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ALEXANDER QUANDLE LOWER BOUNDS FOR LINK GENERA

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ABSTRACT

Every finite field $\mathbb{F}_q$, $q = p^n$, carries several *Alexander quandle* structures $\mathbb{X} = (\mathbb{F}_q, *)$. We denote by $\mathcal{Q}_{\mathcal{F}}$ the family of these quandles, where p and n vary respectively among the odd primes and the positive integers.

For every k-component oriented link L, every partition $\mathcal{P}$ of L into $h := |\mathcal{P}|$ sublinks, and every labeling $\overline{z} \in \mathbb{N}^h$ of such a partition, the number of $\mathbb{X}$-colorings of any diagram of $(L, \overline{z})$ is a well-defined invariant of $(L, \mathcal{P})$, of the form $q^{a_{\mathbb{X}}(L,\mathcal{P},\overline{z})+1}$ for some natural number $a_{\mathbb{X}}(L, \mathcal{P}, \overline{z})$. Letting $\mathbb{X}$ and $\overline{z}$ vary respectively in $\mathcal{Q}_{\mathcal{F}}$ and among the labelings of $\mathcal{P}$, we define the derived invariant $\mathcal{A}_{\mathcal{Q}}(L, \mathcal{P}) := \sup\{a_{\mathbb{X}}(L, \mathcal{P}, \overline{z})\}$.

If $\mathcal{P}_M$ is such that $|\mathcal{P}_M| = k$, we show that $\mathcal{A}_{\mathcal{Q}}(L, \mathcal{P}_M) \le t(L)$, where $t(L)$ is the tunnel number of L, generalizing a result by Ishii. If $\mathcal{P}$ is a "*boundary partition*" of L and $g(L, \mathcal{P})$ denotes the infimum among the sums of the genera of a system of disjoint Seifert surfaces for the L_j's, then we show that $\mathcal{A}_{\mathcal{Q}}(L, \mathcal{P}) \le 2g(L, \mathcal{P}) + 2k - |\mathcal{P}| - 1$. We point out further properties of $\mathcal{A}_{\mathcal{Q}}(L, \mathcal{P})$, mostly in the case of $\mathcal{A}_{\mathcal{Q}}(L) := \mathcal{A}_{\mathcal{Q}}(L, \mathcal{P}_m)$, $|\mathcal{P}_m| = 1$. By elaborating on a suitable version of a result by Inoue, we show that when $L = K$ is a knot then $\mathcal{A}_{\mathcal{Q}}(K) \le \mathcal{A}(K)$, where $\mathcal{A}(K)$ is the breadth of the Alexander polynomial of K. However, for every $g \ge 1$ we exhibit examples of genus-g knots having the same Alexander polynomial but different quandle invariants $\mathcal{A}_{\mathcal{Q}}$. Moreover, in such examples $\mathcal{A}_{\mathcal{Q}}$ provides sharp lower bounds for the genera of the knots. On the other hand, we show that $\mathcal{A}_{\mathcal{Q}}(L)$ can give better lower bounds on the genus than $\mathcal{A}(L)$, when L has $k \ge 2$ components.

We show that in order to compute $\mathcal{A}_{\mathcal{Q}}(L)$ it is enough to consider only colorings with respect to the constant labeling $\overline{z} = 1$. In the case when $L = K$ is a knot, if either $\mathcal{A}_{\mathcal{Q}}(K) = \mathcal{A}(K)$ or $\mathcal{A}_{\mathcal{Q}}(K)$ provides a sharp lower bound for the knot genus, or if $\mathcal{A}_{\mathcal{Q}}(K) = 1$, then $\mathcal{A}_{\mathcal{Q}}(K)$ can be realized by means of the proper subfamily of quandles $\{\mathbb{X} = (\mathbb{F}_p, *)\}$, where p varies among the odd primes.

Keywords: Alexander quandle; quandle colorings; Alexander ideals; genus; tunnel number.

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1. Introduction

A *quandle* $\mathbb{X} = (X, *)$ is a non-empty set X with a binary operation $*$ satisfying the following axioms:

(Q1) $a * a = a$ for every $a \in X$;
(Q2) $(a * b) * c = (a * c) * (b * c)$ for every $a, b, c \in X$;
(Q3) for every $b \in X$, the map $S_b : X \to X$ defined by $S_b(x) = x * b$ is a bijection.

Every set X admits the *trivial quandle* structure $\mathbb{X}(0)$ with the operation defined by $a *^0 b = a$ for every $a, b \in X$. Given a quandle $\mathbb{X} = (X, *) := (X, *^1) := \mathbb{X}(1)$, for every integer $n > 1$ one can define another quandle $\mathbb{X}(n) := (X, *^n)$, where for every $a, b \in X$ one sets $a *^n b = (a *^{n-1} b) * b$. Every finite quandle $\mathbb{X}$ has a well defined *type* $t_{\mathbb{X}} \geq 1$, such that $\mathbb{X}(n) = \mathbb{X}(m)$ if and only if $m = n \mod (t_{\mathbb{X}})$.

We refer the reader e.g. to [16] for a recent survey on many aspects of the theory of quandles.

1.1. *Quandle colorings*

Let $K \subset S^3$ be an oriented (smooth or PL) knot. The *fundamental quandle* of K was defined independently by Joyce [11] and Matveev [14]. They also showed that the fundamental quandle is a classifying invariant of knots. If $\mathbb{X}$ is a *finite* quandle, then for every natural number $z \geq 0$ one can define the invariant $c_{\mathbb{X}}(K, z) \in \mathbb{N}$ which counts the representations of the fundamental quandle of K in $\mathbb{X}(z)$. It turns out that $c_{\mathbb{X}}(K, z)$ can be computed as the number of suitably defined $\mathbb{X}(z)$-*colorings* of any diagram D of K. In order to simplify the notation, we denote by (K, z) a knot labeled by a natural number z. Any label of K obviously defines a label on every diagram of K, and if (D, z) is any diagram of (K, z), then we define an $\mathbb{X}$-coloring of (D, z) to be an $\mathbb{X}(z)$-coloring of D. Of course, if $\mathbb{X}$ has type $t_{\mathbb{X}} \geq 1$, then we may (and we will) actually consider $\mathbb{Z}_{t_{\mathbb{X}}}$-valued (rather than $\mathbb{N}$-valued) labels, where we understand that, for every $j \geq 2$, we identify $\mathbb{Z}_j = \mathbb{Z}/j\mathbb{Z}$ with the set of canonical representatives $\{0, \ldots, j-1\}$. The definition of $c_{\mathbb{X}}(K, z)$ easily extends to the case of oriented labeled links. In fact, let $L = K_1 \cup \cdots \cup K_k$ be an oriented link with k components, and let $\mathcal{P} = (L_1, \ldots, L_h)$ be a partition of L, where the L_i's are disjoint sublinks of L such that $L = L_1 \cup \cdots \cup L_h$. We denote by $|\mathcal{P}| = h$ the number of links in the partition $\mathcal{P}$. A special role is played by the maximal (respectively, minimal) partition $\mathcal{P}_M$ (respectively, $\mathcal{P}_m$) of L, which can be characterized as the unique partition such that $|\mathcal{P}| = k$ (respectively, $|\mathcal{P}| = 1$), so that $L_i = K_i$ for $i = 1, \ldots, k$ (respectively, $L_1 = L$). An ($\mathbb{N}$-valued) $\mathcal{P}$-cycle for L is a map $z : \{1, \ldots, h\} \to \mathbb{N}$ that assigns the non-negative integer $z_i = z(i)$ to every component of the sublink L_i of L. In what follows, we often denote such a cycle $(z_1, \ldots, z_h)$ simply by $\overline{z}$, and we denote by $\overline{0}$ (respectively, by $\overline{1}$) the cycle that assigns the integer 0 (respectively, 1) to every component of L.

If D is a diagram of L, then any $\mathcal{P}$-cycle $\overline{z}$ for L descends to a $\mathcal{P}$-cycle $(D, \overline{z})$ for D. In Sec. 2 we recall the definition of $\mathbb{X}$-coloring of $(D, \overline{z})$. The total number of such colorings is denoted by $c_{\mathbb{X}}(D, \mathcal{P}, \overline{z})$, and turns out to be independent of the chosen diagram, thus defining an invariant $c_{\mathbb{X}}(L, \mathcal{P}, \overline{z})$ of the partitioned and labeled link $(L, \mathcal{P}, \overline{z})$.

1.2. *Alexander quandles*

In this paper we deal with a concrete family $\mathcal{Q}_{\mathcal{F}}$ of finite quandles, that we are now going to introduce. Let us fix some notation we will extensively use from now on. For every odd prime $p \geq 3$, we denote by Λ (respectively, Λ_m) the ring $\mathbb{Z}[t, t^{-1}]$ (respectively, $\mathbb{Z}_m[t, t^{-1}]$). Moreover, $\pi_m \colon \Lambda \to \Lambda_m$ is the ring homomorphism induced by the projection $\mathbb{Z} \to \mathbb{Z}_m$. For every $p(t) \in \Lambda$ (respectively, $p(t) \in \Lambda_m$) we define the *breadth* $\operatorname{br} p(t)$ of $p(t)$ as the difference between the highest and the lowest exponent of the non-null monomials of $p(t)$. In particular, the breadth of any constant polynomial (including the null polynomial) is equal to 0 (the reason why we set $\operatorname{br} 0 = 0$ will be clear soon). If $p(t), q(t)$ are elements of Λ (respectively, of Λ_m), we write $p(t) \doteq q(t)$ if $p(t)$ and $q(t)$ generate the same ideal of Λ (respectively, Λ_m), i.e. if and only if $p(t) = \pm t^k q(t)$, $k \in \mathbb{Z}$ (respectively, $p(t) = a t^k q(t)$, $a \in \mathbb{Z}_m^*$, $k \in \mathbb{Z}$).

Recall that a finite *Alexander quandle* is a pair $(M, *)$, where M is a finite Λ_m-module and the quandle operation is defined (in terms of the module operations) by

$$a * b := ta + (1 - t)b.$$

We now define the family $\mathcal{Q}_{\mathcal{F}}$ of finite Alexander quandles we are interested in. Fix an odd prime p, let $h(t)$ be an irreducible element of Λ_p with positive breadth $\operatorname{br} h(t) = n \geq 1$, and let us define $\mathbb{F}(p, h(t))$ as the quotient ring

$$\mathbb{F}(p, h(t)) = \Lambda_p / (h(t)).$$

If $\hat{h}(t) \in \mathbb{Z}_p[t] \subseteq \Lambda_p$ is such that $\hat{h}(t) \doteq h(t)$ and $h(0) \neq 0$, then it is readily seen that the inclusion $\mathbb{Z}_p[t] \hookrightarrow \Lambda_p$ induces an isomorphism $\mathbb{Z}_p[t]/(\hat{h}(t)) \to \mathbb{F}(p, h(t))$. Since $\deg \hat{h}(t) = \operatorname{br} h(t) = n$, it follows that $\mathbb{F}(p, h(t))$ is a finite field of cardinality $q = p^n$.

We may therefore define the Alexander quandle $\mathbb{X} := (\mathbb{F}(p, h(t)), *)$ by setting

$$a * b := \overline{t} a + (1 - \overline{t}) b \quad \text{for every } a, b \in \mathbb{F}(p, h(t)),$$

where $\overline{t}$ is the class of t in $\mathbb{F}(p, h(t))$. Once $q = p^n$ is fixed, there exists only one finite field $\mathbb{F}_q$ up to field isomorphism. However, even in the case when $h_1(t) \in \Lambda_p$ and $h_2(t) \in \Lambda_p$ have the same breadth, it may happen that the quandles $(\mathbb{F}(p, h_1(t)), *)$ and $(\mathbb{F}(p, h_2(t)), *)$ are not isomorphic (see Remark 2.2).

We now set

$$\mathcal{Q}_{\mathcal{F}}(m) = \{(\mathbb{F}(p, h(t)), *) \mid 1 \leq \operatorname{br} h(t) \leq m\}$$

and

$$\mathcal{Q}_{\mathcal{F}} = \bigcup_{m \geq 1} \mathcal{Q}_{\mathcal{F}}(m).$$

1.3. *The invariant* $\mathcal{A}_{\mathcal{Q}}(L, \mathcal{P})$

Let us fix a quandle $\mathbb{X} = (\mathbb{F}(p, h(t)), *)$. Let D be a diagram of a link L, let $\mathcal{P}$ be a partition of L and $\overline{z}$ be a $\mathcal{P}$-cycle for L. If $\mathbb{X} = (\mathbb{F}(p, h(t)), *)$, then it turns out that the space of the $\mathbb{X}$-colorings of $(D, \overline{z})$ is a $\mathbb{F}(p, h(t))$-vector space of dimension $d_{\mathbb{X}}(L, \mathcal{P}, \overline{z}) \geq 1$. Hence $c_{\mathbb{X}}(L, \mathcal{P}, \overline{z}) = q^{d_{\mathbb{X}}(L,\mathcal{P},\overline{z})}$, so the whole information about $c_{\mathbb{X}}(L, \mathcal{P}, \overline{z})$ is encoded by the integer $a_{\mathbb{X}}(L, \mathcal{P}, \overline{z}) := d_{\mathbb{X}}(L, \mathcal{P}, \overline{z}) - 1 \geq 0$. For instance, if $L = K$ is a knot, then the $\mathcal{A}_{\mathcal{Q}}$*-marked spectrum* of K, that is the set $\{a_{\mathbb{X}}(K, n) \,|\, \mathbb{X} \in \mathcal{Q}_{\mathcal{F}}, n \in \mathbb{Z}_{t_{\mathbb{X}}}\}$, considered as a map defined on a subset of $\mathcal{Q}_{\mathcal{F}} \times \mathbb{N}$, carries the whole information provided by these quandle coloring invariants. In this paper we concentrate our attention on the derived invariant defined by

$$\mathcal{A}_{\mathcal{Q}}(L, \mathcal{P}) := \sup\{a_{\mathbb{X}}(L, \mathcal{P}, \overline{z})\},$$

where $\mathbb{X}$ varies in $\mathcal{Q}_{\mathcal{F}}$ and $\overline{z}$ varies among the $\mathcal{P}$-cycles of L. We show in Lemma 2.5 that $\mathcal{A}_{\mathcal{Q}}(L, \mathcal{P}_M)$ is an invariant of the *unoriented* link L. On the contrary, for a generic partition $\mathcal{P}$ the invariant $\mathcal{A}_{\mathcal{Q}}(L, \mathcal{P})$ can depend on the orientations of the components of L. For every partition $\mathcal{P}$ we have of course

$$\mathcal{A}_{\mathcal{Q}}(L, \mathcal{P}_m) \leq \mathcal{A}_{\mathcal{Q}}(L, \mathcal{P}) \leq \mathcal{A}_{\mathcal{Q}}(L, \mathcal{P}_M).$$

When $L = K$ is a knot, of course there is only one partition ($\mathcal{P}_m = \mathcal{P}_M$) and we simply write $\mathcal{A}_{\mathcal{Q}}(K)$. Moreover, henceforth the invariant $\mathcal{A}_{\mathcal{Q}}(L, \mathcal{P}_m)$ will be denoted simply by $\mathcal{A}_{\mathcal{Q}}(L)$.

1.4. *A lower bound on the tunnel number of links*

Recall that the *tunnel number* $t(L)$ of a link $L \subset S^3$ is the minimum number of properly embedded arcs in $S^3 \backslash L$ to be attached to L in such a way that the regular neighborhood of the resulting connected spatial graph is an *unknotted* handlebody (i.e. it is the regular neighborhood also of a graph lying on a 2-dimensional sphere $S^2 \subseteq S^3$). Of course, the tunnel number is an invariant of *unoriented* links.

The argument of [9, Proposition 6] (originally given for quandles of type 2) easily extends to our situation (see Proposition 2.6) and allows us to prove (in Subsec. 2.3) the following proposition.

Proposition 1.1. *For every link L we have*

$$\mathcal{A}_{\mathcal{Q}}(L, \mathcal{P}_M) \leq t(L).$$

In particular, $\mathcal{A}_{\mathcal{Q}}(L, \mathcal{P})$ is always finite.

1.5. *Lower bounds on genera of links*

We say that $\mathcal{P} = (L_1, \ldots, L_h)$ is a *boundary partition* of $L = K_1 \cup \cdots \cup K_k$ if there exists a system $(\Sigma_1, \ldots, \Sigma_h)$ of *disjoint* connected oriented surfaces such that Σ_i is a Seifert surface of L_i (i.e. $\partial\Sigma_i = L_i$ as oriented 1-manifolds, where $\partial\Sigma_i$ inherits the orientation induced by Σ_i), for every $i = 1, \ldots, h$. If $\mathcal{P}$ is a boundary partition of L, then we define the *genus* of $(L, \mathcal{P})$ by

$$g(L, \mathcal{P}) := \min\left\{\sum_{i=1}^{h} g(\Sigma_i)\right\},$$

where $(\Sigma_1, \ldots, \Sigma_h)$ varies among such systems of Seifert surfaces. If $\mathcal{P}$ is not a boundary partition, we set

$$g(L, \mathcal{P}) = +\infty.$$

Every link admits a connected Seifert surface, so $\mathcal{P}_m$ is always a boundary partition, and the number $g(L) := g(L, \mathcal{P}_m)$ is usually known as the *genus* of L. On the other hand, $\mathcal{P}_M$ is a boundary partition if and only if L is a boundary link. It is immediate that $g(L, \mathcal{P}_M)$ is an invariant of the *unoriented* link L.

The following result provides the fundamental estimate on link genera provided by quandle invariants, and is proved in Sec. 5 (note that the statement below is non-trivial only when $\mathcal{P}$ is a boundary partition).

Theorem 1.2. *Let $(L, \mathcal{P})$ be a k-component partitioned link, and let $\overline{z}_1, \overline{z}_2$ be $\mathcal{P}$-cycles for L. Then we have*:

$$|a_{\mathbb{X}}(L, \mathcal{P}, \overline{z}_1) - a_{\mathbb{X}}(L, \mathcal{P}, \overline{z}_2)| \leq 2g(L, \mathcal{P}) + k - |\mathcal{P}|.$$

Recall that $\overline{0}$ is the cycle that assigns the integer 0 to every component of L. In the hypotheses of the previous theorem, for every partition $\mathcal{P}$ we have $a_{\mathbb{X}}(L, \mathcal{P}, \overline{0}) = k - 1$. Hence Theorem 1.2 immediately implies the following corollary.

