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LONG TIME BEHAVIOUR OF THE SOLUTIONS TO THE NAVIER–STOKES EQUATIONS WITH DIFFUSION

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0. MAIN NOTATION

Ω: an open bounded set in $\mathbb{R}^3$, locally situated on one side of its boundary Γ, a regular (say C^4) manifold.

Q_T: the cylinder $]0, T[\times\Omega$.

$n = n(x)$: unit outward normal to Γ.

$\partial_i, \partial_{ij}, \partial_t$: $\partial/\partial x_i$, $\partial^2/\partial x_i\,\partial x_j$, $\partial/\partial t$.

$\|\ \|$, $(\ ,\ )$: norm and scalar product in $L^2(\Omega)$.

$|\ |_p$: norm in $L^p(\Omega)$, $p \in [1, +\infty]$.

H^k: Sobolev space $H^{k,2}(\Omega)$ with norm

$$\|\sigma\|_k^2 \equiv \sum_{l=0}^{k} \|D^l\sigma\|^2,$$

where

$$\|D^l\sigma\|^2 \equiv \sum_{|\alpha|=l} \|D^\alpha\sigma\|^2.$$

Further,

$$\|D^l\sigma\|_m^2 \equiv \sum_{|\alpha|=l} \|D^\alpha\sigma\|_m^2.$$

H^s, $s \in \mathbb{R}^+$: Sobolev (Bessel) space $H^{s,2}(\Omega)$, see [1].

H_0^1: closure of $C_0^\infty(\Omega)$ in $H^1(\Omega)$.

$|\ |_\infty$: norm in $L^\infty(\Omega)$.

$\mathbf{L}^2, \mathbf{H}^k, \mathbf{H}_0^1$: Hilbert spaces of vectors $v = (v_1, v_2, v_3)$ such that $v_i \in L^2$, $v_i \in H^k$, $v_i \in H_0^1$ $(i = 1, 2, 3)$, respectively. Corresponding notation is used for other spaces of vector fields. Norms are defined in the natural way, and denoted by the symbols used for the scalar fields.

Let us introduce the following functional spaces (see e.g. [2–4] for their properties):

$$H_N^k \equiv \left\{\sigma \in H^k : \frac{\partial\sigma}{\partial n} = 0 \text{ on } \Gamma \text{ and } \int_\Omega \sigma(x)\,dx = 0\right\}, \qquad k \geq 2,$$

$$\mathcal{V} \equiv \{v \in C_0^\infty(\Omega) : \operatorname{div} v = 0 \text{ in } \Omega\},$$

$$H = \{v \in \mathbf{L}^2 : \operatorname{div} v = 0 \text{ in } \Omega,\ v\cdot n = 0 \text{ on } \Gamma\},$$

$$V = \{v \in \mathbf{H}_0^1 : \operatorname{div} v = 0 \text{ on } \Omega\}.$$

H and V are the closures of $\mathcal{V}$ in $L^2(\Omega)$ and H_0^1, respectively. Moreover, $L^2 = H \oplus G$, where $g = \{\nabla p : p \in H^1(\Omega)\}$. Denoting by P the orthogonal projection of L onto H, we define the operator $A \equiv -P\Delta$ on $D(A) \equiv H^2 \cap V$. One has

$$(Au, v) = ((u, v)) \equiv \sum_{i,j} (D_i u_j, D_i v_j), \qquad \forall u \in D(A), \forall v \in V.$$

The norms $\|\sigma\|_2$, $\|\Delta\sigma\|$ are equivalent in H_N^2, $\|\sigma\|_3$, $\|\nabla\Delta\sigma\|$ are equivalent in H_N^3 and $\|v\|_2$, $\|Av\|$ are equivalent in $D(A)$. We define $\|v\|_V^2 \equiv ((v, v))$; the norms $\|v\|_V$, $\|v\|_1$ are equivalent in V.

$L^2(0, T; X)$: Banach space of strongly measurable functions defined in $]0, T[$ with values in (a Banach space) X, for which

$$\|z\|_{L^2(0,T;X)}^2 \equiv \int_0^T \|z(t)\|_X^2 \, dt < +\infty.$$

$C(0, T; X)$: Banach space of X-vector valued continuous functions on $[0, T]$ endowed with the usual norm $\|z\|_{C(0,T;X)}$.

μ: viscosity (a positive constant).

λ: diffusion coefficient (a positive constant).

