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A GEOMETRIC INEQUALITY FOR THE TOTAL CURVATURE OF PLANE CURVES

1. Introduction

Throughout the paper a 'curve' C will be a simple, connected (smooth or simplicial) closed plane curve embedded in $\mathbb{R}^2$ satisfying the following assumptions:

(1) if C is smooth, the inflection points of C are isolated;
(2) if C is simplicial, any two consecutive sides of C are not collinear.

Let us first fix the notations. $K = K(C)$ will be the *total curvature* of C; if C is simplicial, this means the sum of the exterior angles of the polygon bounded by C. $f = f(C)$ will be the *number of inflection points* of C (if C is smooth) and the number of inflection sides of C (if C is simplicial): a side s of a simplicial curve C is said to be an inflection side of C if the two sides of C adjacent to s do not lie in the same half-plane with respect to the straight line containing s. $d = d(C)$ will be the (*geometric*) *degree* of C; that is, $d(C) = \max \# \{r \cap C;\ r \text{ is a straight line transversal to } C\}$.

Note that d and f are always even numbers and that the convexity of C is equivalent to each of the following properties: $K(C) = 2\pi$, $f(C) = 0$, $d(C) = 2$. The aim of this note is to prove the following inequality:

THEOREM. *Let C be a curve for which $f(C) > 0$. Then:*

$$(*) \qquad K < \pi f\left(1 + \frac{d}{2}\right).$$

REMARK 1. The inequality (*) is sharp for $f = 2$ and for any d; to see this, consider for instance the family of curves shown in Figure 1 (rounding off corners gives a smooth example).

REMARK 2. The inequality (*) is not sharp for $d = 4$ and any fixed $f > 2$. In fact, for $d = 4$, (*) reduces to $K < 3\pi f$, while one can prove that

$$(**) \qquad K < \pi(f + 4)$$

(see [1]); note that (**) is increasingly better than (*) when f grows.

REMARK 3. It seems reasonable to conjecture that the inequality (*) is not sharp for any fixed $d \geqslant 4$ and is increasingly far from being sharp when f grows. For example, choose a curve C' with two inflection sides and degree d in the family with 'maximal' curvature defined in Remark 1;

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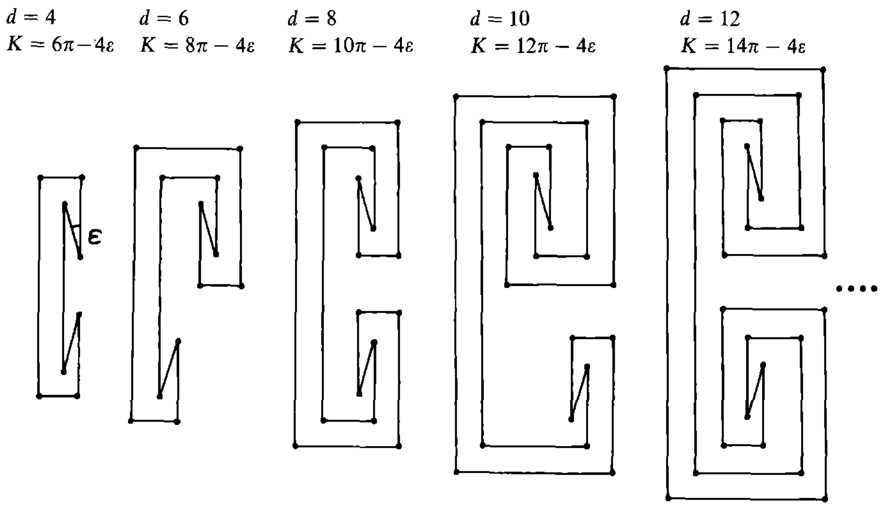
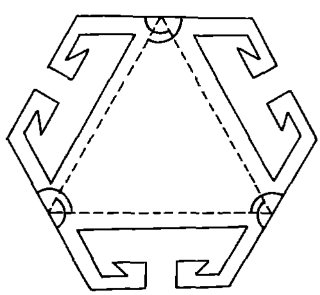


$$K = 4(\pi - \varepsilon) + 2(d - 2)\pi/2 = \pi(d + 2) - 4\varepsilon$$

Fig. 1



$f = 6$; $n = 3$;
$K = 3(10\pi - \varepsilon) - 6\pi \cdot 2/3$

Fig. 2.

