CONCERNING TIME-PERIODIC SOLUTIONS OF THE NAVIER-STOKES EQUATIONS IN CYLINDRICAL DOMAINS UNDER NAVIER BOUNDARY CONDITIONS.

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December 1, 2005

Abstract

The problem of the existence of time-periodic flows in infinite cylindrical pipes in correspondence to any given, time-periodic, total flux, was solved only quite recently in [4]. In this last reference we solved the above problem for flows under the non-slip boundary condition as a corollary of a more general result. Here we want to show that the abstract theorem proved in [4] applies as well for solutions of the well known slip (or Navier) boundary condition (1.7) or to the mixed boundary condition (1.14). Actually, the argument applies for solutions of many other boundary value problems. This paper is a continuation of reference [4], to which the reader is referred for some notation and results.

1 Introduction

Let Ω be a bounded, regular, connected open set in $\mathbb{R}^n$, $n \geq 1$, and consider a cylindrical, $(n+1)-$dimensional infinite pipe $\Lambda = \Omega \times \mathbb{R}$. We denote by Γ the boundary of Ω and by $\Sigma = \Gamma \times \mathbb{R}$ the boundary of Λ. We set $x = (x_1, ..., x_n)$ and denote by z the longitudinal coordinate along the axis of the pipe, say $z = x_{n+1}$. We denote by χ the component of the velocity v in the axial direction z. Note that the physical dimension is $N = n + 1$. In the sequel we consider the problem of the existence of T-periodic flows of the Navier-Stokes equations

$$(1.1) \qquad \begin{cases} \frac{\partial v}{\partial t} - \nu \, \Delta \, v + (v \cdot \nabla) \, v + \nabla p = 0, \\ \\ \nabla \cdot v = 0 \quad \text{in } \Lambda \times \mathbb{R}_t \, , \end{cases}$$

under the slip boundary condition (1.7) or the mixed boundary condition (1.8), in the cylindrical domain Λ. Here the differential operators Δ and ∇ act on all the variables $(x_1, ..., x_n, \, z)$ and there is an (arbitrarily given) T-periodic total flux $g(t)$, i.e.,

$$(1.2) \qquad \int_\Omega \chi(x, z, t) \, dx = g(t)$$

for each cross section $\Omega(z) = \{(x, z) : x \in \Omega\}$. We call the flux $g(t)$, in the cross sections of the pipe, the *total flux*.

In reference [4] we considered the non-slip boundary condition

$$v = 0 \quad \text{on } \Sigma \times \mathbb{R}_t \tag{1.3}$$

and proved that to each T-periodic total flux $g(t)$ there corresponds one and only one T-periodic flow, parallel to the z-axis, independent of z and satisfying the flux constraint (1.2). See Theorem 1 in reference [4]. Particularly related papers are [8] by Galdi and Robertson and [13] by Pileckas. In reference [8], drawing on [4], the authors show an interesting relation between the total flux and the pressure gradient for solutions to the above problem. In reference [13] the author considers the initial-boundary value problem for solutions of (1.1), (1.2), (1.3). We also recall here the classical papers [1], [11] and [19].

Theorem 1 in reference [4] is proved as an application of a more abstract result, Theorem 2 in the same reference (see 2.1 below). In the sequel we show that this last theorem still applies if we replace the adherence (or non-slip) boundary condition (1.3) by the Navier (slip) boundary condition (1.7) or by the mixed boundary condition (1.8). Actually, the above theorem yields the same kind of results under a variety of boundary conditions, as easily verified. To fix ideas we consider here the slip boundary condition, due to its importance in many theoretical and applied problems. Let us recall this boundary condition. Denote by

$$T = -p\, I + \nu(\nabla v + \nabla v^T)$$

the stress tensor and by $\underline{t} = T \cdot \underline{n}$ the stress vector. Hence, with obvious notation,

$$T_{ik} = -\delta_{ik} p + \nu \left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} \right), \tag{1.4}$$

$$t_i = \sum_{k=1}^{3} T_{ik} n_k. \tag{1.5}$$

We also define the linear operators $v_\tau = v - (v \cdot \underline{n})\, \underline{n}$ (the tangential component of v) and the tangential component of $\underline{t}$

$$\underline{\tau}(v) = \underline{t} - (\underline{t} \cdot \underline{n})\underline{n}. \tag{1.6}$$

Note that $\underline{\tau}(v)$ is independent of the pressure p.