$v(t, x)$; $v_0(x)$: mean-volume velocity; initial mean-volume density. Further,

$$m \equiv \inf_{x \in \Omega} \rho_0(x), \qquad M \equiv \sup_{x \in \Omega} \rho_0(x),$$

$$\hat{\rho} \equiv \frac{1}{|\Omega|} \int_\Omega \rho_0(x) \, dx, \qquad \sigma(t, x) \equiv \rho(t, x) - \hat{\rho}.$$

We assume that $m > 0$.

$\pi(t, x)$; $p(t, x)$: pressure; modified pressure

$$p = \pi + \lambda v \cdot \nabla\rho - \lambda^2 \Delta\rho + \lambda(2\mu + \mu') \Delta \log \rho.$$

$f(t, x)$: external mass-force.

We denote by $c, c_0, c_1, c_2, \ldots, \bar{c}, \hat{c}, \tilde{c}$, etc., positive constants depending only on Ω and on the parameters $\mu, \lambda, m, M, \hat{\rho}$. For convenience we sometimes denote different constants by the same symbol c even in the same equation. Otherwise, we use the symbols $\bar{c}$, c_k, $k \in \mathbb{N}$, $\hat{c}$, etc.

1. DESCRIPTION OF THE PROBLEM AND MAIN RESULTS

In this paper we consider the motion of a viscous fluid consisting of two components, for instance, saturated salt water and water. The equations of the model are obtained, for example, in [5–8]. Let us give a brief sketch. Let ρ_1, ρ_2 be the characteristic densities (constants) of the two components, $v^{(1)}(t, x)$ and $v^{(2)}(t, x)$ their velocities and $e(t, x)$, $d(t, x)$ the mass and volume concentration of the first fluid. We define the density $\rho(t, x) \equiv d\rho_1 + (1 - d)\rho_2$, and the mean-volume and mean-mass velocities $v \equiv dv^{(1)} + (1 - d)v^{(2)}$, $w \equiv ev^{(1)} + (1 - e)v^{(2)}$. Then the equations of motion are given by

$$\begin{cases} \rho[\partial_t w + (w \cdot \nabla)w - f] - \mu\,\Delta w - (\mu + \mu')\,\nabla\,\mathrm{div}\, w = -\nabla\pi, \\ \mathrm{div}\, v = 0, \\ \partial_t \rho + \mathrm{div}(\rho w) = 0. \end{cases}$$

On the other hand, Fick's diffusion law (see [5]) gives $w = v - \lambda\rho^{-1}\nabla\rho$. Eliminating w in the preceding equation one gets, after some calculations,

$$\begin{cases} \rho(\partial_t v + (v \cdot \nabla)v) - \mu\,\Delta v - \lambda[(v \cdot \nabla)\,\nabla\rho + (\nabla\rho \cdot \nabla)v] \\ \qquad + \dfrac{\lambda^2}{\rho}\left[(\nabla\rho \cdot \nabla)\,\nabla\rho - \dfrac{1}{\rho}(\nabla\rho \cdot \nabla\rho)\,\nabla\rho + \Delta\rho\,\nabla\rho\right] = -\nabla p + \rho f, \\ \partial_t \rho + v \cdot \nabla\rho - \lambda\,\Delta\rho = 0 \\ \mathrm{div}\, v = 0. \end{cases} \tag{1.1}$$

Here p is the modified pressure. We add to system (1.1) the following initial boundary-value conditions

$$\begin{cases} v = 0 & \text{on }]0, T[\times \Gamma, \\ \dfrac{\partial\rho}{\partial n} = 0 & \text{on }]0, T[\times \Gamma, \\ v|_{t=0} = v_0(x) & \text{in } \Omega, \\ \rho|_{t=0} = \rho_0(x) & \text{in } \Omega. \end{cases} \tag{1.2}$$

The first two conditions mean that there is no flux through the boundary. In [7, 8] Kazhikhov and Smagulov consider the simplified system obtained from $(1.1)_1$ by omitting the term containing λ^2. Moreover, they assume that

$$0 < \lambda < \frac{2\mu}{M - m}. \tag{1.3}$$

Under these conditions, Kazhikhov and Smagulov state the existence of a local solution in time. They also prove, in the two-dimensional case, that the solution is global in time. A challenging open problem is to prove this last result without assumption (1.3) (or to find a counterexample).