consider then the curve C obtained by 'composing' n copies of C' (with an edge deleted) as shown in Figure 2; C has $f = 2n$ inflection sides and degree at least d; even assuming that it could be possible to place the curves C' in such a way that $d(C) = d$, we still get

$$K(C) = nK(C') - (f - 2)\pi < [f(1 + d/2) - (f - 2)]\pi.$$

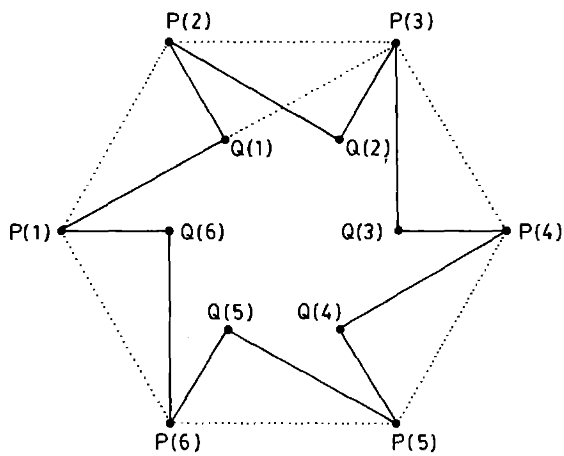
REMARK 4. One could hope that, for fixed d, there could be a bound for f

depending only on d; thus the inequality (*) would give an inequality of the kind

$$(^\circ) \qquad K < h(d).$$

For instance, if C is a non-singular algebraic curve defined by a polynomial of degree n, then clearly $d(C) \leqslant n$ and, on the other hand, one can easily prove that $K(C) < \pi n(n-1)$ (see [3]).

However, the example given in Figure 3 shows that $(^\circ)$ is in general false: here $Q(i)$ ($i = 1, \dots, n$, modulo n) is the midpoint of the diagonal $P(i)P(i+2)$ of a regular n-gon $P(1) \dots P(n)$; rounding off corners gives a smooth example.



$d(C) = 4$; $f(C) = 2n$
$K(C) = \pi(n+2)$

Fig. 3.

In [1] we study more deeply the reasons of the failure of inequality $(^\circ)$.

We refer to the bodies and references of [2] and [3] as regards the motivations for the study of this kind of geometrical invariants (especially in the case of surfaces); even if we do not use their results, we had in mind [4] and the other papers on plane curves listed in its references.

We are grateful to professor N. H. Kuiper and to the referee for useful suggestions.

2. Proof of the Theorem

The proof works in the same way for the smooth and the simplical cases. We shall begin by proving the theorem under a further 'genericity' assumption which allows us to avoid some technical details; at the end we shall prove the theorem in the general case.

Fix an orientation on C; then:

(G. Smooth) *If C is smooth, the straight line through two consecutive inflection points of C is transversal to C.*

(G. Simplicial) *If C is simplicial, any two consecutive inflection sides of C are not collinear.*

Let $w(1), \ldots, w(f)$ be respectively (in the smooth case) the inflection points of C, cyclically ordered with respect to the fixed orientation; and (in the simplicial case) points in the interior of the inflection sides of C (ordered as before) chosen in such a way that the straight line through $w(i)$ and $w(i+1)$ is transversal to C. (Here and in the sequel, we intend that $w(f+1) = w(1)$.)

