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Global Inequalities for Curves and Surfaces in Three-Space (*).

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Summary. – *The aim of the paper is to give upper bounds for the total curvature of smooth curves and surfaces embedded in euclidean space, in terms of other global geometric characters; in particular, for a plane curve γ, we prove the inequality $K(\gamma) < \pi(2 + f(\gamma)d(\gamma)/2)$, where $d(\gamma)$ is the geometric degree of γ and $f(\gamma)$ is the number of its inflection points. In the case of a surface S, a bound is given in terms of the genus $g(S)$, the number of components of the parabolic points on S and the geometry of its apparent contour.*

0. – Introduction.

This paper consists of two parts. The first one is an improvement of the result obtained in [BD]: let $\gamma \subseteq \boldsymbol{R}^3$ be a smooth or simplicial closed simple curve embedded in $\boldsymbol{R}^3$; the *geometric degree* $d(\gamma)$ is defined as

$$d = d(\gamma) = \sup \# \{E \cap \gamma\}$$

where E varies among all the hyperplanes transverse to γ.

If γ is a plane curve, then d is defined in the same way with respect to the transverse lines in $\boldsymbol{R}^2$. Under mild hypotheses of «genericity», (in particular if γ has a finite number $f = f(\gamma)$ of inflection points, or inflection sides in the simplicial case) we have proved in [BD] that, if γ is a simple connected plane curve with $f > 0$, then

$$K < \pi f(1 + d/2) \tag{0.1}$$

where $K = K(\gamma)$ denotes the *total curvature* of γ.

The main result of part I is the following stricter inequality which holds under the same hypothesis of (0.1):

$$K < \pi(2 + fd/2)\,. \tag{0.2}$$

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We shall see that even (0.2) is far to be sharp for general f and d. For instance, we shall show that, if $d = 4$, then $K < \pi(4 + f)$ (see remark 2 of [BD]).

Nevertheless, it is not so easy to pass from (0.1) to (0.2): the main work consists in introducing a quite subtle notion of *depth* of an inflection point (side) of γ (which gives a partial ordering on the set $\mathcal{F}$ of the inflection points); this notion of depth may appear quite complicated: however, we shall show that other ways (at first sight more «natural») to give a partial ordering on $\mathcal{F}$ in fact do not work.

Part I is organized as follows: in § 1 we define and discuss the notion of depth; in § 2 we prove the inequality (0.2); in § 3 we extend it to a plane curve with cusps and normal crossing points; in § 4 we briefly discuss the case of a curve in $\boldsymbol{R}^3$.

In part II, we consider the analogous problem about surfaces in $\boldsymbol{R}^3$; more precisely, let S be a closed connected smooth surface embedded in $\boldsymbol{R}^3$ and $K = K(S)$ be the *total curvature* of S, that is

$$K(S) = \int_S |k(p)|\, dA ,$$

where $k(p)$ denotes the Gaussian curvature of S at the point p and dA is the area element. We shall be interested to bound $K(S)$. Two numbers which «naturally» generalize the geometric degree $d(\gamma)$ of a curve γ are

$$d_1 = d_1(S) = \sup_L \# \{L \cap S\} \qquad \text{and } d_2 = d_2(S) = \sup_H b_0\{H \cap S\} ,$$

where the sup is taken among all the affine lines L (or affine hyperplanes H) which are transverse to S, and $b_0(X)$ denotes the number of connected components of X. As for $f(\gamma)$, a number which is a «natural» analogous in the case of surfaces is

$$F = F(S) = b_0\{\text{parabolic points on } S\} .$$

One could hope then to prove an inequality bounding $K(S)$ in terms of $F(S)$, $d_1(S)$ and/or $d_2(S)$, and, of course, the *genus* $g(S)$ of S. However, this does not work, and one needs other geometric characters of S (less «natural»).

In § 5 we define these characters $a_i(S)$, and we briefly discuss the genericity assumptions under which they are well-defined positive integers; in § 6 we prove some inequalities of the kind

$$K(S) \leqslant f(a_i(S)) \tag{0.3}$$

(see theorems 6.3, 6.5 and 6.10); in § 7 we show, by means of some families of examples, that these geometric characters are actually necessary in order to give a bound to $K(S)$ and there is no hope to get a simpler inequality in terms of g, F, d_1, d_2 only.

It should be noted that, even if the two parts of the paper are strictly related, in fact they can be read quite independently: what one needs, in order to read part II, is only theorem 3.2, which extends (0.2) to the case of a curve with cusps and normal crossings. The reader mainly interested to the case of surfaces can simply assume 3.2 and skip directly to part II; alternatively, he can refer to [BD] to find a considerably simpler proof of the weaker inequality (0.1) and then apply § 3 of part I to (0.1), thus getting to weaker (by qualitatively analogous) inequalities.

The paper is intended to be elementary as much as possible.

PART I CURVES

1. – Depth.

Through all this paragraph γ will denote a simple connected plane smooth curve; we also assume that γ satisfies the following.

1.1 GENERICITY ASSUMPTION. – γ has a finite number of inflection points and no tritangents.

In the spirit of the Fabricius-Bjerre formula (see [FB1] and [B]) we distinguish between *exterior bitangents* and *interior bitangents* to γ, according to figure 1, and we denote by E (resp. I) the set of exterior (resp. interior) bitangents.

Figure 1

Fix an orientation on γ; we can then further distinguish, according to figure 2, between exterior bitangents of the first and of the second kind (the set of which will be denoted, respectively, by E' and E'').

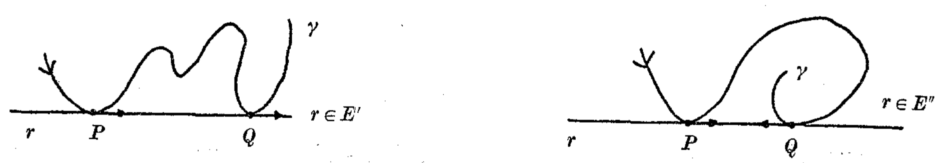

Figure 2

Let r be a bitangent, tangent to γ in points P and Q; we can choose points P' and P'' in γ, close enough to P, in such a way that $P'P''$ is a convex arc and the orthogonal projection of P' (resp. P'') is contained in the interior (resp. in the exterior) of the segment PQ; we can similarly choose points Q' and Q'' close to Q in γ (see figure 3).

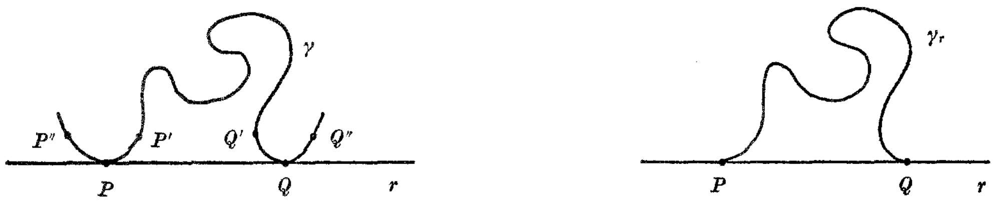

Figure 3

1.2 DEFINITION. – Let $r \in E'$ and P, Q, P', Q' be as above; the *arc associated to* r *(denoted by* γ_r*) is the connected component of* $\gamma \setminus \{P, Q\}$ which contains the points P' and Q'.

Let us denote by $\hat{\alpha}$ the convex hull of α and by $\tilde{\alpha}$ the boundary of $\hat{\alpha}$, where α may be either a closed curve or an arc.

1.3 REMARKS. – If γ is a simple curve, then:

a) $\tilde{\gamma} \cap \gamma$ is the disjoint union of convex subarcs of γ;

b) $\tilde{\gamma} \setminus \gamma$ is the disjoint union of segments (they are disjoint, otherwise γ would have tritangents); each one of these segments is contained in a bitangent r belonging to E'. Let us call E_1 the subset of E' consisting of these bitangents; note that $r \in E_1$ if and only if $r \cap \hat{\gamma} \subset \tilde{\gamma}$.

c) $\gamma \setminus \tilde{\gamma}$ is the disjoint union of subarcs of γ, contained in the interior of $\hat{\gamma}$; let us call C_1 the set of these arcs; note that each α in C_1 is an arc γ_r associated to a bitangent $r \in E_1$.

d) Each $\gamma_r \in C_1$ contains a positive even number of inflection points. In fact, let $\tau\colon \gamma \to S^1$ be the «tangential Gauss map» and P, Q, P', Q' be as in definition 1.2: then, as $r \in E_1$, $\tau(P) = \tau(Q)$ and the arcs $\tau(P)\tau(P')$ and $\tau(Q)\tau(Q')$ are coherently oriented in S^1; thus, the number of points $\tau(X)$ where the orientation of S^1 is reversed must be even. Note that this number cannot be zero, otherwise PQ would not be in $\tilde{\gamma}$.