Corollary 1.3. *If $(L, \mathcal{P})$ is a k-component partitioned link, then*:

$$\mathcal{A}_{\mathcal{Q}}(L, \mathcal{P}) \leq 2g(L, \mathcal{P}) + 2k - |\mathcal{P}| - 1.$$

In particular:

- *If $\mathcal{P}_M$ is the maximal partition of L, then*

$$\mathcal{A}_{\mathcal{Q}}(L, \mathcal{P}_M) \leq 2g(L, \mathcal{P}_M) + k - 1.$$

- *If $\mathcal{P}_m$ is the minimal partition of L, then*

$$\mathcal{A}_{\mathcal{Q}}(L) = \mathcal{A}_{\mathcal{Q}}(L, \mathcal{P}_m) \leq 2g(L) + 2k - 2.$$

- *For every knot K we have*

$$\mathcal{A}_{\mathcal{Q}}(K) \leq 2g(K).$$

Remark 1.4. Let $L = K$ be a knot. Clearly if $\mathcal{A}_{\mathcal{Q}}(K) = 2h$ is even, then $g(K) \geq h$; if $\mathcal{A}_{\mathcal{Q}}(K) = 2h - 1$ is odd, then again $g(K) \geq h$. In particular, if $g(K) = g$, the bound on the genus provided by $\mathcal{A}_{\mathcal{Q}}(K)$ is sharp if and only if $\mathcal{A}_{\mathcal{Q}}(K) \geq 2g - 1$. The very same remark also applies in the general case of partitioned links.

1.6. *Alexander ideals and quandle coloring invariants of links*

Once Theorem 1.2 and Corollary 1.3 are established, we will discuss a bit the performances of the $\mathcal{A}_{\mathcal{Q}}(L,\mathcal{P})$'s as link invariants as well as lower bounds for the link genera. We will mostly concentrate on the case $\mathcal{P}=\mathcal{P}_m$.

The statement of Theorem 1.3 reminds the classical lower bound (see e.g. [4, Theorem 7.2.1])

$$\mathcal{A}(L)\leq 2g(L)+k-1,$$

where $\mathcal{A}(L)$ is the breadth of the Alexander polynomial

$$\Delta(L)(t):=\det(S(L)-tS(L)^T),$$

$S(L)$ being any Seifert matrix of L (of course, the above estimate holds only if we agree that the breadth of the null polynomial is equal to 0).

Let us introduce some notations that will prove useful in describing the relations between Alexander polynomial invariants and quandle coloring invariants of links. We refer to [6, 7] for the definitions and some basic results about Alexander ideals of links and modules. As usual, we denote by p an odd prime number. If K, K' are disjoint oriented knots in S^3, we denote by $\mathrm{lk}(K,K')$ the usual *linking number* of K and K'. For every oriented link $L=K_1\cup\cdots\cup K_k$, let $\widetilde{X}(L)$ be the *total linking number* covering of the complement $\mathsf{C}(L)$ of L in S^3, i.e. the covering associated to the kernel of the homomorphism $\alpha:\pi_1(\mathsf{C}(L))\to\mathbb{Z}$, $\alpha(\gamma)=\sum_{i=1}^k \mathrm{lk}(\gamma,K_i)$. The covering $\widetilde{X}(L)\to\mathsf{C}(L)$ is infinite cyclic, so the homology group $A(L)=H_1(\widetilde{X}(L);\mathbb{Z})$ (respectively, $A^{(p)}(L)=H_1(\widetilde{X}(L);\mathbb{Z}_p)$) admits a natural structure of Λ-module (respectively, Λ_p-module) such that $t\in\Lambda$ (respectively, $t\in\Lambda_p$) acts on $A(L)$ (respectively, $A^{(p)}(L)$) as the map induced by the covering translation corresponding to a loop $\gamma\in\pi_1(\mathsf{C}(L))$ such that $\alpha(\gamma)=1$. Let $E_i(L)\subseteq\Lambda$ (respectively, $E_i^{(p)}(L)\subseteq\Lambda_p$) be the ith elementary ideal of $A(L)$ (respectively, $A^{(p)}(L)$). Since Λ is a U.F.D. (respectively, Λ_p is a P.I.D.), for every $i\geq 1$ it makes sense to define $\Delta_i(L)\in\Lambda$ (respectively, $\Delta_i^{(p)}(L)\in\Lambda_p$) as the generator of the smallest principal ideal containing $E_{i-1}(L)$ (respectively, of the ideal $E_{i-1}^{(p)}(L)$). Then, $\Delta_i(L)(t)$ (respectively, $\Delta_i^{(p)}(L)(t)$) is well-defined only up to invertibles in Λ (respectively, Λ_p), i.e. up to multiplication by $\pm t^k$, $k\in\mathbb{Z}$ (respectively, at^k, $a\in\mathbb{Z}_p^*$, $k\in\mathbb{Z}$). Since $A(L)$ admits the square presentation matrix $S(L)-tS(L)^T$ we have

$$\Delta_1(L)(t)=\det(S(L)-tS(L)^T)=\Delta(L)(t),$$

so $\Delta_1(L)(t)$ coincides with the Alexander polynomial of L. Some relations between $\Delta_i(L)(t)$ and $\Delta_i^{(p)}(L)(t)$ are described in Corollary 6.6 (but see also Remarks 6.8 and 6.9).

Recall that $E_i^{(p)}(L)\subseteq E_{i+1}^{(p)}(L)$ for every $i\in\mathbb{N}$, so either $\Delta_i^{(p)}(L)(t)=\Delta_{i+1}^{(p)}(L)(t)=0$, or $\Delta_{i+1}^{(p)}(L)(t)$ divides $\Delta_i^{(p)}(L)(t)$ in Λ_p. Therefore, it makes sense to define the polynomial $e_i^{(p)}(L)(t) \in \Lambda_p$ (up to invertibles in Λ_p) as follows:

$$e_i^{(p)}(L)(t) = \begin{cases} 0 & \text{if } \Delta_i^{(p)}(L)(t) = \Delta_{i+1}^{(p)}(L)(t) = 0, \\ \dfrac{\Delta_i^{(p)}(L)(t)}{\Delta_{i+1}^{(p)}(L)(t)} & \text{otherwise.} \end{cases}$$

Also recall (see Lemma 6.5) that there exists a minimum $i_0 \in \mathbb{N}$ such that $\Delta_i^{(p)}(L)(t) \doteq 1$ for $i \geq i_0$, whence $e_i^{(p)}(L)(t) \doteq 1$ for every $i \geq i_0$.

In the very same way we can define the family of polynomials with integer coefficients $\{e_i(L)(t)\}$ in Λ.

In Sec. 7 we prove the following result, which is strongly related with the main result of [8], although there is a subtlety in the statement that we will point out below.

Theorem 1.5. *Suppose L is a link, and take a quandle $\mathbb{X} = (\mathbb{F}(p, h(t)), *) \in \mathcal{Q}_{\mathcal{F}}$. Then the space of $\mathbb{X}$-colorings of $(L, \mathcal{P}_m, \overline{z})$ is in bijection with the module*

$$\mathbb{F}(p, h(t)) \oplus \left(\bigoplus_{i=1}^{\infty} \Lambda_p \Big/ (e_i^{(p)}(L)(t^z), h(t)) \right),$$

where $z = \overline{z}(\mathcal{P}_m)$ is the value assigned by $\overline{z}$ to every component of L, and $(e_i^{(p)}(L)(t), h(t)) \subseteq \Lambda_p$ is the ideal generated by $e_i^{(p)}(t)$ and $h(t)$.

Let us compare our result with Inoue's Theorem [8, Theorem 1]. We first observe that in [8, Theorem 1] only the case when $L = K$ is a knot and $\overline{z} = \overline{1}$ is considered. Moreover, our proof of Theorem 1.5 does not make use of Fox differential calculus, and is therefore quite different from Inoue's argument. However, maybe the most interesting feature of the statement of Theorem 1.5 is that

The polynomial $e_i^{(p)}(L)(t) \in \Lambda_p$ is not the reduction mod (p), say $\pi_p(e_i(L)(t))$, of $e_i(L)(t)$

as it could be suggested by the original statement of [8, Theorem 1]. In fact, in Remark 6.8 we show that the statement of Theorem 1.5 does not hold if the $e_i^{(p)}(L)(t)$'s are replaced by the $\pi_p(e_i(L)(t))$'s. In other words, in the following statement from the abstract of [8]:

"*The number of all quandle homomorphisms of a knot quandle of a knot to an Alexander quandle is completely determined by Alexander polynomials of the knot*"

one has to consider not only the Alexander polynomials of the usual Alexander Λ-module $A(L)$, but also the Alexander polynomials associated to the whole family of Λ_p-modules $A^{(p)}(L)$.

In the case of knots, building on Theorem 1.5 we deduce (in Sec. 7) the following theorem.

Theorem 1.6. *For every knot K we have*

$$\mathcal{A}_{\mathcal{Q}}(K) \leq \mathcal{A}(K).$$

Moreover $\mathcal{A}_{\mathcal{Q}}(K) = 0$ if and only if $\mathcal{A}(K) = 0$.

In particular, as a bound on the genus of knots, the invariant $\mathcal{A}_{\mathcal{Q}}(K)$ is dominated by $\mathcal{A}(K)$. Moreover, the following example shows that, when $L = K$ is a knot, the difference between $\mathcal{A}(K)$ and $\mathcal{A}_{\mathcal{Q}}(K)$ may become arbitrarily large.

For any pair p, q of coprime integers, the torus knot $T_{p,q}$ has tunnel number $t(T_{p,q}) = 1$ (and its unknotting tunnels have been classified in [2]). Denoting by $\Delta_{p,q}(t)$ the Alexander polynomial of $T_{p,q}$, it is well-known (see e.g. [3, p. 128]) that:

$$\Delta_{p,q}(t) = \frac{(t^{pq}-1)(t-1)}{(t^p-1)(t^q-1)}, \quad g(T_{p,q}) = \frac{(p-1)(q-1)}{2}.$$

In particular, the bound on the genus of $T_{p,q}$ provided by the Alexander polynomial is sharp, i.e. we have $\mathcal{A}(T_{p,q}) = 2g(T_{p,q})$. As a consequence, we get the following proposition.

Proposition 1.7. *For every $n_0 \in \mathbb{N}$ there exist an integer $n \geq n_0$ and a knot K such that*

$$\mathcal{A}_{\mathcal{Q}}(K) \leq t(K) = 1 < n = 2g(K) = \mathcal{A}(K).$$

While being dominated by $\mathcal{A}(\cdot)$ in the case of knots, the quandle invariant $\mathcal{A}_{\mathcal{Q}}(\cdot)$ may provide a better lower bound on the genus of k-component links, $k \geq 2$. Moreover, $\mathcal{A}_{\mathcal{Q}}(\cdot)$ can provide a sharp lower bound of the knot genus, and can distinguish knots sharing both the genus and the Alexander polynomial. More precisely, in Sec. 9 we prove the following propositions.

Proposition 1.8. *For every $n \in \mathbb{N}$ there exists a link L such that $\mathcal{A}_{\mathcal{Q}}(L) \geq n$ and $\mathcal{A}(L) = 0$.*

Proposition 1.9. *Let us fix $g \geq 1$. Then, for every r_1, r_2 such that $1 \leq r_1 \leq r_2 \leq 2r_1 \leq 2g$, there exist knots K_1 and K_2 such that the following conditions hold*:

$$g(K_1) = g(K_2) = g, \quad \Delta(K_1) = \Delta(K_2) \quad (\text{whence } \mathcal{A}(K_1) = \mathcal{A}(K_2)),$$

while

$$\mathcal{A}_{\mathcal{Q}}(K_1) = r_1, \quad \mathcal{A}_{\mathcal{Q}}(K_2) = r_2.$$

Moreover, we can require that both $\mathcal{A}_{\mathcal{Q}}(K_1)$ and $\mathcal{A}_{\mathcal{Q}}(K_2)$ are realized by means of some dihedral quandle with cycle $\overline{z} = 1$ (*see the end of Subsec.* 2.1 *for the definition of dihedral quandle*).

1.7. *Further properties of the invariant $\mathcal{A}_{\mathcal{Q}}$*

Let $L = K$ be a knot, and let us look for proper subfamilies of $\mathcal{Q}_{\mathcal{F}}$ that carry the relevant information for computing $\mathcal{A}_{\mathcal{Q}}(K)$. In Lemma 7.1 we show that $\mathcal{A}_{\mathcal{Q}}(K)$ is completely determined by the number of colorings relative to the cycle $\overline{z} = 1$: more precisely, we show that for every knot K there exists $\mathbb{X} \in \mathcal{Q}_{\mathcal{F}}$ such that $\mathcal{A}_{\mathcal{Q}}(K) = a_{\mathbb{X}}(K, 1)$. In particular we can set

$$\delta(K) := \inf\{n \in \mathbb{N}^* \mid \mathcal{A}_{\mathcal{Q}}(K) = \sup\{a_{\mathbb{X}}(K,1) \mid \mathbb{X} \in \mathcal{Q}_{\mathcal{F}}(n)\}\} \in \mathbb{N}^*.$$

$$\theta(K) := \inf\{t_{\mathbb{X}} \mid \mathcal{A}_{\mathcal{Q}}(K) = a_{\mathbb{X}}(K,1), \mathbb{X} \in \mathcal{Q}_{\mathcal{F}}(\delta(K))\} \in \mathbb{N}^*.$$

In Subsec. 7.4 we prove the following proposition.

Proposition 1.10. *Let K be a knot.*

(1) *$\theta(K) = 1$ if and only if $\mathcal{A}(K) = \mathcal{A}_{\mathcal{Q}}(K) = 0$. If $\theta(K) > 1$, then $\theta(K) \geq \delta(K) + 1$.*
(2) *If $\mathcal{A}_{\mathcal{Q}}(K) = 1$, then $\delta(K) = 1$.*
(3) *If $\mathcal{A}(K) > 0$, then*

$$\delta(K) \leq \frac{\mathcal{A}(K)}{\max\{2, \mathcal{A}_{\mathcal{Q}}(K)\}}.$$

(4) *Suppose that $\mathcal{A}_{\mathcal{Q}}(K) = \mathcal{A}(K)$ or $\mathcal{A}_{\mathcal{Q}}(K) = \mathcal{A}(K) - 1$. Then $\delta(K) = 1$. Moreover, there exist an odd prime p and an element $a \in \mathbb{Z}_p^*$ such that $(t-a)^{\mathcal{A}_{\mathcal{Q}}(K)}$ divides $\Delta_1^{(p)}(K)(t)$ in Λ_p.*
(5) *If $\mathcal{A}_{\mathcal{Q}}(K) = \mathcal{A}(K)$, then $\delta(K) = 1$ and there exist an odd prime p and an element $a \in \mathbb{Z}_p^*$ such that $\Delta_1^{(p)}(K)(t) \doteq (t-a)^{\mathcal{A}(K)}$ in Λ_p.*

In Sec. 8, Corollary 8.8, we check directly that if $g(K) = 1$ then either $\mathcal{A}_{\mathcal{Q}}(K) = 0$ or $\mathcal{A}_{\mathcal{Q}}(K) \in \{1, 2\}$, and in the last case we have

$$(\delta(K), \theta(K)) = (1, 2)$$

(of course these results also descend from Corollary 1.3 and Proposition 1.10).

Question 1.11. Let $n \in \mathbb{N}$ be fixed. Does a knot K exist such that $\delta(K) \geq n$? (See Remark 7.10 for a brief discussion about this issue.)

Question 1.12. Is $\theta(K)$ bounded from above by an explicit function of $g(K)$ (or $\mathcal{A}(K)$, or $\delta(K)$)?

2. Quandle Invariants

We briefly recall a few details about the definition of quandle invariants of links and about our favourite family $\mathcal{Q}_{\mathcal{F}}$ of finite Alexander quandles.

Let $\mathbb{X} = (X, *)$ be any finite quandle, $|X| = m$. For every $b \in X$, the permutation of X defined by $S_b : a \mapsto a * b$ has order $o(b)$ that divides $m!$. If we denote by $t_{\mathbb{X}}$ the l.c.m. of these orders, then for every $a, b \in X$ we have $S_b^{t_{\mathbb{X}}}(a) = a *^{t_{\mathbb{X}}} b = a$, that is $*^{t_{\mathbb{X}}} = *^0$, and it is readily seen that $t_{\mathbb{X}}$ is in fact the type of $\mathbb{X}$, as defined in Sec. 1.

2.1. *Basic properties of finite Alexander quandles*

Let us now turn to our favourite Alexander quandles $\mathbb{X} = (\mathbb{F}(p, h(t)), *)$ where $h(t)$ is an irreducible polynomial of breadth $n \geq 1$ in Λ_p. Hence $\mathbb{F}(p, h(t))$ is a finite field with $q = p^n$ elements.

