsilon + \|\mathfrak{f} - \mathfrak{f}'\|_{k-1}^2 + |||\mathfrak{f} - \mathfrak{f}'|||_{k-2}^2 + |||(\mathcal{B} - \mathcal{B}')(0)|||_{k-2}^2 \\ & + |||(\mathcal{A} - \mathcal{A}')(0)|||_{k-1}^2 + [\mathcal{F} - \mathcal{F}']_{k-1,t}^2 + [\mathcal{B} - \mathcal{B}']_{k-1,t}^2 \\ & + \tilde{\Lambda}(\varepsilon)[\mathcal{A} - \mathcal{A}']_{k-1,t}^2\}. \end{aligned} \tag{4.10}$$

The results stated in Appendix A and standard calculations easily show that the quantities $|||u(0)|||_{k-1}$, $\|\mathfrak{f}\|_{k-1}$, $\|\mathfrak{f}'\|_{k-1}$, $|||\mathfrak{F}(0)|||_{k-2}$, $|||\mathfrak{F}'(0)|||_{k-2}$, $|||\mathfrak{B}(0)|||_{k-2}$, and $|||\mathfrak{B}'(0)|||_{k-2}$ are bounded by a constant of type $\bar{\psi}$; that $[\mathfrak{F}]_{k-1,T}$ and $[\mathfrak{F}']_{k-1,T}$ are bounded by $\bar{\psi}\, e^{T\hat{\psi}}$; and that $|||\mathfrak{B}|||_{k-1,T}$ and $|||\mathfrak{B}'|||_{k-1,T}$ are bounded by $\hat{\psi}$. Note that $\hat{\Phi} \le \bar{\psi}$, $\hat{\Psi} \le \bar{\psi}$, $\tilde{\Lambda}_0 \le \bar{\psi}\, e^{t\hat{\psi}}$.

Moreover, by taking into account the definitions of $\mathfrak{f}$, $\mathfrak{f}'$, $\mathfrak{F}$, $\mathfrak{F}'$, $\mathfrak{B}$, and $\mathfrak{B}'$, straightforward (but tedious) calculations show that

$$\|\mathfrak{f} - \mathfrak{f}'\|_{k-1} + |||\mathfrak{f} - \mathfrak{f}'|||_{k-2} + |||(\mathfrak{F} - \mathfrak{F}')(0)|||_{k-2} \le \bar{\psi}(\|f - f'\|_k + |||(A - A')(0)|||_{k-1} + |||(B - B')(0)|||_{k-1} + |||(F - F')(0)|||_{k-1} + |||(u - u')(0)|||_{k-1}),$$

$$|||(\mathfrak{F} - \mathfrak{F}')(s)|||^2_{k-1} \le \bar{\psi}\, e^{\hat{\psi}t}(|||(F - F')(s)|||^2_k + |||(B - B')(s)|||^2_k + |||(A - A')(s)|||^2_k + |||(u - u')(s)|||^2_{k-1}),$$

$$|||(\mathfrak{B} - \mathfrak{B}')(0)|||^2_{k-2} \le \hat{\psi}(|||(B - B')(0)|||^2_{k-2} + |||(A - A')(0)|||^2_{k-1}),$$

and

$$|||(\mathfrak{B} - \mathfrak{B}')(s)|||^2_{k-1} \le \hat{\psi}(|||(B - B')(s)|||^2_{k-1} - |||(A - A')(s)|||^2_k).$$

By using these estimates we get from equation (4.10),

$$\sum_{j=1}^{n-1} |||\partial_j(u - u')(t)|||^2_{k-1} + |||\partial_n \vartheta(u - u')(t)|||^2_{k-1} + |||\partial_t(u - u')(t)|||^2_{k-1}$$
$$\le \bar{\psi} \exp(\bar{\psi}\, e^{t\hat{\psi}})(\varepsilon + \|f - f'\|^2_k + |||(F - F')(0)|||^2_{k-1} + |||(A - A')(0)|||^2_{k-1} + |||(B - B')(0)|||^2_{k-1} + |||(u - u')(0)|||^2_{k-1} + [F - F']^2_{k,t} + [B - B']^2_{k,t} + [A - A']^2_{k,t} + [u - u']^2_{k-1,t} + \tilde{\Lambda}(\varepsilon)[A - A']^2_{k-1,T}). \tag{4.11}$$

On the other hand, by using equation (3.14) in order to express $\partial_n u$ as a function of the components of U and by using a corresponding equation to express $\partial_n u'$ in terms of the components of U', we get

$$|||\partial_n(u - u')(t)|||^2_{k-1} \le |||\partial_n[\vartheta(u - u')(t)]|||^2_{k-1} + \bar{\psi}\, e^{t\hat{\psi}}\Bigg(\sum_{j=1}^{n-1} |||\partial_j(u - u')(t)|||^2_{k-1} + |||(A - A')(t)|||^2_{k-1} + |||\partial_t(u - u')(t)|||^2_{k-1} + |||(F - F')(t)|||^2_{k-1} + |||(B - B')(t)|||^2_{k-1} + |||(u - u')(t)|||^2_{k-1}\Bigg).$$

Since $|||(F - F')(t)|||^2_{k-1} \le |||(F - F')(0)|||^2_{k-1} + [F - F']^2_{k,t}$, it follows that $|||\partial_n(u - u')(t)|||^2_{k-1}$ is bounded by the right-hand side of (4.11) (the quantities $\bar{\psi}$ and $\hat{\psi}$ are, if necessary, increased). Hence, the left-hand side of equation (2.12) is bounded by the right-hand side of equation (4.11). The terms $|||(u - u')(0)|||^2_{k-1}$ and $[(u - u')(t)]^2_{k-1,t}$ can be dropped from the right-hand side of (4.11), due to equation (4.7). Hence, (2.12) holds.

