Concerning the regularity of the solutions to the Navier-Stokes equations via the truncation method
(Part II)

Hugo Beirão da Veiga

Dedicated to Jacques-Louis Lions on the occasion of his seventieth birthday

Abstract. The well known differential relation $-\Delta p = \operatorname{div}\,(v\cdot\nabla)v$ concerning the solutions (v,p) to the Navier-Stokes system of equations suggests the possibility of more strict links between the pressure p and the velocity v. Here we prove some rigorous results in this direction which also have applications to the study of the regularity of solutions. The starting point is an integral estimate (see (2.8)) below, proved in the first part of this work (see the references).

Key words: Navier-Stokes, regularity, sufficient conditions.

1 Introduction and results

In the sequel Ω is an open, connected subset of $\mathbf{R}^n$, $n \geq 3$. Our assumptions on Ω are particularly weak. We assume that Ω is bounded, locally located on one side of its boundary Γ and such that $\mathrm{H}_0^1(\Omega) \hookrightarrow \mathrm{L}^{2^*}(\Omega), 1/2^* = 1/2 - 1/n$. We set $Q_T = \Omega \times (0,T)$ and $\Sigma_T = \Gamma \times (0,T)$ where T is a fixed positive real number. We denote by $\|v\|_q$ the canonical norm of v in the space $\mathrm{L}^q = \mathrm{L}^q(\Omega)$, $1 \leq q \leq +\infty$. The symbol $\|v\|_{q,r,T}$ denotes the canonical norm of v in the space $\mathrm{L}^r(0,T;\mathrm{L}^q)$, $1 \leq r, q \leq +\infty$. $\mathrm{H}_0^1 = \mathrm{H}_0^1(\Omega)$ denotes the clososure of $\mathrm{C}_0^\infty(\Omega)$ with respect to the L^2-norm of the gradient. The space H_0^1 is endowed with the norm $\|\nabla v\|_2$. We use here the same notations to indicate functional spaces and norms for scalar and for vector fields.

In the sequel we will use the classical Marcinkiewicz spaces $\mathrm{L}_*^q(E)$, where E is a measurable subset of $\mathbf{R}^m$ and $1 \leq q < +\infty$. In general we denote by $|E|$ the Lebesgue measure of a set E. Let us recall the definition on $\mathrm{L}_*^q(E)$: A measurable function $f(y)$ belongs to $\mathrm{L}_*^q(E)$ if there is a constant $[f]_q$ such that

$$(1.1) \qquad |\{y \in E : |f(y)| > \sigma\}| \leq \left(\frac{[f]_q}{\sigma}\right)^q, \qquad \forall \sigma > 0.$$

The smallest constant $[f]_q$ for which (1.1) holds is called the "norm" of f in $\mathrm{L}_*^q(E)$. Important properties of these spaces are the following algebraic and topological embeddings:

$$(1.2) \qquad \mathrm{L}^q(E) \subset \mathrm{L}_*^q(E) \subset \mathrm{L}^{q-\epsilon}(E), \quad \forall \epsilon > 0,$$

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which shows that L^q_* is "very close" to L^q.

The existence and uniqueness for solutions of the Navier-Stokes equations (and related questions) are among the major problems in the theory of partial differential equations. Fundamental contributions are due to various authors, in particular J.-L. Lions to whom this paper is dedicated (see, for instance, [8], chap. I, commentaires).

Let us start by explaining the motivations that lead us to study the Navier-Stokes equations by the method followed here and in reference [3]. After a long time, the only totally satisfactory estimate that one has been able to prove for solutions of the Navier-Stokes equations (1.3) is the energy estimate, obtained by multiplication by v followed by integration. Hence it seems to us quite natural and (we believe) promising to try to improve the known regularity results by using method which have shown to be sufficiently strong in proving L^p estimates and, in addition, that are based on devices similar to the one providing the energy estimate. These two conditions are satisfied by the truncation method, introduced by E. De Giorgi [5] and further developed by other authors, in particular O.A. Ladyžhenskaja and G. Stampacchia (see [7], [12] and references).