Denote by $A(i)$ the closed subarc of C from $w(i)$ to $w(i+1)$. We say that a subarc A of a curve C is convex (resp. concave) if (in the smooth case) the tangent line to C at any point $P \in A$ is locally outside (resp. inside) the interior of the compact region bounded by C; and if (in the simplicial case) the straight line through any side s of A is locally outside (resp. inside) the interior of the compact polygon bounded by C.

Note that the arcs $A(i)$ are convex for i even and concave for i odd or vice-versa. (See Figure 4, where dotted arcs are concave.)

Let $l(i)$ be the straight line through $w(i)$ and $w(i+1)$, arbitrarily oriented; label the points of $C \cap l(i)$ with a '+' or a '−', according to which of the situations shown in Figure 5 occurs. (The dashed region represents a part of the compact region bounded by C.)

The points of $C \cap l(i)$ can be 'naturally' ordered in two different ways: either according to their position on $l(i)$ (and the fixed orientation on $l(i)$) or, cyclically, according to their position on C (and the fixed orientation on C).

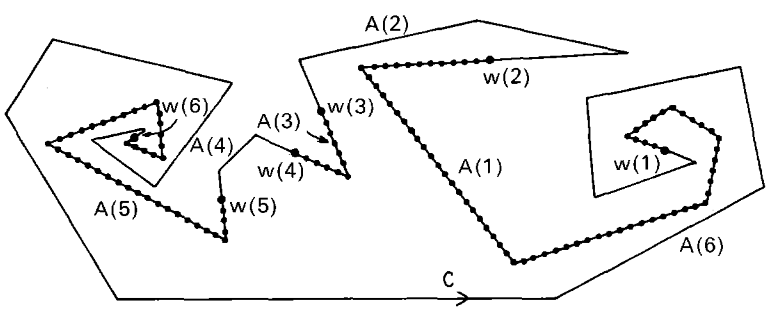


Fig. 4.

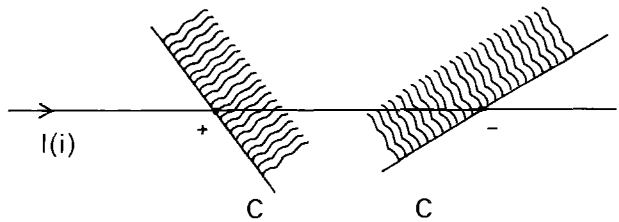


Fig. 5.

Note that:

(1) with respect to both these orderings, the points labelled '+' or '−' are alternating;
(2) with respect to the first ordering, we can speak about a 'first' point and a 'last' point: the first point is labelled '+' and the last one is labelled '−';
(3) these remarks do not depend on the fixed orientations.

CLAIM 1. *Consider the points of $l(i) \cap A(i)$, ordered with respect to their position on $l(i)$. Then the configuration of their labels can only be one of the following:*

(a) *a certain number of points labelled '+', followed by a certain number of points labelled '−'; we denote this configuration by* $(+ \cdots +)(- \cdots -)$;
(b) $(- \cdots -)(+ \cdots +)$;
(c) $(+ \cdots +)(- \cdots -)(+ \cdots +)(- \cdots -)$;
(d) $(- \cdots -)(+ \cdots +)(- \cdots -)(+ \cdots +)$.

Proof of Claim 1. Let $v(1), \ldots, v(m)$ be the points of $l(i) \cap A(i)$, ordered with respect to their position on $A(i)$, so that $v(1) = w(i)$ and $v(m) = w(i+1)$ and their labels are alternating. The points $v(1)$ and $v(2)$ divide $l(i)$ into a segment S and two half-lines $h(1)$ and $h(2)$ with endpoints $v(1)$ and $v(2)$ respectively; the point $v(3)$ can lie either in S or in $h(1)$: this (as all the following arguments) depends on the fact that the arc $A(i)$ is convex (or concave) and C is a simple curve. Consider now the point $v(4)$: if $v(3)$ lies in S, then $v(4)$ is forced to lie in a segment (that is, in the segment with endpoints $v(3)$ and $v(2)$); if $v(3)$ lies in $h(1)$, then we have again two possibilities for $v(4)$, either in a segment (that is, in the segment of endpoints $v(3)$ and $v(1)$) or in a half-line (that is, in $h(2)$).