The (homogeneous) slip boundary condition reads

$$\begin{cases} (v \cdot n)_{|\Sigma} = 0, \\ \beta\, v_\tau + \underline{\tau}(v)_{|\Sigma} = 0, \end{cases} \tag{1.7}$$

where $\beta \geq 0$ is a given constant. When $\beta > 0$ this boundary condition is called slip boundary condition with friction. This type of boundary conditions was proposed by Navier himself, see [12].

For a mathematical study of this boundary condition we refer the reader to the pioneering paper [17] by Solonnikov and Scadilov and to references [3] and [2], where a very complete study is presented. It is worth noting, however, that in the particular case under consideration here we do not need to appeal to the general results proved in these references since, as shown below, under the hypothesis (1.10) the slip boundary conditions become simpler.

For the study of problems under the above, or strongly related, boundary conditions see, for instance, [2], [3], [5], [6], [7], [9], [10], [14], [15], [17], [18], and references.

Below we also consider the case in which the boundary Γ is the union of two disjoint portions

$$\Gamma = \Gamma_1 \cup \Gamma_2 \,,$$

each of them with a non vanishing $(n-1)$-dimensional measure, and the boundary condition (1.7) is replaced by the mixed boundary condition

$$\begin{cases} v_{|\Sigma_1} = 0 \,, \\ (v \cdot n)_{|\Sigma_2} = 0, \\ \beta\, v_\tau + \underline{\tau}(v)_{|\Sigma_2} = 0, \end{cases} \tag{1.8}$$

where $\Sigma_1 = \Gamma_1 \times \mathbb{R}$, $\Sigma_2 = \Gamma_2 \times \mathbb{R}$.

As in reference [4] we assume here that

$$|\Omega| = 1 \,, \quad \text{and that} \quad T = 2\,\pi \,, \tag{1.9}$$

where $|\Omega|$ denotes the Lebesgue measure of Ω. In order to avoid misunderstandings between z and t, we denote by $\mathbb{R}_t$ the real line $\mathbb{R}$ when referred to the time variable t.

One has the following preliminary result.

Lemma 1.1. *Let $g(t)$, $t \in \mathbb{R}_t$, be a given real $2\,\pi$-periodic function. A $2\,\pi$-periodic solution v of the Navier-Stokes equations* (1.1) *in the infinite space-cylinder Λ, v of the form*

$$v(t,x,z) = (0, ..., 0, \chi(t,x)) \tag{1.10}$$

and satisfying the boundary condition (1.7) *and the flux constraint* (1.2)*, exists if and only if χ is a solution to the problem*

$$\begin{cases} \frac{\partial\,\chi}{\partial\,t} - \nu\,\Delta\,\chi + \nu\,\int_\Omega \Delta\,\chi\,dx = g'(t)\,, \quad \textit{in} \quad \Omega \times \mathbb{R}_t\,, \\ \chi(t+T) = \chi(t) \quad \forall t \in \mathbb{R}_t\,, \end{cases} \tag{1.11}$$

which satisfies the boundary condition

$$\beta\,\chi_{|\Gamma} + \nu\,\frac{\partial \chi}{\partial n}_{|\Gamma} = 0, \tag{1.12}$$

and the flux constraint

$$\int_\Omega \chi(x,t)\,dx = g(t)\,. \tag{1.13}$$

The above result still holds if we replace (1.7) *by* (1.8) *and* (1.12) *by*

$$\begin{cases} \chi|_{\Gamma_1} = 0\,, \\ \beta\,\chi_{|\Gamma_2} + \nu\,\frac{\partial \chi}{\partial n}_{|\Gamma_2} = 0\,. \end{cases} \tag{1.14}$$

The reduction of the original problems to the above simplified versions for velocity fields having the form (1.10) is proved here just as was done in section 2 of reference [4] for the boundary condition (1.3).

One has the following existence and uniqueness result. For notation see the next section. The symbol # denotes 2π-periodicity.

