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CONCERNING THE REGULARITY OF THE SOLUTIONS TO THE NAVIER–STOKES EQUATIONS VIA THE TRUNCATION METHOD; PART I

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Abstract. In view of the lack of a global regularity theorem for the solutions (v, p) of the Navier-Stokes equations there has been a great deal of activity in establishing sufficient conditions on the velocity v in order to guarantee the regularity of the solution. However, nontrivial conditions involving the pressure seem not to be available in the literature. In this paper we present a sharp sufficient condition involving a combination of v and p. The proof relies on the truncation method, introduced in reference [3] for studying scalar elliptic equations and developed further by many authors (see, in particular [6] and [4]). In the sequel we use some basic results proved in [4].

1. Introduction and results. In the sequel Ω is an open connected subset of $\mathbb{R}^n$, $n \geq 3$. For convenience we assume that Ω is bounded and locally located on one side of its boundary Γ, a regular manifold. We set $Q_T = \Omega \times (0, T)$ and $\Sigma_T = \Gamma \times (0, T)$ where T is a fixed positive real number. We denote by $\|v\|_p$ the canonical norm in the space $L^p = L^p(\Omega)$, $1 \leq p \leq \infty$, and set $\|v\| = \|v\|_2$. The symbol $\|v\|_{q,r,T}$ denotes the canonical norm in the space $L^r(L^q) = L^r(0, T; L^q)$, $1 \leq r, q \leq \infty$. The symbol $\|v\|_{\ell,q}$ denotes the canonical norm in the Sobolev space $W^{\ell,q} = W^{\ell,q}(\Omega)$. We use the same notations for scalar and for vector fields defined in Ω or in Q_T. The same convention holds for functional spaces and norms. We also use some obvious notations like $\|\nabla v\| = \|\,|\nabla v|\,\|$. If $v = (v_1, \ldots, v_n)$ we define

$$|\nabla v| = \Big[\sum_{i,j=1}^{n} (\partial_j v_i)^2\Big]^{1/2}.$$

Einstein's summation convention on repeated indices is used throughout the paper. Finally, $|E|$ denotes the n-dimensional Lebesque measure of the set E.

This note is concerned with the solutions of the Navier–Stokes equations

$$\begin{cases} \frac{\partial v}{\partial t} - \mu \Delta v + (v \cdot \nabla) v + \nabla p = 0, \\ \operatorname{div} v = 0 \quad \text{in} \quad Q_T, \\ v = 0 \quad \text{on} \quad \Sigma_T, \\ v(x, 0) = v_0(x), \end{cases} \tag{1.1}$$

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where $\operatorname{div} v_0 = 0$ and $v_0 \in H_0^1(\Omega)$. The introduction of an external force f in equation $(1.1)_1$ can be treated in a straightforward way. We leave it to the reader.

We are interested in studying sufficient conditions for regularity of solutions. Since bounded solutions are regular we are concerned with sufficient conditions for boundedness on Q_T. Hence we assume in the sequel that

$$\|v_0\|_\infty \leq k_0 \tag{1.2}$$

for some positive real k_0. There is no difficulty in replacing the boundary condition $v = 0$ by $v = v_1$ on Σ_T, provided that $v_1 \in L^\infty(\Sigma_T)$. In this case k_0 also must satisfy

$$\|v_1\|_{\infty, \Sigma_T} \leq k_0. \tag{1.3}$$

We start by recalling some classical results and by showing the close connection between these results and our work. We will refer to [4] since we follow similar methods. We note that in Chapter VII of this reference the authors also treat systems of equations. However the system (1.1) is not included there. Hence we will just refer to some results for linear scalar equations since this is sufficient to get a better understanding of the known results for the Navier–Stokes equations and to suggest new ones. Consider the scalar equation

$$\frac{\partial v}{\partial t} - \Delta v + b_i \frac{\partial v}{\partial x_i} = \frac{\partial f_i}{\partial x_i} - f \tag{1.4}$$

with initial and boundary conditions like (1.2), (1.3). Clearly, this particular equation can be treated by more classical methods. Assume that

$$b_i, f_i \in L^r(L^q) \quad f \in L^{r/2}(L^{q/2}), \tag{1.5}$$

where

$$\frac{2}{r} + \frac{n}{q} < 1 \quad r \in (2, \infty], \quad q \in (n, \infty]. \tag{1.6}$$