One sees easily that this analysis can be iterated and at each of the following stages we find a situation analogous to the one described above

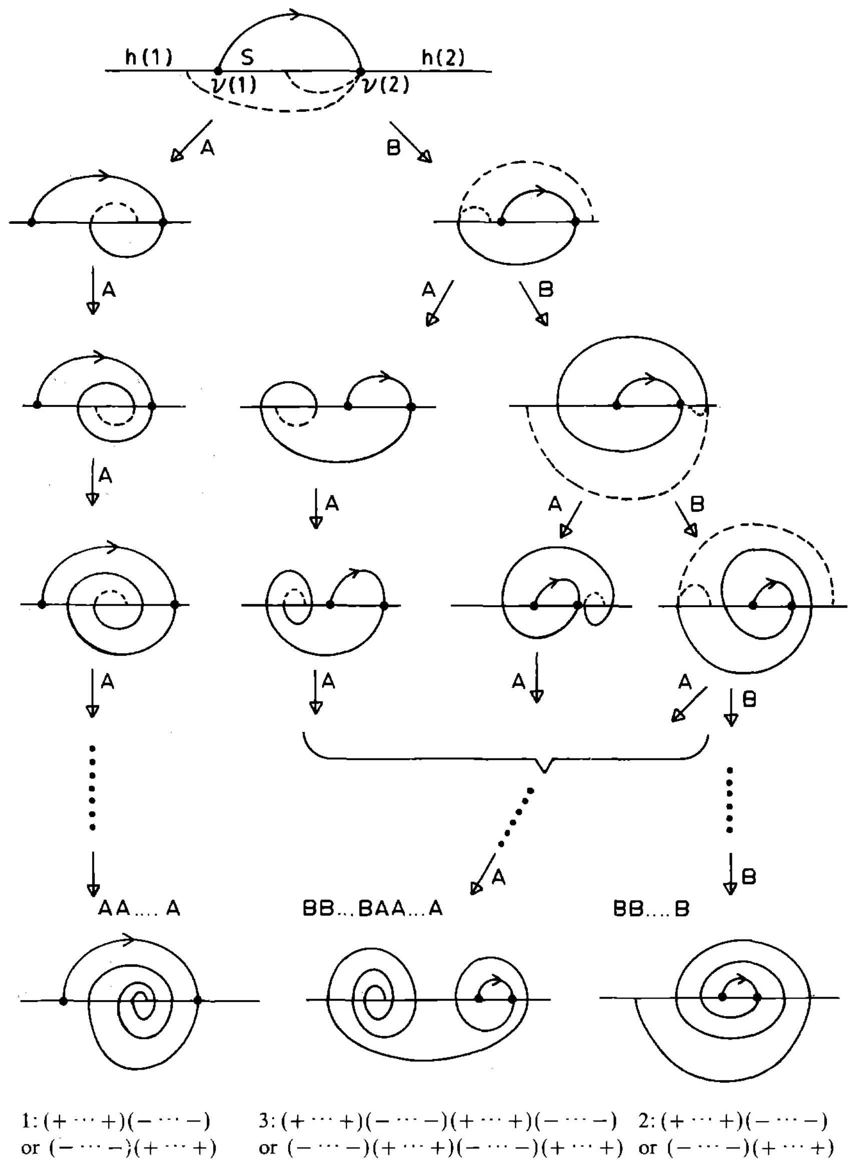


Fig. 6

(see Figure 6); at the end, we get three possible cases:

(1) the point $v(3)$ lies in the segment S, and consequently all the points $v(i)$ are forced to lie in the segment of endpoints $v(i-1)$ and $v(i-2)$; as the points $v(i)$ are alternatively labelled '+' and '−', we get configurations (a) or (b) of the claim; or

(2) at each stage the point $v(i)$ lies in a half-line (referring to the previous

analysis): again we get configurations (a) or (b); or

(3) $v(i)$ lies in a half-line for $3 \leqslant i < i^\circ$ ($i^\circ > 3$), $v(i^\circ)$ lies in a segment and, from then on, the position of all the following points $v(i)$ will be forced in a segment: in this case we get configurations (c) or (d).