This note is concerned with the solutions of the Navier-Stokes equations

$$\tag{1.3} \begin{cases} \dfrac{\partial v}{\partial t} - \mu\Delta v + (v\cdot\nabla)v + \nabla p = 0, & \\ \operatorname{div} v = 0 & \text{in } Q_T, \\ v = 0 & \text{on } \Sigma_T, \\ v(x,0) = v_0(x). & \end{cases}$$

We assume here that $v_0 \in H^1_0$ and $\operatorname{div} v_0 = 0$ in Ω. We point out that the boundary condition $v = 0$ on Σ_T is not at all essential. Similarly, the introduction of an external force f in equation $(1.3)_1$ can be treated in a straightforward way as done in section 3 for the stationary case. We assume here that $v_0 \in L^\infty$ and denote by k_0 a constant such that $\|v_0\|_\infty \le k_0$. For convenience we assume that $k_0 \ge 1$.

In reference [3] we proved that if

$$\tag{1.4} \frac{|p|}{1+|v|} \in L^r(0,T;L^q), \quad \frac{2}{r} + \frac{n}{q} < 1,$$

then the solution v of problem (1.3) is bounded in Q_T. Actually the result in reference [3] is stronger since the assumption (1.4) is replaced by a weaker condition (see [3], Eq. (1.6)). It follows that (v,p) is "smooth". It is worth noting that by replacing in the assumption (1.4) the function $|p|/(1+|v|)$ by $|v|$ one gets the well known statement (pioneering papers are [9] and [10]; for the sharper results see [11]) which establishes that solutions v that belong to $L^r(0,T;L^q)$, $(2/r)+(n/q) < 1$, are "smooth" (the result for v is a little more general, since we may have $(2/r)+(n/q) = 1$, $r > n$. However the same result holds for $|p|/(1+|v|)$; see [4]). This fact suggests that $|p| \simeq |v|^2$, in formal agreement with the well known equation

$$-\Delta p = \sum_{i,j=1}^{n} \partial_i \partial_j (v_i v_j). \tag{1.5}$$

Actually, the lack of a suitable boundary condition on the pressure p prevents from using the equation (1.5). Moreover, the equation (1.5) suggests, at most, that p may be estimated in terms of v but not the opposite, which is just the interesting point. In any case, the formal relation $|p| \simeq |v|^2$ is significant only at the "level of regularity" that corresponds to $v \in \mathrm{L}^r(0,T;\mathrm{L}^q)$, $2/r + n/q = 1$. In the sequel we analyse these and other related problems, in a rigorous way, by turning back to the Navier-Stokes system (1.3).

In the sequel we study the above problems in the particular framework of the spaces of type $\mathrm{L}^r(0,T;\mathrm{L}^q)$ for which $r = q$. We believe that this restriction can be easily droped, however we did not investigate in this direction. Moreover, the basic results will be expressed in the framework of $\mathrm{L}^q_*(Q_T)$ spaces. Then, the embeddings (1.2) provide results concerning the usual L^q spaces. Sufficient conditions for smoothness envolving p and v (case $\theta \neq 0$) or p alone (case $\theta = 0$) follow immediately. For convenience we set

$$N = n + 2. \tag{1.6}$$

This is a significant constant. In fact it is just the minimal exponent (at the light of the results known at present) that guarantees the regularity of solutions. More precisely, it is known that a solution v is regular if it satisfies

$$v \in \mathrm{L}^N(Q_T), \tag{1.7}$$

but a similar result is not known for any exponent less than N.

Our basic result is the following (the proof is postponed to the next section):

Theorem 1.1 *Let (v,p) be a weak solution of problem (1.3). Assume that for some $\theta \in [0,1[$ and some γ such that*

$$\frac{2N}{2\theta + (1-\theta)N} < \gamma < N \tag{1.8}$$

one has

$$\frac{p}{(1+|v|)^\theta} \in \mathrm{L}^\gamma_*(Q_T). \tag{1.9}$$

Then

$$v \in \mathrm{L}^\mu_*(Q_T) \tag{1.10}$$

where

$$\mu = (1-\theta)\frac{N\gamma}{N-\gamma}. \tag{1.11}$$

Moreover, if

$$\frac{p}{1+|v|} \in \mathrm{L}^{\gamma}_{*}(Q_T), \qquad \gamma > N, \tag{1.12}$$

then $v \in \mathrm{L}^{\infty}(Q_T)$.