Finally, according to lemma 2.2, $\tilde{\Lambda}(\varepsilon)$ depends only on ε, μ, T, and on the particular functions $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{f}$, and $\mathfrak{F}$, hence, on ε, μ, T, A, B, f, F, ϑ, and u. However, the solution u of (1.1) depends only on T, A, B, f, F; and ϑ depends only on μ, A, and R. ■

Remark 1. The use of the systems (3.15), (3.15′) is not necessary for proving theorem 2.3. In fact, it is sufficient to apply lemma 2.2 separately to each single pair of equations $(3.7)_j$, $(3.7')_j$, $j = 1, \ldots, n-1$, (3.8), (3.8′), and (3.13), (3.13′), and then use (3.14), (3.14′) to get estimates for $\partial_n(u - u')$.

On the contrary, in Section 2, we need system (3.15) in order to prove existence in $\mathcal{C}_T(H^l)$ as a corollary of existence in $\mathcal{C}_T(H^{l-1})$. However, the recourse to (3.15) is superfluous if we want to establish (2.7) merely as an a priori estimate. In this case we can argue as explained above.

Remark 2. It is worth noting that the proof of lemma 2.2 does not work for $l = k$. In fact, the solution $u_\varepsilon(t)$ of equation (4.2) does not belong to H^{k+1}, since the coefficients $A(t, \cdot)$ do not. A natural device could be to replace, in equation (4.2), the coefficients A_i by coefficients $A_i^\varepsilon \in \mathcal{L}_T^\infty(H^{k+1})$, and such that $\|A_i^\varepsilon - A_i\| = 0(\varepsilon)$, for a suitable norm $\|\cdot\|$. By following this way, we get terms $(A_i^\varepsilon - A_i')\,\partial_i u_\varepsilon$ in the right-hand side of (4.4). These terms give rise to terms $|\!|\!| u_\varepsilon |\!|\!|_{k+1,T}^2 [A_i - A_i']_{k-1,T}^2$ and $|\!|\!| u_\varepsilon |\!|\!|_{k+1,T}^2 [A_i - A_i^\varepsilon]_{k-1,T}^2$, in the right-hand side of equation (4.5). In general we are not able to show that

$$\lim_{\varepsilon \to 0} |\!|\!| u_\varepsilon |\!|\!|_{k+1,T} [A_i - A_i^\varepsilon]_{k,T} = 0. \tag{4.12}$$

In fact, if we assume that $[A_i - A_i^\varepsilon]_{k-1,T}$ is small for small ε, then $[A_i^\varepsilon]_{k+1,T}$ is large. Hence, $|\!|\!| u_\varepsilon |\!|\!|_{k+1,T}$ is large. This adverse behaviour is reinforced, as well, by that of the data $(f_\varepsilon, F_\varepsilon)$, whose behaviour is connected to that of the coefficients A^ε, in particular through the additional compatibility condition of order k.

For nonlinear Cauchy problems it could be possible to prove (4.12), where now $A_i^\varepsilon = A_i(u_\varepsilon)$, by using particular couples $(f_\varepsilon, F_\varepsilon)$. However, this device is not useful for general mixed problems (especially for large values of k) due to the constraints imposed by the compatibility conditions. Moreover, it requires functions $A_i(\cdot)$ of class C^{k+1} instead of C^k.

5. THE NONLINEAR PROBLEM

In this section we prove proposition 2.4 and theorem 2.5 that concern the nonlinear system (1.2), namely

$$\begin{cases} \partial_t u + \sum_{i=1}^{n} A_i(u)\,\partial_i u + B(u)u = F & \text{in } Q_T, \\ Mu = 0 \text{ on } \Sigma_T; \quad u(0) = f. \end{cases} \tag{5.1}$$

Proof of proposition 2.4. Set

$$\mathbb{K}(T, R) = \{v \in \mathcal{L}_T^\infty(H^k) \colon \|v\|_{k,T} \le R,\ Mv|_{\Sigma_T} = 0, \partial_t^j v(0) = f^{(j)},\ j = 0, \ldots, k-1\}. \tag{5.2}$$

The positive constants T, R will be fixed later on. The functions $f^{(j)}$ (see Section 2 and [6]) depend only on the data f and F. They also depend on $A(\cdot)$, $B(\cdot)$, M, but these elements are fixed. Actually, the condition $\partial_t^{k-1} v(0) = f^{(k-1)}$, in definition (5.2), is superfluous.

In order to prove that for suitable values of R and T the set $\mathbb{K}(T, R)$ is not empty we will argue as in [8, lemma A.3], with minor modifications. Consider the system

$$\begin{cases} \partial_t w + \sum_{i=1}^{n} A_i(f)\,\partial_i w + B(f)w = F + G & \text{in } Q_T, \\ Mw = 0 \text{ on } \Sigma_T; \quad w(0) = f, \end{cases} \tag{5.3}$$

and denote by $\bar{f}^{(j)}$, $j = 0, 1, \ldots, k-1$, the value of $\partial_t^j w(0)$ formally obtained from (5.3). The compatibility conditions up to order $k-1$ for this system are $M\bar{f}^{(j)} = 0$ on Γ, for $j = 0, 1, \ldots, k-1$.