For every $m \geq 0$ set $p_{m+1}(t) = \sum_{j=0}^{m} t^j \in \mathbb{Z}[t]$ and $H_m(t) = 1 - t^m \in \mathbb{Z}[t]$, in such a way that $H_m(t) = (1-t)p_m(t)$ for every $m \geq 1$ (when this does not arise ambiguities, we consider $p_m(t)$ (respectively, $H_m(t)$) also as elements of $\mathbb{Z}_p[t]$, Λ and Λ_p). Also recall that $\bar{t}$ denotes the class of t in $\mathbb{F}(p, h(t))$. An easy inductive argument shows that for every $a, b \in \mathbb{F}(p, h(t))$ and every $m \geq 0$ we have

$$a *^m b = \bar{t}^m a + H_m(\bar{t}) b.$$

Lemma 2.1. *Let $\mathbb{X} = (\mathbb{F}(p, h(t)), *)$ be a finite Alexander quandle as above, let $n = \operatorname{br} h(t)$ and set $q = p^n$. Then:*

(1) *$\mathbb{X}$ is trivial if and only if $h(t) \doteq t - 1$.*
(2) *If $\mathbb{X}$ is non-trivial, then $*^m = *^0$ (i.e. m is a multiple of $t_{\mathbb{X}}$) if and only if $h(t)$ divides $p_m(t)$ in Λ_p.*
(3) *$H_m(\bar{t}) = 0$ in $(\mathbb{F}_p, h(t))$ if and only if m is a multiple of $t_{\mathbb{X}}$.*
(4) *Suppose that $\mathbb{X}$ is non-trivial. Then $t_{\mathbb{X}} \geq n + 1$. Moreover, $t_{\mathbb{X}} = n + 1$ if and only if $p_{n+1}(t)$ is irreducible in $\mathbb{Z}_p[t]$ and $h(t) \doteq p_{n+1}(t)$ in Λ_p. If this is the case, then $p_{n+1}(t)$ is irreducible in $\mathbb{Z}[t]$, and $n + 1$ is prime.*

Proof. (1) If $h(t) \doteq t - 1$, then $\bar{t} = 1$ in $\mathbb{F}(p, h(t))$, so $\bar{t}a + (1 - \bar{t})b = a$ for every $a, b \in \mathbb{F}(p, h(t))$. On the other hand, if $\mathbb{X}$ is trivial, then $(1 - \bar{t})(b - a) = 0$ for every $a, b \in \mathbb{F}(p, h(t))$, so $1 - \bar{t} = 0$. This implies that $t - 1$ divides $h(t)$ in Λ_p, so $h(t) \doteq t - 1$ by irreducibility of $h(t)$.

(2) We have $a *^m b = \bar{t}^m a + (1 - \bar{t}) p_m(\bar{t}) b = a$ if and only if $(\bar{t} - 1) p_m(\bar{t})(a - b) = 0$. By point (1) this equality holds for every a, b if and only if $p_m(\bar{t}) = 0$ in $(\mathbb{F}_p, h(t))$, i.e. if and only if $h(t)$ divides $p_m(t)$ in Λ_p.

(3) By point (1), $\mathbb{X}$ has type 1 (i.e. it is trivial) if and only if $H_1(\bar{t}) = 0$, so we may suppose that $\mathbb{X}$ is non-trivial. In this case, since $H_m(\bar{t}) = (1 - \bar{t}) p_m(\bar{t})$ and $1 - \bar{t} \neq 0$ in $(\mathbb{F}_p, h(t))$, point (3) is an immediate consequence of (2).

(4) By point (2) the polynomial $h(t)$ divides $p_{t_{\mathbb{X}}}(t)$ in Λ_p, so $n = \operatorname{br} h(t) \leq \operatorname{br} p_{t_{\mathbb{X}}} = t_{\mathbb{X}} - 1$, whence the first statement. Moreover, $t_{\mathbb{X}} = n + 1$ if and only if $h(t)$ divides $p_{n+1}(t)$ in Λ_p. Since $\operatorname{br} h(t) = \operatorname{br} p_{n+1}(t)$, this condition holds if and only if $p_{n+1}(t) \doteq h(t)$, and this implies that $p_{n+1}(t)$ is irreducible in Λ_p, whence in $\mathbb{Z}_p[t]$. Being monic, if $p_{n+1}(t)$ is irreducible in $\mathbb{Z}_p[t]$, then it is irreducible also in $\mathbb{Z}[t]$, and this implies in turn that $n + 1$ is prime. □

The simplest non-trivial quandles in our family $\mathcal{Q}_{\mathcal{F}}$ are the *dihedral quandles* $\mathcal{D}_p = (\mathbb{F}(p, 1+t), *)$. In this case the quandle operation takes the form $a*b = 2b-a$, in terms of the field operations of $\mathbb{F}(p, h(t)) = \mathbb{Z}_p$. Dihedral quandles are *involutory*, i.e. their type is equal to 2.

Remark 2.2. If $q = p^n$, the finite field $\mathbb{F}_q$, which is unique up to isomorphism, supports in general non-isomorphic quandle structures. This phenomenon shows up already when $n = 1$, i.e. when considering Alexander quandles in $\mathcal{Q}_{\mathcal{F}}(1)$. For every odd prime p and every $a \in \mathbb{Z}_p^*$, let $h_a(t) = a + t$, and let $\mathbb{X}_{p,a} = (\mathbb{F}(p, h_a(t)), *)$ be the corresponding Alexander quandle. We have seen in Lemma 2.1(1) that $\mathbb{X}_{p,a}$ is trivial if and only if $a = p - 1$. On the other hand, if $a = 1$ then $\mathbb{X}_{p,a}$ is a dihedral quandle, and its type is equal to 2. By Lemma 2.1(4), if $a \notin \{1, p-1\}$ then $t_{\mathbb{X}_{p,a}} > 2$, so the quandles $\mathbb{X}_{p,a}$, $\mathbb{X}_{p,1}$ and $\mathbb{X}_{p,p-1}$ are pairwise non-isomorphic. For example, Lemma 2.1(2) implies that $t_{\mathbb{X}_{p,a}} = 3$ if and only if $a \neq 0, -1$ and $-a$ is a root of $t^2 + t + 1$, i.e. if and only if $p \neq 3$ and the equation $a^2 - a + 1 = 0$ has a root in $\mathbb{Z}_p$ (such a root is necessarily distinct from $0, -1$). The discriminant of this quadratic equation is equal to -3, so we can conclude that $t_{\mathbb{X}_{p,a}} = 3$ if and only if $p \neq 3$, the element $p - 3$ admits a square root c in $\mathbb{Z}_p$, and $a = (1 \pm c)/2$ (recall that 2 in invertible in $\mathbb{Z}_p$).

Also observe that, if $p > n(n-1)/2 + 1$, then there exists $a \in \mathbb{Z}_p \backslash \{0, -1\}$ such that $-a \in \mathbb{Z}_p$ is not a root of $p_i(t) \in \mathbb{Z}_p[t]$ for every $i = 1, \ldots, n$. By Lemma 2.1(2), this implies that the type of $\mathbb{X}_{p,a}$ exceeds n, and this shows that $\mathcal{Q}_{\mathcal{F}}(1)$ contains quandles of arbitrarily large type.

Here is another construction of non-isomorphic quandles supported by the same finite field $\mathbb{F}_q$. Assume for example that both $1 + t^m$ and $p_{m+1}(t)$ are irreducible in $\mathbb{Z}_p[t]$. By Lemma 2.1(4), the type of $(\mathbb{F}(p, p_{m+1}(t)), *)$ is equal to $m + 1$. On the other hand, since $(1 + t^m)p_{m+1}(t) = p_{2m}(t)$, points (4) and (2) of Lemma 2.1 imply respectively that the type of $(\mathbb{F}(p, 1 + t^m), *)$ is bigger than m and divides $2m$, and is therefore equal to $2m$. An example of this kind is obtained by taking $m = 2$ and $p = 11$, so that we have two non-isomorphic quandle structures (of type 3 and 4, respectively) on $\mathbb{F}_q$, where $q = 11^2$.

2.2. *Quandle colorings of links*

Let $L = L_1 \cup \cdots \cup L_h = K_1 \cup \cdots \cup K_k$ be an oriented partitioned link with k components, where $\mathcal{P} = (L_1, \ldots, L_h)$ is a partition of L into sublinks, and let D be any diagram of L. A ($\mathbb{Z}_{t_{\mathbb{X}}}$-valued) $\mathcal{P}$-cycle on L is a map $\overline{z}\colon \{1, \ldots, h\} \to \mathbb{Z}_{t_{\mathbb{X}}}$, where $\overline{z}(i)$ labels every component of the sublink L_i. Such a cycle naturally descends to D. An *arc* of D is any embedded open interval in D whose endpoints are undercrossings. An $\mathbb{X}$-coloring of $(D, \mathcal{P}, \overline{z})$ assigns to each arc of D a "color" belonging to $\mathbb{X}$ in such a way that at every crossing we see the local configuration shown in Fig. 1. Here $a, b \in \mathbb{X}$ are colors, and z refers to the value assigned by $\overline{z}$ to the sublink that contains the overcrossing arc.

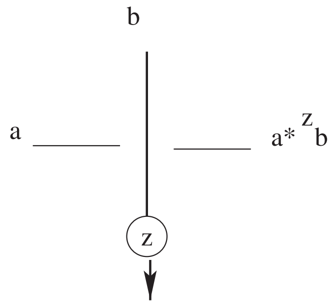

Fig. 1. The local configuration of a quandle coloring.

Remark 2.3. The case when $\mathbb{X}$ is a dihedral quandle is particularly simple to handle because in this case orientations become immaterial from the very beginning, in the sense that the rule of Fig. 1 is well-defined even if one forgets the orientation of the overcrossing arc.

The following proposition shows that

$$c_{\mathbb{X}}(L, \mathcal{P}, \overline{z}) := c_{\mathbb{X}}(D, \mathcal{P}, \overline{z})$$

is a well defined invariant of $(L, \mathcal{P}, \overline{z})$ (up to isotopy of oriented, partitioned and labeled links), where $c_{\mathbb{X}}(D, \mathcal{P}, \overline{z})$ is the number of $\mathbb{X}$-colorings of $(D, \mathcal{P}, \overline{z})$.

Proposition 2.4. *Let $(L, \mathcal{P}, \overline{z})$ be a partitioned link endowed with a fixed $\mathbb{Z}_{t_{\mathbb{X}}}$-cycle, and let D, D' be diagrams of L. Then we have*

$$c_{\mathbb{X}}(D, \mathcal{P}, \overline{z}) = c_{\mathbb{X}}(D', \mathcal{P}, \overline{z}).$$

Proof. Let us briefly describe how our statement can be deduced from the results proved in [9, 10] (in [9] only the case of involutory quandles is considered, but such a restriction is overcome in [10]). In order to check that $c_{\mathbb{X}}(D, \mathcal{P}, \overline{z})$ is independent of D it is sufficient to prove the statement in the case when D and D' are related to each other by a classical Reidemeister move on oriented link diagrams. In the cited papers the authors consider indeed a more general situation, where D and D' are trivalent spatial graphs, and D' is obtained from D either via a Reidemeister move, or via a Whitehead's move (by the way, this ensures that D and D' have ambient-isotopic regular neighborhoods in S^3 — see also the discussion in Subsec. 2.3 below). In our case we have to deal only with the usual Reidemeister moves. Moreover, every $\mathbb{Z}_{t_{\mathbb{X}}}$-cycle on D canonically defines a $\mathbb{Z}_{t_{\mathbb{X}}}$-cycle on D', so the arguments in [9, 10] prove the claimed result. □

Let $\mathbb{X} \in \mathcal{Q}_{\mathcal{F}}$ be a quandle of type k supported by the field $\mathbb{F}_q$. It is clear that the $\mathbb{X}$-colorings of a diagram $(D, \mathcal{P}, \overline{z})$ as above correspond to the solutions of a linear system over $\mathbb{F}_q$. Therefore, the space of such colorings (which contains all the constant colorings) is a $\mathbb{F}_q$-vector space of dimension $d_{\mathbb{X}}(D, \mathcal{P}, \overline{z}) \geq 1$, so the whole information about $c_{\mathbb{X}}(D, \mathcal{P}, \overline{z})$ is encoded by the natural number

$$a_{\mathbb{X}}(D, \mathcal{P}, \overline{z}) := d_{\mathbb{X}}(D, \mathcal{P}, \overline{z}) - 1.$$

By Proposition 2.4, this number is a well defined isotopy invariant of oriented and $\mathbb{Z}_{t_{\mathbb{X}}}$-labeled partitioned links and will be denoted by $a_{\mathbb{X}}(L, \mathcal{P}, \overline{z})$. As a consequence, the following polynomial, that collects all such "monomial" invariants, is an invariant of oriented partitioned links:

$$\Phi_{\mathbb{X}}(L, \mathcal{P})(t) := \sum_{\overline{z}} t^{a_{\mathbb{X}}(L, \mathcal{P}, \overline{z})} \in \mathbb{N}[t].$$

Also observe that by the very definitions we have

$$\deg \Phi_{\mathbb{X}}(L, \mathcal{P})(t) = \sup_{\overline{z}} a_{\mathbb{X}}(L, \mathcal{P}, \overline{z}),$$

whence

$$\mathcal{A}_{\mathcal{Q}}(L, \mathcal{P}) = \sup_{\mathbb{X} \in \mathcal{Q}_{\mathcal{F}}} \deg \Phi_{\mathbb{X}}(L, \mathcal{P})(t). \tag{2.1}$$

Lemma 2.5. *Let L be an oriented link, and let $\mathcal{P}_M$ be its maximal partition. Then the polynomial $\Phi_{\mathbb{X}}(L, \mathcal{P}_M)(t)$ is an invariant of L as an* unoriented *link. As a consequence, $\mathcal{A}_{\mathcal{Q}}(L, \mathcal{P}_M)$ is an invariant of L as an* unoriented *link.*

Proof. Let $\mathcal{P}_M = (K_1, \ldots, K_h)$ be the maximal partition of L, and for every $\epsilon : \{1, \cdots, h\} \to \{\pm 1\}$ let us denote by ϵL the link $\epsilon(1)K_1 \cup \cdots \cup \epsilon(h)K_h$, where as usual the symbols K and $-K$ denote knots having the same support and opposite orientations. We also define the cycle $\epsilon\overline{z}$ by setting $(\epsilon\overline{z})(j) = \epsilon(j)\overline{z}(j)$. It is not hard to verify that for every cycle $\overline{z}$ and every ϵ we have

$$a_{\mathbb{X}}(\epsilon L, \mathcal{P}_M, \overline{z}) = a_{\mathbb{X}}(L, \mathcal{P}_M, \epsilon\overline{z}).$$

We now say that two cycles $\overline{z}$ and $\overline{z}'$ are *equivalent* if and only if there exists ϵ such that $\overline{z}' = \epsilon\overline{z}$, and we denote by $[\overline{z}]$ the equivalence class of $\overline{z}$. The previous discussion shows that the polynomials

$$\Phi_{\mathbb{X}}(L, \mathcal{P}, [\overline{z}])(t) := \sum_{\overline{z}' \in [\overline{z}]} t^{a_{\mathbb{X}}(L, \mathcal{P}, \overline{z}')}$$

do not depend on the orientation of the components of L. The conclusion now follows from the obvious equality

$$\Phi_{\mathbb{X}}(L, \mathcal{P}_M)(t) := \sum_{[\overline{z}]} \Phi_{\mathbb{X}}(L, \mathcal{P}_M, [\overline{z}])(t). \qquad \square$$

2.3. *Quandle invariants and tunnel number*

For every finite quandle $\mathbb{X}$, the number $c_{\mathbb{X}}(L, \mathcal{P}_m, \overline{1})$ (that is the number of colorings associated to the cycle assigning the value 1 to every component of L) is in a sense the most widely considered quandle coloring invariant of classical links. The *multiset* of invariants obtained by varying the $\mathbb{Z}_{t_{\mathbb{X}}}$-cycles (when $\mathbb{X} \in \mathcal{Q}_{\mathcal{F}}$, such a multiset is encoded by the polynomial $\Phi_{\mathbb{X}}(L, \mathcal{P}_M)(t)$) has been introduced in [9, 10] in order to extend quandle coloring invariants to *spatial graphs* and even to *spatial handlebodies*. It turns out that this approach is useful also in the case of links. An interesting application of these extended invariants is given in [9, Proposition 6], where only quandles of type 2 are considered. However, Ishii's argument applies verbatim to our (more general) case, thus giving the following proposition.

Proposition 2.6. *For every $\mathbb{X} \in \mathcal{Q}_{\mathcal{F}}$ and every (unoriented) link L we have*

$$t(L) \geq \mathcal{A}_{\mathcal{Q}}(L, \mathcal{P}_M).$$

Proof. We sketch the proof for the sake of completeness. By equality (2.1), it is sufficient to show that $t(L) \geq \deg \Phi_{\mathbb{X}}(L, \mathcal{P}_M)(t)$ for every quandle $\mathbb{X} \in \mathcal{Q}_{\mathcal{F}}$. Set $m = t(L)$. Then there is a sequence $L := G_0 \subset G_1 \subset G_2 \subset \cdots \subset G_m$, where G_j is a spatial graph with trivalent vertices obtained by attaching an arc to G_{j-1}, and G_m is a spine of an unknotted handlebody. According to [9], for every quandle $\mathbb{X}$ the $\mathbb{X}$-colorings of any diagram of a trivalent graph like G_j verify (in addition to the rule already described in Fig. 1) the further vertex condition described on the left of Fig. 2 (here a refers to a color). With such a definition of coloring, the number of $\mathbb{X}$-colorings of the diagram of a spatial graph does depend only on the isotopy class of a regular neighborhood of the graph, which is a spatial handlebody (the proof of [9, Theorem 5] does not really makes use of condition (K2') stated there, that is equivalent to asking that the considered quandle has type 2).

We can assume that G_{j-1} and G_j admit, respectively diagrams D and D' that differ from each other only by the local configurations shown on the right of Fig. 2. Every cycle on G_{j-1} extends to a cycle on G_j that assigns the value 0 to the added arc. Then it is easy to show that

$$\deg \Phi_{\mathbb{X}}(G_j)(t) \geq \deg \Phi_{\mathbb{X}}(G_{j-1})(t) - 1.$$

Moreover, since a regular neighborhood of G_m is an unknotted handlebody, we have

$$\deg \Phi_{\mathbb{X}}(G_m)(t) = 0,$$

hence

$$0 \geq \deg \Phi_{\mathbb{X}}(L)(t) - t(L). \qquad \square$$

Proposition 1.1 is now an easy consequence of Proposition 2.6. In [1] we have used Ishii's quandle coloring invariants of graphs (only exploiting the dihedral case) in order to detect different *level of knottings* of spatial handlebodies.