Remark 1.2 The solution v is "smooth" if $\gamma > N/(2-\theta)$, $\theta \in [0,1]$, since in this case $\mu > N$. —

Remark 1.3 From our proofs it follows that in the assumptions (1.9) and (1.12) (as well as in (1.13) below) the set Q_T can be replaced by a set $B(k_1)$ (see definition (2.12)) for an arbitrarily large value k_1. —

Remark 1.4 The case $\gamma > N$ falls within the range of application of the Theorem I in reference [3] by setting there $q = r = N$ (see also [4]). However, the proof here is much easier to handle. —

Remark 1.5 The condition $\gamma > 2N/[2\theta + N(1-\theta)]$ is assumed here only in order that $\mu > m$, where $m = 2(1+2/n)$. In fact, by (2.9), weak solutions of Navier-Stokes equations necessarily belong to $\mathrm{L}^m(Q_T)$. Hence (1.10) is obvious when $\mu \leq m$. —

Remark 1.6 At the light of the formal relation $p \simeq |v|^2$, the assumption (1.9) corresponds to

$$v \in \mathrm{L}^{(2-\theta)\gamma}_{*}(Q_T).$$

Hence there would be an increase of regularity if $\mu > (2-\theta)\gamma$ and a loss if $\mu < (2-\theta)\gamma$. This means $N/(2-\theta) < \gamma$ and $\gamma < N(2-\theta)$, respectively. There would be no change in the level of regularity if $\gamma = N/(2-\theta)$. In this last case one has $\mu = (2-\theta)\gamma = N$ where $N = n+2$ is the minimal exponent that guarantees smoothness of solutions. —

At last, let us consider by itself the case $\theta = 0$ since this is just the case in which the pressure p appears alone in the hypothesis (1.9).

Corollary 1.7 *Let (v,p) be a weak solution to problem* (1.3). *Assume, moreover, that*

$$p \in \mathrm{L}^{\gamma}_{*}(Q_T) \tag{1.13}$$

for some $\gamma \in]2, N[$. *Then*

$$v \in \mathrm{L}^{\mu}_{*}(Q_T), \quad \mu = \frac{N\gamma}{N-\gamma}. \tag{1.14}$$

In particular if $p \in \mathrm{L}^{N/2}_{*}(Q_T)$ *then* $v \in \mathrm{L}^{N}_{*}(Q_T)$ *and if* $p \in \mathrm{L}^{\gamma/2}_{*}(Q_T)$, $\gamma > N$, *then* v *is "smooth".*

It would be interesting to know if $p \in L^{N/2}(Q_T)$ (or even if $v \in L^N_*(Q_T)$) is sufficient to guarantee the "smoothness" of the solution.

Remark 1.8 It is worth noting that, in spite of the presence of the nonlinear term $(v \cdot \nabla)v$, the regularity result stated in the Corollary 1.7 is just that obtained for solutions of the linear equation $\partial v/\partial t - \Delta v = \nabla p$, where p is given. However this comparison with the heat equation is still not completely satisfactory, since in equation (1.3) p is an unknown. For instance, if $p \in L^2(Q_T)$ the solution of the above linear equation belongs precisely to $L^\mu(Q_T)$, $\mu = 2 + n/4$ (which formally corresponds to the limit case $\gamma = 2$ in Corollary I). However this same result holds for the Navier-Stokes equation whether knowing if p belongs to $L^2(Q_T)$ or not.

Remark 1.9 In reference [6], S. Kaniel proves that if $p \in L^\infty(0, T; L^q)$ with $q > 12/5$, then the solution is smooth (here $n = 3$). This result looks weaker then ours since it requires $2/r + 3/q < 5/4$, although $2/r + 3/q < 2$ should be sufficient. A result in this last direction was recently obtained by L. Berselli. —

Finally, for results concerning the stationary case

$$\begin{cases} -\mu\Delta v + (v \cdot \nabla)v + \nabla p = f, & \\ \operatorname{div} v = 0 & \text{in } \Omega, \\ v = \varphi & \text{on } \Gamma, \end{cases} \tag{1.15}$$

the reader is referred to section 3.