The pairs (r, q) can be different for distinct coefficients. Under these assumptions the solution of (1.4) is bounded in Q_T. See [4], Chap. III, Sect. 7, Theorem 7.1. Moreover, just for the coefficients b_i, (1.6) can be replaced by

$$\frac{2}{r} + \frac{n}{q} = 1, \quad q \in (n, \infty], \tag{1.7}$$

as follows from the Remark 7.3, loc. cit.

Next we apply, in a formal way, the above results for solutions of (1.4) to solutions of (1.1) and we compare these (fictitious) results to the real known results for (1.1). We remark right now that we are interested here in obtaining sufficient conditions on the pressure to guarantee the regularity (boundedness) of the solution (the typical results concern conditions on the velocity alone). We show in the sequel that we can obtain better results than those directly suggested by the regularity results for (1.4). Let us start by assuming that the solution v of (1.1) satisfies $|v|^2 \in L^r(L^q)$, for some values r, q. Since $(v \cdot \nabla)v = \partial_{x_i}(v_i v)$, $|v|^2$ plays the role of the f_i's in equation (1.4). This suggests that v should be bounded whenever $|v|^2 \in L^r(L^q)$ for a pair (r, q) satisfying (1.6). However, the known results for the system (1.1) establish that v is regular if $v \in L^r(L^q)$, (r, q) satisfying (1.7). This stronger result is suggested as well by the results for equation (1.4) by considering the equation

$$\frac{\partial u}{\partial t} - \Delta u + (v \cdot \nabla)u + \nabla p = 0 \tag{1.8}$$

and by associating the v_i's to the b_i's (and by treating $u = v$ as an "unknown"). The main point here is changing $|v|^2$ by $|v|$ and not (1.6) by (1.7). However, it is not out of interest remarking that also this last substitution entirely fits with the results obtained for (1.4) (see the Remark 7.3 quoted above).

Let us now consider the pressure term as the "known regular term." In this case *the device consisting of using the equation* (1.8) *has no counterpart.* This is a crucial point to keep in mind. On the other hand, comparison of (1.1) with (1.4) shows that p corresponds to the f_i's. This suggests that the solution of (1.1) is regular if

$$p \in L^r(0, T; L^q) \tag{1.9}$$

with (r, q) satisfying (1.6). However the well-known equation

$$-\Delta p = \partial_i \partial_j (v_i v_j) \tag{1.10}$$

suggests some correspondence between $|p|$ and $|v|^2$. The sharp result would be to prove that the solution v of (1.1) is regular if p satisfies (1.9) with r and q satisfying

$$\frac{2}{r} + \frac{n}{q} < 2. \tag{1.11}$$

We ignore if this result holds. However, it is worth noting that the equation (1.10) only "shows" that $|p| \lesssim |v|^2$. There are even no heuristic reasons to believe that $|v|^2 \lesssim |p|$ (hence, sufficient conditions on the pressure seems harder to prove). Nevertheless we are able to prove that v is necessarily bounded in Q_T if

$$\frac{|p|}{1 + |v|} \in L^r(0, T; L^q) \tag{1.12}$$

with (r, q) satisfying (1.6). The assumption (1.12) still corresponds to the strong sufficient condition $v \in L^r(0, T; L^q)$ with (r, q) satisfying (1.6), *if* we suppose that $|p| \simeq |v|^2$. However, as claimed above, there are no arguments supporting that $|p| \lesssim |v|^2$; hence, our condition (1.12) looks stronger.