Local existence in the general case (i.e. with the λ^2-term and, clearly, without assuming (1.3)) was proved for *inviscid* fluids ($\mu = 0$) in [9], by partially following some ideas from [10]. As remarked in [9], the above result leads us to believe that, in the viscous case, the assumption (1.3) is superfluous. This was shown, in [11], for $\Omega = \mathbb{R}^3$, by following the proof for inviscid fluids given in [9]. In [12] we gave a proof completely different from that in [9]. This proof, specific for *viscous* flows, relies essentially on a balanced estimate obtained by taking the inner product in H, of the projection of the main equation $(1.1)_1$ into H with $\partial_t v + \varepsilon_0 A v$, and choosing $\varepsilon_0 > 0$ in a convenient way. This estimate together with a suitable use of the continuity method allows us to reduce the existence of the solution of the main linearized equation (equation (2.2) in [12]) to that of the Stokes evolution problem. These devices had been successfully used also in [13].

In theorems 1.1–1.3 below, we summarize results proved in [12] that will be useful in the sequel. We set $\sigma_0 \equiv \rho_0 - \hat{\rho}$.

THEOREM 1.1. Let $v_0 \in V$, $\sigma_0 \in H_n^2$, $f \in L^2(0, T; L^2)$. Then, there exists a $T_1 \in]0, T]$ such that problem (1.1), (1.2) is uniquely solvable in Q_{T_1}. Moreover, $v \in L^2(0, T_1; H^2) \cap C(0, T_1; V)$, $\partial_t v \in L^2(0, T_1; H)$, $\sigma \in L^2(0, T_1; H_N^3) \cap C(0, T_1; H_N^2)$, $\partial_t \rho \in L^2(0, T; H^1)$ and $m \le \rho(t, x) \le M$.

We denote by $c_0 = c_0(\Omega)$ a positive constant such that $|\sigma|_\infty \le c_0\|\sigma\|_2$, for each $\sigma \in H_N^2(\Omega)$.

THEOREM 1.2. Assume that $f \in L^2_{\text{loc}}(0, +\infty; L^2)$ and that

$$\|\sigma_0\|_2 \le (2c_0)^{-1}\hat{\rho}. \tag{1.4}$$

Then, there is a constant $\bar{c}$ such that the local strong solution in theorem 1.1 satisfies

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mu}{2}\|v\|_V^2 + \|\Delta\sigma\|^2\right) + \frac{m}{2}\|\partial_t v\|^2 + \frac{m\mu^2}{16M^2}\|Av\|^2 + \frac{\lambda}{2}\|\nabla\,\Delta\sigma\|^2 \\ \le \bar{c}(\|v\|_V^6 + \|\Delta\sigma\|^6 + \|f\|^2),\end{aligned} \tag{1.5}$$

as long as the solution exists and belongs to the functional spaces in theorem 1.1.

Equations (1.4) and (1.5) are, respectively, equation (4.1) and the equation between (4.3) and (4.4), in [12].

THEOREM 1.3. There are constants c_1, c_2 and c_3 such that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|v\|_V^2 + \|\Delta\sigma\|^2) \le -[c_1 - c_2(\|v\|_V^2 + \|\Delta\sigma\|^2)^2 \cdot (\|v\|_V^2 + \|\Delta\sigma\|^2)] + c_3\|f\|^2. \tag{1.6}$$

In particular, if

$$\begin{cases} c_2(\|v_0\|_V^2 + \|\Delta\sigma_0\|^2)^2 < c_1/2, \\ c_3\|f\|^2_{L^\infty(0,+\infty;L^2)} \le (c_1/2)\sqrt{c_1/2c_2}, \end{cases} \tag{1.7}$$

then the solution is global in time. Moreover,

$$c_2(\|v(t)\|_V^2 + \|\Delta\sigma(t)\|^2)^2 < c_1/2, \qquad \forall t \ge 0. \tag{1.8}$$

Equations (1.6) and (1.7) correspond (respectively) to equations (4.4) and (4.5) in [12]. Equation (1.8) can be found in [12], just after equation (4.4).

In the sequel we will prove the following theorems.

THEOREM 1.4. Under the assumptions (1.7), and if necessary, choosing a smaller constant c_1, one has

$$\|\sigma(t)\|_2^2 \le c_4\,\mathrm{e}^{-c_5 t}\|\sigma_0\|_2^2. \tag{1.9}$$

If, moreover, $(\bar{v}, \bar{\sigma})$ is another solution of problem (1.1), (1.2) with initial data $(\bar{v}_0, \bar{\sigma}_0)$ satisfying $(1.7)_1$ then

$$\|v(t) - \bar{v}(t)\|_s \le c_6\,\mathrm{e}^{-c_7 t}, \qquad \forall t \ge 0, \tag{1.10}$$

for each fixed $s \in [0, 1[$. Here the positive constants c_6 and c_7 may also depend on s. In particular, if $(w, 0)$ is a solution of the homogeneous Navier–Stokes equations