2 Proof of theorem 1.1

We start by proving a lemma that extends a result due to G. Stampacchia [12], where $\theta = 0$. The proof is similar.

Lemma 2.1 *Let $\varphi : [k_0, +\infty[\to [0, +\infty[$, $k_0 > 0$, be a decreasing (not necessarily strictly) function such that*

$$\varphi(h) \le c \frac{k^\theta}{(h-k)^\alpha} [\varphi(k)]^\beta, \qquad \forall h > k \ge k_0, \tag{2.1}$$

where c and α are positive constants, $0 \le \theta < \alpha$ and $0 < \beta < 1$. Then

$$\varphi(k) \le \frac{C}{k^\mu}, \qquad fk \ge k_0, \tag{2.2}$$

where

$$\mu = \frac{\alpha - \theta}{1 - \beta} \tag{2.3}$$

and

$$C = 2^{\mu/(1-\beta)} \left[c^{1/(1-\beta)} + (2k_0)^\mu \varphi(k_0) \right].$$

PROOF :

For each $h \geq k_0$ set

$$\psi(h) = \left(\frac{1}{c}\right)^{\frac{1}{1-\beta}} h^{\mu}\varphi(h). \tag{2.4}$$

From (2.1) it follows that

$$\psi(h) \leq k^{\theta-\beta\mu}\frac{h^{\mu}}{(h-k)^{\alpha}}[\varphi(k)]^{\beta},$$

for $h > k \geq k_0$. By setting $h = 2k$ one easily gets

$$\psi(2k) \leq 2^{\mu}[\psi(k)]^{\beta}.$$

Hence

$$\psi(2^j k) \leq \left(2^{\mu}\right)^{\sum_{i=0}^{j-1}\beta^i}[\psi(k)]^{\beta^j},$$

for each positive integer j. It readily follows that

$$\psi(2^j k) \leq 2^{\frac{\mu}{1-\beta}}[1+\psi(k)]. \tag{2.5}$$

Hence ψ is bounded. Since any real M, $M \geq k_0$, can be written as $M = 2^j k_1$, where $k_1 \in [k_0, 2k_0]$ and j is an integer, it easily follows that

$$\psi(M) \leq 2^{\frac{\mu}{1-\beta}}[1+\psi(k_1)]. \tag{2.6}$$

On the other hand

$$\psi(k_1) \leq \sup_{k_0\leq k\leq 2k_0}\psi(k) \leq \left(\frac{1}{c}\right)^{\frac{1}{1-\beta}} \sup_{k_0\leq k\leq 2k_0} k^{\mu}\varphi(k).$$

Hence

$$\psi(k_1) \leq \left(\frac{1}{c}\right)^{\frac{1}{1-\beta}}(2k_0)^{\mu}\varphi(k_0). \tag{2.7}$$

From (2.4), (2.6) and (2.8) one gets (2.2). —— QED (Theorem 1.1)

In the sequel we assume that the hypotheses in theorem 1.1 are satisfied. We start by introducing the fundamental estimate (2.8) below. We set

$$A_k(t) = \{x \in \Omega : |v(x,t)| > k\}$$

and define

$$v^k(x,t) = \max\{|v(x,t)| - k, 0\},$$

for each $k > 0$. In [3] (see [3], Eq. (2.21)) we prove that

$$\|v^k\|^2_{2,\infty,T} + \|\nabla v^k\|^2_{2,2,T} \le \frac{64}{\mu} k^2 \int_0^T \int_{A_k(t)} \frac{|p|^2}{1+|v|^2} \, dx \, dt, \tag{2.8}$$

for each $k \ge k_0 = \max\{1, \|v_0\|_\infty\}$. We note that there is a slight difference between notations here and in part I, where the scalar quantity v^k is denoted by $|v|^{(k)}$ and the symbol v^k is used to denote the vector field $v^{(k)} = (|v|^{(k)}/|v|)v$. One easily shows that

$$\|u\|^2_{q,r,T} \le c(\|u\|^2_{2,\infty,T} + \|\nabla u\|^2_{2,2,T}), \tag{2.9}$$

where

$$\frac{2}{r} + \frac{n}{q} = \frac{n}{2}. \tag{2.10}$$

See [7], chap. II, Eq. (3.4). From (2.8) and (2.9) one shows that

$$\|v^k\|^2_{q,r,T} \le ck^2 \int_{B(k)} \frac{|p|^2}{(1+|v|)^2} \, dx \, dt \tag{2.11}$$

where

$$B(k) = \{(x,t) \in Q_T : |v(x,t)| > k\}. \tag{2.12}$$

In particular, by setting $m = q = r$, i.e.