Actually, our sufficient condition is stronger than (1.12). We prove that v is bounded in Q_T if (1.12) holds just on the subset where $|v(x,t)| > k$ for some arbitrarily large k. More precisely, let k be any positive real and define

$$\phi_k(x,t) = \begin{cases} |p(x,t)|/(1+|v(x,t)|) & \text{if } |v(x,t)| > k, \\ 0 & \text{otherwise.} \end{cases} \tag{1.13}$$

We prove the following result.

Theorem 1.1. *Let v be a weak solution of problem* (1.1), *where* v_0 *satisfies* (1.2). *Assume that for some positive real number k the function* ϕ_k *belongs to* $L^r(0,T;L^q)$, *for some pair* (r,q) *satisfying* (1.6). *Then* v *is bounded in* Q_T. *In particular,* v *is bounded in* Q_T, *if* (1.12) *holds.*

2. Proof of Theorem 1.1. Without loss of generality we assume in the sequel that q and r are finite. Theorem 1.1 will be proved by using the truncation method, introduced in reference [3] by E. De Giorgi and further developed by many authors, in particular by G. Stampacchia and by O.A. Ladyženskaja and N.N. Ural'ceva (see [6] and [4]). This method was first applied to variational inequalities in reference [2]. On the other hand, in reference [5], J. Moser gives a different proof of De Giorgi's theorem. One can prove results similar to those obtained by the truncation method by using Moser's approach. This was done, in particular, by D.G. Aronson and J. Serrin (see [1]) for parabolic equations.

We set, for each $k > 0$,

$$v^{(k)} = \begin{cases} (1 - \frac{k}{|v|})v & \text{if } |v| > k, \\ 0 & \text{otherwise,} \end{cases} \tag{2.1}$$

where v is a vector field defined in some suitable domain. Note that in reference [4], Chap. VII, a different truncation is used. Definition (2.1) corresponds to setting

$$v^{(k)} = |v|^{(k)} \frac{v}{|v|}, \tag{2.2}$$

where the truncation of a scalar function f is defined by

$$f^{(k)} = \max\{f - k, 0\}. \tag{2.3}$$

Note that

$$|v^{(k)}| = |v|^{(k)}. \tag{2.4}$$

On applying the truncation method, the first step consists in using the function $v^{(k)}$ as test function in the weak (variational) formulation of the problem under consideration. The justification of the necessary manipulations is standard. However, since the solution of problem (1.1) is locally (in time) regular for an $L^\infty \cap H_0^1$ initial data and, moreover, is regular as long as it is bounded, it is sufficient to prove here that solutions in Q_T that are regular in $Q_{T-\epsilon}$ for each $\epsilon > 0$ and satisfy our assumptions in Q_T are necessarily bounded in Q_T. Concerning the regularity of the truncated functions, it is well known that if a scalar function f belongs to $W_0^{1,2}$ then the truncated function $f^{(k)}$ still belongs to the same space (see, for instance, [6], [4]). The equation (2.2) shows that the same result holds for vector fields, since $k > 0$. Here

$$\partial_i v^{(k)} = \partial_i v - k\partial_i \frac{v}{|v|} \text{ in } A_k, \tag{2.5}$$

and $\partial_i v^{(k)} = 0$ in Ω/A_k, where

$$A_k(t) = \{x \in \Omega : |v(x,t)| > k\}. \tag{2.6}$$

If $\operatorname{div} v = 0$ in Ω, one gets from (2.4)

$$\operatorname{div} v^{(k)} = k\frac{v_i v_j}{|v|^3}\partial_i v_j \text{ in } A_k, \tag{2.7}$$

$\operatorname{div} v^{(k)} = 0$ in Ω/A_k. In particular

$$|\operatorname{div} v^{(k)}| \leq \frac{k}{|v|}|\nabla v| \text{ in } A_k. \tag{2.8}$$