Now, we look for a function $G \in \mathcal{L}_T^\infty(H^k)$ such that $\bar{f}^{(j)} = f^{(j)}$, for $j = 0, \ldots, k-1$. Then, the compatibility conditions for system (5.3) are satisfied, since $M\bar{f}^{(j)} = Mf^{(j)}$. The above equations hold if

$$\partial_t^j G(0) = b_j \equiv -\left\{ \sum_{i=1}^n [\partial_t^j, A_i(u)]\, \partial_i u + [\partial_t^j, B(u)]u \right\}\Bigg|_{t=0}, \tag{5.4}$$

for $j = 0, \ldots, k-1$;† here, $b_0 = 0$. The right-hand sides of (5.4) are expressions in terms of f and F, through the functions $f^{(j)}$.

Since the functions $A(\cdot)$ and $B(\cdot)$ are fixed and of class C^k one shows, by taking into account equation (B.4) in Appendix B and corollary A.2 in Appendix A, that

$$\|b_j\|_{k-j} \leq \lambda(\|f\|_k, |||F(0)|||_{k-1}), \qquad \text{for } 0 \leq j \leq k-1. \tag{5.5}$$

Hence, there is a function $G \in \mathcal{C}_T(H^k)$ such that (5.4) holds, moreover,

$$|||G|||_{k,T} \leq \lambda(\|f\|_k, |||F(0)|||_{k-1}). \tag{5.6}$$

One can assume that $\partial_t^k G(0) = 0$. Note that the function λ, in equation (5.6), does not depend on T. In particular, $[G + F]_{k,T}^2 \leq [F]_{k,T}^2 + T\lambda(\|f\|_k, |||F(0)|||_{k-1})$. Consequently, by using (2.8), we get

$$|||w|||_{k,T}^2 \leq e^{T\lambda_4(\|f\|_k)}\{\lambda_5(\|f\|_k, |||F(0)|||_{k-1})(1 + T) + \lambda_6(\|f\|_k)[F]_{k,T}^2\}. \tag{5.7}$$

From now on impose on R the condition

$$R \geq \text{``right-hand side of equation (5.7)''}. \tag{5.8}$$

Let $v \in \mathbb{K}(T, R)$ and consider the solution $u \in \mathcal{C}_T(H^k)$ of problem

$$\begin{cases} \partial_t u + \sum_{i=1}^n A_i(v)\, \partial_i u + B(v)u = F & \text{in } Q_T, \\ Mu = 0 \text{ on } \Sigma_T; \qquad u(0) = f. \end{cases} \tag{5.9}$$

Note that the compatibility conditions up to order $k-1$ are satisfied, since $\partial_t^j v(0) = f^{(j)}$. Set $u = Sv$, for each $v \in \mathbb{K}$. Clearly, $v \in C(Q_T)$, moreover,

$$\|v\|_{C(Q_T)}^2 \leq c\|v\|_{k-1,T}^2 \leq c(\|f\|_{k-1}^2 + TR^2). \tag{5.10}$$

Hence, from hypothesis (N.2), one gets

$$\mu(v) \geq \lambda(\|f\|_{k-1}^2 + TR^2)^{-1}, \qquad \forall\, v \in \mathbb{K}(T, R). \tag{5.11}$$

Since the range, in $\mathbb{R}^m$, of the functions $v(t, x)$ is contained in a sphere of radius equal to the right-hand side of equation (5.11) and centre in the origin, it follows that the norm of $A(v)$ in $\mathcal{L}_T^\infty(H^k)$ and the norm of $B(v)$ in $\mathcal{L}_T^\infty(H^{k-1})$ are bounded by $\lambda(R)$ and by $\lambda(\|f\|_{k-1}^2 + TR^2)$, respectively. By using these estimates, (5.10), and (5.11), one gets from (2.8) the estimate

$$|||u|||_{k,T}^2 \leq e^{T\lambda_7(R)}\{\lambda_8(\|f\|_{k-1}^2 + TR^2)(\|f\|_k^2 + |||F(0)|||_{k-1}^2 + \lambda_9(R)[F]_{k,T}^2\}. \tag{5.12}$$

† The value $j = k-1$ is superfluous.

Fix

$$R = e\{\lambda_5(\|f\|_k, |||F(0)|||_{k-1}) + \lambda_8(1 + \|f\|_{k-1}^2)(\|f\|_k^2 + |||F(0)|||_{k-1}^2) + 3\}$$

and, after that, fix $T > 0$ in such a way that, in equations (5.7) and (5.12), all terms that have T or $[F]_{k,T}^2$ as a factor became less or equal to 1. For these values of R and T, the set $\mathbb{K}(T, R)$ is not empty, moreover, $|||u|||_{k,T}^2 \leq R$. Hence, $S(\mathbb{K}) \subset \mathbb{K}$. Note that $R = \lambda(\|f\|_k, |||F(0)|||_{k-1})$, and that the hypotheses on T are of type (2.16).

The set $\mathbb{K} = \mathbb{K}(T, R)$ is closed and convex in $\mathcal{C}_T(H^2)$, since bounded sets in $\mathcal{L}_T^\infty(H^k)$ are weak-* relatively compact in this space. In order to show that S has a fixed point in $\mathbb{K}$ (which is the solution of problem (5.1)) it remains to prove that S is a strict contraction. Let $u = S(v)$ and $u' = S(v')$, for $v, v' \in \mathbb{K}$. Then, $L(v)(u - u') = \sum_i (A_i(v') - A_i(v))\,\partial_i u' + (B(v') - B(v))u'$ in Q_T, $M(u - u') = 0$ on Σ_T, $(u - u')(0) = 0$. Equation (2.8), applied for $l = 2$, shows that

$$|||(u - u')(t)|||_2^2 \leq \lambda_{10}(R)\, e^{T\lambda_{11}(R)} T\, |||(v - v')(t)|||_2^2.$$