Theorem 1.2. *Let Ω be an open, bounded and connected, set in $\mathbb{R}^n$ and consider the infinite cylinder $\Lambda = \Omega \times \mathbb{R}$. Let $g \in H^1_{\#}(\mathbb{R}_t)$ be given. There is a unique solution v of the Navier-Stokes equations* (1.1) *in Λ which, for each t, satisfies one of the the boundary conditions* (1.7) *or* (1.8), *and such that:*

v is (2π)-time periodic.

v has the form (1.10), *namely*

$$v(t,x,z) = (0, ..., 0, \chi(t,x))\,.$$

The total flux satisfies (1.13).

Moreover χ satisfies the estimates

$$\|\Delta\,\chi\|^2_{L^2_{\#}(\mathbb{R}_t;H)} \leq c\,\|g\|^2_{L^2_{\#}(\mathbb{R}_t)} + \frac{c}{\nu^2}\,\|g'\|^2_{L^2_{\#}(\mathbb{R}_t)}\,, \tag{1.15}$$

$$\|\chi'\|^2_{L^2_{\#}(\mathbb{R}_t;H)} \leq c\,\nu^2\,\|g\|^2_{L^2_{\#}(\mathbb{R}_t)} + c\,\|g'\|^2_{L^2_{\#}(\mathbb{R}_t)}\,, \tag{1.16}$$

and

$$\|\chi\|^2_{C_{\#}(\mathbb{R}_t;V)} \leq c\,\nu\,\|g\|^2_{L^2_{\#}(\mathbb{R}_t)} + \frac{c}{\nu}\,\|g'\|^2_{L^2_{\#}(\mathbb{R}_t)}\,. \tag{1.17}$$

As shown in the sequel this result follows as a corollary of Theorem 2 (see Theorem 2.1 below) in reference [4].

Remark 1.1. *If $\beta = 0$ the resolution of the problem* (1.1), (1.2), (1.7) *is trivial since the integral on the right hand side of the first equation* (1.11) *vanishes identically. For this reason, in considering the boundary condition* (1.7), *we will assume that $\beta > 0$.*

Under the Navier boundary condition (1.7), if Ω is locally situated on one side of Γ and if Γ is a differentiable manifold of class $C^{1,1}$, well known elliptic regularity results show that

$$\|\Delta\,\chi\|^2_{L^2_{\#}(\mathbb{R}_t;H)} \simeq \|\chi\|^2_{L^2_{\#}(\mathbb{R}_t;H^1_0\cap H^2)}\,.$$

Moreover, regularity results for the heat equation yield regularity results for χ and v, depending on the regularity of Γ and $g(t)$. In particular, if Γ and $g(t)$ are infinitely differentiable, so is v in $\overline{\Lambda} \times \mathbb{R}$.

Finally we note that, by doing the change of variables $\tau = \frac{2\pi}{T}\,t$, it readily follows that all the results proved below still hold if we replace the period 2π by an arbitrary period T. It is worth noting that the estimates (2.4), (2.5) and (2.6) (hence (1.15), (1.16) and (1.17)) hold with the same constants on the right hand side. In particular these constants do not depend on the period T.

2 The abstract theorem. Proof of theorem 1.2

Problem (1.11), (1.13), under one of the boundary conditions (1.12) or (1.14) can be easily seen, and solved, as a particular case of a more general problem. In the sequel we state an abstract result proved in reference [4] (see sections 3, 4 and 5 of this last reference). Let H and V be real separable Hilbert spaces, with V densely and compactly embedded in H and denote respectively by $(\ ,\ )$ and $\|\ \|$ the scalar product and the norm in H. We identify H with its dual H'. We have $V \subset H \simeq H' \subset V'$ where V' denotes the dual of V. Define an operator A by means of $(Au, v) = a(u, v)$, where a is any symmetric, continuous, bilinear, V-elliptic form over $V \times V$. We take $a(u, v)$ as the scalar product in V, and set $((u, v)) = a(u, v)$. Hence, $(Au, v) = ((u, v))$; moreover

$$(Av, v) = \|v\|_V^2 \,,$$

for each $v \in V$. A is an isomorphism between $D(A)$ and H, where

$$D(A) = \{v \in V : A\,v \in H\}$$

and the norm of an element $v \in D(A)$ is given by $\|A\,v\|$.