By multiplication of both sides of $(1.1)_1$ by $v^{(k)}$ followed by integration over Ω one gets

$$\int_\Omega (\partial_t v)v^k\,dx + \mu\int_\Omega \nabla v\cdot\nabla v^k\,dx + \int_\Omega (v\cdot\nabla)v\cdot v^k\,dx + \int_\Omega \nabla p\cdot v^k\,dx = 0, \tag{2.9}$$

where, for convenience, we set from now on $v^k = v^{(k)}$. Let us study separately each of the above integrals. From (2.2) it follows that $v^k\partial_t v = |v|^{(k)}(v/|v|)\partial_t v = |v|^{(k)}\partial_t|v|$ in A_k. On the other hand $\partial_t|v|^{(k)} = \partial_t|v|$ on A_k since the scalar field $|v|^{(k)}$ is the truncation of $|v|$ (see [4] for details). It follows that

$$\int_\Omega (\partial_t v)v_k\,dx = \frac{1}{2}\frac{d}{dt}\int_\Omega [|v|^{(k)}]^2\,dx = \frac{1}{2}\frac{d}{dt}\int_\Omega |v^{(k)}|^2\,dx, \tag{2.10}$$

by (2.4). Note that integrals over Ω/A_k vanish. Next we show that

$$\int_\Omega \nabla v\cdot\nabla v^k\,dx \geq \int_\Omega |\nabla v^k|^2\,dx. \tag{2.11}$$

Since both integrals vanish on Ω/A_k we only take into account the domain A_k. One has, almost everywhere in A_k,

$$\nabla v \cdot \nabla v^k = |\nabla v^k|^2 + k\nabla\Big(\frac{v^k}{|v^k|}\Big) \cdot \nabla v^k, \tag{2.12}$$

where (2.5) was used together with $v/|v| = v^k/|v^k|$. Denoting v^k simply by w, straightforward calculations show that

$$\nabla w \cdot \nabla \frac{w}{|w|} = \frac{1}{|w|}|\nabla w|^2 - \frac{1}{|w|^3}\sum_{j=1}^{n}\Big(w_i\frac{\partial w_i}{\partial x_j}\Big)^2. \tag{2.13}$$

Since

$$\Big(w_i\frac{\partial w_i}{\partial x_j}\Big)^2 \le |w|^2 \sum_{i=1}^{n}\Big(\frac{\partial w_i}{\partial x_j}\Big)^2$$

it readily follows that the left-hand side of (2.13) is nonnegative. This fact together with (2.12) proves (2.11).

Next, we consider the "nonlinear term." We will show that

$$\int_\Omega (v\cdot\nabla)v \cdot v^k\,dx = \int_{A_k} (v\cdot\nabla)v\cdot v^k\,dx = 0. \tag{2.14}$$

One has, almost everywhere in A_k,

$$v_i(\partial_i v_j)v_j^k = v_i\partial_i(v_j^k + kv_j/|v|)v_j^k = (1/2)v_i\partial_i|v^k|^2 + kv_i\partial_i(v_j^k/|v^k|)v_j^k. \tag{2.15}$$

Moreover

$$\int_{A_k} v_i\partial_i|v^k|^2\,dx = \int_\Omega v_i\partial_i|v^k|^2\,dx = 0$$

since $\operatorname{div} v = 0$ in Ω and $v \in H_0^1(\Omega)$. By taking into account (2.15) and the identity $w_j\partial_i(w_j/|w|) = 0$, we prove (2.14). From (2.9), (2.10), (2.11) and (2.14) it follows that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |v^k|^2\,dx + \mu\int_\Omega |\nabla v^k|^2\,dx + \int_\Omega \nabla p\cdot v^k\,dx = 0, \tag{2.16}$$

for each $k > 0$. Clearly,

$$\Big|\int_\Omega \nabla p\cdot v^k\,dx\Big| \le \int_{A_k} |p|\,|\nabla v^k|\,dx.$$

However this estimate is too crude for our purpose. We will improve it by showing that

$$\Big|\int_\Omega \nabla p\cdot v^k\,dx\Big| \le k\int_{A_k}\frac{|p|}{|v|}|\nabla v^k|\,dx. \tag{2.17}$$

One has, from (2.7),

$$\int_\Omega \nabla p \cdot v^k \, dx = -k \int_{A_k} p \frac{v_i v_j}{|v|^3} \partial_i v_j \, dx. \tag{2.18}$$