$$\begin{cases} \hat{\rho}(\partial_t w + (w \cdot \nabla)w) - \mu \Delta w = -\nabla\pi + \hat{\rho}f, \\ \operatorname{div} w = 0 & \text{in } Q_{+\infty}, \\ w = 0 & \text{on }]0, T[\times \Gamma, \\ w|_{t=0} = w_0(x) & \text{in } \Omega, \end{cases} \tag{1.11}$$

with initial data w_0 satisfying $c_2\|w_0\|_V^4 \le c_1/2$, then

$$\|v(t) - w(t)\|_s^2 + \|\rho(t) - \hat{\rho}\|_2^2 \le c_8\, \mathrm{e}^{-c_9 t}, \tag{1.12}$$

for each $t \ge 0$.

THEOREM 1.5. Assume that v_0, σ_0 and f satisfy (1.7) and assume that f is T-periodic for some $T > 0$. Then (1.12) holds, where now $w(t)$ is a periodic solution to the homogeneous Navier–Stokes equation $(1.11)_{1,2,3}$.

2. PRELIMINARIES

For the reader's convenience we briefly show how to establish (1.5) as an *a priori* estimate. This is the main point in the proof of theorem 1.1. From (1.5) we deduce theorems 1.2 and 1.3, as done in [12].

An application of the maximum principle to the solution of the parabolic equation $(1.1)_2$, with boundary and initial conditions $(1.2)_2$, $(1.2)_4$ shows that

$$m \le \rho(t, x) \le M. \tag{1.13}$$

Note, moreover, that the derivative with respect to time of the integral of $\rho(t, x)$ over Ω vanishes. Hence

$$\frac{1}{|\Omega|}\int_\Omega \rho(t, x)\, \mathrm{d}x = \hat{\rho}, \qquad \text{for each } t \ge 0.$$

Next, apply the operator Δ to both sides of $(1.1)_2$, multiply by $\Delta\sigma$, and integrate over Ω. This shows that $(1/2)(\mathrm{d}/\mathrm{d}t)\|\Delta\sigma\|^2 + (\nabla(\lambda\,\Delta\sigma - v \cdot \nabla\sigma), \nabla\,\Delta\sigma) = 0$ since $(\partial/\partial n)(\lambda\,\Delta\sigma - v \cdot \nabla\sigma) = 0$ on Γ. Hence

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta\sigma\|^2 + \lambda\|\nabla\,\Delta\sigma\|^2 \le c(\|Dv\, D\sigma\| + \|v\, D^2\sigma\|)\|\nabla\,\Delta\sigma\|. \tag{1.14}$$

Using Sobolev's embedding theorem $H^1 \hookrightarrow L^6$ and Hölder's inequality we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta\sigma\|^2 + \lambda\|\nabla\,\Delta\sigma\|^2$$
$$\le c_{10}(\|Dv\|^{1/2}\|Dv\|_1^{1/2}\|\nabla\sigma\|_1 + \|v\|_1\|D\sigma\|_1^{1/2}\|\nabla\,\Delta\sigma\|^{1/2})\|\nabla\,\Delta\sigma\|.$$

Since $abd \le a^2b^2/\varepsilon + \varepsilon d^2/4 \le a^4/2\varepsilon^3 + \varepsilon b^4/2 + \varepsilon d^2/4$ it readily follows that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta\sigma\|^2 + \lambda\|\nabla\,\Delta\sigma\|^2 \le \frac{c\varepsilon}{2}\|Dv\|_1^2 + \frac{c}{2\varepsilon^3}\|Dv\|^2\|\nabla\sigma\|_1^4 + c\varepsilon\|\nabla\,\Delta\sigma\|^2 + \frac{c}{2\varepsilon^3}\|v\|_1^4\|\nabla\sigma\|_1^2,$$

where $c = c_{10}$. By assuming that ε satisfies the condition

$$\varepsilon \le \lambda/(2c_{10}) \tag{1.15}$$

and by taking into account that $\|Dv\|_1^2 \le c\|Av\|^2$, one finds

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta\sigma\|^2 + \lambda\|\nabla\,\Delta\sigma\|^2 \le \frac{c_{10}}{\varepsilon^3}(\|v\|_1^2 + \|\nabla\sigma\|_1^2)\cdot\|v\|_1^2\|\nabla\sigma\|_1^2 + c_{11}\varepsilon\|Av\|^2. \tag{1.16}$$

Note that $\|\sigma\|_2 \le \varepsilon\|\Delta\sigma\|$, $\|\sigma\|_3 \le c\|\nabla\,\Delta\sigma\|$.