1.4 DEFINITION. – We say that a bitangent r is a *bitangent of depth* 1 if $r \in E_1$.

1.5 A SPECIAL CASE. – Assume that, for each $r \in E_1$, γ_r contains exactly two inflection points; set $f = \# \mathcal{F}$, $e = \# E$, $i = \# I$, $e_1 = \# E_1$ and note that (see figure 4):

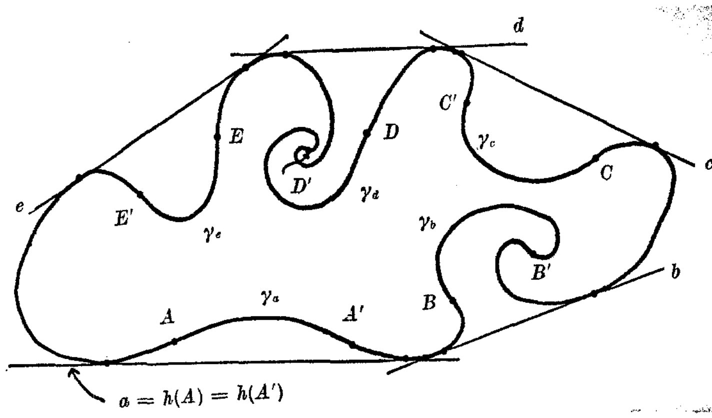

Figure 4

a) there is a natural $2-1$ map $h\colon \mathcal{F} \to E_1$ such that $h^{-1}(r)$ consists of the two inflections points in γ_r;

b) in particular, $f/2 = e_1$; as the Fabricius-Bjerre formula in this case reduces to $e = f/2 + i$, we also have $e - i = e_1$;

c) let γ' be the curve obtained from γ by replacing each of the arcs γ_r with the segment having the same endpoints: γ' is a convex (C^1, piecewise C^∞) curve and

$$K(\gamma) = K(\gamma') + \sum_{r \in E} K(\gamma_r) = 2\pi + \sum_{r \in E_1} K(\gamma_r)\,.$$

($K(\alpha)$ denotes here the total curvature of the arc α).

We want to generalize the above situation to any curve γ; that is, we want to prove the following:

1.6 PROPOSITION. – *For any γ one can choose*:

1) *a subset $\tilde{E} \subseteq E'$*;

2) *subsets $E_i \subseteq \tilde{E}$ ($r \in E_i$ will be called a* bitangent of depth *i*);

3) *subsets $F_i \subseteq \mathcal{F}$ ($P \in F_i$ will be called an* inflection point of depth *i*);

4) *two families of arcs C_i and C'_i*;

such that:

a) $\mathcal{F} = F_1 \sqcup \ldots \sqcup F_k$ *and* $\tilde{E} = E_1 \sqcup \ldots \sqcup E_k$;

b) *there exists a natural* $2-1$ map $h_i\colon F_i \to E_i$; *in particular (setting, as usual, $\tilde{e} = \# \tilde{E}$, $e_i = \# E_i$, $f_i = \# F_i$) we have that $e_i = f_i/2$ and $f/2 = \tilde{e} = e - i$*;

c) C_i *is the family of arcs* γ_r *associated to* $r \in E_i$; *for each* γ_r *we can construct an* auxiliary arc γ_r' *such that* γ_r' *contains exactly the two inflection points of* γ *which are in* $h_i^{-1}(r)$; C_i' *will be the family of such auxiliary arcs*;

d) $K(\gamma) = 2\pi + \sum_{r \in \tilde{E}} K(\gamma_r')$;

e) if, for each $r \in E_i$, γ_r *contains exactly two inflection points, then* $C_i' = C_i$ *and* $F_{i+1} = E_{i+1} = \emptyset$.

Note: 1.5 *above corresponds to the case* $k = 1$.

PROOF. – The proof consists in constructing inductively the families F_i, C_i', E_{i+1} and C_{i+1}, starting from E_i and C_i, provided they satisfy

$$(1.7) \qquad a)\ E_i \subseteq E'; \qquad b)\ C_i = \{\gamma_r : r \in E_i\}; \qquad c)\ r \cap \hat{\gamma}_r \subset \tilde{\gamma}_r, \quad \forall r \in E_i .$$

We give the details for $i = 1$: the construction will be the same in the general inductive step.

Note that, by remarks 1.3, E_1 and C_1 satisfy (1.7) above.

Fix $r \in E_1$ and let P_r and Q_r be the endpoints of the associated arc γ_r; we want to choose

1) two elements of F_1 which will be the elements of $h_1^{-1}(r)$ (and will be called *privileged* for r);

2) two subsets $E_{2,r} \subseteq E_2$ and $C_{2,r} \subseteq C_2$ satisfying (1.7) above;

3) one element γ_r' of C_1' such that γ_r' contains the two privileged inflection points for r and $K(\gamma_r) = K(\gamma_r') + \sum_{s \in E_{2,r}} K(\gamma_s)$.

If γ_r contains exactly two inflections points A_r and B_r, then $h_1^{-1}(r) = \{A_r, B_r\}$, $E_{2,r} = C_{2,r} = \emptyset$ and $\gamma_r' = \gamma_r$.

Otherwise, consider $\hat{\gamma}_r$ and $\tilde{\gamma}_r$: the situation is similar to the one we discussed in 1.3 for $\hat{\gamma}$ and $\tilde{\gamma}$, with the only difference that γ_r is now an arc instead of a closed curve; however, recalling that $r \in E'$, we can say that:

a) $\tilde{\gamma}_r \cap \gamma_r$ is the disjoint union of convex subarcs of γ_r;

b) $\tilde{\gamma}_r \setminus \gamma_r$ is the disjoint union of a simplicial arc $P_r^{(1)} P_r Q_r Q_r^{(1)}$ and of some segments; each one of these segments is contained in a bitangent s belonging to E'; while the segment $P_r P_r^{(1)}$ (resp. $Q_r Q_r^{(1)}$) is contained in a tangent through P_r (resp. Q_r) to γ_r at $P_r^{(1)}$ (resp. $Q_r^{(1)}$) (see figure 5);

c) $\gamma_r \setminus \tilde{\gamma}_r$ is the disjoint union of the two arcs α_r and β_r with endpoints P_r, $P_r^{(1)}$ and Q_r, $Q_r^{(1)}$ and of some other subarcs of γ_r: each one of these other subarcs is the arc γ_s associated to one of the exterior bitangents s found in *b)* above;

d) each γ_s contains a positive even number of inflection points and is such that $s \cap \hat{\gamma}_s \subset \tilde{\gamma}_s$; on the other hand, both α_r and β_r contain an odd number of inflection points.

This can be proved as in remark 1.3 *d*): note that the situation at the endpoints of the arc α_r looks like in figure 6.

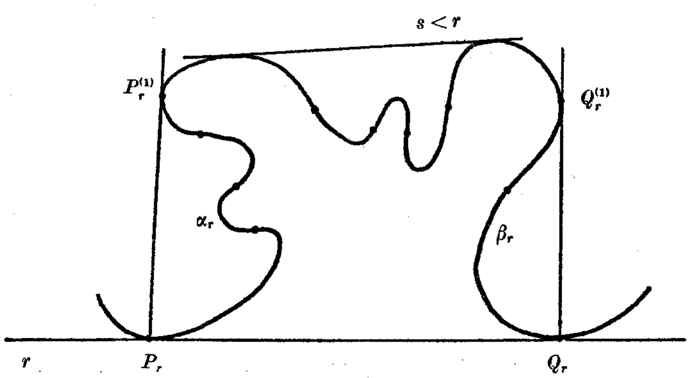

Figure 5

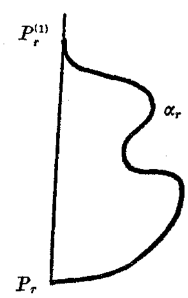

Figure 6

We say that the bitangents s found as above are *derived from* r (and we write $s < r$), and the arcs γ_s found as above are *derived from* γ_r (and we write $\gamma_s < \gamma_r$).

If now both α_r and β_r contain exactly one inflection point (let them be A_r and B_r respectively), then we can finish the construction as follows:

1) $h_1^{-1}(r) = \{A_r, B_r\} \subseteq F_1$;

2) γ_r' is the arc obtained from γ_r by replacing γ_s with the segment having the same endpoints, for each $\gamma_s < \gamma_r$;

3) $E_{2,r} = \{s: s < r\}$ and $C_{2,r} = \{\gamma_s: \gamma_s < \gamma_r\}$.

Otherwise, we need to iterate this construction on the arcs α_r (and/or β_r) in order to choose on each one of them one «privileged» inflection point and «cut off» the remaining ones by pairs, by means of exterior bitangents which will be further elements of $E_{2,r}$.

The only difference between the arc α_r (or β_r) and the arc γ_r is the situation at the endpoints, which looks as in figure 7. Thus:

a) $\hat{\alpha}_r \cap \alpha_r$ is the disjoint union of convex subarcs of γ_r;

b) $\tilde{\alpha}_r \setminus \alpha_r$ is the disjoint union of a simplicial arc $P_r P_r^{(1)} P_r^{(2)}$ and of some segments; each one of these segments is contained in a bitangent s belonging to E', while the segment $P_r^{(1)} P_r^{(2)}$ is contained in a tangent through $P_r^{(1)}$ to α_r at $P_r^{(2)}$; again we shall say that s is derived from r (and we write $s < r$);

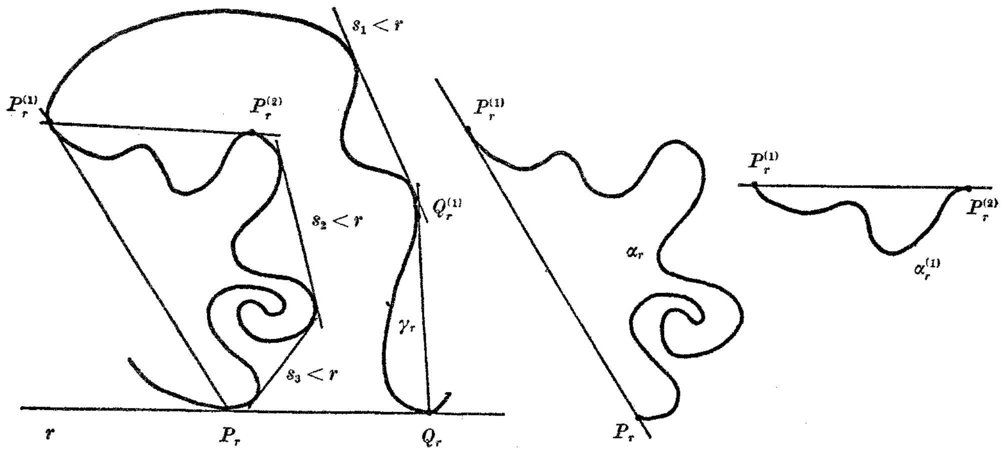

Figure 7

c) $\alpha_r \setminus \tilde{\alpha}_r$ is the disjoint union of the subarc $\alpha_r^{(1)}$ with endpoints $P_r^{(1)}$ and $P_r^{(2)}$ and of some other arcs: each one of these other arcs is the arc γ_s associated to one of the bitangents s found in *b*) above;

d) each one of the arcs γ_s contains an even number of inflection points and is such that $s \cap \hat{\gamma}_s \subset \tilde{\gamma}_s$; while $\alpha_r^{(1)}$ contains an odd number of inflection points;

e) the arc $\alpha_r^{(1)}$ at its endpoints looks like the arc α_r, so that we can (eventually) iterate the construction without any further difference (see figure 7).