$$m = 2(1 + \frac{2}{n})$$

one gets

$$\|v^k\|^2_{m,Q_T} \le ck^2 \int_{B(k)} \frac{|p|^2}{(1+|v|)^2} \, dx \, dt. \tag{2.13}$$

On the other hand

$$\|v^k\|^2_{m,Q_T} \ge \left(\int_{B(h)} |v^k|^m \, dx \, dt \right)^{2/m} \ge (h-k)^2 |B(h)|^{2/m},$$

for $h > k$. Hence

$$|B(h)| \le c \frac{k^m}{(h-k)^m} \left(\int_{B(k)} \frac{|p|^2}{(1+|v|)^2} \, dx \, dt \right)^{m/2},$$

where (here and in the sequel) c denotes any positive "numerical constant". Since $k < |v|$ on $B(k)$, it readily follows that

$$|B(h)| \le c \frac{k^{m\theta}}{(h-k)^m} \left(\int_{B(k)} \frac{|p|^2}{(1+|v|)^{2\theta}} \, dx \, dt \right)^{m/2}.$$

Next we prove that (1.9) implies

$$\int_{B(k)} \frac{|p|^2}{(1+|v|)^{2\theta}}\,\mathrm{d}x\,\mathrm{d}t \le \frac{\gamma}{\gamma-2} M^2 |B(k)|^{1-\frac{2}{\gamma}} \tag{2.14}$$

where, for convenience, we set $M = [|p|/(1+|v|)^\theta]_\gamma$. In order to prove (2.14) we show that if $g \in \mathrm{L}^\gamma_*(E)$, where $\gamma > 2$, and $B \subset E$ then

$$\int_B g^2\,\mathrm{d}y \le C_0 |B|^{1-\frac{2}{\gamma}}, \tag{2.15}$$

and we apply this result to $g = |p|/(1+|v|)^\theta$, $E = Q_T$ and $B = B(k)$. This yields (2.14).

Let us prove (2.15). One has

$$\begin{aligned}\int_B g^2\,\mathrm{d}y &= 2\int_0^{+\infty} |\{y \in E : |g| > \sigma\} \cap B|\sigma\,\mathrm{d}\sigma \\ &\le 2\int_0^a |B|\sigma\,\mathrm{d}\sigma + 2\int_a^{+\infty} \left(\frac{[g]_\gamma}{\sigma}\right)^\gamma \sigma\,\mathrm{d}\sigma \\ &= a^2|B| + 2[g]_\gamma^\gamma a^{2-\gamma}(\gamma-2)^{-1}.\end{aligned}$$

By setting $a = [g]_\gamma |B|^{-1/\gamma}$ it follows that

$$\int_B g^2\,\mathrm{d}y \le [g]^2_\gamma \frac{\gamma}{\gamma-2}|B|^{1-\frac{2}{\gamma}}.$$

Next, from (2.12) and (2.14) one obtains

$$|B(h)| \le cM^m \frac{k^{m\theta}}{(h-k)^m}|B(k)|^{m(\frac{1}{2}-\frac{1}{\gamma})}, \tag{2.16}$$

for $h > k \ge k_0$. Now, we apply the lemma 2.1 to the function $\varphi(k) = |B(k)|$, with θ replaced by θm, with $\alpha = m$ and with $\beta = m(\gamma-2)/2\gamma$. This shows that

$$|B(k)| \le \frac{C_0}{k^\mu} \qquad fk > k_0, \tag{2.17}$$

where

$$C_0 = 2^{\mu\lambda}\left(c^\lambda M^{\frac{N\gamma}{N-\gamma}} + (2k_0)^\mu|\Omega|\right), \tag{2.18}$$

and $\lambda = (1-\beta)^{-1} = \gamma(N-2)/2(N-\gamma)$. Recall that M is the norm of $|p|/(1+|v|)^\theta$ in the space $\mathrm{L}^\gamma_*(Q_T)$.