The claim is proved.

CLAIM 2. *If $A(i)$ is convex (resp. concave), then only configurations* (a) *or* (c) *(resp. (b) or (d)) of Claim* 1 *can occur.*

Proof of Claim 2. Suppose, for instance, that $A(i)$ is convex and let Q be the first point of $l(i) \cap A(i)$, according to the ordering on $l(i)$; if configuration (b) or (d) occurs, then Q is labelled with a '$-$'; let $l-$ (resp. $l+$) be the half-line of origin Q preceding (resp. following) Q according to the fixed orientation on $l(i)$.

The fact that $l(i)$ is the straight line joining the two endpoints of $A(i)$ implies that there is at least another point, besides Q, in $l(i) \cap A(i)$; the fact that $A(i)$ is convex implies that there is at least one in $l-$; this is absurd for the choice of Q. The other cases are analogous.

Let $m(i)$ and $h(i)$ be the numbers of points

$$m(i) = \#\{l(i) \cap A(i)\} \quad \text{and} \quad h(i) = \#\{l(i) \cap (C - A(i))\}.$$

Clearly,

$$m(i) + h(i) = \#\{l(i) \cap C\} \leqslant d.$$

CLAIM 3. $(h(1) + \cdots + h(f)) \geqslant (m(1) + \cdots + m(f)) - 2f.$

Proof of Claim 3. Claim 1 gives us four possible cases for the configuration of the labelled points of $l(i) \cap A(i)$, ordered on the line $l(i)$; in each of these cases, consider the minimal number of points which must be added in order to get the possible configuration of the labelled points of $l(i) \cap C$, which is necessarily of the form $+ - + - \cdots + -$ (see Remarks 1 and 2 preceding Claim 1). In this way we get an inequality on $h(i)$, in terms of $m(i)$, that is:

Case (a): $h(i) \geqslant m(i) - 2;$

Case (b): $h(i) \geqslant m(i);$

Case (c): $h(i) \geqslant m(i) - 4;$

Case (d): $h(i) \geqslant m(i) - 2.$

In fact, we shall prove that in Case (d) the following stricter inequality holds:

Case (d): $h(i) \geqslant m(i).$

Assuming this for the moment, we can conclude by noting that $h(i) \geqslant m(i) - 4$, for each $i = 1, \ldots, f$, and $h(i) \geqslant m(i)$ for at least half of the indices i (due to Claim 2). The required inequality follows.

In order to conclude the proof of the claim, consider now the configuration of Case (d), that is $(- \cdots -)(+ \cdots +)(- \cdots -)(+ \cdots +)$, and let Q (resp. Q') be the last point labelled '+' in the first block of '+' (resp. the first point labelled '−' in the second block of '−'); to prove that $h(i) \geqslant m(i)$, it is enough to show that there are at least two other points of $l(i) \cap (C - A(i))$ in the segment S with endpoints Q and Q': in fact, otherwise, the whole segment S would be contained in the interior of the compact region bounded by C and this is absurd, because C is a connected curve. (See Figure 7, where dotted arcs represent part of the arcs of $C - A(i)$ and the dashed region is part of the compact region bounded by C.)