Since there is only a finite number of inflection points, the construction ends in a finite number of steps.

Set $\alpha_r^{(0)} = \alpha_r$, $\beta_r^{(0)} = \beta_r$ and let n (resp. m) be the smallest integer $n \geqslant 0$ such that $\alpha_r^{(n)}$ (resp. $\beta_r^{(m)}$) contains exactly one inflection point A_r (resp. B_r); then:

1) the privileged inflection points for r are $h_1^{-1}(r) = \{A_r, B_r\} \subseteq F_1$;

2) $E_{2,r} = \{s : s < r\}$ is the set of all the bitangents derived from r found during the construction;

3) $C_{2,r} = \{\gamma_s : \gamma_s < \gamma_r\}$ is the set of all the arcs derived from γ_r found during the construction;

4) the auxiliary arc $\gamma_r' \in C_1'$ is the arc obtained from γ_r by replacing each subarc $\gamma_s < \gamma_r$ with the segment having the same endpoints.

Note that γ_r' contains exactly two inflection points, that is A_r and B_r; moreover, for each $s \in E_{2,r}$, one has $s \cap \hat{\gamma}_s \subset \tilde{\gamma}_s$.

Finally, we have:

$$F_1 = \{P \in h_1^{-1}(r), \text{ for some } r \in E_1\};$$

$$E_2 = \{s \in E_{2,r}, \text{ for some } r \in E_1\};$$

$$C_2 = \{\alpha \in C_{2,r}, \text{ for some } r \in E_1\};$$

$$C_1' = \{\gamma_r', \text{ for some } r \in E_1\}.$$

In the example shown in figure 8 we have that r (resp. γ_r) is the only element of E_1 (resp. of C_1), $F_1 = \{A_r, B_r\}$, $E_2 = \{a, b, c, d\}$, $C_2 = \{\gamma_a, \gamma_b, \gamma_c, \gamma_d\}$ and C_1' consists of the curve γ_r' in figure 9.

Note that $K(\gamma) = 2\pi + \sum_{r \in E_1} K(\gamma_r') + \sum_{s<r} K(\gamma_s)$; moreover, it is clear that E_2 and C_2 satisfy (1.7), so that we can iterate the construction. As the number of inflection points is finite, there must exist a $h \geqslant 0$ such that $F_h = \emptyset$, that is the construction ends in a finite number of steps: we define the *depth of* γ (written dep (γ)) as the biggest integer n such that $F_n \neq \emptyset$; if dep $(\gamma) = n$, then $\mathcal{F} = F_1 \sqcup \ldots \sqcup F_n$; set $\tilde{E} = E_1 \sqcup \ldots \sqcup E_n$, and note that the properties stated are satisfied. In particular, the identity (d) on $K(\gamma)$ can easily be proved by induction on dep (γ). $\square$

1.8 REMARKS. – *a*) The above construction gives a partial ordering on the set $\tilde{E}$ (or, equivalently, on the set of couples $h^{-1}(r)$ in $\mathcal{F}$); the associated graph is the disjoint union of e_1 trees: in fact, for each $s \in E_j$ $(j \geqslant 2)$ there exists a unique $r \in E_{j-1}$ such that $s < r$.

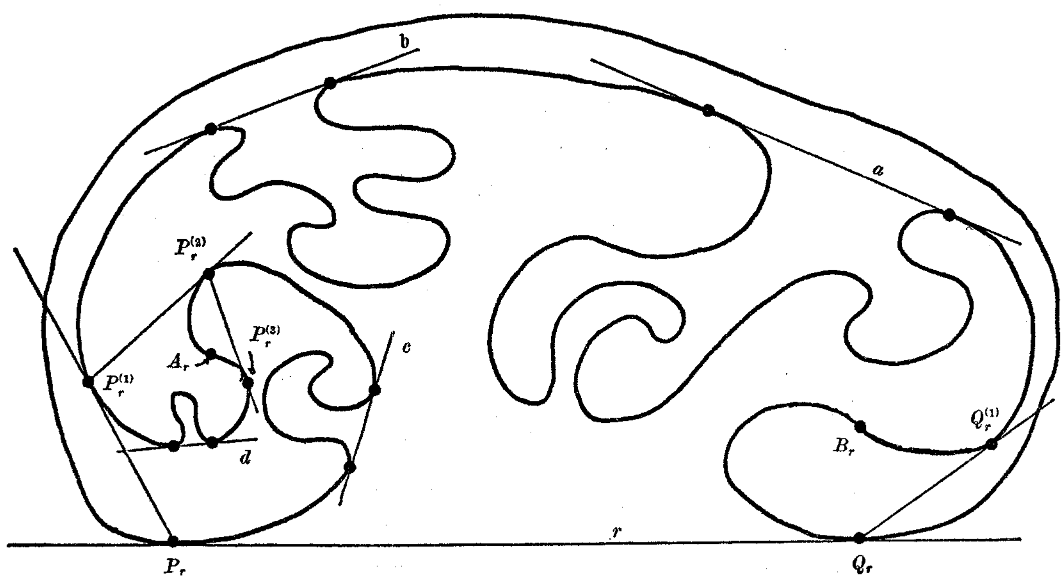

Figure 8

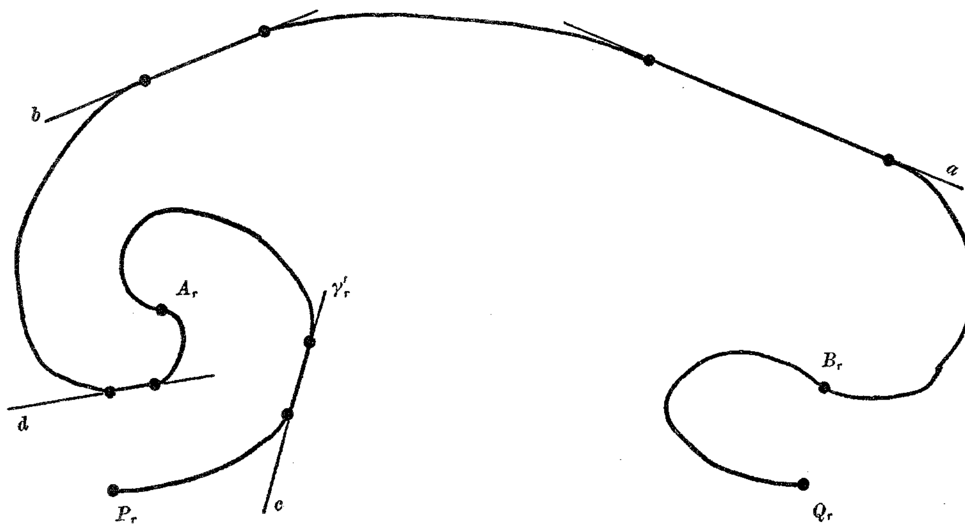

Figure 9

b) The definition of depth may seem very complicated and one could hope to find a simpler way to get the same notion (that is, families satisfying 1.6); for example, a more « natural » way to associate to an exterior bitangent of E' a couple of inflection points of γ could be to consider the first and the last one on the arc associated to r (with respect to a fixed orientation on γ); however, by examining the following figure 10, the reader can convince himself that this way does not work: note that there does not exist an exterior bitangent s to γ such that γ_s contains the inflection points B and C; following 1.6, instead, we have $F_1 = \{A, B\}$, $F_2 = \{C, D\}$ and dep $(\gamma) = 2$.

1.9. EXTENSION TO THE SIMPLICIAL CASE. – All the above discussion applies, with slight changes, to a simplicial curve (see also [BD]); for such a curve γ, simple closed and connected, $f = f(\gamma)$ is the number of the *inflection sides* of γ; note that, in this case, a line r can be « tangent » to γ either at a vertex or at a side; the same applies to bitangents (see figure 11). The analogous of (1.1) is in this case:

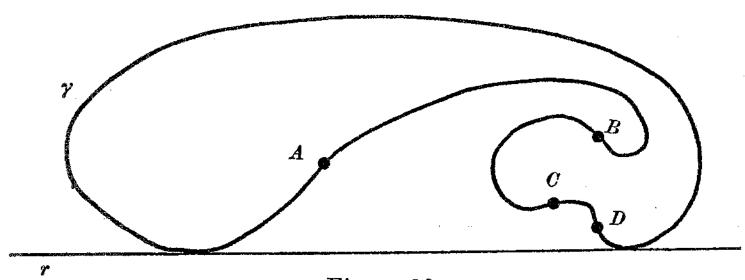

Figure 10

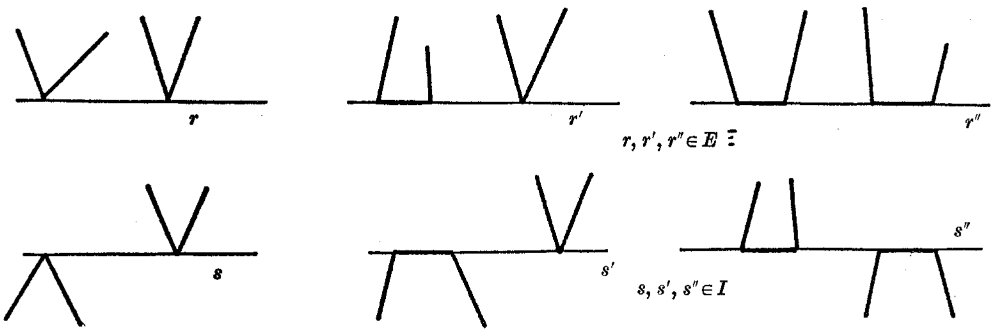

Figure 11

1.10 SIMPLICIAL GENERICITY ASSUMPTION. – No two inflection sides of γ are on the same line; γ has no tritangents; γ has no tangent line at an inflection side.