Denote by e a fixed element $e \in H$, such that $e \notin V$. Without any loss of generality, we normalize e by assuming that $\|e\| = 1$. Finally, we define $w \in D(A)$ as the unique solution of the equation

$$A\,w = e\,. \tag{2.1}$$

We set

$$C_1^2 = (Aw, w) = \|w\|_V^2 \,,$$

and

$$C_0^2 = \|w\|^2 \,.$$

Consider the following problem:

Problem. Let H, V, A and e be as above and let $g(t)$ be a given real, $2\,\pi$-time periodic function. We look for solutions χ of the linear problem

$$\begin{cases} \chi' + \nu\,A\,\chi - \nu\,(A\,\chi, e)\,e = g'(t)\,e\,, \\ \\ \chi(t+T) = \chi(t)\,, \end{cases} \tag{2.2}$$

such that

$$(\chi(t), e) = g(t)\,. \tag{2.3}$$

We set $L^2_\#(\mathbb{R}_t) = L^2_\#(\mathbb{R}_t; \mathbb{R})$ and $H^1_\#(\mathbb{R}_t) = H^1_\#(\mathbb{R}_t; \mathbb{R})$. Recall that the symbol # means $2\,\pi$-periodicity with respect to time.

One has the following result (Theorem 2 in reference [4]).

Theorem 2.1. *Let $g \in H^1_\#(\mathbb{R}_t)$ and $e \in H$, $\|e\| = 1$ and $e \notin V$, be given. Then there is a unique solution χ to the problem (2.2) such that (2.3) holds. One has $\chi \in L^2_\#(\mathbb{R}_t; D(A)) \cap C_\#(\mathbb{R}_t; V)$, $\chi' \in, L^2_\#(\mathbb{R}_t; H)$ and*

$$\|\chi\|^2_{L^2_\#(\mathbb{R}_t; D(A))} \leq \widetilde{C}_0\,\|g\|^2_{L^2_\#(\mathbb{R}_t)} + \frac{\widetilde{C}}{\nu^2}\,\|g'\|^2_{L^2_\#(\mathbb{R}_t)}\,. \tag{2.4}$$

Moreover,

$$\|\chi'\|^2_{L^2_\#(\mathbb{R}_t;H)} \le 8\,\widetilde{C}_0\,\nu^2\,\|g\|^2_{L^2_\#(\mathbb{R}_t)} + (2+8\,\widetilde{C})\,\|g'\|^2_{L^2_\#(\mathbb{R}_t)}\,, \tag{2.5}$$

where the constants $\widetilde{C}$ and $\widetilde{C}_0$ depend only on C_0 and C_1. In particular,

$$\|\chi\|^2_{C_\#(\mathbb{R}_t;V)} \le c\,\nu\,\|g\|^2_{L^2_\#(\mathbb{R}_t)} + \frac{c}{\nu}\,\|g'\|^2_{L^2_\#(\mathbb{R}_t)}\,. \tag{2.6}$$

For the proof see [4].

The problem (1.11), (1.13), under a large class of boundary conditions, is a particular case of problem (2.2), (2.3) by setting $H = L^2(\Omega)$ and by choosing V and $a(u,v)$ in dependence on the particular boundary condition under consideration. Moreover, we define e as being the constant function $e(x) = 1$, for each $x \in \Omega$.

Let us consider the two cases under consideration here. The proof of Theorem 1.2 follows immediately from Lemma 1.1 and Theorem 2.1. If the boundary condition is (1.7) or, equivalently (after appeal to the above lemma) (1.12), we apply Theorem 2.1 with $V = H^1$ and

$$a(u,v) = \int_\Omega \nabla u\cdot\nabla v\,dx + \beta\int_\Gamma u\,v\,d\Gamma\,. \tag{2.7}$$

If the boundary condition is the mixed boundary condition (1.8) or, equivalently (1.14), we apply the Theorem 2.1 with

$$V = \{v : v = 0 \quad \text{on}\,\Gamma_1\}$$

and

$$a(u,v) = \int_\Omega \nabla u\cdot\nabla v\,dx + \beta\int_{\Gamma_1} u\,v\,d\Gamma\,. \tag{2.8}$$

Note that in all the above cases $A = -\Delta$ with domain $D(A) = \{v \in V : A\,v \in H\}$. We may extend the above types of results to other operators and boundary conditions (as well as to suitable constraints depending on the choice of e). In particular, see [4], if the boundary condition is (1.3) we set $V = H^1_0(\Omega)$ and

$$a(u,v) = \int_\Omega \nabla u\cdot\nabla v\,dx\,.$$

As remarked in section 1, under the slip boundary condition without friction (i.e.,$\beta = 0$) the problem becomes trivial and appeal to theorem 2.1 is superfluous (actually, this theorem is not directly applicable since the equation (2.1) has no solution).