Note that $[A(v') - A(v)]_{2,T} \leq \lambda(R)[v' - v]_{2,T}$. By imposing on T the additional conditions $eT\lambda_{10}(R) \leq 1/2$ and $T\lambda_{11}(R) \leq 1$, one gets the desired property for the map S. ■

Proof of theorem 2.5. Owing to (2.18), the norms $[F_\nu]_{k,T_0}$, $|||F_\nu|||_{k-1,T_0}$, and $\|f_\nu\|_k$ are uniformly bounded. Hence, we can assume in theorem 2.4 that λ_1, λ_2, λ_3 have the same (constant) value for all problems $(2.17)_\nu$, $\nu \in \mathbb{N}$, and for problem (1.2). Moreover, they do not depend on the intial time (i.e. we can replace the initial time 0 by any $t_0 \in [0, T_0]$). Fix $T > 0$ such that $T\lambda_1 \leq 1$, $[F_\nu]_{k,T}\lambda_2 \leq 1$, and $[F]_{k,T}\lambda_2 \leq 1$. Recall that $|||u_\nu|||_{k,T} \leq \lambda_3$.

We start by showing that

$$|||u - u_\nu|||_{k-1,T}^2 \leq c(\|f - f_\nu\|_{k-1}^2 + |||(F_\nu - F_\nu)(0)|||_{k-2}^2 + [F - F_\nu]_{k-1,T}^2). \tag{5.13}$$

By taking the difference side by side, between equations (1.2) and $(2.17)_\nu$, we get $L(u)(u - u_\nu) = [A(u_\nu) - A(u)]\,\partial u_\nu + [B(u_\nu) - B(u)]u_\nu + F - F_\nu$ in Q_T; $M(u - u_\nu) = 0$ on Σ_T; $(u - u_\nu)(0) = f - f_n$. By applying (2.8) for $l = k - 1$, and by doing straightforward calculations, we show that

$$|||(u - u_\nu)(t)|||_{k-1}^2 \leq c(\|f - f_\nu\|_{k-1}^2 + |||(F - F_\nu)(0)|||_{k-2}^2 + [F - F_\nu]_{k-1,T}^2 + [u - u_\nu]_{k-1,t}^2).$$

By using Gronwall's lemma, we prove (5.13).

Now, we apply (2.12) to the couple of systems (1.2), $(2.17)_\nu$, i.e. u' is replaced by u_ν and so on in (2.9). It readily follows that, given a positive ε, one has the estimate

$$|||(u - u_\nu)(t)|||_k^2 \leq c\varepsilon + c\{\|f - f_\nu\|_k^2 + |||(F - F_\nu)(0)|||_{k-1}^2 + |||(u - u_\nu)(0)|||_{k-1}^2 + [F - F_\nu]_{k,T}^2\}$$
$$+ c\tilde{\Lambda}(\varepsilon)[u - u_\nu]_{k-1,T}^2 + c[u - u_\nu]_{k,t}^2,$$

for each $\nu \in N$. By taking into account equation (5.13) and by using Gronwall's lemma, we get

$$|||(u - u_\nu)(t)|||_k^2 \leq c\{\varepsilon + \|f - f_\nu\|_k^2 + |||(F - F_\nu)(0)|||_{k-1}^2 + [F - F_\nu]_{k,t}^2\}$$
$$+ c\Lambda(\varepsilon)\{\|f - f_\nu\|_{k-1}^2 + |||(F - F_\nu)(0)|||_{k-2}^2 + [F - F_\nu]_{k-1,t}^2\}. \tag{5.14}$$

Hence, $u_\nu \to u$ in $\mathcal{C}_T(H^k)$. Now, we extend the result to $[0, T_0]$. By the remarks at the beginning of the proof, the above argument applies to any interval $[t_0, t_0 + T]$ contained in $[0, T_0]$. Since $u_\nu(0) \to u(0)$ in H^k it follows that $u_\nu \to u$ in $\mathcal{C}_T(H^k)$. In particular, $u_\nu(T) \to u(T)$ in H^k. Moreover, the compatibility conditions up to order $k - 1$ are satisfied, at time T, as follows from the equations. By applying the convergence result to the interval $[T, 2T]$, we show that $u_\nu \to u$ in $\mathcal{C}_{[T,2T]}(H^k)$. And so on, up to T_0.

The last assertion in theorem 2.5 follows by adapting the above proof to the more general case under consideration here. In the sequel we show the main points and leave the details to the reader. For brevity, we refer below to the As. Similar devices are also effective for the Bs. Now, additional terms $A^\nu(u_\nu) - A(u)$ are present in the above estimates. These terms should be decomposed as follows: $A^\nu(u_\nu) - A(u) = [A^\nu(u_\nu) - A(u_\nu)] + [A(u_\nu) - A(u)]$. The last term on the right-hand side is not new. The first one should be estimated by taking into account that

$$|||A^\nu(u_\nu) - A(u_\nu)|||_l \leq \lambda(|||u_\nu|||_l)\|A^\nu - A\|_{C^l(\Lambda_\nu)}, \tag{5.15}$$

for each $l \leq k$, where Λ_ν is any subset of $\mathbb{R}^m$ that contains the range $u_\nu(Q_T)$ of u_ν.

We start by proving our thesis by assuming that $A^\nu(\cdot)$ converges to $A(\cdot)$ with respect to the $C^k(\mathbb{R}^m)$ norm. In this case, the functions λ_1, λ_2 and λ_3 (in proposition 2.4) can be chosen independently of ν, since they depend only on a uniform upper bound for the $C^k(\mathbb{R}^m)$ norms of the $A^\nu(\cdot)$s. Hence, T is uniformly bounded from below (with respect to ν) and the norms $|||u_\nu|||_{k,T}$ are uniformly bounded from above. In particular, the right-hand side of (5.15) tends to zero as ν tends to infinity.