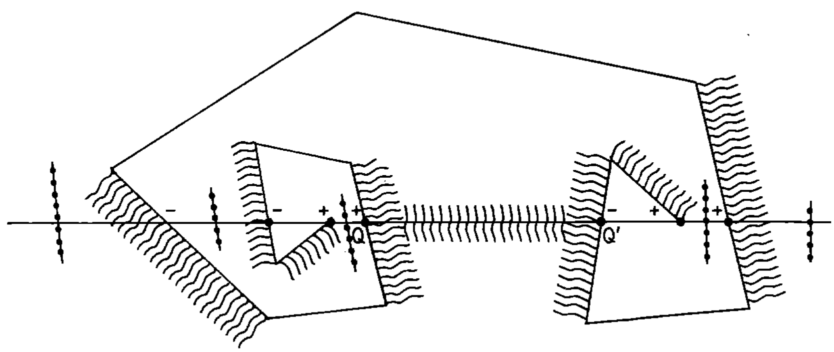


Fig. 7.

End of the proof (*in the 'generic' case*). Let $K(i)$ be the contribution to the curvature given by the arc $A(i)$. If C is simplicial, this is the sum of the exterior angles (with respect to C) between two consecutive oriented sides of $A(i)$. Clearly

$$K = K(1) + \cdots + K(f).$$

For each i, $K(i)$ can be estimated in terms of the number $m(i)$, using the fact that the arc $A(i)$ is convex (or concave) (see Figure 8); that is,

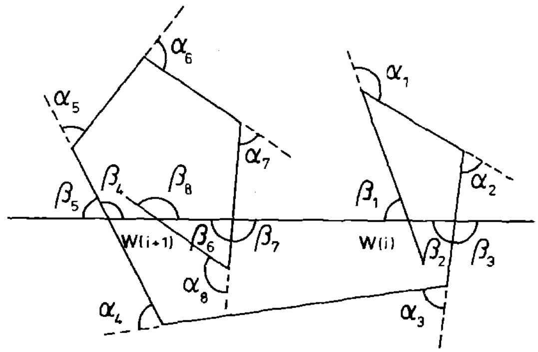
$$K(i) < 2\pi(m(i) - 1) - \pi(m(i) - 2) = \pi m(i).$$

On the other hand, Claim 3 gives us

$$2(m(1) + \cdots + m(f)) \leqslant$$
$$\leqslant (m(1) + \cdots + m(f)) + (h(1) + \cdots + h(f)) + 2f \leqslant f(d + 2).$$

Thus

$$K = K(1) + \cdots + K(f) < \pi f\left(1 + \frac{d}{2}\right).$$



$$
\begin{aligned}
&m(i) = 5\\
&K(i) = \Sigma\alpha_j =\\
&4\cdot 2\pi - \Sigma\beta_j =\\
&8\pi - 3\pi - \beta_1 - \beta_8 <\\
&< 5\pi
\end{aligned}
$$

Fig. 8.

The theorem is proved.

Proof of the general case. Let $w(i)$, $A(i)$, $K(i)$, $l(i)$ be defined as above and assume that the straight line $l(i)$ is not transversal to C; the different situations which may occur are sketched in Figure 9. (Note that, in the simplicial case, (c) necessarily occurs, eventually together with (a) and (b).)

Moreover, we can always assume that (a) does not occur: in fact, the modifications outlined in Figure 10 give a curve C' for which (a) does not occur and such that $K(C') = K(C)$, $f(C') = f(C)$, $d(C') \leqslant d(C)$.

Assume then only (b) and (c) occur. Let $m(i) = m'(i) + m''(i)$, $h(i) = h'(i) + h''(i)$, where $m'(i) = \# M'(i)$, $m''(i) = \# M''(i)$ and $M'(i) = \{P$: $P \in (l(i) \cap A(i))$, $l(i)$ intersects $A(i)$ transversally in $P\}$.