Next

$$|v|^{-1} v_j \partial_i v_j = \partial_i |v| = \partial_i (|v^k| + k) = \partial_i |v^k| = |v^k|^{-1} v_j^k \partial_i v_j^k$$

in A_k. It readily follows that

$$\big| |v|^{-1} v_i v_j \partial_i v_j \big| = \big| v_i |v^k|^{-1} v_j^k \partial_i v^k \big| \le |v| \, |\nabla v^k|.$$

This estimate together with (2.18) proves (2.17). By using (2.16) one obtains

$$\frac{1}{2} \frac{d}{dt} \int_\Omega |v^k|^2 \, dx + \frac{\mu}{2} \int_\Omega |\nabla v^k|^2 \, dx \le \frac{2}{\mu} k^2 \int_{A_k} \frac{|p|^2}{|v|^2} \, dx. \tag{2.19}$$

Since $|\nabla |w||^2 \le |\nabla w|^2$, where w is a vector field, and $|v^k|^2 = (|v|^{(k)})^2$, one has

$$\frac{1}{2} \frac{d}{dt} \int_\Omega (|v|^{k)})^2 \, dx + \frac{\mu}{2} \int_\Omega |\nabla |v|^{(k)}|^2 \, dx \le \frac{2}{\mu} k^2 \int_{A_k} \frac{|p|^2}{|v|^2} \, dx. \tag{2.20}$$

In the sequel we assume that the real parameter k satisfies $k \ge \hat{k} = \max\{k_0, k_1, 1\}$. By integration of (2.20) on $(0, T)^{(*)}$ it readily follows our main estimate

$$| |v|^{(k)} |_{Q_T}^2 \le \frac{64}{\mu} k^2 \int_0^T \int_{A_k} \frac{|p|^2}{1 + |v|^2} \, dx \, dt, \tag{2.21}$$

where $| \cdot |_{Q_T}^2$ is defined by (see [4], Chapter I, Section 1, Equation (1.5))

$$|f|_{Q_T} \equiv \|f\|_{2,\infty,T} + \|\nabla f\|_{2,2,T}.$$

By denoting the function $\phi_{\hat{k}}$ (see (1.13)) simply by ϕ, by taking into account the hypotheses made in Theorem 1.1, and by using Hölder's inequalities, one easily shows that

$$| |v|^{(k)} |_{Q_T}^2 \le \frac{64}{\mu} k^2 \|\phi\|_{q,r,T}^2 \Big(\int_0^T |A_k(t)|^{\frac{r(q-2)}{q(r-2)}} dt \Big)^{\frac{r-2}{2r}}. \tag{2.22}$$

Hence, for each $k \ge \hat{k}$,

$$| |v|^{(k)} |_{Q_T} \le \frac{8}{\sqrt{\mu}} \|\phi\|_{q,r,T} k \Big(\int_0^T |A_k(t)|^{\frac{\bar{r}(1+\chi)}{\bar{q}(1+\chi)}} dt \Big)^{\frac{1+\chi}{\bar{r}(1+\chi)}}, \tag{2.23}$$

(*) More precisely, we integrate on $(0, T - \epsilon)$ and we prove boundedness in $Q_{T-\epsilon}$, uniformly with respect to ϵ.

where $\chi = 2\epsilon/n$, $\epsilon = 1 - (2/r + n/q)$, and $\bar{r} = 2r/(r-2)$, $\bar{q} = 2q/(q-2)$. Note that $\bar{r} \in (2, \infty)$. Moreover,

$$\frac{1}{\bar{r}(1+\chi)} + \frac{n}{2\bar{q}(1+\chi)} = \frac{n}{4}. \tag{2.24}$$

Since $\bar{r}(1+\chi) \in (2, \infty)$, (2.24) shows that $\bar{q}(1+\chi) \in (2, 2n/(n-2))$. Theorem 6.1, Chapter II in [4] shows that $|v|$ is bounded in Q_T.

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