The first part of theorem I is proved since (2.17) means that $v \in \mathrm{L}^\mu_*(Q_T)$. Moreover $[v]_\mu = C_0^{1/\mu}$.

Remark 2.2 It is worth noting that the basic estimates (2.8) can be improved. In particular (2.16) holds with $B(k)$ replaced by $B(k)\backslash B(h)$. We have not yet investigated on the possible consequences of this fact. —

Finally we prove (1.12). Let us recall the following result proved by us in a previous paper. For the readers convenience we present here the proof.

Lemma 2.3 *Let* $\varphi : [k_0, +\infty[\to [0, +\infty[$, $k_0 \geq 0$, *be a decreasing (not necessarily strictly) function such that* (2.1) *holds, where* $c, \alpha, \theta, \beta = 1+\chi$ *are constants such that* $c \geq 0$, $0 \leq \theta < \alpha\beta$ *and* $\beta > 1$. *Then* $\varphi(2d) = 0$ *where* $d > k_0$ *is the root of the equation*

$$d = k_0 + \lambda k_0^{\theta/\alpha} d^{(\theta-\alpha)/\chi^\alpha} \tag{2.19}$$

and

$$\lambda^\alpha = 2^{\frac{\alpha+\theta}{\chi}+\frac{\alpha}{\chi^2}} c^{1+\frac{1}{\chi}} [\varphi(k_0)]^\beta. \tag{2.20}$$

PROOF :

Set $k_j = 2d(1 - 2^{-j})$, j positive integer. We want to show that

$$\varphi(k_j) \leq \left[\frac{d^{\alpha-\theta}}{2^{\alpha(j+1/\chi)+\theta} c} \right]^{1/\chi}. \tag{2.21}$$

Since $\lim\limits_{j\to\infty} k_j = 2d$, (2.21) implies $\varphi(2d) = 0$.

Equation (2.1) for $h = k_1$ and $k = k_0$ shows that

$$\varphi(k_1) \leq \frac{c k_0^\theta}{(d - k_0)^\alpha} [\varphi(k_0)]^{1+\chi}. \tag{2.22}$$

By replacing $(d - k_0)^\alpha$ by the value obtained from equation (2.19) it readily follows that the right hand side of (2.22) is equal to the right hand side of (2.21) for $j = 1$. Next, by assuming that (2.21) holds for some $j \geq 1$ and by using (2.1) we prove that

$$\varphi(k_{j+1}) \leq \frac{2^{\theta+j\alpha} c}{d^{\alpha-\theta}} \left[\frac{d^{\alpha-\theta}}{2^{\alpha(j+1/\chi)+\theta} c} \right]^{1+1/\chi}. \tag{2.23}$$

Straightforward calculations show that the right hand side of (2.23) is equal to that of (2.21) if here we replace j by $j+1$. —— QED (Lemma 2.3)

Next, we consider the equation (2.16) for $\theta = 1$, i.e.

$$|B(h)| \leq c M^m \frac{k^m}{(h-k)^m} |B(k)|^{m(\frac{1}{2}-\frac{1}{\gamma})}, \tag{2.24}$$

where $\gamma > N$, and we apply the lemma 2.2 with $\theta = \alpha = m$ and $1+\chi = m(\frac{1}{2}-\frac{1}{\gamma})$. This yields $|B(2d)| = 0$, hence $\|v\|_{L^\infty(Q_T)} \leq 2d$. —— QED (Theorem)

3 The stationary case.

For the stationary case (1.15) the same techniques apply. Instead of equation (2.20) in [3] one has

$$\frac{\mu}{2}\int_\Omega |\nabla v^k|^2 \,\mathrm{d}x \le \frac{2}{\mu}k^2\int_{A_k}\frac{|p|^2}{|v|^2}\,\mathrm{d}x + \int_\Omega f\cdot v^{(k)}\,\mathrm{d}x. \tag{3.1}$$