Now let $A^\nu(\cdot)$ be as in theorem 2.5. Since u is bounded, there is a compact subset Λ of $\mathbb{R}^m$ that contains the range $u(Q_T)$ of u. Let Λ' and Λ'' be compact subsets such that $\Lambda \subset\subset \Lambda' \subset\subset \Lambda''$. We fix elements $\bar{A}^\nu(\cdot)$ which coincide with $A^\nu(\cdot)$ on Λ' and with $A(\cdot)$ outside Λ''. By the previous result, the solutions $\bar{u}_\nu$ (obvious notation) converge in $\mathcal{C}_T(H^k)$ to u. Hence, for sufficiently large ν, the ranges of the $\bar{u}^\nu$s are contained in Λ'. Hence, $\bar{u}^\nu = u^\nu$.

Note that the data f_ν, F_ν satisfy the compatibility conditions with respect to the coefficients $\bar{A}^\nu$, since (for large ν) the range of the f_νs is contained in Λ'. ■

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APPENDIX A

In the sequel Ω denotes an open, bounded subset of $\mathbb{R}^n$, $\mathbb{R}^n$ itself, $\mathbb{R}^n_+$, or more generally, an open set for which the Sobolev embedding theorems used below hold. We denote by r an integer such that $r > n/2$. The symbols l, s, α, β

denote nonnegative integers. Below, the symbol c denotes positive constants that depend only on Ω and on r. Here H^l denotes $H^l(\Omega)$, and so on.

LEMMA A.1. Let $0 \le s \le r$, $f \in H^{r-s}$, $g \in H^s$. Then

$$\|fg\| \le c\|f\|_{r-s}\|g\|_s. \tag{A.1}$$

Proof. If $r - s > n/2$ then $H^{r-s} \hookrightarrow L^\infty$, and (1) follows. If $r - s = n/2$, then $s > 0$. Hence, $H^s \hookrightarrow L^p$, for some $p > 2$. Moreover, $H^{r-s} \hookrightarrow L^{2p/(p-2)}$. Consequently, $\|fg\| \le \|f\|_{2p/(p-2)}\|g\|_p$. If $s \ge n/2$, we argue as above by replacing s by $r - s$. Finally, if $r - s < n/2$ and $s < n/2$, set $1/p = (1/2) - (r - s)/n$, $1/q = (1/2) - (s/n)$. Since $p, q \in [2, +\infty[$ and $(1/p) + (1/q) < 1/2$, there are reals $p_0, q_0 \in [2, +\infty[$ such that $(1/p_0) + (1/q_0) = 1/2$, $H^{r-s} \hookrightarrow L^{p_0}$, $H^s \hookrightarrow L^{q_0}$. This yields (A.1). ■

PROPOSITION A.1. Let $0 \le l \le r$, and $0 \le s \le r - l$. Then

$$\|fg\|_l \le c\|f\|_{r-s}\|g\|_{l+s}. \tag{A.2}$$

Proof. Each derivative $D^l(fg)$ of fg, of order l, is a linear combination of terms $(D^{l-q}f)(D^q g)$, for $0 \le q \le l$. For fixed q, apply the lemma A.1 for $\bar{s} = s + l - q$. One has $0 \le s \le s + l - q = \bar{s} \le s + l \le r$. Hence, the lemma can be used. One gets

$$\|(D^{l-q}f)(D^q g)\| \le c\|D^{l-q}f\|_{r-(s+l-q)}\|D^q g\|_{s+l-q} \le \|f\|_{r-s}\|g\|_{l+s}.$$

A similar estimate for derivatives of fg of order less than l is immediate. Alternatively, one can use the estimates $\|fg\| \le c\|f\|_{r-s}\|g\|_s \le c\|f\|_{r-s}\|g\|_{l+s}$ and $\|fg\|_l \le c\|D^l(fg)\| + c\|fg\|$. ■

COROLLARY A.1. Let $0 \le l \le \alpha, \beta \le r$, and $\alpha + \beta = l + r$. Then

$$\|fg\|_l \le c\|f\|_\alpha\|g\|_\beta. \tag{A.3}$$

COROLLARY A.2. Under the hypothesis of corollary A.1, one has

$$|||fg|||_l \le c|||f|||_\alpha|||g|||_\beta. \tag{A.4}$$

Proof. By the Leibnitz formula,

$$|||fg|||_l \le c\sum_{j=0}^{l}\|\partial_t^j(fg)\|_{l-j} \le c\sum_{j=0}^{l}\sum_{p=0}^{j}\|(\partial_t^p f)(\partial_t^{j-p}g)\|_{l-j} \le c\sum_{j=0}^{l}\sum_{p=0}^{j}\|\partial_t^p f\|_{\alpha-p}\|\partial_t^{j-p}g\|_{\beta-(j-p)}$$

which is bounded by $c|||f|||_\alpha|||g|||_\beta$. In fact

$$|||fg|||_l \le c\left(\sum_{p=0}^{l}\|\partial_t^p f\|_{\alpha-p}\right)\left(\sum_{p=0}^{l}\|\partial_t^p g\|_{\beta-p}\right). \ ■ \tag{A.5}$$

The estimate (A.4) can be generalized as follows. Define

$$|||f|||_l^{[q]} \equiv \sum_{p=0}^{q}\|\partial_t^p f\|_{l-p}.$$

Then, equation (A.5) easily shows the following result.