Next, we consider equation $(1.1)_1$. Set

$$F(\rho, v) = P\left\{-\rho(v\cdot\nabla v) + \lambda[(v\cdot\nabla)\rho + (\nabla\rho\cdot\nabla)v] + \frac{\lambda^2}{\rho}\left[(\nabla\rho\cdot\nabla)\,\nabla\rho - \frac{1}{\rho}(\nabla\rho\cdot\nabla\rho)\,\nabla\rho + (\Delta\rho)\,\nabla\rho\right]\right\} + \rho f. \tag{1.17}$$

By using Sobolev's embedding theorem $H^1 \hookrightarrow L^6$ and Hölder's inequality one easily obtains

$$\|F(\rho, v)\|^2 \le c\|v\|_1^3\|Dv\|_1 + c\|v\|_1^2\|D^2\sigma\|\,\|D^2\sigma\|_1 + c\|v\|_1\|Dv\|_1\|D\sigma\|_1^2 + c\|D\sigma\|_1^3\|D^2\sigma\|_1 + c\|D\sigma\|_1^6 + c\|f\|^2.$$

Hence,

$$\|F(\rho, v)\|^2 \le c(\|v\|_1^3 + \|\sigma\|_2^3)(\|v\|_2 + \|\sigma\|_3) + c\|\sigma\|_2^6 + c\|f\|^2. \tag{1.18}$$

By taking into account $(1.1)_3$ and the boundary condition $(1.2)_3$, we write equation $(1.1)_1$ in the form

$$P(\rho\partial_t v) + \mu Av = F(\rho, v). \tag{1.19}$$

Next, take the inner product in H of (1.19) with $\partial_t v + \varepsilon_0 Av$, $\varepsilon_0 > 0$. Since $(\partial_t v, Av) = (1/2)(\mathrm{d}/\mathrm{d}t)\|v\|_V^2$ one is led to

$$m\|\partial_t v\|^2 + \frac{\mu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|_V^2 + \varepsilon_0\mu\|Av\|^2 \le \|F\|\,\|\partial_t v\| + \varepsilon_0\|F\|\,\|Av\| + \varepsilon_0 M\|\partial_t v\|\,\|Av\|.$$

By using the inequalities $\|F\|\,\|\partial_t v\| \le 4^{-1}m\|\partial_t v\|^2 + m^{-1}\|F\|^2$, $\|F\|\,\|Av\| \le 4^{-1}\mu\|Av\|^2 + \mu^{-1}\|F\|^2$ and $\|\partial_t v\|\,\|Av\| \le 4M^{-1}\mu\|Av\|^2 + \mu^{-1}M\|\partial_t v\|^2$ one shows that

$$\frac{3}{4}m\|\partial_t v\|^2 + \frac{\mu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|_V^2 + \varepsilon_0\frac{\mu}{2}\|Av\|^2 \le \left(\frac{1}{m} + \frac{\varepsilon_0}{\mu}\right)\|F\|^2 + \frac{\varepsilon_0 M^2}{\mu}\|\partial_t v\|^2. \tag{1.20}$$

By setting $\varepsilon_0 = m\mu/4M^2$ and by using (1.18) it readily follows that

$$\frac{\mu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|_V^2 + \frac{m}{2}\|\partial_t v\|^2 + \frac{m\mu^2}{8M^2}\|Av\|^2$$

$$\leq \frac{c^2}{2\delta}(\|v\|_1^6 + \|\sigma\|_2^6) + \delta(\|Av\|^2 + \|\nabla\,\Delta\sigma\|^2) + c\|\sigma\|_2^6 + c\|f\|^2. \tag{1.21}$$

We have used that $\|v\|_2 \leq c\|Av\|$ and $\|\sigma\|_3 \leq c\|\nabla\,\Delta\sigma\|$. Here, $\delta > 0$.

Finally, by choosing $\varepsilon = \min\{\lambda/2c_0,\ m\mu^2/32M^2c_{11}\}$ in (1.16), $\delta = \min\{m\mu^2/32M^2, \lambda/2\}$ in (1.21) and by adding side by side the two equations, one proves (1.5). This yields (1.6).

Next, assume that (1.7) holds. Clearly, $(\mathrm{d}/\mathrm{d}t)(\|v(t)\|_V^2 + \|\Delta\sigma(t)\|^2) < 0$ for $t = 0$. Let us show that (1.8) holds. If not, let t_0 be the smallest $t > 0$ for which $c_2(\|v(t)\|_V^2 + \|\Delta\sigma(t)\|^2)^2 = c_1/2$. This equality together with (1.6) and $(1.7)_2$ implies that $(\mathrm{d}/\mathrm{d}t)(\|v(t)\|_V^2 + \|\Delta\sigma(t)\|^2) < 0$ for $t = t_0$. This contradicts the above assumption.