With these changes, the construction works as in the proof of 1.6.

1.11. RELATIONSHIP BETWEEN dep (γ) AND $d(\gamma)$. – Let γ be either smooth or simplicial; $d(\gamma)$ denotes the geometric degree of γ (as defined in the introduction). Note that:

a) $d(\gamma) = 2 \Leftrightarrow \gamma$ is a convex curve $\Leftrightarrow$ dep $(\gamma) = 0$.

b) If $r \in E_k$ and $k \geqslant 2$, then r is tangent to γ at two points which lie in the interior of $\hat{\gamma}$; thus, there exists a line r' (parallel and closed to r, transverse to γ) such that $\# \{r' \cap \gamma\} \geqslant 6$. This proves that $d(\gamma) = 4 \Rightarrow$ dep $(\gamma) \leqslant 1$.

c) The curve in figure 12 has degree 6 and depth 3.

This suggests the following problem, which, in spite of the elementarity of its formulation, seems us to be quite subtle:

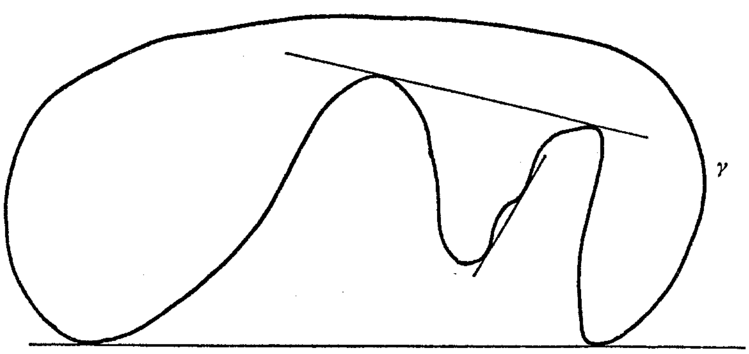

Figure 12

1.12 PROBLEM. – Does there exist a function h, depending only on $d = d(\gamma)$, such that, for every γ as before, $\operatorname{dep}(\gamma) \leqslant h(d(\gamma))$?

1.13 EXERCISE. – The simplest non trivial case to consider is, probably, the one of the simplicial curves γ constructed inductively as follows: γ_0 is a convex polygon; the convex hull Γ_i of $\gamma_{i+1} \setminus \gamma_i$ intersects γ_i along one side s_i of γ_i; $\overline{\gamma_{i+1} \setminus \gamma_i} = \overline{\Gamma_i \setminus s_i}$ (see figure 13). Let $s_\gamma = \#\{s_i : s_i \text{ is not an inflection side of } \gamma_i\}$.

Then:

a) (easy) $\operatorname{dep}(\gamma) = s_\gamma$;

b) (we do not know the solution) find (if any) an inequality of the kind $s_\gamma \leqslant h(d(\gamma))$.

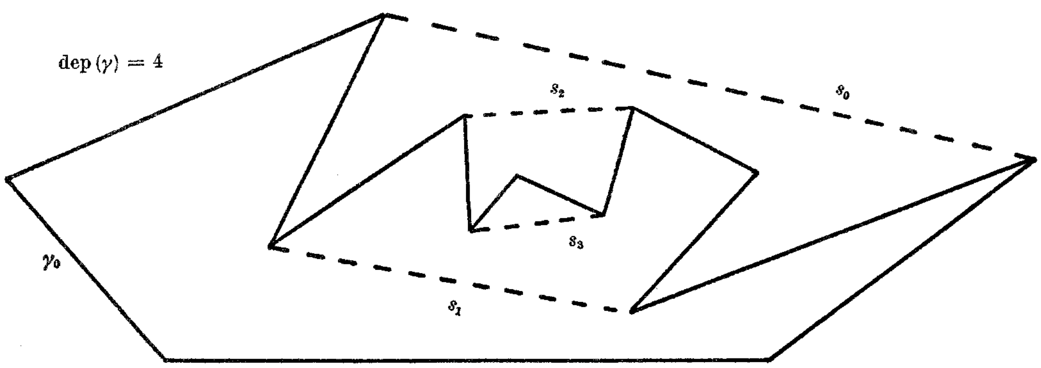

Figure 13

2. – The geometric inequality.

2.1 THEOREM. *Let γ be a simple connected plane curve with $f = f(\gamma) > 0$; then $K < \pi(2 + fd/2)$.*

PROOF. – Recall that $K(\gamma) = 2\pi + \sum_{r \in \tilde{E}} K(\gamma'_r)$ (see 1.6 *d*)).

Since $\tilde{e} = f/2$, it is enough to show that, $\forall r \in \tilde{E}$, $K(\gamma'_r) < \pi d$. Let us fix $r \in \tilde{E}$ and let γ'_r be the auxiliary arc containing the two inflection points A and B of $h^{-1}(r)$; let a (resp. b) be the tangent line to γ'_r at A (resp. B) and s be the line through A and B (see figure 14); assume that s is transverse to γ'_r. Clearly

$$n = \#\{s \cap \gamma'_r\} \leqslant d - 2 .$$

Moreover, if α, β, α' and β' are the angles marked in figure 16, then $\alpha + \beta = \alpha' + \beta'$. We can compute $K(\gamma'_r)$ as the sum of the total curvatures of the convex arcs between

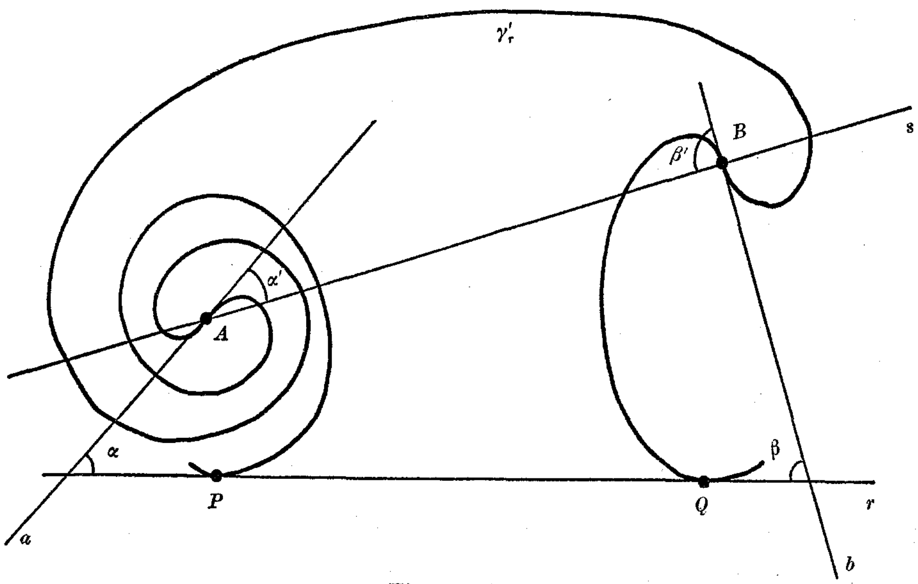

Figure 14

two consecutive (along γ_r') points of $\gamma_r' \cap s$, thus obtaining:

$$K(\gamma_r') = 2(\alpha + \beta) + (n-2)\pi < 4\pi + (n-2)\pi \leqslant d\pi .$$

If s is not transverse to γ, then the same argument given in [BD] (pag. 113) shows that the same inequality holds a fortiori. The theorem is proved. □

2.2. REMARKS. – *a*) Let L be a line and p_L the orthogonal projection onto L; we say that L is generic for γ if $p_L|\gamma$ is a Morse function; for such an L, set $\mu(L) = \#\{\text{critical points of } p_L|\gamma\}$ and denote by sb the *superbridge index* of γ (so called in analogy with knot theory: see [K]), that is $\mathrm{sb} = \mathrm{sb}(\gamma) = \sup_L \mu(L)$.

Recall that the total curvature of γ may be interpreted as the *mean* or *expectation value* of $\mu(L)$, that is $K(\gamma) = 1/2\int_{S^1} \mu(L)$ (where the lines L are intended here to be oriented); thus we obviously have $K(\gamma) \leqslant \pi\, \mathrm{sb}(\gamma)$.

b) It is not difficult to see that the proof of 2.1 gives in fact

$$\mathrm{sb}(\gamma) < 2 + df/2 \tag{2.3}$$

(if $f > 0$; similarly in [BD] we had actually proved that:

$$\mathrm{sb}(\gamma) < f(2 + d/2) \tag{2.4}$$

c) For $f = 2$ and any d the inequality proved in 2.1 is sharp: see [BD], remark 1.

d) The inequality $K(\gamma_r') < \pi d$ is sharp for a single arc $\gamma_r' \in C_1'$: in fact, the same family of examples used in [BD] (figure 1) to show that 2.4 is sharp for $f = 2$ gives, $\forall \varepsilon > 0$, a curve γ and a bitangent $r \in E_1$ such that $K(\gamma_r) = \pi d - \varepsilon$.

e) For $f > 2$, the inequality in 2.1 is NOT sharp (see, for instance, the following proposition 2.5); the main reason of this fact is that, as we noted in remark *b*) above, we actually proved (2.4) and derived from it the inequality in 2.1.