Finally, note that under the slip boundary condition, if Ω is of class $C^{1,1}$, or convex, one has

$$D(A) = H^2(\Omega)\cap H^1_0(\Omega)\,.$$

References

[1] C.J. Amick, *Steady solutions of the Navier-Stokes equations in unbounded channels and pipes*, Ann. Sc. Norm. Sup. Pisa, **4** (1977), 473-513.

[2] H. Beirão da Veiga, *Regularity for Stokes and generalized Stokes systems under nonhomogeneous slip-tipe boundary conditions*, Advances Diff. Eq., **9** (2004), 1079-1114.

[3] H. Beirão da Veiga, *Regularity of solutions to a nonhomogeneous boundary value problem for general Stokes systems in* $\mathbb{R}^n_+$, Math. Annalen, **328** (2004), 173-192.

[4] H. Beirão da Veiga, *On the time-periodic solutions of the Navier-Stokes equations in unbouded domains. Leray's problem*, Archive Rat. Mech. Analysis, **178** (2005), 301-325.

[5] L.C. Berselli, and M. Romito, *On the existence and uniquiness of weak solutions for a vorticity seeding model*, SIAM J. Math. Analysis, in press.

[6] C. Conca, *On the application of the homogenization theory to a class of problems arising in fluid mechanics*, J. Math. Pures Appl., **64** (1985), 31-75.

[7] G.P. Galdi, W. Layton, *Approximation of the larger eddies in fluid motion: A model for space filtered flow*, Math. Models and Meth. in Appl. Sciences, **3** (2000), 343-350.

[8] G.P. Galdi, A.M. Robertson, *The relation between flow rate and axial pressure gradient for time-periodic Poiseuille flow in a pipe*, J.Math. Fluid Mechanics, **7** suppl.2 (2005), 215-223.

[9] V. John, *Slip with friction and penetration with resistence boundary conditions for the Navier-Stokes equations-numerical tests and aspects of the implementations*, J. Comp. Appl. Math., **147** (2002), no. 2, 287-300.

[10] V. John, *Large eddy simulation of turbulent incompressible flows. Analytical and numerical results for a class of LES models*, Habilitationsschrift, Otto-von-Guericke-Universität Magdeburg, April 2002.

[11] O.A. Ladyzhenskaya and V.A. Solonnikov, *On the solvability of boundary value and initial-boundary value problems for the Navier-Stokes equations in regions with noncompact boundaries*, English Trans., Vestnik Leningrad Univ. Math., **10** (1977), 271-279.

[12] C.L.M.H. Navier, *Memoire sur les lois du mouvement des fluides*, Mem. Acad. Sci. Ist. de France (2), **6** (1823), 389-440.

[13] K. Pileckas, *Existence of solutions with the prescribed flux of the Navier-Stokes system in an infinite cylinder*, preprint.

[14] A. Liakos, *Discretization of the Navier-Stokes equations with slip boundary condition*, Num. Meth. for Partial Diff. Eq.,**1** (2001), 1-18.

[15] C. Pare's, *Existence, uniquiness and regularity of solutions of the equations of a turbulence model for incompressible fluids*, Appl. Analysis, **43** (1992), 245-296.

[16] J. Serrin, *Mathematical Principles of Classical Fluid Mechanics*, in *Encyclopedia of Physics* VIII, p.p. 125-263, Springer-Verlag, Berlin, 1959.

[17] V.A. Solonnikov and V.E. Ščadilov, *On a boundary value problem for a stationary system of Navier-Stokes equations*, Proc. Steklov Inst. Math.,**125** (1973), 186-199.

[18] R. Verfűrth, *Finite element approximation of incompressible Navier-Stokes equations with slip boundary conditions*, Numer. Math., **50** (1987), 697-721.

[19] J.R. Womersley, *Method for the calculation of velocity, rate of flow and viscous drag in arteries when the pressure gradient is known*, J. Physiol., **127** (1955), 553-56.