Remark 1.5. It is immediate that the statement of theorem 1.3 holds by replacing c_1 by any positive constant $\tilde{c}_1$ smaller than c_1.

2. PROOF OF THEOREMS 1.4 AND 1.5

Write the equation $(1.1)_2$ in the equivalent form

$$\partial_t\sigma + v\cdot\nabla\sigma - \lambda\,\Delta\sigma = 0, \tag{2.1}$$

multiply both sides by σ and integrate over Ω. This shows that

$$\|\sigma(t)\|^2 \leq \mathrm{e}^{-ct}\|\sigma_0\|^2, \qquad \forall t \geq 0. \tag{2.2}$$

Next, by multiplying (2.1) by $-\Delta\sigma$, by integrating over Ω and by using Sobolev's embedding theorems $H^1 \hookrightarrow L^6$, $H^{1/2} \hookrightarrow L^3$, and Hölder's inequality one shows that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\sigma\|^2 + \lambda\|\Delta\sigma\|^2 \leq c\|v\|_V\|\nabla\sigma\|_{1/2}\|\nabla\sigma\|. \tag{2.3}$$

Since $\|v\|_V \leq c$ and $\|\nabla\sigma\|_{1/2} \leq c\|\sigma\|^{1/4}\|\Delta\sigma\|^{3/4}$, it readily follows that the right-hand side of (2.3) is bounded by $c\|\sigma\|^2 + (\lambda/2)\|\Delta\sigma\|^2$. Here, we have used a well-known inequality with the exponents 8 and 8/7. Hence, by taking into account (2.2), one finds

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\sigma\|^2 + c\frac{\lambda}{2}\|\nabla\sigma\|^2 \leq c\,\mathrm{e}^{-ct}\|\sigma_0\|^2,$$

where the positive constants c should not necessarily be the same. By integration of this differential inequality it follows that

$$\|\sigma(t)\|_1^2 \leq c\,\mathrm{e}^{-ct}\|\sigma_0\|_1^2. \tag{2.4}$$

Next, we study the behaviour of $\|\sigma(t)\|_2$. Since

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta\sigma\|^2 = 2(\Delta\partial_t\sigma, \Delta\sigma) = -2(\nabla(\partial_t\sigma), \nabla\,\Delta\sigma)$$

(note that $\partial_t(\partial\sigma/\partial n) = 0$ on the lateral boundary $]0, +\infty[\times \Gamma$, in a standard weak sense) one gets, from equation (2.1),

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta\sigma\|^2 + 2(\nabla(\lambda\,\Delta\sigma - v\cdot\nabla\sigma), \nabla\,\Delta\sigma) = 0.$$

Hence

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta\sigma\|^2 + \lambda\|\nabla\,\Delta\sigma\|^2 \le c(\|Dv\||D\sigma|_\infty + |v|_6|D^2\sigma|_3)\|\nabla\,\Delta\sigma\|. \tag{2.5}$$

By taking into account that $|v|_6 \le c\|Dv\|$, that $|D\sigma|_\infty \le c\|D\sigma\|_{5/3}$, that $|D^2\sigma|_3 \le c\|D\sigma\|_{3/2}$, and that $\|D\sigma\|_{5/3} \le c\|\nabla\sigma\|^{1/6}\|\nabla\sigma\|_2^{5/6}$, it readily follows that the left-hand side of (2.5) is bounded by $c\|\nabla\sigma\|^{1/6}\|\nabla\,\Delta\sigma\|^{11/6}$. By using a classical inequality with exponents 12 and 12/11 one easily shows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta\sigma\|^2 + c\|\Delta\sigma\|^2 \le c\|\sigma\|_1^2.$$

Integration of this differential inequality together with (2.4) proves (1.9).