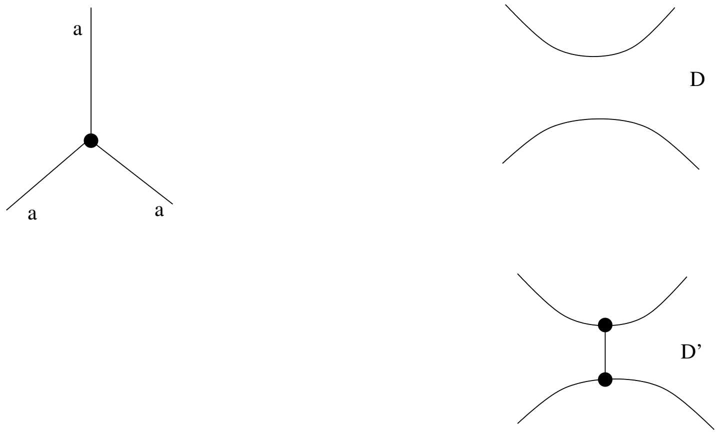

Fig. 2. Quandle colorings at vertices of trivalent graphs.

3. Ribbon Tangles

Let us now fix a quandle $\mathbb{X} \in \mathcal{Q}_{\mathcal{F}}$ of type $t_{\mathbb{X}}$. The following simple lemma (it is a straightforward computation) plays a crucial role in the proof of our main results. Consider the local configurations of Fig. 3. Here a, b, c, b_1, b_2 are colors belonging to some $\mathbb{X}$-coloring, where we understand that $z \in \{0, \ldots, t_{\mathbb{X}} - 1\}$ is the same value of the cycle on *both* the overcrossing strands.

Lemma 3.1. *For the diagram on the left of Fig.* 3 *we have*:

$$c = a + \overline{t}^{-z} H_z(\overline{t})(b_1 - b_2).$$

For the diagram on the right we have:

$$c = a + H_z(\overline{t})(b_2 - b_1).$$

Let us consider a decorated tangle diagram T as suggested in Fig. 4.

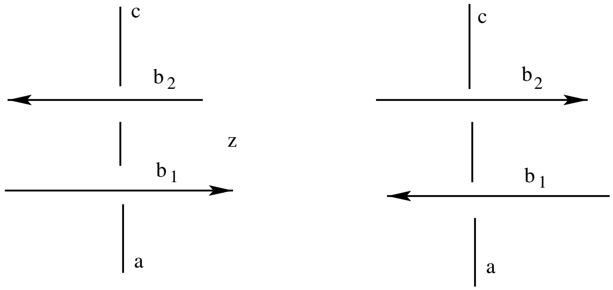

Fig. 3. The quandle colorings described in Lemma 3.1.

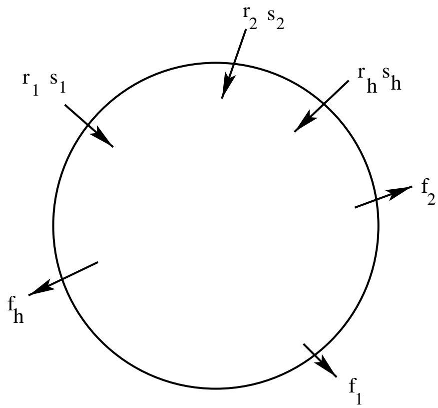

Fig. 4. A string tangle T.

It is understood that the circular box contains h oriented strings, each of which has an "input" and an "output" endpoint. Moreover, the jth string is decorated with a sign $s_j \in \{\pm 1\}$, and its endpoints are endowed with an input color r_j and an output color f_j.

We use such a string tangle to encode an associated *ribbon tangle* $R(T)$ with oriented *ribbon boundary tangle* $D(T)$, by applying the doubling rules suggested in Fig. 5, where the left (right) side refers to the string sign $s = 1$ ($s = -1$). Every ribbon component has two oriented boundary components, that are two copies of the corresponding string of T with opposite orientations. These boundary components are also ordered by taking first the component which shares the same orientation as the corresponding string of T.

If $\overline{z}\colon \{1, \ldots, h\} \to \mathbb{Z}_{t_{\mathbb{X}}}$ is any cycle defined on the strings of T, we define the associated *ribbon boundary cycle* $\hat{z}$ on $D(T)$ by assigning the same value $\overline{z}(j)$ to both boundary components of the ribbon associated to the jth string of T. In this way we have obtained a $\mathbb{Z}_{t_{\mathbb{X}}}$*-labeled ribbon boundary tangle* $(D(T), \hat{z})$. Arcs of T and

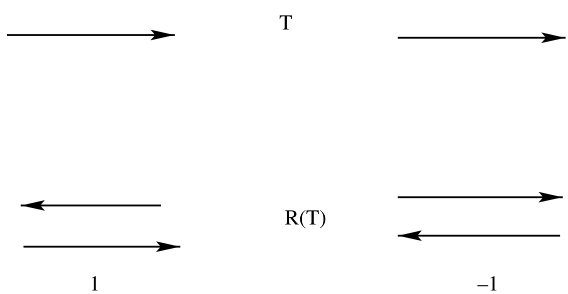

Fig. 5. From a string tangle to a ribbon tangle.

of $D(T)$ are defined as usual, provided now that also the endpoints of the strings of T and $D(T)$ have to be considered as endpoints of arcs of T and $D(T)$.

The notion of $\mathbb{X}$-coloring extends obviously to any $\mathbb{Z}_{t_{\mathbb{X}}}$-labeled ribbon boundary tangle $(D(T), \hat{z})$. For every such a coloring, along every arc of T we see a couple of ordered arcs of $D(T)$ carrying an ordered couple of colors, say (a, b). The following result is an immediate consequence of Lemma 3.1.

Lemma 3.2. *For every $\mathbb{X}$-coloring of $(D(T), \hat{z})$, the color difference $d = b - a$ is constant along every string of T.*

Then every such $\mathbb{X}$-coloring can be described as follows. At the input point of the jth string of T we have an ordered couple of colors $(a_j, a_j + d_j)$. Along every arc α of T belonging to the jth string, we have a couple of colors of the form $(a_j + r_\alpha, a_j + d_j + r_\alpha)$.

For obvious reasons, we say that the r_α's define an $\mathbb{X}_{\text{diff}}$-*coloring* of the arcs of T, which vanishes at the input points of the strings of T (observe that the definition of difference between colors relies on the fact that X is a module, and is not related to the quandle operation of $\mathbb{X}$). We now deduce from Lemma 3.1 the rule governing these $\mathbb{X}_{\text{diff}}$-colorings at crossings. We refer to Fig. 6. Here r_I, r_F are $\mathbb{X}_{\text{diff}}$-colors, d, z, s are respectively the constant $\mathbb{X}$-color difference, the value of the cycle and the sign of the string that contains the overcrossing strand, and $\epsilon = \pm 1$ is the usual sign of the crossing.

With notation as in Fig. 6, Lemma 3.1 readily implies that

$$r_F = r_I - \epsilon H_z(\overline{t})\overline{t}^{\sigma(s)z} d, \quad \text{where } \sigma(s) = \frac{-s-1}{2}. \tag{3.1}$$

As a consequence, every $\mathbb{X}_{\text{diff}}$-coloring of $(T, \overline{z})$ (in particular the corresponding set of output colors $\{f_j\}$) is completely determined by the input data $\{d_j\}$, and every $\mathbb{X}$-coloring of $(D(T), \hat{z})$ is completely determined by the input data $\{(a_j, d_j)\}$. In fact, every $\mathbb{X}_{\text{diff}}$-coloring of $(T, \overline{z})$ can be constructed as follows: we run along every string of T from its input point to its output point, and at every undercrossing we add to the local input value r_I a suitable term according to Eq. (3.1).

Given an ordered couple (i, j) of string indices, let $n^+_{i,j}$ (respectively, $n^-_{i,j}$) be the number of times the ith string passes under the jth string at a positive (respectively,

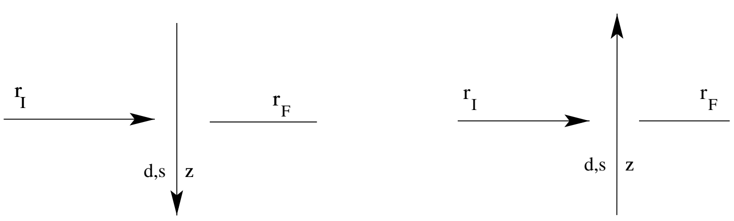

Fig. 6. The behavior of $\mathbb{X}_{\text{diff}}$ at crossings. On the top, a positive crossing. On the bottom, a negative crossing.

negative) crossing, and let us set

$$M_{i,j} = n^+_{i,j} - n^-_{i,j}.$$

The following proposition summarizes the discussion carried out in this section.

Proposition 3.3. *Let T be a decorated tangle with associated ribbon boundary tangle $D(T)$. Then, every $\mathbb{X}_{\text{diff}}$-coloring of $(T, \overline{z})$ (in particular the corresponding set of output colors $\{f_j\}$) is completely determined by the input data $\{d_j\}$, and every $\mathbb{X}$-coloring of $(D(T), \hat{z})$ is completely determined by the input data $\{(a_j, d_j)\}$. In particular, the f_j's can be computed in terms of the d_i's by means of the formula*

$$f_i = -\sum_{j=1}^{h} M_{i,j} \overline{t}^{\sigma(s(j))\overline{z}(j)} H_{\overline{z}(j)}(\overline{t}) d_j.$$

4. Seifert Surfaces and Special Diagrams

Let us consider a compact oriented surface $\Sigma_{g,s}$ of genus g having $s \geq 1$ boundary components. Clearly $g + s \geq 1$, and $g + s = 1$ if and only if $g = 0$ and $s = 1$, i.e. if $\Sigma_{g,s}$ is a disk. Let us assume that $g + s > 1$. It is well known that $\Sigma_{g,s}$ is homeomorphic to the model shown in Fig. 7, where the case $g = 2$, $s = 3$ is considered. The picture stresses also the fact that $\Sigma_{g,s}$ is the regular neighborhood of a 1-dimensional trivalent graph $P_{g,s}$, which is therefore a *spine* of $\Sigma_{g,s}$.

Let now L be an oriented link endowed with a Seifert surface Σ of genus g, and let s be the number of components of L. Assume first that $g + s > 1$. Then the pair (Σ, L) is the image of a suitable embedding of the corresponding model $(\Sigma_{g,s}, \partial\Sigma_{g,s})$ in S^3. As a consequence, L admits a *special diagram* $\mathcal{D}(T)$ as described in Fig. 8: on the top there is a suitable decorated tangle T with $2g + s - 1$ strings (see Sec. 3), where we understand that all the strings have positive sign; on the bottom we see a standard *closing tangle* C which closes the ribbon boundary tangle

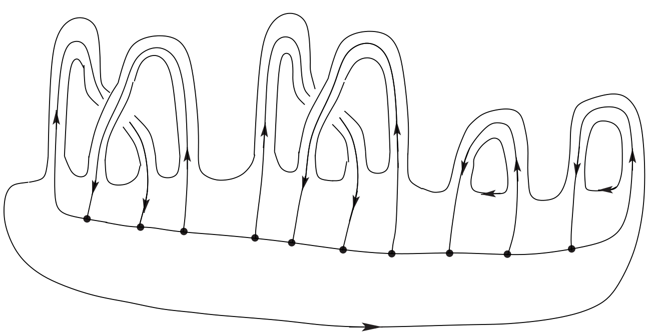

Fig. 7. The surface $\Sigma_{2,3}$ and the spine $P_{2,3}$.

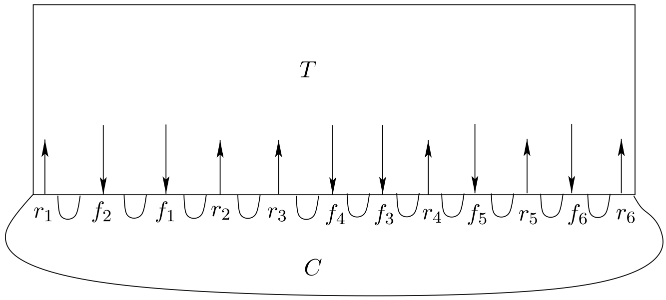

Fig. 8. A special diagram of a 3-component link endowed with a Seifert surface of genus 2.

$D(T)$ associated to T. The strings of T correspond to a generic projection of the image (via the embedding $\Sigma_{g,s} \hookrightarrow S^3$) of some oriented edges of the spine $P_{g,s}$ of $\Sigma_{g,s}$. We say that T is the *primary tangle* of the special diagram $\mathcal{D}(T)$. If $g+s=1$, then L is a trivial knot and Σ is a spanning disk of L; in this case we understand that the only special diagram of L is given by the trivial diagram D of L, and we agree that the closing tangle C coincides with D, while the primary tangle T is empty.

Let us now consider an oriented link L endowed with a boundary partition $\mathcal{P} = (L_1, \ldots, L_h)$, and let $\Sigma_1, \ldots, \Sigma_h$ be a system of disjoint Seifert surfaces such that $\partial\Sigma_i = L_i$ (as *oriented* 1-manifolds). If g_i and s_i are the genus and the number of boundary components of Σ_i, then the pair $(\Sigma_1 \cup \cdots \cup \Sigma_h, L)$ is the image of a suitable embedding in S^3 of the disjoint union $\bigsqcup_{i=1}^h (\Sigma_{g_i,s_i}, \partial\Sigma_{g_i,s_i})$. It readily follows that L admits a special diagram as described in Fig. 9, where the closing tangle C decomposes into the union of h closing tangles $C_1, \ldots, C_h$. Of course, strings of T corresponding to distinct Σ_i's may be linked to each other.

Such a special diagram is *adapted* to $\mathcal{P}$, in the sense that every arc of the primary tangle T gives rise to a pair of arcs of $D(T)$ that belong to the same link of the partition $\mathcal{P}$. Therefore, every $\mathcal{P}$-cycle on L descends to a well-defined cycle on T.

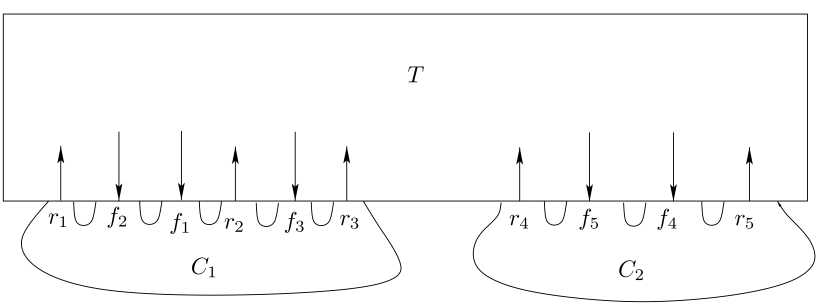

Fig. 9. A special diagram for the link $L = L_1 \cup L_2$, where L_1 is a 2-component link bounding a Seifert surface of genus 1, and L_2 is a knot bounding a Seifert surface of genus 1.

Remark 4.1. Suppose that $\mathcal{P}$ is a boundary partition of a k-component link L. The procedure described in this section provides a special diagram of L adapted to $\mathcal{P}$ whose primary tangle has exactly $2g(\mathcal{P}) + k - |\mathcal{P}|$ strings.

5. Lower Bounds for Link Genera

We are now ready to give the following proof.

Proof of Theorem 1.2. Let $(L, \mathcal{P})$ be a k-component partitioned link, and let us set

$$\alpha = 2g(\mathcal{P}) + k - |\mathcal{P}|.$$

As pointed out in Remark 4.1, L admits a special diagram $\mathcal{D}(T)$ adapted to $\mathcal{P}$ whose primary tangle T has exactly α strings.

Let us take a quandle $\mathbb{X} \in \mathcal{Q}_{\mathcal{F}}$ and a $\mathcal{P}$-cycle $\overline{z}: \mathcal{P} \to \mathbb{Z}_{t_{\mathbb{X}}}$. Such a cycle descends to the diagram $\mathcal{D}(T)$, whence to the boundary ribbon tangle $D(T) \subset \mathcal{D}(T)$. What is more, since $\mathcal{D}(T)$ is adapted to $\mathcal{P}$, the cycle $\overline{z}$ induces a cycle on T, which will also be denoted by $\overline{z}$. The $\mathbb{X}$-colorings of $(\mathcal{D}(T), \overline{z})$ are the $\mathbb{X}$-colorings of $(D(T), \hat{z})$ that extend to the whole $(\mathcal{D}(T), \overline{z})$.

In order to study the space of $\mathbb{X}$-colorings of $(D(T), \overline{z})$ we exploit the results obtained in Sec. 3. The space of $\mathbb{X}$-colorings of $(\mathcal{D}(T), \overline{z})$ is then obtained by imposing the conditions corresponding to the fact that colors have to match along the closing tangle C of $\mathcal{D}(T)$.

Let us associate to every string of T four variables (a_i, d_i, b_i, f_i), $i = 1, \ldots, \alpha$. As usual, the pair $(a_i, a_i + d_i)$ refers to the values of an $\mathbb{X}$-coloring on the arcs of $D(T)$ originating at the input point of the ith string of T, while $(a_i + f_i, a_i + d_i + f_i)$ refers to the values of such a coloring on the arcs of $D(T)$ ending at the output point. Finally, the auxiliary variable b_i encodes the change that an arc of $D(T)$ undergoes whenever it undercrosses the band corresponding to the ith string. Therefore, the value of b_i depends both on d_i and on the value assigned by $\overline{z}$ to the ith string of T. Henceforth, we denote such a value by z_i (so $z_i = \overline{z}(j(i))$ when the ith string of T corresponds to a band of $D(T)$ whose boundary lies on $L_{j(i)}$).

Let us write down the system that computes the space of colorings we are interested in. Proposition 3.3 implies that the space of $\mathbb{X}$-colorings of $(D(T), \overline{z})$ is identified with the space of the solutions of the linear system

$$b_i = -\overline{t}^{-z_i} H_{z_i}(\overline{t}) d_i, \quad i = 1, \ldots, \alpha. \tag{5.1}$$

$$f_i = \sum_{j=1}^{\alpha} M_{i,j} b_j, \quad i = 1, \ldots, \alpha. \tag{5.2}$$

In order to obtain the space of $\mathbb{X}$-colorings of $(\mathcal{D}(L), \overline{z})$, we have to add to these equations also the conditions arising from the fact that colors must match along the strings of the closing tangle C. These conditions can be translated into a linear system

$$S(\{a_i\},\{d_i\},\{f_i\})=0, \tag{5.3}$$

and we stress that such a system does not involve the b_i's (this system is written down in Subsec. 6.1, but this is not relevant to our purposes here).