COROLLARY A.3. Let $0 \le q \le l \le r$, where $r \ge n/2$, and let $\alpha_1, \dots, \alpha_m, \beta \in [l, r]$ satisfy $\beta + \sum_{i=1}^m \alpha_i = mr + l$. Then

$$|||f_1 \cdots f_m g|||_l^{[q]} \le c|||g|||_l^{[q]}\prod_{i=1}^{m}|||f_i|||_{\alpha_i}^{[q]}. \tag{A.6}$$

COROLLARY A.4. Let $k > 1 + (n/2)$ and $|\alpha| = l$, where $1 \le l \le k$. Assume that $\nabla a \in H^{k-1}$ and that $h \in H^{l-1}$. Then

$$\|\partial^\alpha(ah) - a(\partial^\alpha h)\| \le c\|\nabla a\|_{k-1}\|h\|_{l-1}. \tag{A.7}$$

The proof follows from lemma A.1. ■

APPENDIX B

In this appendix we prove lemma B.1 below (used in Section 3) and proposition 4.1. The following notations and definitions are those used in [6].

Set $G \equiv \sum_{j=1}^{n} A_j \partial_j + B$, $G_0 \equiv G(0)$, and $G_1 \equiv [\partial_t, G_{i-1}](0)$, for $1 \leq i \leq k$. Hence, for $1 \leq i \leq k$,

$$G_i = \sum_{j=1}^{n} (\partial_t^i A_j(0))\, \partial_j + \partial_t^i B(0). \tag{B.1}$$

Define $f^{(0)} = f$ and

$$f^{(p)} \equiv \sum_{i=0}^{p-1} \binom{p-1}{i} G_i f^{(p-i-1)} + \partial_t^{p-1} F(0), \tag{B.2}$$

if $1 \leq p \leq k$. Recall that the compatibility condition of order p is given by $Mf^{(p)} = 0$ on Γ.

LEMMA B.1. Let $0 \leq l < k$ and assume that $f \in H^l$ and $F \in \mathcal{L}_T^2(H^l)$. Then $f^{(p)} \in H^{l-p}$, moreover

$$\|f^{(p)}\|_{l-p} \leq c(1 + \|\!|A(0)|\!\|_{k-1} + \|\!|B(0)|\!\|_{k-2})^p \cdot (\|f\|_l + \|\!|F(0)|\!\|_{l-1}), \tag{B.3}$$

for each p such that $0 \leq p \leq l$.

If $0 \leq l \leq k$ and if (2.3) holds, then $f^{(p)} \in H^{l-p}$, moreover

$$\|f^{(p)}\|_{l-p} \leq c(1 + \|\!|A(0)|\!\|_{k-1} + \|\!|B(0)|\!\|_{k-1})^p \cdot (\|f\|_l + \|\!|F(0)|\!\|_{l-1}), \tag{B.4}$$

for each p such that $0 \leq p \leq l$. In particular

$$\|\!|f|\!\|_l^2 \leq \phi(\|\!|A(0)|\!\|_{k-1}, \|\!|B(0)|\!\|_{k-2}) \cdot (\|f\|_l^2 + \|\!|F(0)|\!\|_{l-1}^2), \tag{B.5}$$

where $k - 2$ should be replaced by $k - 1$, if $l = k$.

Proof. Let l be fixed, $0 \leq l < k$. We will prove the result by induction on p. If $p = 0$, then $f_0 = f$, and (4.4) holds. We assume that the thesis holds for each $p \in [0, q - 1]$, for some q satisfying $1 \leq q \leq l$, and we prove the thesis for the function $f^{(q)}$. By using the corollary A.1 for $r = k - 1$, we prove that

$$\|f^{(q)}\|_{l-q} \leq c \sum_{i=0}^{q-1} (\|\partial_t^i A_j(0)\|_{k-i-1} \|\partial_j f^{(q-i-1)}\|_{l-q+i} + \|\partial_t^i B(0)\|_{k-i-2} \|f^{(q-i-1)}\|_{l-q+i+1}) + \|F(0)\|_{l-1}. \tag{B.6}$$

Hence,

$$\|f^{(q)}\|_{l-q} \leq c(\|\!|A(0)|\!\|_{k-1} + \|\!|B(0)|\!\|_{k-2}) \sum_{i=0}^{q-1} \|f^{(q-i-1)}\|_{l-q+i+1} + \|F(0)\|_{l-1}.$$

By using the induction hypothesis it follows (with simplified notations) that the left-hand side of the last inequality is bounded by

$$c(A + B)[(1 + A + B)^{q-1} + \cdots + (1 + A + B)^0](\|f\|_l + \|\!|F(0)|\!\|_{l-1}) + \|F(0)\|_{l-1}.$$

The thesis follows.

If $l = k$ the above calculation fails for the B term, in the right-hand side of (B.6); here, we now use the decomposition $k - i - 1, l - q + i$ instead of $k - i - 2, l - q + i + 1$. ■

Now we prove proposition 4.1. As remarked in Section 4, the proof is inspired by that of lemma 3.3 of Rauch and Massey [6]. Here, we have to argue more carefully, due to the lack of regularity of the coefficients A and B. Note, for instance, that proposition 4.1 loses sense if $l \geq k$ (compare with lemma 3.3 in [6]). This new situation compels us to investigate carefully, and to develop, some points treated in [6]. We start by proving some auxiliary lemmas.

By using corollary A.1 in Appendix A, we easily prove the following result (we denote by $\mathcal{L}[X; Y]$ the set of all bounded linear operators on X to Y).

LEMMA B.2. Assume that (2.1) holds and let $1 \leq i \leq l \leq k$, and $i \leq k - 1$. Then

$$G_i \in \mathcal{L}[H^l; H^{l-i}]. \tag{B.7}$$

If (2.3) holds, the value $i = k$ can be attained. Moreover, $G_0 \in \mathcal{L}[H^l; H^{l-1}]$ for $1 \leq l \leq k$. If (2.3) holds, the value $l = k + 1$ can be attained.