Next, we prove (1.10). Let f and the two couple of initial data (v_0, σ_0) and $(\bar{v}_0, \bar{\sigma}_0)$ satisfy (1.7), and set (see [12, Section 5])

$$u = v - \bar{v}, \qquad \eta = \rho - \bar{\rho}. \tag{2.6}$$

By subtracting the equation (1.19) written for $(\bar{\rho}, \bar{v})$ from the same equation written for (ρ, v), and by taking the inner product with u in H of both sides of the equation obtained in that way, one gets

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\rho u, u) + \mu\|u\|_V^2 = -\frac{1}{2}(v\cdot\nabla\sigma, u^2) + \frac{\lambda}{2}(\Delta\sigma, u^2) - (\eta, \partial_t\bar{v}\cdot u) + (F - \bar{F}, u), \tag{2.7}$$

where $\bar{F} = F(\bar{\rho}, \bar{v})$. From (1.8) it follows that $\|v(t)\|_V$, $\|\bar{v}(t)\|_V$ and $\|u(t)\|_V$ are uniformly bounded. So are the L^6-norms of $v(t)$, $\bar{v}(t)$ and $u(t)$. By using these properties, Hölder's inequality and (1.9) it readily follows that

$$\left|-\frac{1}{2}(v\cdot\nabla\sigma, u^2) + \frac{\lambda}{2}(\Delta\sigma, u^2)\right| \le c\,\mathrm{e}^{-ct}, \qquad \forall t \ge 0. \tag{2.8}$$

Similarly,

$$|(\eta, \partial_t\bar{v}\cdot u)| \le c\,\mathrm{e}^{-ct}, \qquad \forall t \ge 0. \tag{2.9}$$

For convenience, we denote simply by $F_\lambda(\rho, v)$ the part on the right-hand side of the definition of $F(\rho, v)$ (see (1.17)) that depends on λ and λ^2. Similarly, for $\bar{F}$. One easily shows that

$$|(F_\lambda, u)| \le c(|v|_3\|D^2\sigma\| + \|v\|_1|\nabla\sigma|_3 + |\nabla\sigma|_3\|D^2\sigma\| + |\nabla\sigma|_6^3)|u|_6$$

and, similarly, for $|(\bar{F}_\lambda, u)|$. Hence,

$$|(\bar{F}_\lambda - F_\lambda, u)| \le c\,\mathrm{e}^{-ct}. \tag{2.10}$$

On the other hand,

$$|(P(\rho f) - P(\bar{\rho}f), u)| \le c\|\sigma\|_1\|f\|\|u\|_1 \le c\,\mathrm{e}^{-ct}. \tag{2.11}$$

Furthermore,

$$|((\rho - \bar{\rho})(\bar{v} \cdot \nabla)\bar{v}, u)| \le c|\rho - \bar{\rho}|_6 |\bar{v}|_6 \|\nabla \bar{v}\| |u|_6 \le c\,e^{-ct}, \tag{2.12}$$

and

$$|(\rho[(v \cdot \nabla)u + (u \cdot \nabla)\bar{v}], u)| \le \tfrac{1}{2}|(\nabla\sigma, vu^2)| + M|u|_6^2 \|\nabla \bar{v}\|.$$

The right-hand side of this last inequality is bounded by $c(|\nabla\sigma|_3 \|\nabla v\| |u|_6^2 + |u|_6^2 \|\nabla \bar{v}\|)$ which, in turn, is bounded by $c\,e^{-ct} + c\|u\|_V^2 \|\bar{v}\|_V$. This fact together with (2.12) shows that

$$|(\rho(v \cdot \nabla)v - \bar{\rho}(\bar{v} \cdot \nabla)\bar{v}, u)| \le c\,e^{-ct} + c\|\bar{v}\|_V \|u\|_V^2. \tag{2.13}$$

From (2.10), (2.11) and (2.13) we obtain the estimate

$$|(F - \bar{F}, u)| \le c\,e^{-ct} + c\|\bar{v}\|_V \|u\|_V^2. \tag{2.14}$$

Finally, (2.7), (2.8), (2.9) and (2.14) yield

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\rho u, u) + \mu\|u\|_V^2 \le c\,e^{-ct}(1 + \|\partial_t \bar{v}\|) + c_{12}\|\bar{v}\| \|u\|_V^2.$$

Choose the constant c_1 in (1.7) in such a way as to have $c_{12}(c_1/2c_2)^{1/4} \le \mu/2$. By using (1.8) one shows that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\rho u, u) + \tilde{c}(\rho u, u) \le c_{13}\,e^{-\hat{c}t}(1 + \|\partial_t \bar{v}\|) \tag{2.15}$$

since $(\rho u, u) \le c\|u\|_V^2$.