Other reasons are that: i) it could be possible to get for $K(\gamma_r')$ a stricter bound, when r is a bitangent of depth greater than 1; ii) we took into account each γ_r' one by one and not their reciprocal positions (see also the following proposition).

2.5 PROPOSITION. – *If* $d(\gamma) = 4$, *then* $K < \pi(4 + f)$.

PROOF. – Note that $\operatorname{dep}(\gamma) = 1$ and $\tilde{E} = E = E_1$. For each $r_i \in E_1$ $(i = 1, \dots, m = f/2)$, let α_i, β_i, n_i be defined as in the proof of 2.1; note that $n_i = 2$ for each i; hence:

$$K(\gamma) = 2\pi + \sum_{i=1,\dots,m} K(\gamma_i) = 2\pi + \sum_{i=1,\dots,m} 2(\alpha_i + \beta_i) .$$

Using the fact that $d = 4$, it is not difficult to show that:

$$\alpha_i + \beta_i \geqslant \pi \Rightarrow \alpha_j + \beta_j < \pi \qquad \forall j \neq i .$$

This implies that $K(\gamma) < 2\pi + 4\pi + (m - 1)2\pi = 4\pi + 2m\pi = 4\pi + \pi f$. $\square$

2.6 REMARK. – The inequality of (2.5) is sharp for $f = 2$ and for $f = 4$ (see figure 15); it is likely that it is not sharp for $f \geqslant 6$.

2.7 PROBLEM. – Give a sharp inequality for $K(\gamma)$ in terms of $d(\gamma)$ and $f(\gamma)$.

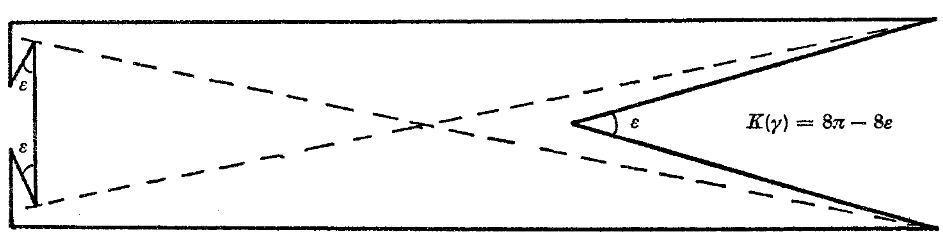

Figure 15

3. – Generalization to curves with cusps and normal crossings.

We consider curves $\gamma \subseteq \boldsymbol{R}^2$ such that:

1) there exists a local homeomorphism $g\colon S^1 \to \gamma$ onto γ;

2) there is a finite set $\mathcal{C}(\gamma)$ of *cusp points* of γ;

3) $g|\{S^1 \setminus g^{-1}(\mathcal{C}(\gamma))\}$ is a *smooth normal immersion* with a finite number of *simple normal crossings*.

We distinguish between cusps of the first and of the second kind (denoted by $\mathcal{C}'(\gamma)$ and $\mathcal{C}''(\gamma)$), according to figure 16.

Figure 16

We call such a γ an *irreducible plane curve with cusps and normal crossings*. Let us denote:

$$\mathcal{N}(\gamma) = \{\text{normal crossings}\}\,; \quad n(\gamma) = \#\,\mathcal{N}(\gamma)\,; \quad c'(\gamma) = \#\,\mathcal{C}'(\gamma)\,;$$

$$c''(\gamma) = \#\,\mathcal{C}''(\gamma)\,; \quad c(\gamma) = \#\,\mathcal{C}(\gamma) = c'(\gamma) + c''(\gamma)\,.$$

3.1 REMARK. – Recall that the generalization of the Fabricius-Bjerre formula to this case gives (see [FB2]):

$$e = i + f/2 + n + c' + c''/2\,.$$

This suggests that the quantity $f/2 = e - i$ of 2.1 should be replaced in this case by $f/2 + n + c' + c''/2$. In fact:

3.2 THEOREM. – *Let γ be an irreducible plane curve with cusps and normal crossings and assume that it is «generic» (in the sense of Fabricius-Bjerre); then*:

$$K(\gamma) < 2\pi + \pi d(\gamma)[f(\gamma)/2 + n(\gamma) + c'(\gamma) + c''(\gamma)/2]\,.$$

PROOF. – The idea is to construct a smooth connected simple curve γ' such that:

$$\begin{cases} K(\gamma') \geqslant K(\gamma) , \\ d(\gamma') \leqslant d(\gamma) , \\ f(\gamma') \leqslant f(\gamma) + 2n(\gamma) + 2c'(\gamma) + c''(\gamma) . \end{cases}$$

The « moves » in order to pass from γ to γ' are shown in figure 17: note that they can be done in such a way that γ' is a C^∞ curve.

It is easy to show, by induction on $n(\gamma)$, that we can choose one of the two moves described for the crossing points in such a way that γ' is connected. If no crossing point is an inflection point, then the moves add two inflection points in cases A and B and one in case C (see figure 17). Thus

$$f(\gamma') = f(\gamma) + 2n(\gamma) + 2c'(\gamma) + c''(\gamma) .$$

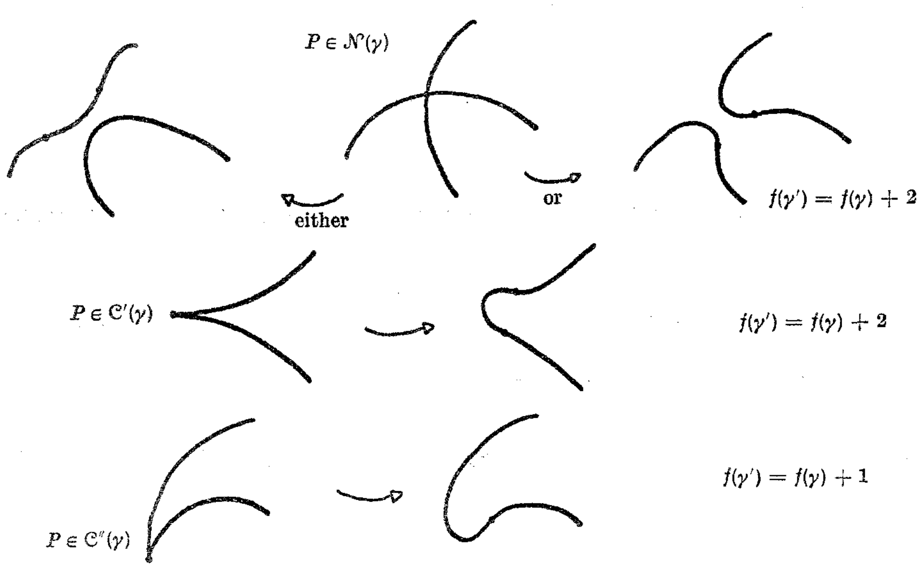

Figure 17

If, instead, some crossing point is also an inflection point, it is easy (see figure 18) to show that « two » is replaced by « at most two », so that « = » is replaced by « $\leqslant$ » above.

In order to see that $d(\gamma') \leqslant d(\gamma)$ it is enough to note that every move replaces two arcs (say α' and α'') of γ by one convex arc (say α) whose convexity is directed towards the singular point (say P): see figure 19.

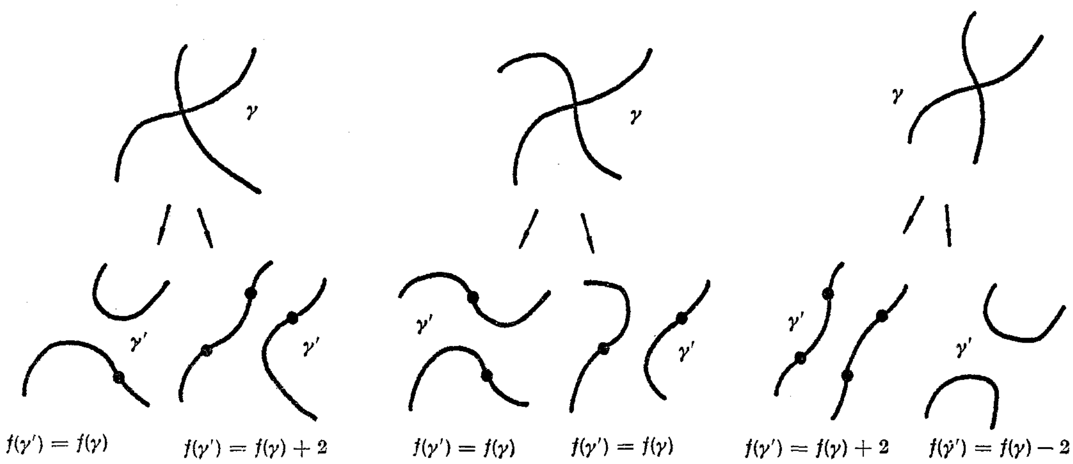

Figure 18

To see that $K(\gamma') \geqslant K(\gamma)$, let the endpoints of α tend towards P; then $K(\alpha') + K(\alpha'') \to 0$ (see figure 19), while $K(\alpha) \to \alpha_0 > 0$ (note that if P is a cusp then $\alpha_0 = \pi$). The theorem is proved. $\square$

3.3 Remarks. – *a*) 3.2 holds also in the simplicial case; in this case there are no cusps, and the moves at a crossing point are the natural analogous to the previous ones; the inequality becomes

$$K(\gamma) < 2\pi + \pi d(\gamma)(f(\gamma)/2 + n(\gamma)) .$$

b) As in the simple case (see remark 2.2 *b*)), 3.2 gives in fact

$$\mathrm{sb}\,(\gamma) < 2 + d(\gamma)[f(\gamma)/2 + n(\gamma) + c'(\gamma) + c''(\gamma)/2] \tag{3.4}$$

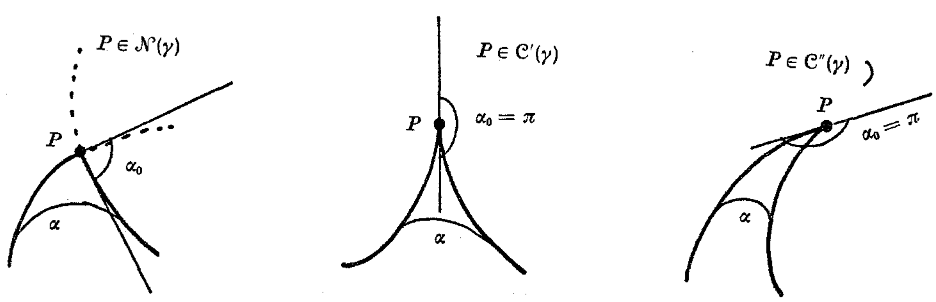

Figure 19

where sb (γ) can be defined as in the simple case, with respect to lines L generic for γ, that is: no «tangent» to γ at a singular point is orthogonal to L and $p_L|[\gamma\setminus(\mathcal{C}(\gamma)\sqcup\mathcal{N}(\gamma))]$ is a Morse function (with necessarily a finite number of critical points).