Let now $\overline{z}'$ be another $\mathcal{P}$-cycle, and let us concentrate on the difference

$$|a_{\mathbb{X}}(L,\mathcal{P},\overline{z})-a_{\mathbb{X}}(L,\mathcal{P},\overline{z}')|.$$

We have just seen that the linear system that computes the space $c_{\mathbb{X}}(L,\mathcal{P},\overline{z})$ is given by the union of Eqs. (5.1)–(5.3). Now, the argument above shows that the system computing the space $c_{\mathbb{X}}(L,\mathcal{P},\overline{z}')$ is given by the union of the systems (5.2) and (5.3) with the following:

$$b_i=-\overline{t}^{-z_i'}H_{z_i'}(\overline{t})d_i,\quad i=1,\dots,\alpha, \tag{5.4}$$

where z_i' is the value assigned by $\overline{z}'$ to the ith string of T. Therefore, the system computing $c_{\mathbb{X}}(L,\mathcal{P},\overline{z}')$ is obtained from the system computing $c_{\mathbb{X}}(L,\mathcal{P},\overline{z}')$ just by replacing (5.1) with (5.4). Since such equations are in number of $\alpha=2g(\mathcal{P})+k-|\mathcal{P}|$ we finally obtain

$$|a_{\mathbb{X}}(L,\mathcal{P},\overline{z})-a_{\mathbb{X}}(L,\mathcal{P},\overline{z}')|\leq 2g(\mathcal{P})+k-|\mathcal{P}|.$$

This concludes the proof of Theorem 1.2. □

Finally we note that the very same argument of the above proof gives the following improvement of Theorem 1.2.

Theorem 5.1. *Let $\mathcal{P}=(L_1,\dots,L_h)$ be a boundary partition of L, where L_i is a k_i-component link, let $\overline{z}$ and $\overline{z}'$ be two $\mathcal{P}$-cycles on L, and let $I=\{i\in\{1,\dots,h\}\mid \overline{z}(i)\neq \overline{z}'(i)\}$. Let also $(\Sigma_1,\dots,\Sigma_h)$ be a system of disjoint Seifert surfaces for the L_i's. Then the following inequality holds*:

$$|a_{\mathbb{X}}(L,\mathcal{P},\overline{z})-a_{\mathbb{X}}(L,\mathcal{P},\overline{z}')|\leq 2\sum_{i\in I}g(\Sigma_i)+\sum_{i\in I}k_i-|I|.$$

5.1. *An example*

The following example shows that Theorem 5.1 could prove more effective than Theorem 1.2 in providing bounds on the genus of links.

Let us consider the tangle B showed in Fig. 10. Recall that $\mathcal{D}_p$ is the dihedral quandle of order p, let $\overline{1}$ be the cycle that assigns the value $1\in\mathbb{Z}_{t_{\mathbb{D}_p}}=\mathbb{Z}_2$ to every arc of B, and let us denote by $C_p(a,b,c,d)$ the number of $\mathcal{D}_p$-colorings of B (relative to the cycle $\overline{1}$) which extend the colors a,b,c,d assigned on the "corners" of the diagram.

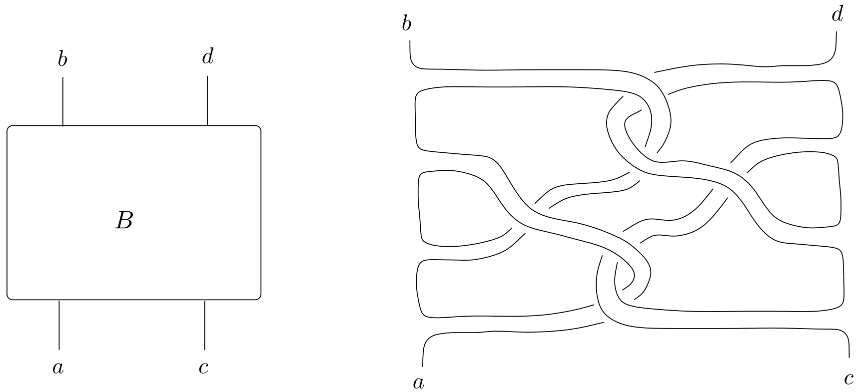

Fig. 10. The tangle B.

The following Lemma is proved in [1]:

Lemma 5.2. *We have*

$$\begin{cases} C_p(a,b,c,d) = p^2 & \text{if } a=b, c=d \quad \text{and} \quad p=3, \\ C_p(a,b,c,d) = 1 & \text{if } a=b, c=d \quad \text{and} \quad p\neq 3, \\ C_p(a,b,c,d) = 0 & \text{otherwise.} \end{cases}$$

For every $q \geq 1$, let L_q be the link described in Fig. 11.

It is obvious from the picture that L_q is a boundary link such that $g(L, \mathcal{P}_M) \leq 2q$. Let K_q (respectively, K_q') be the component of L_q on the top half (respectively, the bottom half) of the diagram shown on the top of Fig. 11. We denote every $\mathcal{P}_M$-cycle $\overline{z}\colon \mathcal{P}_M \to \mathbb{Z}_2$ simply by the pair $(\overline{z}(\{K_q\}), \overline{z}(\{K_q'\}))$, and the integers $a_{\mathbb{D}_3}(L_q, \mathcal{P}_M, (z_1, z_2))$ simply by $a_{\mathbb{D}_3}(L_q, (z_1, z_2))$.

Proposition 5.3. *For every $q \geq 1$ we have*

$$a_{\mathcal{D}_3}(L_q, (1,1)) = 2q+1,$$
$$a_{\mathcal{D}_3}(L_q, (1,0)) = a_{\mathcal{D}_3}(L_q, (0,1)) = a_{\mathcal{D}_3}(L_q, (0,0)) = 1.$$

Proof. As usual, the only $(0,0)$-colorings of L_q are those which are constant on every component of L_q, so $a_{\mathbb{D}_3}(L_q, (0,0)) = 1$.

Let us now concentrate on $(1,0)$-colorings of L_q. It is immediate to observe that K_q and K_q' are both trivial. Since the cycle $(1,0)$ vanishes on K_q', it is immediate to realize that any $(1,0)$-coloring of L_q restricts to a coloring of $K_q(1)$. Since K_q is trivial, this implies that every $(1,0)$-coloring of L_q is constant on K_q. The discussion in Sec. 3 now implies that the colorings of K_q' are not affected by the crossings between the bands of K_q' and the bands of K_q. Then, every $(1,0)$-coloring of L_q restricts to a 0-coloring (i.e. to a constant coloring) of K_q'. We have proved that the only $(1,0)$-colorings are the ones which are constant on every component of

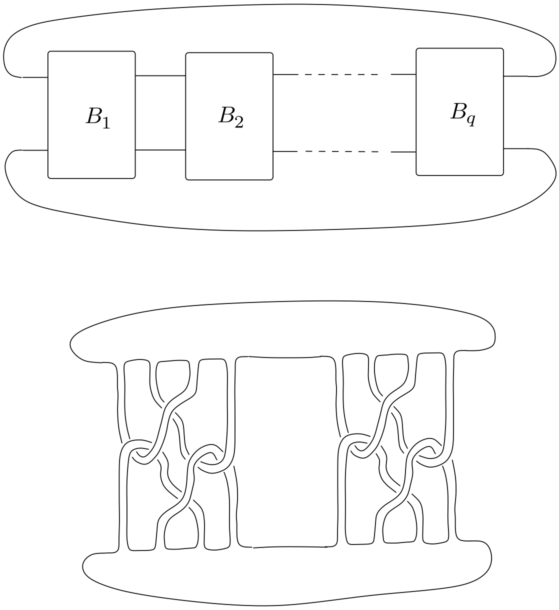

Fig. 11. On the top: the link L_q; every B_i is a copy of the tangle B. On the bottom: the case $q = 2$.

L_q, so $a_{\mathbb{D}_3}(L, (1, 0)) = 1$. The same is true (by the very same argument) also for $(0, 1)$-colorings, so $a_{\mathbb{D}_3}(L, (0, 1)) = 1$.

Let us now fix two colors $a, b \in \mathbb{D}_3$. An easy application of Lemma 5.2 shows that the number of the colorings of L_q which take the value a (respectively, b) on the arc of K_q (respectively, of K'_q) joining the tangles B_1 and B_q is equal to 3^{2q}. Therefore, the number of $(1, 1)$-colorings of L_q is equal to 3^{2q+2}, whence the conclusion. □

Corollary 5.4. *For every $q \geq 1$ we have*

$$g(L_q, \mathcal{P}_M) = 2q.$$

Proof. Let (Σ_q, Σ'_q) be a system of disjoint Seifert surfaces for K_q, K'_q. We have to show that $g(\Sigma_q) + g(\Sigma'_q) \geq 2q$. By Theorem 5.1 we have

$$2q = |a_{\mathbb{D}_3}(L_q, (1, 1)) - a_{\mathbb{D}_3}(L_q, (0, 1))| \leq 2g(\Sigma_q),$$
$$2q = |a_{\mathbb{D}_3}(L_q, (1, 1)) - a_{\mathbb{D}_3}(L_q, (1, 0))| \leq 2g(\Sigma'_q),$$

so $g(\Sigma_q) \geq q$ and $g(\Sigma'_q) \geq q$, whence the conclusion. □

It is maybe worth mentioning that the bound provided by Corollary 1.3 is less effective in order to compute $g(L, \mathcal{P}_M)$. In fact, Proposition 5.3 implies that $\mathcal{A}_{\mathcal{Q}}(L, \mathcal{P}_M) = 2q + 1$, so the inequality $\mathcal{A}_{\mathcal{Q}}(L, \mathcal{P}_M) \leq 2g(L, \mathcal{P}_M) + 1$ only implies $g(L, \mathcal{P}_M) \geq q$.

6. A Proof of Theorem 1.5

With notations as in the preceding section, let us describe more explicitly the system computing the $\mathbb{X}$-colorings of $(L, \mathcal{P}_m, \overline{z})$, where $\mathbb{X} = \mathbb{F}(p, h(t))$ is a quandle in $\mathcal{Q}_{\mathcal{F}}$.

6.1. *More details on the system associated to a special diagram*

Let us now concentrate on the case $\mathcal{P} = \mathcal{P}_m$, so that there exists $z \in \mathbb{N}$ such that $z = \overline{z}(i)$ for every $i = 1, \ldots, k$ (recall that k is the number of components of L). We also set $g = g(L) = g(L, \mathcal{P}_m)$.

Then, the linear system described by Eqs. (5.1)–(5.3) reduces to the system

$$f_i = (1 - \overline{t}^{-z}) \sum_{j=1}^{\alpha} M_{i,j} d_j, \quad S(\{a_i\}, \{d_i\}, \{f_i\}) = 0, \tag{6.1}$$

where $\alpha = 2g + k - 1$ and a_i, d_i, f_i have to be considered as variables in $\mathbb{F}(p, h(t))$. Moreover, the system $S(\{a_i\}, \{d_i\}, \{f_i\}) = 0$ has integer coefficients.

Let us look more closely at the closing conditions $S\{a_i\}, \{d_i\}, \{f_i\}) = 0$. By looking at the definition of special diagram for L, one can easily show that such closing conditions reduce, after easy simplifications, to the system

$$\begin{cases} a_{2i-1} = a_{2i} - d_{2i-1}, d_{2i} = f_{2i-1}, f_{2i} = -d_{2i-1}, & i = 1, \ldots, g, \\ a_{2i} = a_{2i+1} + d_{2i+1}, & i = 1, \ldots, g-1, \\ a_i = a_{2g}, f_i = 0, & i = 2g+1, \ldots, \alpha, \\ a_{2g} = a_1 + d_1. \end{cases}$$

An easy inductive argument shows that the condition $a_{2g} = a_1 + d_1$ is a consequence of equations $a_{2i-1} = a_{2i} - d_{2i-1}$, $i = 1, \ldots, g$, and $a_{2i} = a_{2i+1} + d_{2i+1}$, $i = 1, \ldots, g-1$. Therefore, the system (6.1) is equivalent to the system

$$\begin{cases} a_{2i-1} = a_{2i} - d_{2i-1}, & i = 1, \ldots, g, \\ a_{2i} = a_{2i+1} + d_{2i+1}, & i = 1, \ldots, g-1, \\ a_i = a_{2g}, & i = 2g+1, \ldots, \alpha, \\ -d_{2i} + (1 - \overline{t}^{-z}) \displaystyle\sum_{j=1}^{\alpha} M_{2i-1,j} d_j = 0, & i = 1, \ldots, g, \\ d_{2i-1} + (1 - \overline{t}^{-z}) \displaystyle\sum_{j=1}^{\alpha} M_{2i,j} d_j = 0, & i = 1, \ldots, g, \\ (1 - \overline{t}^{-z}) \displaystyle\sum_{j=1}^{\alpha} M_{i,j} d_j = 0, & i = 2g+1, \ldots, \alpha, \end{cases} \tag{6.2}$$

where we have eliminated the f_i's from the variables.

Let us now define two square matrices $N(z)$ and J of order α with coefficients in Λ as follows:

$$J_{i,j} = \begin{cases} -1 & \text{if } i = 2h-1, j = 2h, h \leq 2g, \\ 1 & \text{if } i = 2h, j = 2h-1, h \leq 2g, \\ 0 & \text{otherwise,} \end{cases}$$

(so J has in fact integer coefficients), and

$$N(z) = (t^z - 1)M + t^z J.$$

We also denote by $N(z,p)$ the matrix obtained by replacing each coefficient of $N(z)$ by its image via $\pi_p : \Lambda \to \Lambda_p$, and by $N(z,p,h(t))$ the matrix obtained by further projecting each coefficient of $N(z,p)$ onto $\mathbb{F}(p,h(t))$.

We are now ready to prove the following lemma.

Lemma 6.1. *The space of $\mathbb{X}$-colorings of $(L, \mathcal{P}_m, \overline{z})$ is in natural bijection with the direct sum*

$$\mathbb{F}(p,h(t)) \oplus \ker N(z,p,h(t)),$$

so

$$a_{\mathbb{X}}(L, \mathcal{P}_m, \overline{z}) = \dim \ker N(z,p,h(t)).$$

Proof. The previous discussion shows that the space of colorings we are considering is in natural bijection with the solutions of the system (6.2). It is immediate to realize that, for every such solution, each a_i, $i \geq 2$, is uniquely determined by a_1 and the d_j's. Moreover, once a solution of the system (6.2) is fixed, we can obtain another solution just by adding a constant term to every a_i.

Therefore, the space of the solutions of (6.2) is isomorphic to the direct sum of $\mathbb{F}(p,h(t))$ with the space of the solutions of the system

$$\begin{aligned} & -d_{2i} + (1 - \overline{t}^{-z}) \sum_{j=1}^{\alpha} M_{2i-1,j} d_j = 0, \quad && i = 1, \ldots, g, \\ & d_{2i-1} + (1 - \overline{t}^{-z}) \sum_{j=1}^{\alpha} M_{2i,j} d_j = 0, && i = 1, \ldots, g, \\ & (1 - \overline{t}^{-z}) \sum_{j=1}^{\alpha} M_{i,j} d_j = 0, && i = 2g+1, \ldots, \alpha. \end{aligned}$$

The matrix encoding this system is equal to $\overline{t}^{-z} N(z,p,h(t))$, and $\overline{t}^{-z}$ is invertible in $\mathbb{F}(p,h(t))$, whence the conclusion. □

6.2. *Some relations between M and the Seifert matrix of L*

Let Σ be the Seifert surface of L encoded by the fixed special diagram $\mathcal{D}(T)$ we are considering, and observe that each (oriented) string of T canonically defines an (oriented) arc lying on Σ. The module $H_1(\Sigma;\mathbb{Z})$ admits a special geometric basis $\{\beta_1,\dots,\beta_\alpha\}$, where β_j is obtained by closing the jth string of T in the portion of Σ carried by the closing tangle C, in such a way that we introduce just one intersection point between β_{2i-1} and β_{2i}, $i=1,\dots,g$, while β_i is disjoint from β_j for every $i=2g+1,\dots,\alpha$, $j=1,\dots,\alpha$. Recall that the Seifert matrix $S(L)$ of L is the square matrix with integer coefficients defined by $S(L)_{i,j}=\mathrm{lk}(\beta_i,\beta_j^+)$, $i,j=1,\dots,\alpha$, where $\mathrm{lk}(\beta_i,\beta_j^+)$ is the linking number (in S^3) between β_i and the knot β_j^+ obtained by slightly pushing β_j to the positive side of Σ. From the very definition of linking number we readily obtain the following lemma.

Lemma 6.2. *We have*

$$S(L)=\frac{M+M^T+J}{2}.$$

Let us point out another interesting property of M that will prove useful later.

Lemma 6.3. *We have*

$$M-M^T=-J.$$

Proof. Since both $M-M^T$ and J are antisymmetric, it is sufficient to show that for every $i<j$ we have $M_{i,j}-M_{j,i}=1$ if $j=i+1$, and $M_{i,j}-M_{j,i}=0$ otherwise. However, it follows from the definition of M that the number $M_{i,j}-M_{j,i}$ is equal to the algebraic intersection number between the projections of the jth and the ith string of T (taken in this order) onto the plane containing the special diagram. If $j>i+1$ (respectively, $j=i+1$), such number is equal to the algebraic intersection number between the projections of β_j and of β_i (respectively, is equal to 1 plus the algebraic intersection number between the projections of β_j and of β_i). But the algebraic intersection number between the projections of β_j and of β_i is obviously null, whence the conclusion. □

Putting together Lemmas 6.2 and 6.3 we get the following corollary.

Corollary 6.4. *We have*

$$S(L)=M+J,\quad t^zS(L)-S(L)^T=(t^z-1)M+t^zJ=N(z).$$

6.3. *Proof of Theorem* 1.5

Let $\widetilde{X}(L)$ and $A^{(p)}(L)$ be the cyclic covering and the Λ_p-module defined in Sec. 1. We have the following lemma.