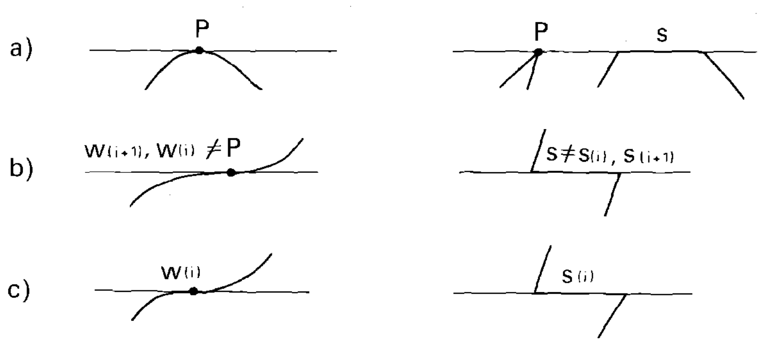


Fig. 9.

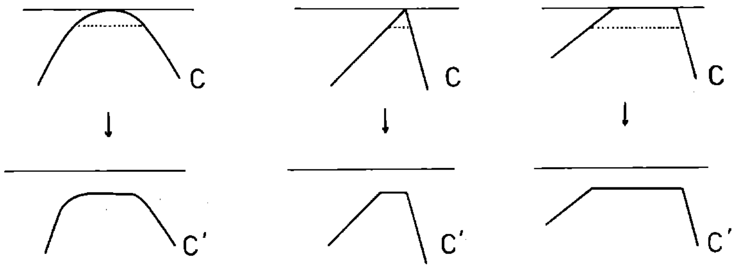


Fig. 10.

(Simplicial case)

$$M''(i) = \{s\colon s \text{ is a side of } C \text{ contained in } l(i)\};$$

(Smooth case)

$M''(i) = \{P\colon \quad P \in (l(i) \cap A(i))$ and the intersection is not transverse$\}$.

$h(i)$ will be defined in the same way, referring to the intersections of $l(i)$ with $C - A(i)$. Note that:

(i) the points (resp. sides) in $M''(i)$ or $H''(i)$ are necessarily inflection points (resp. inflection sides) of C;

(ii) $m''(i) \leqslant 2$: in fact, (b) cannot occur for a point P (or a side s) in the interior of the arc $A(i)$, as one easily sees with a reasoning analogous to the proof of Claim 1;

(iii) $h(i) + m(i) \leqslant d - 2$: in fact, let P be a point (resp. s be a side) in $M''(i)$ or in $H''(i)$ and consider a line l transversal to C, near to $l(i)$ and such that l intersects C in three points in a neighbourhood of P (resp. s) (see Figure 11); then $h(i) + m(i) = \#\{l \cap C\} - 2 \leqslant d - 2$;

(iv) as all the intersection points (or sides) are either transversal or inflection points (or sides), one can label them with '+' or '−' signs exactly as before and claims 1, 2 and 3 hold with the same proof.

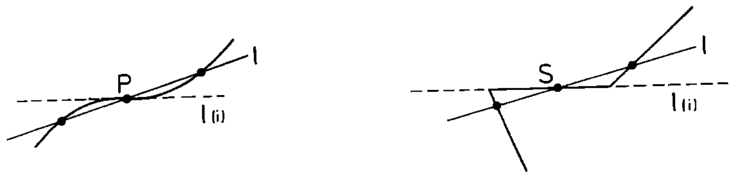


Fig. 11.

For (ii) above, one can compute the curvature $K(i)$ exactly as in the 'generic' case; however, we only get $K(i) \leqslant \pi m(i)$, instead of the strict inequality, because the angles between C and $l(i)$ in the points $w(i)$ and $w(i+1)$ (β_1 and β_8 in Figure 8) may be zero. However, when we consider the sum $K = K(1) + \cdots + K(f)$, either one at least of the straight lines $l(i)$ is transversal to C (and in this case $K < \pi(m(1) + \cdots m(f)) \leqslant \pi f(1 + d/2)$) or for each one of them we can use (iii), thus getting $K \leqslant \pi(m(1) + \cdots m(f)) < \pi f(1 + d/2)$: in both cases the inequality is proved.

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