4. – An application to knots.

Let γ be a smooth knot in $\boldsymbol{R}^3$ with nowhere vanishing torsion $\tau > 0$; the total curvature and the total torsion are defined as $K(\gamma) = \int_\gamma |k(s)|\, ds$ and $T(\gamma) = \int_\gamma |\tau(s)|\, ds$. An hyperplane E of $\boldsymbol{R}^3$ is generic for γ if $p_E|_\gamma$ is a normal immersion.

Set $N(\gamma) = \sup\limits_{E \text{ generic}} \#\{\text{normal crossings of } p_E|_\gamma\}$. Then

4.1 PROPOSITION

$$K(\gamma) < 2\pi + d(\gamma)[4\pi N(\gamma) + T(\gamma)]/8 .$$

PROOF. – It is an immediate consequence of 3.2 and the results of [Mi]. A similar result holds for simplicial knots. $\square$

PART II SURFACES

5. – The geometric characters.

From now on, S will denote a smooth closed connected surface embedded in $\boldsymbol{R}^3$.

Let us first fix some notations: one uses $\boldsymbol{RP}^2$ to parametrize both the lines and the hyperplanes through the origin in $\boldsymbol{R}^3$; to distinguish these two cases we shall adopt the notation $\boldsymbol{RP}^2_L$ and $\boldsymbol{RP}^2_H$ respectively. If $E \in \boldsymbol{RP}^2$, $p_E\colon \boldsymbol{R}^3 \to E$ will denote the orthogonal projection onto E (or its restriction to S). We say that $E \in \boldsymbol{RP}^2_L$ is *generic* for S if $p_E|S$ is a Morse function; we say that $E \in \boldsymbol{RP}^2_H$ is *generic* for S if $p_E|S$ is *excellent in the sense of Whitney* (that is, it is a stable map). In particular, if E is generic, the *critical set* Σ_E of p_E is a smooth closed curve in S and the *apparent contour* of S in E, via p_E, that is $\gamma_E = p_E(\Sigma_E)$, is a *curve with cusps and simple normal crossing points*.

Let M be a surface diffeomorphic to S (by means, for example, of the identity map; let us denote by Emb $(M, \boldsymbol{R}^3)$ the open set of $C^\infty(M, \boldsymbol{R}^3)$ (with the Whitney topology) consisting of smooth embeddings of M in $\boldsymbol{R}^3$; if $h \in \operatorname{Emb}(M, \boldsymbol{R}^3)$ we write M_h for $h(M)$.

5.1 THEOREM. – *There exists an open dense subset A of* $\operatorname{Emb}(M, \boldsymbol{R}^3)$ *such that $\forall h \in A$ the (normal) Gauss map $n_h\colon M_h \to S^2$ is a* stable map, *that is it is excellent in the sense of Whitney.*

PROOF. – We refer to [BW], [Br], [McS]. □

We shall say that S is *generic* if $S = M_h$, for some $h \in A$. 5.1 implies, in particular, that, if S is generic, then the critical set Σ_n of the Gauss map $n\colon S \to S^2$ is a smooth closed curve in S.

As in the case of curves (see 2.2 *a*)), we can define the *superbridge index* of S as

$$\mathrm{sb} = \mathrm{sb}\,(S) = \sup \mu(E)$$

where $\mu(E) = \#\{\text{critical points of } p_E|S\}$, and the sup is taken among all the hyperplanes $E \in \boldsymbol{RP}^2_L$ which are generic for S. Again, sb (S) is related to the *total curvature* of S

$$K = K(S) = \int_S |k(p)|\, dA$$

by the inequality

$$K(S) = 1/2 \int_{S^2} \mu(E) \leqslant 2\pi\, \mathrm{sb}\,(S)\,.$$

Two geometric characters of S which certainly play a role in bounding $K(S)$ are:

$$g = g(S) \qquad (\text{the } genus \text{ of } S)$$

$$F = F(S) = b_0(\Sigma_n)\,.$$

Note that, by 5.1, F is a well-defined positive integer: moreover, $g > 0 \Rightarrow F > 0$ and $F = g = 0 \Leftrightarrow S$ is a convex sphere. For each $E \in \boldsymbol{RP}^2_H$, E generic for S, set:

$$n(E) = \#\{\text{normal crossing points in } p_E(\Sigma_E)\}\,;$$

$$c(E) = \#\{\text{cusps in } p_E(\Sigma_E)\}\,; \qquad r(E) = b_0(\Sigma_E)\,.$$

If $E \in \boldsymbol{RP}^2_H$ is such that Σ_E and Σ_n are transverse curves on S, set $t(E) = \sup \#\{\Sigma_0 \cap \Sigma_E\}$, where the sup is taken among all the connected components Σ_0 of Σ_n.

We can now introduce the geometric characters associated to S, in terms of which we shall give a bound to $K(S)$:

$$N = N(S) = \sup_{E \text{ generic}} n(E)$$

$$C = C(S) = \sup_{E \text{ generic}} c(E)$$

$$R = R(S) = \sup_{E \text{ generic}} r(E)$$

$$D = D(S) = \sup_{E \text{ generic}} d(\Sigma_E)$$

$$T = T(S) = \sup_{E \text{ generic}} t(E)\,.$$

We want first to justify the fact that these geometric characters (as long as sb) are well-defined and «generically» finite.

As regards D, the geometric degree of a smooth curve γ in $\boldsymbol{R}^3$ (or of a plane curve γ) can be seen to be well-defined by means of elementary transversality arguments.

In order to see that N, C, R, and sb are well-defined, we must show that the set of lines $E \in \boldsymbol{RP}^2_L$ (or hyperplanes $E \in \boldsymbol{RP}^2_H$) generic for S is not empty: this is an easy application of a general result of Mather ([M]):

5.2 PROPOSITION. – *For any S (not necessarily S generic)*:

a) there exists an open dense subset $W \subseteq \boldsymbol{RP}^2_L$ such that, $\forall E \in W$, $p_E|S$ is a Morse function;

b) there exists an open dense subset $\Omega \subseteq \boldsymbol{RP}^2_H$ such that, $\forall E \in \Omega$, $p_E|S$ is a stable map (excellent in the sense of Whitney). $\square$

Thus, N, C, R, sb are well-defined.

Note that the analogous of the first statement in 5.2 holds also for smooth closed curves γ in $\boldsymbol{R}^3$ (or for plane curves γ with respect to lines in $\boldsymbol{RP}^1$).

As regards T, we must also show that it is not empty the set of E such that Σ_E is transverse to Σ_n: thus, we need a sort of «relative version» of 5.2 (applied to the curve Σ_n in S); again, it is not hard to prove it in our simple case, using the techniques of [M]:

5.3 LEMMA. – *There exists an open dense subset $\Omega' \subseteq \Omega$ such that, $\forall E \in \Omega'$, the curves Σ_E and Σ_n are transverse in S; moreover, Σ_n does not meet Σ_E at any cusp or normal crossing of p_E and $p_E|\Sigma_n$ is a normal immersion (with simple normal crossings) of Σ_n in E.* $\square$

Note that, a priori, the geometric characters we have defined could be ∞; in this case, the inequalities we are concerned with would be trivial. However, they are

« generically » finite: one can see this, for example, by approximating the surface S with some non singular algebraic surface V: for such a surface V, the set of generic lines (or hyperplanes) is a semi-algebraic set and thus has a finite number of connected components; as the geometric characters we have defined are constant on each of these connected components, they are finite.

From now on, we shall always assume, implicitely, that S is generic and that all its geometric characters are finite.

6. – The inequalities.