Now we exploit the decomposition (B.8) below.

LEMMA B.3. Let $1 \le p \le k$. Then

$$f^{(p)} = B_p f + E_p F, \tag{B.8}$$

where the operators B_p and E_p have the following structure

$$B_p f = G_0^p f + \sum G_0^\vartheta G_{i_1} \cdots G_{i_q} f, \tag{B.9}$$

where "$\sum$" means "sum of terms of the form", $\vartheta + i_1 + \cdots + i_q \le p - 1$, and $i_s \ne 0$ for $s = 1, \ldots, q$.

$$E_p F = \sum G_0^\lambda G_{j_1} \cdots G_{j_r} \partial_t^\sigma F(0), \tag{B.10}$$

where $\lambda + j_1 + \cdots + j_r + \sigma \le p - 1$, and $j_s \ne 0$ for $s = 1, \ldots, r$.

Proof. By induction on p. For $p = 1$ the thesis holds since $f^{(1)} = G_0 f + F(0)$. Let $1 \le p < k$ and assume that the thesis holds for every index p' such that $1 \le p' \le p$. We want to prove the thesis for the value $p + 1$. From (4.3) one gets

$$f^{(p+1)} = G_0 f^{(p)} + \sum_{i=0}^{p-1} \binom{p}{i} G_i (B_{p-i} f + E_{p-i} F) + G_p F + \partial_t^p F(0). \tag{B.11}$$

It is not difficult to verify that each of the terms on the right-hand side of (B.11) has a particular form which is admissible to our purposes. Let us show it for the terms $G_i B_{p-1} f$, where i is fixed and satisfies $0 \le i \le p - 1$. By the induction hypothesis one has

$$B_{p-i} f = G_0^{p-i} f + \sum G_0^\vartheta G_{i_1} \cdots G_{i_q} f,$$

where $\vartheta + i_1 + \cdots + i_q \le p - i - 1$. Hence,

$$G_i B_{p-i} f = G_i G_0^{p-i} f + \sum G_0^\vartheta G_i G_{i_1} \cdots G_{i_q} f.$$

Since $i + (p - i) = p$ and $i = \vartheta + i_1 + \cdots + i_q \le p - 1$, the thesis follows. ■

LEMMA B.4. Let $1 \le p \le l \le k$. Then $B_p \in \mathcal{L}(H^l; H^{l-p})$, $B_p - G_0^p \in \mathcal{L}(H^l; H^{l-p+1})$, and $E_p \in \mathcal{L}(\mathcal{C}_T(H^l); H^{l-p+1})$.

The proof of lemma B.4 follows from lemmas B.2 and B.3.

Proof of proposition 4.1. In order to fix the ideas, we assume that $l = k - 1$. We point out that this is the more difficult case, due to the loss of regularity of the coefficients.

Let f and F be as in proposition 4.1. In order to prove this proposition it is sufficient to show that there are sequences of functions $F_\nu \in \mathcal{L}_T^2(H^k)$ and $f_\nu \in H^k$ such that $F_\nu \to F$ in $\mathcal{L}_T^2(H^{k-1})$ and $f_\nu \to f$ in H^{k-1}, as $\nu \to \infty$. Moreover, $Mf_\nu^{(p)} = 0$ on Γ for $0 \le p \le k - 1$. Let F_ν be as above, and let $g_\nu \in H^k$ converge to f in H^{k-1} as $\nu \to \infty$. We set $f_\nu = g_\nu - h_\nu$ and we look for $h_\nu \in H^k$ such that $h_\nu \to 0$ in H^{k-1} and $Mf_\nu^{(p)} = 0$, i.e. $MB_p h_\nu = M(B_p g_\nu + E_p F_\nu)$ for $0 \le p \le k - 1$ (for convenience, $B_0 = I$, $E_0 = 0$). As in [6], we denote by T the inverse of M when it is restricted to the orthogonal complement of the kernel of M, and we set $a_{p,\nu} \equiv TM(B_p g_\nu + E_p F_\nu)$. The above equations are then written in the form

$$B_p h_\nu = a_{p,\nu} \text{ on } \Gamma, \qquad \text{for } p = 0, \ldots, k - 1. \tag{B.12}$$

Note that $a_{p,\nu} \in H^{k-p-1/2}(\Gamma)$. Since $g_\nu \to f$ in H^{k-1} and $F_\nu \to F$ in $\mathcal{L}_T^2(H^{k-1}) \hookrightarrow \mathcal{C}_T(H^{k-2})$, it readily follows, from lemma B.4, that $a_{p\nu} \to TM(B_p f + E_p f)$ in $H^{k-1-p-1/2}(\Gamma)$, for each p such that $0 \le p \le k - 2$. Since the pair f, F satisfies the compatibility conditions up to order $k - 2$, it follows then that

$$a_{p\nu} \to 0 \text{ in } H^{k-1-p-1/2}(\Gamma), \qquad p = 0, \ldots, k - 2. \tag{B.13}$$