Moreover,

$$\left|\int_\Omega f\cdot v^{(k)}\,\mathrm{d}x\right| \le \frac{\mu}{4}\int_\Omega \left|\nabla |v|^k\right|^2\,\mathrm{d}x + \frac{c}{\mu}\left(\int_{A_k}|f|^s\,\mathrm{d}x\right)^{2/s}, \tag{3.2}$$

where $s=(2^*)'=2n/(n+2)$. By assuming that

$$f\in \mathrm{L}^\delta(\Omega) \tag{3.3}$$

where $\delta > s$, one has

$$\left(\int_{A_k}|f|^s\,\mathrm{d}x\right)^{2/s} \le \|f\|_\delta^2 |A_k|^{\frac{2}{s}-\frac{2}{\delta}}. \tag{3.4}$$

As in section 2, we may replace (3.3) by

$$f\in \mathrm{L}^\delta_*(\Omega). \tag{3.5}$$

From (3.1), (3.2) and (3.4) it follows (this corresponds to equation (2.13)) that

$$\|v^k\|^2_{2^*}\le ck^2\int_{A_k}\frac{|p|^2}{(1+|v|)^2}\,\mathrm{d}x + c\|f\|^2_\delta|A_k|^{\frac{2}{s}-\frac{2}{\delta}}.$$

By arguing as in section 2 with Q_T replaced by Ω, $B(k)$ replaced by A_k, $N=n+2$ replaced by n and $m=2^*$ we get (in correspondence to (2.16))

$$|A_h|\le C\frac{k^{m\theta}}{(h-k)^m}|A_k|^{m(\frac{1}{2}-\frac{1}{\gamma})} + \frac{C}{(h-k)^m}|A_k|^{m(\frac{1}{s}-\frac{1}{\delta})} \tag{3.6}$$

where, for convenience, the dependence on M and on the norm of f is contained in the constant C. Next, we choose δ such that

$$\frac{\delta n}{n-2\delta}=\mu \tag{3.7}$$

where now μ is defined in equation (3.10) below (compare to (1.14)). Note that if $\theta=0$ the exponents of $|A_k|$ in equation (3.6) coincide. Moreover $1/\mu=1/\delta-2/n$ as expected, since $W^{2,\delta}\hookrightarrow \mathrm{L}^\mu$.

Next if $\theta=0$ we apply the lemmas 2.1 and 2.2 to the inequality (3.6). If $\theta\neq 0$ a generalization of these lemmas is needed. In this way the following result holds:

Theorem 3.1 *Let* (v,p) *be a weak solution to problem* (1.15) *where the boundary condition is given by assuming that* $v-\varphi \in \mathrm{H}_0^1(\Omega)$, *for some* $\varphi \in \mathrm{H}^1(\Omega) \cap \mathrm{L}^\infty(\Omega)$, $\|\varphi\|_\infty \leq k_0$. *If* Γ *is sufficiently regular this means that* $\varphi \in \mathrm{H}^{1/2}(\Gamma) \cap \mathrm{L}^\infty(\Gamma)$, $\|\varphi\|_{\mathrm{L}^\infty(\Gamma)} \leq k_0$. *Assume that for some* $\theta \in [0,1[$ *and some* γ *such that*

$$\frac{2n}{2\theta+(1-\theta)n} < \gamma < n, \tag{3.8}$$

one has

$$\frac{p}{(1+|v|)^\theta} \in \mathrm{L}_*^\gamma(\Omega) \tag{3.9}$$

and, moreover, that f *satisfies* (3.5) *where* δ *is defined by* (3.7). *Then*

$$v \in \mathrm{L}_*^\mu(\Omega)\ ,\ \ \mu = (1-\theta)\frac{n\gamma}{n-\gamma}. \tag{3.10}$$

If

$$\frac{p}{1+|v|} \in \mathrm{L}_*^\gamma(\Omega), \quad \gamma > n,$$

and if (3.5) *holds for some* $\delta > n/2$ *then* $v \in \mathrm{L}^\infty(\Omega)$.

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Hugo Beirão da Veiga,
Department of Applied Mathematics
Pisa University – Via Bonanno, 25/B,
56100 Pisa, Italy
bveiga@mail.dm.unipi.it