Next, by integration of (1.5) one gets

$$\frac{\mu}{2}\|v(t)\|_V^2 + \|\Delta\sigma(t)\|^2 + c\int_0^t (\|\partial_t v\|^2 + \|Av\|^2 + \|\nabla\,\Delta\sigma\|^2)\,\mathrm{d}s$$

$$\le \frac{\mu}{2}\|v_0\|_V^2 + \|\Delta\sigma_0\|^2 + \bar{c}\int_0^t (\|v\|_V^6 + \|\Delta\sigma\|^6 + \|f\|^2)\,\mathrm{d}s.$$

In particular,

$$\int_0^t (\|\partial_t v\|^2 + \|Av\|^2 + \|\nabla\,\Delta\sigma\|^2)\,\mathrm{d}s \le c + ct, \qquad \forall t \ge 0. \tag{2.16}$$

Integration of (2.15) shows that, for each $t \ge 0$,

$$(\rho u, u)(t) \le e^{-\tilde{c}t}(\rho u, u)(0) + c_{13}\,e^{-\tilde{c}t}\int_0^t e^{-(\hat{c}-\tilde{c})s}(1 + \|\partial_t \bar{v}\|)\,\mathrm{d}s.$$

Hence,

$$m\|u(t)\|^2 \le M\,e^{-\tilde{c}t}\|u(0)\|^2 + \Lambda(t), \tag{2.17}$$

where

$$\Lambda(t) \le c_{13}\,e^{-\tilde{c}t}\left(\int_0^t e^{-2(\hat{c}-\tilde{c})s}\,\mathrm{d}s\right)^{1/2}\left(\int_0^t (1 + \|\partial_t \bar{v}\|)^2\,\mathrm{d}s\right)^{1/2}.$$

By using (2.16) it readily follows that

$$\Lambda(t) \le c\,\mathrm{e}^{-ct}. \tag{2.18}$$

The estimates (2.17) and (2.18) show that $\|u(t)\|^2 \le c\exp(-ct)$. Since $\|u(t)\|_V \le c$, (1.10) follows by interpolation. In order to prove (1.12), under the assumptions of theorem 1.4, it is sufficient to note that the solutions of problem (1.12) are just particular solutions of problem $(1.11)_{1,2}$, corresponding to the case in which the initial density $\rho_0(x)$ is constant (equal to $\hat\rho$).

Next, we prove theorem 1.5. Let (ρ, v) be a solution of problem (1.1), let (1.7) be satisfied and let $f(t+T) = f(t)$ for $t \ge 0$. In view of (1.8), the couple $(\bar\rho(t), \bar v(t)) \equiv (\rho(t+T), v(t+T))$ is a solution of equation (1.1) and, moreover, the initial data $(\bar\rho_0, \bar v_0) \equiv (\rho(T), v(T))$ satisfies $(1.7)_1$ (recall that we set $\rho = \hat\rho + \sigma$, and so on). From (1.9), (1.10) it follows that

$$\|\rho(t) - \hat\rho\|_2^2 + \|v(t+T) - v(t)\|_s^2 \le c\,\mathrm{e}^{-ct}, \qquad \forall t \ge 0. \tag{2.19}$$

Let $t_0 \in [0, T]$. Then (2.19) shows that

$$\|v(t_0 + (n+1)T) - v(t_0 + nT)\|_s \le c_{13}\,\mathrm{e}^{-cnT}, \qquad \forall n \in \mathbb{N}.$$

In particular, for each pair $k, n \in \mathbb{N}$, one has

$$\|v(t_0 + (n+k)T) - v(t_0 + nT)\|_s \le c_{13}\,\mathrm{e}^{-cnT} \int_0^{+\infty} \mathrm{e}^{-cT\lambda}\,\mathrm{d}\lambda,$$

and, hence, the left-hand side is bounded by $(c_{13}/cT)\cdot\exp(-cnT)$. It follows that $v(t_0 + nT)$ is a Cauchy sequence in H^s which converges uniformly with respect to $t_0 \in [0, T]$. Consequently, there is a periodic function $w \in C(0, T; H^s)$, $w(t+T) = w(t)$, such that (1.12) holds. Clearly, by (1.8),

$$c_2\|w(t)\|_V^4 \le c_1/2, \qquad \forall t \ge 0.$$

Moreover, as $n \to +\infty$, $v(nT + t)$ converges to $w(t)$ weakly in $L^2(0, T; D(A))$ and $\partial_t v$ converges to $\partial_t w$ weakly in $L^2(0, T; H)$. Similarly, besides (1.12), ρ converges weakly to $\hat\rho$ in $L^2(0, T; H^3)$ and $\partial_t \rho$ converges weakly to 0 in $L^2(0, T; H^1)$. By passing to the limit in equation (1.1) it readily follows that $(\hat\rho, w(t))$ is a periodic solution to the homogeneous Navier–Stokes equations (1.11). In particular, $w \in C(0, +\infty; V)$.

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