On the other hand, (B.9) shows that the operator B_p has the form

$$B_p = A_n^p \partial_n^p + \sum_{i=0}^{p-1} C_{p-1}^{(p)} \partial_n^i, \qquad p = 1, \ldots, k - 1, \tag{B.14}$$

where $C^{(p)}_{p-i}$ is a linear differential operator of order $p - i$ that contains only tangential derivatives. Hence, for each $i \in [0, p-1]$, one has $C^{(p)}_{p-i} \in \mathcal{L}(H^{k-i}; H^{k-p})$. On the boundary, one has,

$$C^{(p)}_{p-i} \in \mathcal{L}(H^{k-i-1/2}(\Gamma); H^{k-p-1/2}(\Gamma)), \qquad i = 0, \ldots, p-1, \tag{B.15}$$

moreover,

$$C^{(p)}_{p-i} \in \mathcal{L}(H^{k-i-3/2}(\Gamma); H^{k-p-3/2}(\Gamma)), \qquad i = 0, \ldots, p-1, \tag{B.16}$$

for $0 \le p \le k-2$. By setting $b_{0,\nu} = TMg_\nu$, by defining

$$b_{p,\nu} = A_n^{-p}\left(a_{p,\nu} - \sum_{i=0}^{p-1} C^{(p)}_{p-i} b_{i,\nu}\right), \qquad 1 \le p \le k-1,$$

and by taking into account (B.13), equation (B.12) can be written in the form

$$\partial_n^p h_\nu = b_{p,\nu} \text{ on } \Gamma, \qquad \text{for } 0 \le p \le k-1. \tag{B.17}$$

Now, we want to prove that

$$b_{p,\nu} \to 0 \text{ in } H^{k-p-1-1/2}(\Gamma), \qquad \text{for } p = 0, \ldots, k-2. \tag{B.18}$$

The proof is done by induction. For $p = 0$, the thesis is obvious since $g_\nu \to f$ in H^{k-1} and $TMf = 0$ on Γ. Assume now that (B.18) holds for each $i \in [0, p-1]$. Then it readily follows from (B.13), from the definition of $b_{p,\nu}$, and from (3.12) that (B.18) holds for the value p. By using (B.15) instead of (B.16) and by using $a_{p,\nu} \in H^{k-p-1/2}(\Gamma)$ instead of (B.13), one shows that $b_{p,\nu} \in H^{k-p-1/2}(\Gamma)$ for $p \in [0, k-1]$.

At this point, we are able to solve equation (B.17). Let $R \in \mathcal{L}(\prod_{p=0}^{k-1} H^{k-p-1/2}(\Gamma); H^k)$ be such that $\partial_n^p R(b_0, \ldots, b_{k-1}) = b_p$ on Γ for $p \in [0, k-1]$ and that the map $(b_0, \ldots, b_{k-2}) \to R(b_0, \ldots, b_{k-2}, 0)$ is continuous from $\prod_{p=0}^{k-2} H^{k-p-3/2}(\Gamma)$ into H^{k-1}. Such a map exists, see [12, theorem 2.5.7]. Now, for each ν, fix a function $b'_{k-1,\nu} \in C_0^\infty(\Gamma)$ such that $\|b'_{k-1,\nu} - b_{k-1,\nu}\|_{1/2,\Gamma} \le 1/\nu$, where the norm is that in $H^{1/2}(\Gamma)$. Set

$$h_\nu = h_\nu^{(1)} + h_\nu^{(2)} + w_\nu$$

where $h_\nu^{(1)} = R(b_{0,\nu}, \ldots, b_{k-2,\nu}, 0)$ and $h_\nu^{(2)} = R(0, \ldots, 0, b_{k-1,\nu} - b'_{k-1,\nu})$. Both functions belong to H^k. One has

$$\partial_n^p[h_\nu^{(1)} + h_\nu^{(2)}] = b_{p,\nu}, \qquad \text{for } 0 \le p \le k-2,$$

$$\partial_n^{k-1}[h_\nu^{(1)} + h_\nu^{(2)}] = b_{k-1,\nu} - b'_{k-1,\nu} \in H^{1/2}(\Gamma).$$

Moreover, (B.18) shows that $h_\nu^{(1)} \to 0$ in H^{k-1}. Furthermore, $h_\nu^{(2)} \to 0$ in H^k. Hence, $h_\nu^{(1)} + h_\nu^{(2)} \to 0$ in H^{k-1}.

In order to accomplish the proof, it suffices to show that there exist functions $w_\nu \in H^k$ such that $w_\nu \to 0$ in H^{k-1} as $\nu \to \infty$, $\partial_n^p w_\nu = 0$ on Γ for $p \in [0, k-2]$, and $\partial_n^{k-1} w_\nu = b'_{k-1,\nu}$ on Γ.

Assume, without loss of generality (repeat, if necessary, some elements) that $\|b'_{k-1,\nu}\|^2_{k-1,\Gamma} \le$ const. ν, where the norm is that of $H^{k-1}(\Gamma)$. Let $\phi \in C_0^\infty([0, \infty))$ be a function such that $\Phi(t) = 1$ if t belongs to some neighbourhood of the origin. Set $\psi_\nu(x_n) = (1/k!)x_n^k \phi(\nu x_n)$ and define $w_\nu(x) = \psi_\nu(x_n) b'_{k-1,\nu}(x_1, \ldots, x_{n-1})$. Clearly $w_\nu \in C_0^\infty(\bar{\mathbb{R}}^n_+)$. Moreover,

$$\|w_\nu\|^2_{k-1} \le \|\psi_\nu(x_n)\|^2_{C^{k-1}} \|b'_{k-1,\nu}\|^2_{H^{k-1}}.$$

Since $\|\psi_\nu(x_n)\|_{C^{k-1}} \le$ const. ν^{-1}, one has $\|w_\nu\|^2_{k-1} \le$ const. ν^{-1}. Finally, $\psi_\nu^{(p)}(0) = 0$ if $0 \le p \le k-2$, $\psi_\nu^{(k-1)}(0) = 1$. Consequently, $\partial_n^p w_\nu = 0$ for $0 \le p \le k-2$ and $\partial_n^{k-1} w_\nu = b'_{k-1,\nu}$ on Γ. ■