Lemma 6.5. *The module $A^{(p)}(L)$ admits the square presentation matrix*

$$tS(L)^{(p)}-(S(L)^{(p)})^T=N(1,p).$$

In particular, $\Delta_i^{(p)}(L)(t)=e_i^{(p)}(L)(t)=1$ for every $i>\alpha$.

Proof. The usual proof that $tS(L) - S(L)^T$ is a presentation of $H_1(\widetilde{X}(L);\mathbb{Z})$ over Λ relies on some standard Mayer–Vietoris argument and on Alexander–Lefschetz duality, which ensures that, if $\{\beta_1,\ldots,\beta_{2g+k-1}\}$ is any base of the first homology group of a Seifert surface Σ for L, then the first homology group of $S^3\backslash\Sigma$ admits a dual base $\{\gamma_1,\ldots,\gamma_\alpha\}$ such that $\mathrm{lk}(\beta_i,\gamma_j) = \delta_{ij}$ (see e.g. [3, Chap. 8]). Both these tools may still be exploited when $\mathbb{Z}$ is replaced by $\mathbb{Z}_p$, and this readily implies the conclusion.

An alternative proof can be obtained as follows. An easy application of the Universal Coefficient Theorem for homology shows that $A_p(L) \cong A(L) \otimes_{\mathbb{Z}} \mathbb{Z}_p$, and this easily implies that any presentation

$$0 \longrightarrow \Lambda^r \longrightarrow \Lambda^s \longrightarrow A(L) \longrightarrow 0$$

induces a presentation

$$\begin{array}{ccccccc}
\Lambda_p^r & \longrightarrow & \Lambda_p^s & \longrightarrow & A_p(L) & \longrightarrow & 0 \\
\updownarrow & & \updownarrow & & \updownarrow & & \\
\Lambda^r \otimes_{\mathbb{Z}} \mathbb{Z}_p & & \Lambda^s \otimes_{\mathbb{Z}} \mathbb{Z}_p & & A(L) \otimes_{\mathbb{Z}} \mathbb{Z}_p, & &
\end{array}$$

whence the conclusion. □

The following result describes some relations between $\Delta_i(L)(t)$ and $\Delta_i^{(p)}(L)(t)$, where $i \in \mathbb{N}$.

Corollary 6.6. (1) *For every $i \in \mathbb{N}$ we have $E_i^{(p)}(L) = \pi_p(E_i(L))$.*

(2) *For every $i \in \mathbb{N}$ the polynomial $\pi_p(\Delta_i(L)(t))$ divides $\Delta_i^{(p)}(L)(t)$ in Λ_p.*

(3) *We have $\Delta_1^{(p)}(L)(t) = \pi_p(\Delta(L)(t))$.*

(4) *If $f(t) \in E_i(L)$, then $\Delta_i^{(p)}(L)(t)$ divides $\pi_p(f(t))$ in Λ_p.*

Proof. By Lemma 6.5, π_p maps a set of generators (over Λ) of the ideal $E_i(L)$ onto a set of generators (over Λ_p) of the ideal $E_i^{(p)}(L)$. Since π_p is surjective, this readily implies point (1).

By point (1), the polynomial $\pi_p(\Delta_i(L)(t))$ divides every element of $E_{i-1}^{(p)}(L)$, whence point (2).

Since $A(L)$ admits the square presentation matrix $S(L) - tS(L)^T$, the ideal $E_0(L)$ is principal. Together with (1), this immediately gives (3).

Point (4) is an easy consequence of point (1). □

Let us now consider the Λ_p-linear map $\psi_z : \Lambda_p^\alpha \to \Lambda_p^\alpha$ such that $\psi_z(x) = N(z,p)\cdot x$ for every $x \in \Lambda_p^\alpha$. Of course, ψ_z induces a quotient map $\overline{\psi}_z : \mathbb{F}(p,h(t))^\alpha \to \mathbb{F}(p,h(t))^\alpha$ such that $\overline{\psi}_z(\overline{x}) = N(z,p,h(t))\cdot\overline{x}$ for every $\overline{x} \in \mathbb{F}(p,h(t))^\alpha$. Let now $\overline{z}$ be a $\mathcal{P}$-cycle for L, and denote by $z = \overline{z}(\mathcal{P}_m)$ the value assigned by $\overline{z}$ to every component of L. By Lemma 6.1, the space of $\mathbb{X}$-colorings of $(L,\mathcal{P}_m,\overline{z})$ is in bijection with $\mathbb{F}(p,h(t))\oplus\ker\overline{\psi}_z$, whence to $\mathbb{F}(p,h(t))\oplus\operatorname{coker}\overline{\psi}_z$ (here we use that $\mathbb{F}(p,h(t))^\alpha$ is finite).

Since Λ_p is a P.I.D., there exist square univalent matrices $U(1), V(1)$ with coefficients in Λ_p such that

$$U(1)\cdot N(1,p)\cdot V(1)=\operatorname{diag}(e_1^{(p)}(L)(t),\dots,e_\alpha^{(p)}(L)(t)),$$

where $\operatorname{diag}(\gamma_1,\dots,\gamma_\alpha)$ denotes the diagonal matrix with the γ_i's on the diagonal. Let $U(z)$ (respectively, $V(z)$) be the matrix obtained by applying to every coefficient of $U(1)$ (respectively, $V(1)$) the ring endomorphism of Λ_p that maps t to t^z. Then we obviously have

$$U(z)\cdot N(z,p)\cdot V(z)=\operatorname{diag}(e_1^{(p)}(L)(t^z),\dots,e_\alpha^{(p)}(L)(t^z)).$$

After reducing the coefficients modulo $h(t)$, this equality translates into the equality

$$\overline{U}(z)\cdot N(z,p,h(t))\cdot\overline{V}(z)=\operatorname{diag}(e_1^{(p)}(L)(t^z),\dots,e_\alpha^{(p)}(L)(t^z)),$$

where $\overline{U}(z),\overline{V}(z)$ are invertible over $\mathbb{F}(p,h(t))$, and we denote the class of $e_i^{(p)}(L)(t)$ in $\mathbb{F}(p,h(t))$ simply by $e_i^{(p)}(L)(t)$. This implies that $\operatorname{coker}\overline{\psi}_z$ is isomorphic to

$$\bigoplus_{i=1}^{\alpha}\mathbb{F}(p,h(t))/(e_i^{(p)}(L)(t^z))\cong\bigoplus_{i=1}^{\alpha}\Lambda_p/(e_i^{(p)}(L)(t^z),h(t))$$
$$\cong\bigoplus_{i=1}^{\infty}\Lambda_p/(e_i^{(p)}(L)(t^z),h(t)),$$

where the last equality is due to Lemma 6.5. This concludes the proof of Theorem 1.5. For later purposes we point out the following corollary.

Corollary 6.7. *We have*

$$a_{\mathbb{X}}(L,\mathcal{P}_m,\overline{z})>0$$

if and only if $h(t)$ *divides* $\pi_p(\Delta(L)(t^z))$ *in* Λ_p.

Proof. Theorem 1.5 implies that $a_{\mathbb{X}}(L,\mathcal{P}_m,\overline{z})>0$ if and only if $h(t)$ divides $e_i^{(p)}(L)(t^z)$ for some $i\geq 1$. The conclusion follows from the fact that

$$\pi_p(\Delta(L)(t^z))=\Delta_1^{(p)}(L)(t^z)=\prod_{i=1}^{\infty}e_i^{(p)}(L)(t^z).\qquad\square$$

Remark 6.8. On may wonder if the equality $\pi_p(e_i(L)(t))\doteq e_i^{(p)}(L)(t)$ holds for every $i\geq 1$, so that in the statement of Theorem 1.5 we could replace the summand

$$\Lambda_p/(e_i^{(p)}(L)(t^z),h(t))$$

with the module

$$\Lambda_p/(\pi_p(e_i(L)(t^z)), h(t)).$$

Such a claim seems also suggested, at least when L is a knot and $\overline{z} = 1$, by the original statement of [8, Theorem 1]. However, this is not the case, as the following construction shows.

In fact, let $k_1(t) = t - 1 + t^{-1}$ and $k_2(t) = -2t + 5 - 2t^{-1}$, and observe that $k_i(t^{-1}) = k_i(t)$, $i = 1, 2$. It is proved in [12, Theorem 2.5] that a knot K exists such that $A(K)$ is presented by the matrix

$$\mathrm{diag}(k_1(t), k_1(t), k_2(t), k_2(t)).$$

This readily implies that

$$E_0(K) = (k_1(t)^2 k_2(t)^2), \qquad E_1(K) = (k_1(t)^2 k_2(t), k_1(t)k_2(t)^2),$$
$$E_2(K) = (k_1(t)^2, k_2(t)^2, k_1(t)k_2(t)), \quad E_3(K) = (k_1(t), k_2(t))$$

and $E_i(K) = \Lambda$ for every $i \geq 4$, whence

$$\Delta_1(K)(t) = k_1(t)^2 k_2(t)^2, \quad \Delta_2(K)(t) = k_1(t)k_2(t),$$

and $\Delta_i(K)(t) = 1$ for every $i \geq 3$. As a consequence we get

$$e_1(K)(t) = e_2(K)(t) = k_1(t)k_2(t), \quad e_i(K)(t) = 1 \quad \text{for every } i \geq 2.$$

On the other hand, let us fix $p = 3$, and observe that in this case $\pi_3(k_1(t)) = \pi_3(k_2(t)) = k(t) \in \Lambda_3$, where $k(t) = (t+1)^2$. Therefore, from the equality $E_i^{(3)}(K) = \pi_p(E_i(K)(t))$ (see Corollary 6.6) we easily deduce that

$$\Delta_1^{(3)}(K) = k(t)^4, \quad \Delta_2^{(3)}(K) = k(t)^3, \quad \Delta_3^{(3)}(K) = k(t)^2, \quad \Delta_4^{(3)}(K) = k(t),$$

and $\Delta_i^{(3)}(K) = 1$ for every $i \geq 5$, so

$$e_1^{(3)}(K)(t) = e_2^{(3)}(K)(t) = e_3^{(3)}(K)(t) = e^{(4)}(K)(t) = k(t),$$
$$e_i^{(3)}(K)(t) = 1 \quad \text{for every } i \geq 5.$$

Therefore, if $h(t) = t + 1 \in \Lambda_3$, then we have

$$\bigoplus_{i=1}^{\infty} \Lambda_3/(e_i^{(3)}(K)(t), h(t)) \cong \mathbb{F}(3, h(t))^4,$$

while

$$\bigoplus_{i=1}^{\infty} \Lambda_3/(\pi_3(e_i(K)(t)), h(t)) \cong \mathbb{F}(3, h(t))^2.$$

Remark 6.9. Let $k_1(t), k_2(t) \in \Lambda$ and $k(t) \in \Lambda_3$ be the polynomials introduced in the previous remark. It is proved in [13] that a knot K' exists whose module $A(K')$ is isomorphic to $\Lambda/(k_1(t)k_2(t))$ (see also [15, Theorem 7.C.5]). Let $K'' = K' + K'$. Then we have $A(K'') = A(K') \oplus A(K')$ (see e.g. [15, Theorem 7.E.1]), and this readily implies that

$$E_0(K'') = (k_1(t)^2 k_2(t)^2), \quad E_1(K'') = (k_1(t)k_2(t)),$$

and $E_i(K'') = \Lambda$ for every $i \geq 2$. Therefore,

$$\Delta_1(K'')(t) = k_1(t)^2 k_2(t)^2, \quad \Delta_2(K'')(t) = k_1(t)k_2(t),$$

and $\Delta_i(K'')(t) = 1$ for every $i \geq 3$. Moreover, since the elementary ideals of K'' are principal, we also have

$$\Delta_1^{(3)}(K'')(t) = \pi_3(\Delta_1(K)(t)) = k(t)^4, \quad \Delta_2^{(3)}(K'')(t) = \pi_3(\Delta_2(K)(t)) = k(t)^2,$$

and $\Delta_i^{(3)}(K'')(t) = \pi_3(\Delta_i(K)(t)) = 1$ for every $i \geq 3$.

Therefore, the knot K'' and the knot K introduced in the previous remark satisfy the condition $\Delta_i(K)(t) = \Delta_i(K'')(t)$ for every $i \geq 1$, but have a different number of $\mathcal{D}_3$-colorings with respect to the cycle $\overline{z} = \overline{1}$. What is more, since for every p the Λ_p-module $A^{(p)}(K)$ (respectively, $A^{(p)}(K'')$) admits a square presentation matrix of order 4 (respectively, of order 2), Theorem 1.5 readily implies that $\mathcal{A}_{\mathcal{Q}}(K) \leq 4$ (respectively, $\mathcal{A}_{\mathcal{Q}}(K'') \leq 2$). Our computations imply now that $\mathcal{A}_{\mathcal{Q}}(K) = 4$ and $\mathcal{A}_{\mathcal{Q}}(K'') = 2$. Therefore, even if they share every Alexander polynomial $\Delta_i(K)(t) = \Delta_i(K'')(t)$, $i \geq 1$, the knots K, K'' are distinguished from each other by the invariant $\mathcal{A}_{\mathcal{Q}}$.

7. Comparing $\mathcal{A}_{\mathcal{Q}}$ with $\mathcal{A}$

Let us keep notation from the preceding Section. Of course, since $\mathbb{F}(p, h(t))$ is a field, the quotient $\Lambda_p/(e_i^{(p)}(L)(t^z), h(t))$ of $\mathbb{F}(p, h(t))$ is null (respectively, isomorphic to $\mathbb{F}(p, h(t))$) if and only if $h(t)$ divides (respectively, does not divide) $e_i^{(p)}(L)(t^z)$ in Λ_p. Therefore, if we set

$$I(z, p, h(t), L) = \{i \in \mathbb{N}^* \mid h(t) \text{ divides } e_i^{(p)}(L)(t^z)\},$$

$$|I(z, L)| = \sup_{p, h(t)} |I(z, p, h(t), L)|,$$

$$|I(L)| = \sup_{z} |I(z, L)|,$$

then we easily obtain that

$$\mathcal{A}_{\mathcal{Q}}(L) = |I(L)|.$$

Therefore, in order to prove Theorem 1.6 it is sufficient to show that, if $L = K$ is a knot, then:

- $I(K) \leq \mathcal{A}(K)$,
- $I(K) = 0$ if and only if $\mathcal{A}(K) = 0$.

7.1. *Reduction to the cycle $\overline{z} = 1$*

We first prove that, in order to compute $\mathcal{A}_{\mathcal{Q}}(K)$, it is sufficient to restrict our attention to colorings relative to the cycle $\overline{z} = \overline{1}$.

Lemma 7.1. *We have*

$$I(L) = I(1, L).$$

In the proof we use the following elementary lemma.

Lemma 7.2. *Let $p_1(t), \ldots, p_n(t)$ be polynomials in Λ_p, and let $d(t)$ be their G.C.D. in Λ_p. For every integer $z \geq 1$, the polynomial $d(t^z) \in \Lambda_p$ is the G.C.D. of $p_1(t^z), \ldots, p_n(t^z)$ in Λ_p.*

Proof. For every $i = 1, \ldots, n$, the fact that $d(t)$ divides $p_i(t)$ readily implies that $d(t^z)$ divides $p_i(t^z)$. On the other hand, Λ_p is P.I.D., so Bezout's Identity implies that there exist $\lambda_1(t), \ldots, \lambda_n(t) \in \Lambda_p$ such that

$$d(t) = \lambda_1(t)p_1(t) + \cdots + \lambda_n(t)p_n(t),$$

whence

$$d(t^z) = \lambda_1(t^z)p_1(t^z) + \cdots + \lambda_n(t^z)p_n(t^z).$$

Therefore, if $d'(t)$ divides every $p_i(t^z)$, then $d'(t)$ also divides $d(t^z)$, whence the conclusion. $\square$

Proof of Lemma 7.1. It is sufficient to show that, for every odd prime p, every positive integer z and every irreducible polynomial $h(t) \in \Lambda_p$, there exists an irreducible polynomial $h'(t) \in \Lambda_p$ such that

$$|I(z, p, h(t), L)| \leq |I(1, p, h'(t), L)|.$$

Let $d(t) \in \Lambda_p$ be the G.C.D. of the polynomials $\{e_i^{(p)}(t), i \in I(z, p, h(t), L)\}$. By the very definitions, $h(t)$ divides $e_i^{(p)}(t^z)$ for every $i \in I(z, p, h(t), L)$, so by Lemma 7.2 we have that $h(t)$ divides $d(t^z)$. This implies that the breadth of $d(t)$ is positive, so $d(t)$ admits an irreducible factor $h'(t)$ of positive breadth. By construction we have that $h'(t)$ divides $e_i^{(p)}(t)$ for every $i \in I(z, p, h(t), L)$, so $I(z, p, h(t), L) \subseteq I(1, p, h'(t), L)$, whence the conclusion. $\square$

7.2. *More details on Alexander ideals of links*

Recall that $\widetilde{X}(L)$ is the total linking number covering of the complement of L, and that k denotes the number of components of L. If $x_0 \in \mathsf{C}(L)$ is any basepoint and $\widetilde{X}_0$ is the preimage of x_0 in $\widetilde{X}(L)$, then the relative homology module $A'(L) = H_1(\widetilde{X}(L), \widetilde{X}_0; \mathbb{Z})$ also admits a natural structure of Λ-module. Moreover, it is not difficult to show that $A'(L) \cong A(L) \oplus \Lambda$ (as Λ-modules), so $E_i(A(L)) = E_{i+1}(A'(L))$ for every $i \in \mathbb{N}$, and $\Delta_i(L)(t) \in \Lambda$ is the generator of the smallest principal ideal containing $E_i(A'(L))$. If $L = K$ is a knot, this immediately implies that $\Delta_i(L)(t) \in \Lambda$ coincides with the so called *ith Alexander polynomial of K*.