Let $E \in \Omega'$ (as in 5.3); then the critical set Σ_E of $p_E|S$ is a smooth closed curve in S, while the apparent contour of S in E, that is $\gamma_E = p_E(\Sigma_E)$, is a curve with (a finite number of) cusps and simple normal crossings; moreover, if $\Sigma_1, \dots, \Sigma_k$ are the connected components of Σ_E, then $\gamma_i = p_E(\Sigma_i)$ $(i = 1, \dots, k)$ are the irreducible components of γ_E. Clearly, we want to apply the results of § 3 to the curve γ_E: the fact that this is possible is an easy consequence of the genericity assumption on S:

6.1 LEMMA. – *There exists an open dense subset $\Omega'' \subseteq \Omega'$ such that, $\forall E \in \Omega''$, any irreducible component γ_i of γ_E is a generic curve with cusps and normal crossings.* □

Recall now that a line L in $\boldsymbol{P}(E)$ is generic for γ_E if p_E is a Morse function outside the singular points of γ_E and no tangent to γ in a singular point is orthogonal to L. In particular, if L is generic for γ_E, then it is generic for each γ_i. It is not hard to see that the set of lines generic for γ_E is an open dense subset of $\boldsymbol{P}(E)$; if L is such a line, then $p_L \circ p_E \colon S \to L$ is a Morse function on S and there is a natural bijection (via $p_E|$) between the critical points of $p_L \circ p_E|S$ and the critical points of $p_L|(\gamma_E \setminus \operatorname{Sing}(\gamma_E))$. It is clear from the above discussion that there exist E and L as before such that

$$\operatorname{sb}(S) = \#\{\text{critical points of } p_L \circ p_E|S\}\,.$$

Thus we have, for such an E:

$$(*) \qquad \operatorname{sb}(S) = \operatorname{sb}(\gamma_E) = \operatorname{sb}(\gamma_1) + \dots + \operatorname{sb}(\gamma_k);$$

Moreover, as we can assume E in Ω'':

$$(**) \qquad \operatorname{sb}(\gamma_i) \leqslant 2 + d_i(n_i + c_i + f_i/2)$$

where $d_i = d(\gamma_i)$, $n_i = n(\gamma_i)$, $c_i = c(\gamma_i)$, $f_i = f(\gamma_i)$.

It is obvious, by definitions, that:

$$n_1 + \dots + n_k \leqslant N(S); \qquad c_1 + \dots + c_k \leqslant C(S); \qquad k \leqslant R(S)\,.$$

As regards d_i, note that, if L is an affine line in $E \in \Omega''$, transverse to γ_E, then the hyperplane H through L orthogonal to E is transverse to Σ_E and there is a natural bijection between $\{L \cap \gamma_E\}$ and $\{H \cap \Sigma_E\}$; thus, $d_i \leqslant D(S)$, $\forall i$.

This means that, from the inequalities $(*)$ and $(**)$, we can get

$$(***) \qquad \mathrm{sb}\,(S) \leqslant 2R + D[N + C + f(\gamma_E)/2]\,.$$

6.2 LEMMA. – *For each E in Ω'', $f(\gamma_E) \leqslant F(S)\,T(S)$.*

PROOF. – Let $y \in \Sigma_E$ be a point such that $p_E(y)$ is not a singular point of γ_E: then, as Σ_E and Σ_n are transverse, there exists a neighbourhood of y in Σ_E such that its image is a planar arc σ where the curvature never vanishes; note that in a neighbourhood of y in S the gaussian curvature k of S is positive (resp. negative) if S folds towards the convex (resp. concave) part of α (see figure 20).

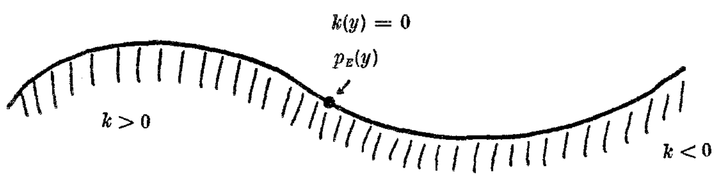

Figure 20

Thus, if x_0 is an inflection point of γ_E, then $x_0 = p_E(y_0)$, where $k(y_0) = 0$, that is $y_0 \in \Sigma_E \cap \Sigma_n$. The thesis now follows from the definitions of $F(S)$ and $T(S)$. $\square$

6.3 COROLLARY.

$$K(S) \leqslant 4\pi R + \pi D[2N + 2C + TF]\,. \quad \square$$

We want now to show that it is possible to eliminate either R or C from the inequality in 6.3, by adding the genus $g(S)$.

The possibility of eliminating C follows from a nice result of Pignoni (see [P]).

6.4 LEMMA. – *For every generic E*

$$c(E)/2 \leqslant n(E) + r(E) + g - 1\,.$$

PROOF. – In fact Pignoni proves a quite stronger result: he shows how to associate to every component Γ_i of Σ_E a sign $\mu_i = \pm 1$ and similarly to every normal crossing a sign σ_j; then, setting $N^+ = \#\{j: \sigma_j = 1\}$ and $N^- = \#\{j: \sigma_j = -1\}$, one has the *identity*:

$$c(E)/2 = (\sum \mu_i) + N^+ - N^- + g - 1 .$$

Of course the inequality follows trivially. □

6.5 COROLLARY. - *For every generic surface S in $\mathbf{R}^3$,*

$$K(S) \leqslant 4\pi R + \pi D[6N + 4R + 4g - 4 + TF] . \quad \square$$

We are now going to see how one can eliminate R from 6.3.

Let S^+ and S^- be the subsets of S where k is positive and negative respectively ($S \setminus \Sigma_n = S^+ \sqcup S^-$). Note that:

a) every cusp point of Σ_E must be contained in S^-: in fact it is not on Σ_n and the cusp in E looks like in figure 21;

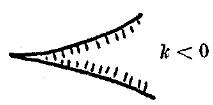

Figure 21

b) if $\Gamma \cap \Sigma_n = \emptyset$, and Γ does not contain cusps, then $p_E|\Gamma$ is a normal *convex immersion*, that is $f(p_E(\Gamma)) = 0$;

c) if $\Gamma \cap \Sigma_n = \emptyset$, and $\Gamma \subseteq S^-$, then Γ could contain cusps, but again $f(p_E(\Gamma)) = 0$. We call $p_E|\Gamma$ a *convex immersion with cups.*

Let X be the set of the connected components of Σ_E; set:

$$V = \{\Gamma \in X: \Gamma \cap \Sigma_n \neq \emptyset\};$$

$$W = \{\Gamma \in X: \Gamma \cap \Sigma_n = \emptyset \text{ and } p_E|\Gamma \text{ is a convex immersion with cusps}\};$$

$$Z = \{\Gamma \in X: \Gamma \cap \Sigma_n = \emptyset \text{ and } p_E|\Gamma \text{ is a normal convex immersion}\} .$$

It is immediate to see that

$$b_0(V) \leqslant TF$$

while, recalling that (by [H]) every component of W has an even number of cusps, we also have

$$b_0(W) \leqslant C/2 .$$

Thus, we are left with the problem to give a bound to $b_0(Z)$. In order to do this, we need some lemmas.

6.6 LEMMA. – *Let $\Gamma \in Z$ and assume that it bounds a disk D in $S^+ \sqcup S^-$ such that D does not intersect any other component of Σ_E then $\Gamma \subseteq S^+$ and $\gamma = p_E(\Gamma)$ is a simple convex curve.*

PROOF. – Choose a small collar of Γ inside D. This is an annulus A, $\partial A = \Gamma \sqcup \Gamma'$, and we may assume (if Γ' is close to Γ) that $p_E|\Gamma'$ is a normal convex immersion and $\gamma' = p_E(\Gamma')$ looks like γ.

Moreover, it is clear that $p_E|\overline{D \setminus A}$ is an immersion; by a result of [H], the *rotation number* of $p_E|\Gamma'$ must be 1. If Γ is in S^-, the rotation number should be negative; while, if Γ is in S^+ and γ is not simple, the rotation number should be >1 (see figure 22).

This proves the lemma. □

6.7 LEMMA. – *Let Γ, Γ' be in Z and assume that they are both in S^+ or both in S^-. Then $\Gamma \sqcup \Gamma'$ cannot be the boundary of an annulus A in $S \setminus \Sigma_n$ such that A does not intersect any other component of Σ_E.*

PROOF. – Use the same argument as before and the fact that the rotation number must be 0. □

6.8 LEMMA. – *If $\Gamma \in Z$ is like in 6.6, and S is not the 2-sphere, then there exists a component Γ' of Σ_n which bounds a disk $D' \subseteq S^+ \cup \Gamma'$ such that $\Gamma \subseteq D'$.*

PROOF. – By 6.8, $\gamma = p_E(\Gamma)$ is a simple convex curve and $\Gamma \subseteq S^+$; if there does not exist a Γ' like in the thesis, then S should close up to give a sphere. □

6.9 LEMMA. – *Let S be a non convex surface; if $(g+1)F > 1$, then $b_0(Z) \leqslant$ $\leqslant 3g + 3F + C - 3$.*

If $(g+1)F = 1$, then $b_0(Z) \leqslant 2$.

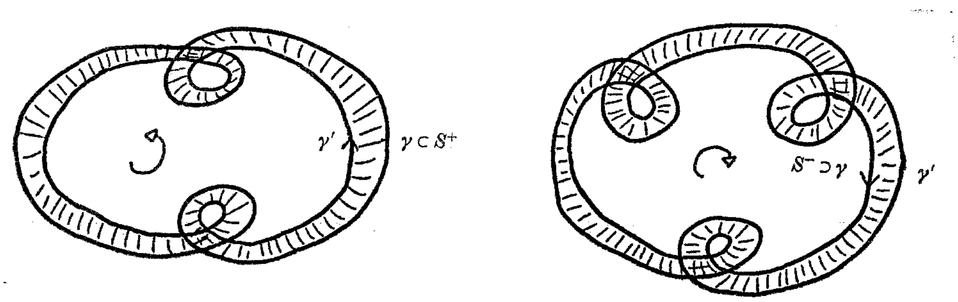

Figure 22

PROOF. – Let M denote the surface obtained by cutting S along the $k \leqslant F + C/2$ curves of $\Sigma_n \sqcup W$; note that $\chi(M) = \chi(S) = 2 - 2g$ and M has $2k > 0$ boundary components (if k where 0, S would be a convex surface).

By the above lemmas, we are interested to bound the maximal number $\alpha(M)$ of connected simple closed curves on $M \setminus \partial M$ such that:

1) no one of these curves bounds a disk in $M \setminus \partial M$,

2) no two of these curves bound an annulus in $M \setminus \partial M$.