Lemma 7.3. *We have*

$$\Delta_i^{(p)}(L)(t) \neq 0 \quad \text{for every } i \geq k.$$

Proof. Let $\widehat{X}(L)$ be the maximal abelian covering of $\mathsf{C}(L)$ and let $\widehat{X}^0$ be the preimage of x_0 in $\widehat{X}(L)$. Then the homology group $\widehat{A}(L) = H_1(\widehat{X}(L), \widehat{X}^0; \mathbb{Z})$ admits a natural structure of $\mathbb{Z}[t_1, t_1^{-1}, \ldots, t_k, t_k^{-1}]$-module (see e.g. [6, 7]). Just as in the case of the total linking number covering, one may define the ith elementary ideal $\widehat{E}_i(L) \subseteq \mathbb{Z}[t_1, t_1^{-1}, \ldots, t_k, t_k^{-1}]$ of this module. If $\tau : \mathbb{Z}[t_1, t_1^{-1}, \ldots, t_k, t_k^{-1}] \to \Lambda$ is the ring homomorphism that sends each $t_i^{\pm 1}$ into $t^{\pm 1}$, it is not difficult to show that

$$E_i(L) = E_{i+1}(A'(L)) = \tau(E_{i+1}(\widehat{A}(L)))$$

(see e.g. [7, p. 106]).

Let now $\varepsilon : \mathbb{Z}[t_1, t_1^{-1}, \ldots, t_k, t_k^{-1}] \to \mathbb{Z}$ be the *augmentation homomorphism* defined by $\varepsilon(f(t_1, \ldots, t_k)) = f(1, \ldots, 1)$. A classical result about Alexander ideals of links (see e.g. [7, Lemma 4.1]) ensures that $\varepsilon(E_k(\widehat{A}(L))) = \mathbb{Z}$. In particular, there exists $g(t) \in E_k(\widehat{A}(L))$ such that $g(1, \ldots, 1) = 1$. Let us set $f(t) = \tau(g(t)) \in E_{k-1}(L)$ and $f^{(p)}(t) = \pi_p(f(t))$. Our choices readily imply that $f^{(p)}(1) = 1$ in $\mathbb{Z}_p$, so $f^{(p)}(t) \neq 0$ in Λ_p. By Corollary 6.6(4), the polynomial $\Delta_k^{(p)}(L)(t)$ divides $f^{(p)}(t)$ in Λ_p, so $\Delta_i^{(p)}(t)$ is not null for every $i \geq k$. □

Corollary 7.4. *We have*

$$e_i^{(p)}(L)(t) \neq 0 \quad \text{for every } i \geq k.$$

7.3. *Proof of Theorem* 1.6

The key step for proving Theorem 1.6 is the following proposition.

Proposition 7.5. *If L is a k-component link, then*

$$I(1, p, h(t), L) \leq \frac{\operatorname{br} \Delta_k^{(p)}(L)(t)}{\operatorname{br} h(t)} + k - 1.$$

Proof. By the very definitions we have

$$\Delta_k^{(p)}(L)(t) = \prod_{i \geq k} e_i^{(p)}(L)(t),$$

so (since $\Delta_k^{(p)}(L)(t) \neq 0$ by Lemma 7.3)

$$\operatorname{br} \Delta_k^{(p)}(t) = \sum_{i \geq k} \operatorname{br} e_i^{(p)}(L)(t).$$

Let us now set $I'(1, p, h(t), L) = I(1, p, h(t), L) \cap \{i \in \mathbb{N} \mid i \geq k\}$. In order to conclude it is sufficient to show that

$$|I'(1, p, h(t), L)| \leq \frac{\operatorname{br} \Delta_k^{(p)}(t)}{\operatorname{br} h(t)}.$$

By Corollary 7.4, if $i \in I'(1, p, h(t), L)$ then $e_i^{(p)}(L)(t)$ is not null and divisible by $h(t)$, so $\operatorname{br} e_i^{(p)}(L)(t) \geq \operatorname{br} h(t)$. This readily implies that

$$\operatorname{br} \Delta_k^{(p)}(t) \geq \sum_{i \in I'(1,p,h(t),L)} \operatorname{br} e_i^{(p)}(L)(t) \geq |I'(1, p, h(t), L)| \cdot \operatorname{br} h(t),$$

whence the conclusion. □

Let us now point out the following lemma.

Lemma 7.6. *If L is any link, then*

$$\mathcal{A}_{\mathcal{Q}}(L) = 0 \Rightarrow \mathcal{A}(L) = 0.$$

Proof. Recall from Corollary 6.6 that $\Delta_1^{(p)}(L)(t) = \pi_p(\Delta(L)(t))$. As a consequence, if $\mathcal{A}(L) > 0$, then $\operatorname{br} \Delta^{(p)}(L)(t) > 0$ for some odd prime p (just choose p to be larger than the absolute value of all the coefficients of $\Delta(L)(t)$). This implies that $\operatorname{br} e_{i_0}^{(p)}(t) > 0$ for some $i_0 \in \mathbb{N}$. If $h(t)$ is any irreducible factor of $e_{i_0}^{(p)}(t)$ in Λ_p, then $i_0 \in I(1, p, h(t), L)$, so $\mathcal{A}_{\mathcal{Q}}(L) \geq I(1, p, h(t), L) = 1$. □

The following corollary readily implies Theorem 1.6.

Corollary 7.7. *If $L = K$ is a knot, then*

$$I(1, p, h(t), K) \leq \frac{\mathcal{A}(K)}{\operatorname{br} h(t)} \leq \mathcal{A}(K),$$

$$\mathcal{A}_{\mathcal{Q}}(K) = I(K) = I(1, K) \leq \mathcal{A}(K),$$

$$\mathcal{A}_{\mathcal{Q}}(K) = 0 \Leftrightarrow \mathcal{A}(K) = 0.$$

Proof. By Corollary 6.6(3) we have $\operatorname{br} \Delta_1^{(p)}(K)(t) \leq \operatorname{br} \Delta(K)(t) = \mathcal{A}(K)$, so the first inequality follows immediately from Proposition 7.5. As a consequence, we have $I(1, K) \leq \mathcal{A}(K)$, so the second inequality is a consequence of Lemma 7.1. The fact that $\mathcal{A}_{\mathcal{Q}}(K) = 0$ if and only if $\mathcal{A}(K) = 0$ easily follows from the second inequality and Lemma 7.6. □

7.4. *Computing $\mathcal{A}_\mathcal{Q}$ via proper subfamilies of $\mathcal{Q}_\mathcal{F}$*

This subsection is devoted to determine proper subfamilies of $\mathcal{Q}_\mathcal{F}$ that carry the whole information about the invariant $\mathcal{A}_\mathcal{Q}$. We will be mainly interested in the case when $L = K$ is a knot (some considerations below hold more generally for $(L, \mathcal{P}_m)$).

Let K be a knot, and recall that $\delta(K)$ has been defined in Subsec. 1.7. We begin with the following lemma.

Lemma 7.8. *Let $f(t) \in \mathbb{Z}[t]$ be a polynomial and suppose that there exist prime numbers $p_1, \ldots, p_k$ such that*

$$f(n) = \pm p_1^{\alpha_1(n)} \cdot \ldots \cdot p_k^{\alpha_k(n)} \quad \text{for every } n \geq n_0,$$

where $n_0 \in \mathbb{N}$ is fixed. Then $f(t)$ is constant.

Proof. Let $d = \deg f(t)$, and take $h > 0$ such that $|f(n)| \leq hn^d$. Then $\alpha_1(n) \ln p_1 + \cdots + \alpha_k(n) \ln p_k \leq d \ln n + \ln h$ for every $n \geq n_0$. In particular, there exists a constant $w \geq 1$ such that $\alpha_i(n) \leq w \ln n$ for every $n \geq n_0$. Therefore, if n_1 is such that $(n_1 - n_0) > 2(d + 1)(w \ln n_1)^k$, then the interval $[n_0, n_1]$ contains (at least) $2(d+1)$ integers $m_1, \ldots, m_{2(d+1)}$ such that $\alpha_i(m_j) = \alpha_i(m_{j'})$ for every $i = 1, \ldots, k$, $j, j' = 1, \ldots, 2(d + 1)$, whence $f(m_i) = \pm f(m_{j'})$ for every $j, j' = 1, \ldots, 2(d + 1)$. It follows that f takes the same value on at least $d + 1$ distinct integers. Since $\deg f = d$, this implies in turn that f is constant. □

We now prove Proposition 1.10, which we recall here for the convenience of the reader.

Proposition 7.9. *Let K be a knot.*

(1) *$\theta(K) = 1$ if and only if $\mathcal{A}(K) = \mathcal{A}_\mathcal{Q}(K) = 0$. If $\theta(K) > 1$, then $\theta(K) \geq \delta(K) + 1$.*
(2) *If $\mathcal{A}_\mathcal{Q}(K) = 1$, then $\delta(K) = 1$.*
(3) *If $\mathcal{A}(K) > 0$, then*

$$\delta(K) \leq \frac{\mathcal{A}(K)}{\max\{2, \mathcal{A}_\mathcal{Q}(K)\}}.$$

(4) *Suppose that $\mathcal{A}_\mathcal{Q}(K) = \mathcal{A}(K)$ or $\mathcal{A}_\mathcal{Q}(K) = \mathcal{A}(K) - 1$. Then $\delta(K) = 1$. Moreover, there exist an odd prime p and an element $a \in \mathbb{Z}_p^*$ such that $(t - a)^{\mathcal{A}_\mathcal{Q}(K)}$ divides $\Delta_1^{(p)}(K)(t)$ in Λ_p.*
(5) *If $\mathcal{A}_\mathcal{Q}(K) = \mathcal{A}(K)$, then $\delta(K) = 1$ and there exist an odd prime p and an element $a \in \mathbb{Z}_p^*$ such that $\Delta_1^{(p)}(K)(t) \doteq (t - a)^{\mathcal{A}(K)}$ in Λ_p.*

Proof. (1) By definition, $\theta(K) = 1$ if and only if $\mathcal{A}_\mathcal{Q}(K) = a_\mathbb{X}(K, 1)$ for a trivial quandle $\mathbb{X}$. However, if $\mathbb{X}$ is trivial, then $a_\mathbb{X}(K, 1) = 0$, so if $\theta(K) = 1$ then $\mathcal{A}_\mathcal{Q}(K) = 0$ (and $\mathcal{A}(K) = 0$ by Theorem 1.6). On the other hand, if $\mathcal{A}_\mathcal{Q}(K) = 0$

then $\mathcal{A}_{\mathcal{Q}}(K) = a_{\mathbb{X}}(K,1)$ for any trivial quandle $\mathbb{X}$, so $\theta(K) = 1$. Suppose now that $\theta(K) > 1$, and that $\mathcal{A}_{\mathcal{Q}}(K) = |I(1,p,h(t),K)|$, where $\operatorname{br} h(t) = \delta(K)$. Since $\theta(K) > 1$, the quandle $\mathbb{X} = (\mathbb{F}(p,h(t)),*)$ cannot be trivial, so Lemma 2.1 implies that $t_{\mathbb{X}} \geq \delta(K)+1$.

(2) By Theorem 1.6 we have $\mathcal{A}(K) > 0$, whence $\operatorname{br}\Delta(K)(t) > 0$. Let $f(t) \in \mathbb{Z}[t] \subseteq \Lambda$ be such that $f(t) \doteq \Delta(K)(t)$ and $f(0) \neq 0$, so that $\deg f(t) = \operatorname{br}\Delta(K)(t) = d > 0$. By Lemma 7.8, there exists $n > |f(0)|$ such that $f(n)$ is divided by a prime number $p > |f(0)|$. Let a be the class of n in $\mathbb{Z}_p$, and let us set $h(t) = t - a \in \mathbb{Z}_p[t] \subset \Lambda_p$. Since p divides $f(n)$, we have that $h(t)$ divides $\Delta_1^{(p)}(L)(t) = \pi_p(\Delta(L)(t))$ in Λ_p. Also observe that p does not divide $f(0)$, so p does not divide $f(n) - f(0)$, and this readily implies that $a \neq 0$ in $\mathbb{Z}_p$. It follows that $h(t)$ is irreducible of positive breadth in Λ_p, so we may set $\mathbb{X} = (\mathbb{F}(p,h(t)),*)$. By construction we have $a_{\mathbb{X}}(K,1) \geq 1 = \mathcal{A}_{\mathcal{Q}}(K)$, so $\delta(K) = 1$.

(3) It is well-known that $\mathcal{A}(K) = \operatorname{br}\Delta(K)(t)$ is even, so $\mathcal{A}(K) > 0$ implies that $\mathcal{A}(K) \geq 2$. Together with the inequality $\mathcal{A}_{\mathcal{Q}}(K) \leq \mathcal{A}(K)$, this implies that $\mathcal{A}(K)/\max\{2, \mathcal{A}_{\mathcal{Q}}(K)\} \geq 1$, so we may suppose $\delta(K) > 1$, whence $\mathcal{A}_{\mathcal{Q}}(K) \geq 2$ (see point (2)).

Suppose now that $\mathcal{A}_{\mathcal{Q}}(K) = I(1,p,h(t),K)$, where $\delta(K) = \operatorname{br} h(t)$. Corollary 7.7 implies that $\mathcal{A}_{\mathcal{Q}}(K) = I(1,p,h(t),K) \leq \mathcal{A}(K)/\delta(K)$, whence the conclusion.

(4) The case $\mathcal{A}(K) = 0$ is trivial, so the first statement is an immediate consequence of (3). Then, we may choose $h(t) = (t-a) \in \Lambda_p$, $a \in \mathbb{Z}_p^*$, in such a way that $\mathcal{A}_{\mathcal{Q}}(K) = |I(1,p,h(t),K)|$. Now $h(t)$ divides $e_i^{(p)}(K)(t)$ for every $i \in I(1,p,h(t),K)$, so $h(t)^{\mathcal{A}_{\mathcal{Q}}(K)}$ divides $\prod_{i \in I(1,p,h(t),K)} e_i^{(p)}(K)(t)$, which divides in turn $\Delta_1^{(p)}(K)(t)$.

(5) is an immediate consequence of (4). □

Remark 7.10. As mentioned in Question 1.11, we are not able to prove that $\delta(K)$ may be arbitrarily large. Let us point out some difficulties that one has to face in order to prove (or disprove) such a statement. If one tries to construct a knot K with $\delta(K) \geq n$, one has to find K and $\mathbb{X} = (\mathbb{F}(p,h(t),*)$ such that $\mathcal{A}_{\mathcal{Q}}(K) = a_{\mathbb{X}}(K,1) = |I(1,p,h(t),K)|$ and $\operatorname{br} h(t) = n$. Once this has been established, the inequality $\delta(K) \leq n$ is proved. In order to show that $\delta(K) = n$ we are left to prove that for every odd prime q the number of polynomials $e_i^{(q)}(K)(t)$, $i \in \mathbb{N}$, admitting a common factor of breadth at most $n-1$ is strictly less than $\mathcal{A}_{\mathcal{Q}}(K)$. One may probably start with a knot K whose Alexander polynomial $\Delta(K)(t)$ decomposes as the product of irreducible factors of large breadth. This would ensure that also the $e_i(K)(t)$'s have large breadth. However, it is not clear how to control the breadth (and the existence of common divisors) of the $e_i^{(q)}(K)(t)$'s, when q is a generic prime, even under the hypothesis that $I(1,p,h(t),K)$ collects a maximal subset of indices such that the corresponding $e_i^{(p)}(K)(t)$'s have a non-trivial common divisor (and such a divisor has breadth n).

8. Genus-1 Knots

In this section we fully describe the case of knots that admit a Seifert surface of genus 1 (that is, knots of genus 1 or the unknot).

Let us fix a quandle $\mathbb{X} = (\mathbb{F}(p, h(t)), *)$, take $z \in \mathbb{N}$ and consider a special diagram $(\mathcal{D}(T), z)$ of (K, z) (see Sec. 3).

By Lemma 6.1, the integer $a_{\mathbb{X}}(K, z)$ is equal to the dimension of $\ker N(z, p, h(t))$, where

$$N(z,p,h(t)) = \begin{pmatrix} (\overline{t}^z - 1)M_{1,1} & \overline{t}^z(M_{1,2} - 1) - M_{1,2} \\ \overline{t}^z(M_{2,1} + 1) - M_{2,1} & (\overline{t}^z - 1)M_{2,2} \end{pmatrix}$$

(recall that $\overline{t}$ denotes the class of t in $\mathbb{F}(p, h(t))$).

Recall that the *determinant* $\det K$ of K is defined as $\det K = |\Delta(K)(-1)| = |\det(S(L) + S(L)^T)|$, where $S(L)$ is a Seifert matrix for K.

Lemma 8.1. *We have*

$$M_{1,2} - M_{2,1} = 1, \quad \det K = |4 \det M - 1|.$$

Proof. The first equality is an immediate consequence of Lemma 6.3. Putting together Lemmas 6.2 and 6.3 we also obtain $S(L) + S(L)^T = 2M + J$, whence the conclusion. □

Proposition 8.2. *Let K be a knot such that $g(K) = 1$, let us take $z \in \mathbb{N}$ and a quandle $\mathbb{X} = (\mathbb{F}(p, h(t)), *) \in \mathcal{Q}_{\mathcal{F}}$. Then*

$$a_{\mathbb{X}}(K, z) = 2$$

if and only if for (one, and hence for) every special diagram $(\mathcal{D}(T), z)$ of (K, z) the following conditions hold:

$$M_{1,1} = M_{2,2} = 0, \quad M_{1,2} = \frac{p+1}{2}, \quad M_{2,1} = \frac{p-1}{2} \quad \text{in } \mathbb{Z}_p$$

and

$$h(t) \,|\, (1 + t^z) \quad \text{in } \Lambda_p.$$

Proof. If $\mathbb{X}$ is trivial, then for every z we have $a_{\mathbb{X}}(K, z) = 0$. Moreover, $h(t) = t - 1$ does not divide $1 + t^z$, so we may suppose that $\mathbb{X}$ is non-trivial. Suppose now that $z = 0$ in $\mathbb{Z}_{t_{\mathbb{X}}}$. Then we have $a_{\mathbb{X}}(K, z) = 0$, and $h(t)$ has to divide $1 - t^z$ by Lemma 2.1(2). As a consequence, $h(t)$ cannot divide $1 + t^z$, so the conclusion holds also in this case. We may therefore assume that $z \neq 0$ in $\mathbb{Z}_{t_{\mathbb{X}}}$.