One easily shows (by computation on the Euler-Poincaré characteristic), that, if M' is a *connected* surface with genus g' and $k' > 0$ boundary components, then $\alpha(M') = 3g' + 2k' - 3$, except for the case $g' = 0$ and $k' = 1$ (that is a disk), where $\alpha(M') = 0$; this exception implies that, if M is as above, then the maximal $\alpha(M)$ occurs when all the curves along which S is cutted to get M bound disks in S, that is M is the disjoint union of k disks and a surface M' with k boundary components and genus $g(M') = g(S)$. Thus $\alpha(M) = \alpha(M') = 3g + 2k - 3$. As before, the only exception is for $g = 0$ and $k = 1$ (that is, as S is not convex, a sphere with $F = 1$ and $C = 0$), where $\alpha(M) = 0$.

In order to get $b_0(Z)$, it is enough to note that (for lemma 6.8) Z could have one component more for each component of Σ_n: that is $b_0(Z) \leqslant \alpha(M) + F \leqslant 3g + 2(F + C/2) - 3 + F = 3g + 3F + C - 3$ if $(g+1)F > 1$, while $b_0(Z) \leqslant 2$ if $g = 0$ and $F = 1$. □

6.10 THEOREM. – *Let S be a generic, non convex, surface in $\mathbf{R}^3$. Then:*

$$K(S) \leqslant 2\pi[(2T + 4)F + 3C + 6g - 6] + \pi D[2N + 2C + FT] \qquad \textit{if } (g+1)F > 1;$$

and

$$K(S) \leqslant 2\pi[2T + 2] + \pi D[2N + T] \qquad \textit{if } (g+1)F = 1\,.$$

PROOF. – Of course, $R \leqslant b_0(V) + b_0(W) + b_0(Z)$; moreover, if Γ is a component of Σ_n such that there exists $\Gamma' \in V$ with $\Gamma \cap \Gamma' \neq \emptyset$, then Γ can not be among the components of Σ_n occurring in lemma 6.8; thus, if $(g+1)F > 1$.

$$R \leqslant b_0(V) + b_0(W) + b_0(Z) - F =$$
$$= TF + C/2 + 3g + 3F + C - 3 - F = (T + 2)F + 3C/2 + 3g - 3\,.$$

If $(g+1)F = 1$ and S is not convex, then $g = 0$, $F = 1$ and the worse case occurs when $C = 0$ (see 6.9); thus $R \leqslant T + 1$. In both cases, the thesis follows then immediately from 6.3. □

6.11 REMARK. – A component Γ in W may actually bound a disk or an annulus in S^- (see lemmas 6.7 and 6.8): an example is given in figure 23.

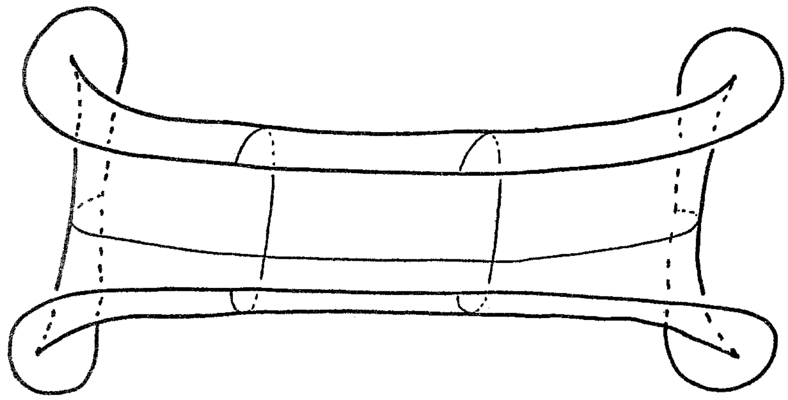

Figure 23

7. – Remarks and examples.

We defined in the introduction two other geometric characters associated to a surface S, that is $d_1 = d_1(S)$ and $d_2 = d_2(S)$, which generalize, more directly than $D(S)$, the geometric degree of a curve. Note first that (« Morse theory » in dimension 1)

$$d_1 \leqslant D/2 \quad \text{ and } d_2 \leqslant D/2 .$$

We are going to show that d_1 and d_2 are essentially independent and that D is not bound by d_1 and d_2.

7.1 EXAMPLE (d_2 is not bound by d_1). – Consider a smooth arc σ which spires « going down » (see figure 24) on the surface of revolution obtained from the graph of $y = x^{\frac{1}{2}}$. If we choose a suitably small smooth « regular » neighbourhood of σ in $\boldsymbol{R}^3$, then the boundary S of such a neighbourhood is a surface whose d_2 can be made arbitrarily large, while $d_1 = 4$. Note that one can arrange the example in such a way that $F = 1$ (and $g = 0$).

7.2 EXAMPLE (d_1 is not bound by d_2). – Consider in the plane xy in $\boldsymbol{R}^3$ an arc σ looking as in figure 25; consider then a suitably small tubular neighbourhood U of σ in the closed half space $\{z \geqslant 0\}$ in such a way that the boundary of $\{z > 0\} \cup U$ is a smooth surface S' (after eventually « smoothing » the corners) whose d_1 can be made arbitrarily large while $d_2 = 2$; S' is not compact, but it is not hard to arrange the example in order to get a compact surface S with the same properties; moreover, one can get also $F = 2$ (and $g = 0$).

7.3 EXAMPLE (D is not bound by d_1 and d_2). – Consider in the yz plane of $\boldsymbol{R}^3$ a simple smooth curve γ, symmetric with respect to the z-axis, looking as in figure 26.

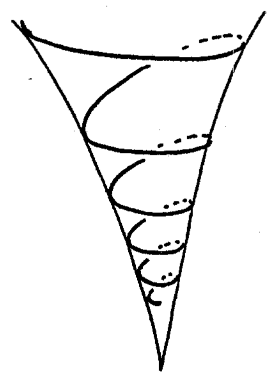

Figure 24

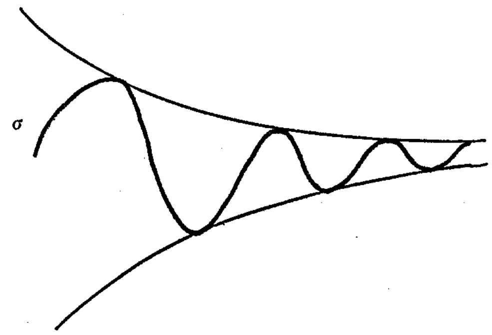

Figure 25

We can arrange γ in such a way that $d(\gamma) = 4$, but the number of critical points of the orthogonal projection onto the z-axis is arbitrarily large. The surface of revolution S obtained from γ has $d_1 = 4$ and $d_2 = 2$, but D can be made arbitrarily large.

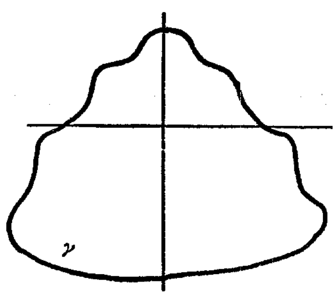

Figure 26

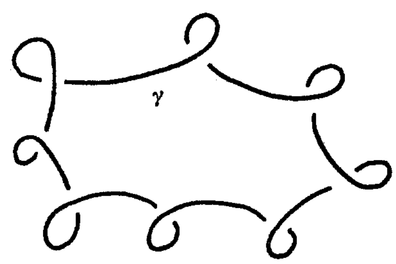

Figure 27

7.4 PROBLEM. – Do they hold inequalities of the kind:

$$K \leqslant h(F, N, C, T, d_1, d_2, g) \quad \text{or} \quad K \leqslant h(F, N, R, T, d_1, d_2, g)$$

To solve the problem, it is enough to control D in terms of d_1, d_2 and some other geometric character out of F, N, C, R, T.

The following examples show that D and F are not enough in order to bound $n(\gamma_E)$ and $f(\gamma_E)$.

7.5 EXAMPLE (D and F do not bound $n(\gamma_E)$). – Consider a suitable knot γ in $\boldsymbol{R}^3$ projecting into the xy-plane as in figure 27. By taking the boundary of a small tubular neighbourhood of γ, we can obtain a surface S such that $D = 4$ and $F = 2$, while $K(S)$ (together with $N(S)$), can be arbitrarily large.

7.6 EXAMPLE (D and F do not bound $f(\gamma_E)$). – Consider n arcs, meeting at a point, drawn on a sphere like in figure 28; let S be the surface obtained from the sphere by « pushing down » a small neighbourhood of this graph (see figure 29); S has $F = R = 1$, while $T(S) = 2n$ $(= f(\gamma_E))$ can be made arbitrarily large (see figure 30). This shows that F is not enough in order to bound the number of inflection points of γ_E. Note that we can arrange the example in order to have $D(S) = 4$.

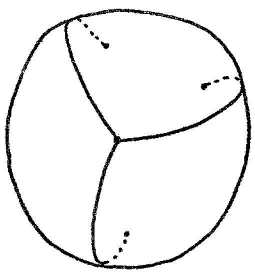

Figure 28

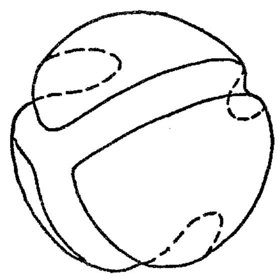

Figure 29

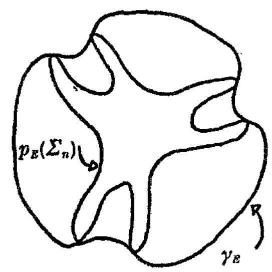

Figure 30

7.7 PROBLEM. – Find a good analogue of F for a simplicial surface, in order to adapt all the above discussion to the P.L. case.

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