Observe that $a_{\mathbb{X}}(K, z) = 2$ if and only if $N(z, p, h(t)) = 0$, i.e. if and only if

$$(\overline{t}^z - 1)M_{1,1} = 0, \qquad (\overline{t}^z - 1)M_{2,2} = 0,$$
$$\overline{t}^z(M_{1,2} - 1) - M_{1,2} = 0, \quad \overline{t}^z(M_{2,1} + 1) - M_{2,1} = 0.$$

Since $z \neq 0$ in $\mathbb{Z}_{t_{\mathbb{X}}}$, we have $\overline{t}^z - 1 \neq 0$, so the first and the second relations give $M_{1,1} = M_{2,2} = 0$. By Lemma 8.1, the third equation can be rewritten as

$M_{2,1}\overline{t}^z = M_{2,1}+1$. Together with the fourth equation, this immediately implies that $2M_{2,1} = -1$, whence $M_{2,1} = (p-1)/2$, and $M_{1,2} = (p+1)/2$ again by Lemma 8.1. Under these conditions, the third and the fourth equations are equivalent to the fact $\overline{t}^z + 1 = 0$, i.e. to the fact that $h(t)$ divides $t^z + 1$ in Λ_p. □

Proposition 8.2 readily implies the following corollary.

Corollary 8.3. *With notations as in Proposition* 8.2, *we have*

$$\mathcal{A}_{\mathcal{Q}}(K) = 2$$

if and only if there exist a special diagram $\mathcal{D}(T)$ *of* K *and a prime number* $p \geq 3$ *such that the following equalities hold in* $\mathbb{Z}_p$:

$$M_{1,1} = M_{2,2} = 0, \quad M_{1,2} = \frac{p+1}{2}, \quad M_{2,1} = \frac{p-1}{2}.$$

Moreover, in this case there exists a dihedral quandle $\mathbb{X} \in \mathcal{Q}_{\mathcal{F}}(1)$ *such that* $a_{\mathbb{X}}(K,1) = 2$, *so that*

$$\delta(K) = 1, \quad \theta(K) = 2.$$

Remark 8.4. Notice that if a special diagram $(\mathcal{D}(T), z)$ verifies the conditions of Corollary 8.3 that involve $M_{1,2}$ and $M_{2,1}$, then we can easily realize also the conditions on $M_{1,1}$ and $M_{2,2}$ via suitable Reidemeister moves of the first type (i.e. by "adding kinks") on the two strings of T.

Let us now discuss the conditions under which $a_{\mathbb{X}}(K,z) = 0$ for every $\mathbb{X} \in \mathcal{Q}_{\mathcal{F}}$, $z \in \mathbb{Z}$. We begin with the following lemmas.

Lemma 8.5. *We have*

$$\Delta(K)(t) \doteq \det M + (1 - 2\det M)t + (\det M)t^2.$$

Proof. Corollary 6.4 implies that

$$\Delta(K)(t) = \det((t-1)M + tJ) = (t-1)^2 M_{1,1}M_{2,2} - ((t-1)M_{1,2} - t)((t-1)M_{2,1} + t).$$

Since $M_{1,2} - M_{2,1} = 1$, the conclusion follows. □

Corollary 8.6. *The integer*

$$W(K) = \det M$$

is a well-defined invariant of K, *i.e. it does not depend on the special diagram of* K *defining* M. *Moreover*, $\Delta(K)(t) \doteq \Delta(K')(t)$ *if and only if* $W(K) = W(K')$.

Proof. Suppose that $\mathcal{D}'(T)$ is a special diagram of K of genus 1, and let M' be the matrix encoding the linking numbers of the strings of T. By Lemma 8.5 we have

$$\det M + (1 - 2\det M)t + (\det M)t^2 \doteq \det M' + (1 - 2\det M')t + (\det M')t^2,$$

so $\det M = \pm \det M'$, $1 - 2\det M = \pm(1 - 2\det M')$, and $\det M = \det M'$. □

Putting together Lemmas 8.5 and 6.7 we readily get the following proposition.

Proposition 8.7. *Let K be a knot such that $g(K) = 1$, let us take $z \in \mathbb{N}$ and a quandle $\mathbb{X} = (\mathbb{F}(p, h(t)), *) \in \mathcal{Q}_{\mathcal{F}}$. Then*

$$a_{\mathbb{X}}(K, z) \geq 1$$

if and only if

$$h(t) \,|\, W(K) + (1 - 2W(K))t^z + W(K)t^{2z} \quad \text{in } \Lambda_p.$$

In particular, if $W(K) = 0$ then $a_{\mathbb{X}}(K, z) = 0$.

Corollary 8.8. *Let K be a knot, such that $g(K) \leq 1$. Then $\mathcal{A}_{\mathcal{Q}}(K) = 0$ if and only if $W(K) = 0$ (due to Lemma 8.1, this condition is equivalent to $\det K = 1$). In all the other cases there exists a dihedral quandle $\mathbb{X} \in \mathcal{Q}_{\mathcal{F}}(1)$ such that $\mathcal{A}_{\mathcal{Q}}(K) = a_{\mathbb{X}}(K, 1) \geq 1$. We have in particular*

$$\delta(K) = 1, \quad \theta(K) = 2.$$

Proof. By Proposition 8.7, it is sufficient to show that, if $W(K) \neq 0$, then there exists a dihedral quandle $\mathbb{X} \in \mathcal{Q}_{\mathcal{F}}(1)$ such that $\mathcal{A}_{\mathcal{Q}}(K) = a_{\mathbb{X}}(K, 1) \geq 1$.

In fact, if $W(K) \neq 0$ we may choose an odd prime p be dividing $1 - 4W(K)$. Then the polynomial $1 + t$ divides $W(K)t^2 + (1 - 2W(K))t + W(K)$ in Λ_p. By Proposition 8.7, this implies that $a_{\mathbb{X}}(K, 1) \geq 1$, where $\mathbb{X}$ is the dihedral quandle $\mathbb{X} = (F(p, 1 + t), *)$. □

Remark 8.9. By Corollary 8.8, every genus-1 knot such that $\mathcal{A}_{\mathcal{Q}}(K) \geq 1$ is such that $\delta(K) = 1$. Since for every such knot we obviously have $\mathcal{A}(K) \leq 2g(K) = 2$, this fact is also a consequence of Proposition 1.10.

8.1. *A few manipulations on special diagrams*

Here below we describe a few simple manipulations on (genus-1) special diagrams, which are useful to construct large families of examples.

Lemma 8.10. *Let K be a genus-1 knot with $a_{\mathbb{X}}(K, z) = 2$ for some quandle $\mathbb{X} \in \mathcal{Q}_{\mathcal{F}}$, and let $(\mathcal{D}(T), z)$ be a special diagram of (K, z). Then by adding kinks to only one of the two strings of T we can arbitrarily modify either $M_{1,1}$ or $M_{2,2}$, so that the resulting $(D(T'), z)$ is a special diagram of some (K', z) such that $a_{\mathbb{X}}(K', z) = 1$.*

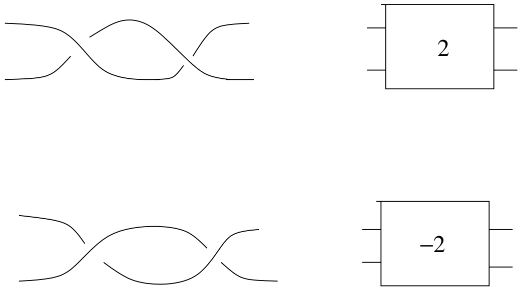

Fig. 12. Linking moves.

Lemma 8.11. *Let K be a genus-1 knot with $a_{\mathbb{X}}(K, z) \geq 1$ for some quandle $\mathbb{X} = (\mathbb{F}(p, h(t)), *)$, and let $(\mathcal{D}(T), z)$ be a special diagram of (K, z). Let us modify T by means of any sequence of usual second and third Reidemeister moves, of first Reidemeister moves provided that $M_{1,1}$ and $M_{2,2}$ are kept constant mod (p), and of positive (negative) linking moves between the two strings, (see Fig. 12; here the actual sign of the move depends also on the omitted orientations of the strings), provided that their number is equal to 0 mod (p). Then we get a special diagram $(\mathcal{D}(T'), z)$ of some (K', z) such that $a_{\mathbb{X}}(K, z) = a_{\mathbb{X}}(K', z) \geq 1$.*

8.2. *The dihedral case*

Let us specialize the results above to the simplest case of dihedral quandles, i.e. to the case when $\mathbb{X} = \mathbb{D}_p = (\mathbb{F}(p, 1+t), *)$ and $z = 1$. In such a case we simply write

$$a_p(K) = a_{\mathbb{D}_p}(K, 1).$$

The following result is an immediate consequence of Propositions 8.2 and 8.7.

Lemma 8.12. *Let K be a knot such that $g(K) = 1$, represented by a special diagram $\mathcal{D}(T)$.*

(1) *$a_p(K) = 2$ if and only the following system of relations is satisfied in $\mathbb{Z}_p$:*

$$M_{1,2} = \frac{p+1}{2}, \quad M_{2,1} = \frac{p-1}{2}, \quad M_{2,2} = 0, \quad M_{1,1} = 0.$$

(2) *$a_p(K) \geq 1$ if and only if*

$$1 - 4W(K) = 0 \quad \text{in } \mathbb{Z}_p.$$

By Lemma 8.1, this condition holds if and only if p divides $\det K$.

Now we want to show that, for every $p > 2$, the first set of conditions in the last lemma can be actually realized by a special diagram $\mathcal{D}(T_p)$ of some knot K_p. Let us consider the tangle of Fig. 13. Here $p = 2k+1$ and there are k (respectively,

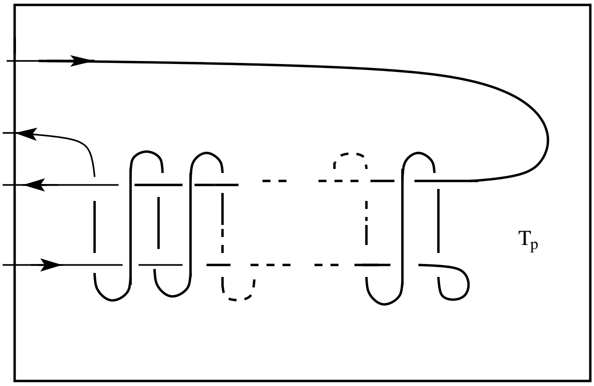

Fig. 13. The tangle $T(p)$.

$k+1$) overcrossing (respectively, undercrossing) vertical strands. So it is immediate to verify that (in $\mathbb{Z}$):

$$M_{1,1} = 0, \quad M_{2,2} = p, \quad M_{1,2} = -\frac{p-1}{2}, \quad M_{2,1} = -\frac{p+1}{2},$$

whence

$$1 - 4W(K) = 1 - 4 \det M = p^2.$$

Hence have the following lemma.

Lemma 8.13. *The family* $\{K_p\}$ *of genus*-1 *knots constructed above is such that* $a_p(K_p) = 2$ *while* $a_{p'}(K_p) = 0$ *for every* $p' \neq p$. *In particular*, K_p *is not isotopic to* $K_{p'}$ *if* $p \neq p'$, *and* $\mathcal{A}_{\mathcal{Q}}(K_p) = 2$ *for every* p.

Let us now modify T_p into a tangle T'_p by adding one positive kink to the second string of T, and let us denote by K'_p the knot described by the special diagram $\mathcal{D}(T'_p)$.

Lemma 8.14. *The family* $\{K'_p\}$ *of genus*-1 *knots just constructed is such that* $a_p(K'_p) = 1$ *while* $a_{p'}(K'_p) = 0$ *for every* $p' \neq p$. *In particular*, K'_p *is not isotopic to* $K'_{p'}$ *if* $p \neq p'$, *and* $\mathcal{A}_{\mathcal{Q}}(K'_p) = 1$ *for every* p. *Moreover*, $\Delta(K_p)(t) \doteq \Delta(K'_p)(t)$ *for every odd prime* p.

Proof. It is readily seen that $W(K'_p) = W(K_p) = (p^2 - 1)/4$, so $\Delta(K'_p)(t) = \Delta(K_p)(t)$ and $a_p(K'_p) \geq 1$, while $a_{p'}(K'_p) = 0$ for every $p' \neq p$. Moreover, $a_p(K'_p) \neq 2$ since for the tangle T'_p the value of $M_{2,2}$ is equal to $p+1$ which is not null in $\mathbb{Z}_p$. □

Lemmas 8.13 and 8.14 imply that the quandle invariant $\mathcal{A}_{\mathcal{Q}}$ can be more effective than the Alexander polynomial in distinguishing knots, and this phenomenon shows up already in the case of genus-1 knots.

Corollary 8.15. *There exist genus-1 knots K, K' such that $\Delta(K)(t) \doteq \Delta(K')(t)$, while $\mathcal{A}_{\mathcal{Q}}(K) = 2$ and $\mathcal{A}_{\mathcal{Q}}(K') = 1$. Moreover, $\mathcal{A}_{\mathcal{Q}}(K)$ and $\mathcal{A}_{\mathcal{Q}}(K')$ may be realized by the same dihedral quandle and the same cocycle $z = 1$.*

8.3. *An example involving a quandle of order p^2*

All the previous explicit examples are obtained by using some Alexander quandle structure on $\mathbb{F}_p$. By Corollary 8.8, such quandle structures encode the relevant information about the invariant $\mathcal{A}_{\mathcal{Q}}$ of genus-1 knots.

Let us show anyway also an example based on a quandle of order p^2. Consider the tangle $T(h,k)$ of Fig. 14, encoding a knot $K(h,k)$. One can verify that $a_{\mathbb{X}}(K(5,3),2) = 2$, when $\mathbb{X} = (\mathbb{F}(11, 1+t^2), *)$, which is of type $t_{\mathbb{X}} > 2$ (here $q = 11^2$).

8.4. *The general picture of genus-1 knots*

The following proposition summarizes the discussion carried out in the preceding subsections.

Proposition 8.16. *Let K and K' be knots of genus $g \leq 1$. Then*:

(1) *Let M be the matrix associated to a special diagram of K. Then the integer $W(K) = \det M$ is a well-defined invariant of K (i.e. it does not depend on the chosen diagram).*
(2) *$\Delta(K)(t) \doteq \Delta(K')(t)$ if and only if $W(K) = W(K')$.*
(3) *$\mathcal{A}(K) = 0$ if and only if $\mathcal{A}_{\mathcal{Q}}(K) = 0$ if and only if $W(K) = 0$.*
(4) *$\mathcal{A}(K) \in \{0,2\}$, while $\mathcal{A}_{\mathcal{Q}}(K) \in \{0,1,2\}$. More precisely, for every $\eta \in \{0,1,2\}$ there exists a genus-1 knot K such that $\mathcal{A}_{\mathcal{Q}}(K) = \eta$.*
(5) *There exist K and K' such that $\Delta(K)(t) = \Delta(K')(t)$, while $\mathcal{A}_{\mathcal{Q}}(K) = 2$ and $\mathcal{A}_{\mathcal{Q}}(K') = 1$.*

Proof. By Lemmas 8.5, Corollaries 8.6, 8.8, 8.15, we are only left to prove that there exists a genus-1 knot K such that $\mathcal{A}_{\mathcal{Q}}(K) = 0$. As an example of such a

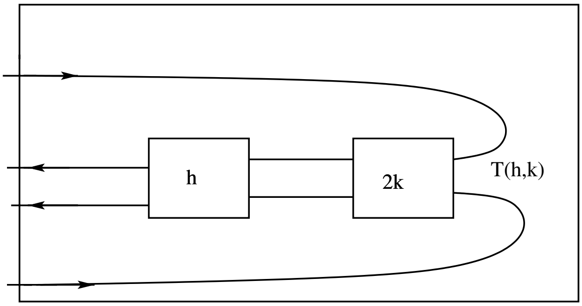

Fig. 14. The tangle $T(h,k)$. The rectangular boxes refer to the tangles described in Fig. 12.

knot, one may take any genus-1 knot with trivial Alexander polynomial, such as the Whitehead double of the figure-eight knot. □

8.5. *On genus-1 knots with minimal Seifert rank*

Recall that a Seifert surface Σ of a knot K is said to have *minimal Seifert rank* if the rank of its Seifert form S equals the genus $g = g(\Sigma)$. Moreover, a knot has *minimal Seifert rank* if it admits a Seifert surface (of arbitrary genus) having minimal Seifert rank. It is a well-known fact that every knot K with minimal Seifert rank has trivial Alexander polynomial $\Delta(K)(t) \doteq 1$ (i.e. $\mathcal{A}(K) = 0$). We claim that:

If K is a knot such that $g(K) \leq 1$ and $\mathcal{A}(K) = 0$, then every genus-1 Seifert surface for K has minimal Seifert rank. It follows that a genus-1 knot has trivial Alexander polynomial if and only if it has minimal Seifert rank.

In fact, let $\mathcal{D}$ be a special diagram for K associated to a given genus-1 Seifert surface Σ, let M be the matrix associated to $\mathcal{D}$, and let S be the matrix representing the Seifert form on Σ with respect to the geometric basis carried by $\mathcal{D}$. Corollary 6.4 implies that

$$S = \begin{pmatrix} M_{1,1} & M_{1,2} - 1 \\ M_{2,1} + 1 & M_{2,2} \end{pmatrix},$$

so S has rank equal to 1 if and only if $0 = \det S = \det M - M_{1,2} + M_{2,1} + 1 = \det M = W(K)$. By Proposition 8.16-(3), if $\mathcal{A}(K) = 0$ then $W(K) = 0$, so S has minimal rank.

It is a non-trivial fact proved in [5] that the last statement of the claim does not hold in general for knots of genus ≥ 2.

9. Sums of Genus-1 Knots

We can use genus-1 knots as building blocks for the construction of examples of arbitrary genus. Let us first observe that, if K and K' are (oriented) knots endowed respectively with special diagrams $\mathcal{D}(T)$ and $\mathcal{D}(T')$, then the knot $K + K'$ admits an obvious special diagram $\mathcal{D}(T + T')$ (see Figs. 15 and 16).

Let $N(z)$, $N'(z)$, $N''(z)$ be the matrices associated to the special diagrams $\mathcal{D}(T)$, $\mathcal{D}(T')$, $\mathcal{D}(T+T')$ as in Sec. 5, where z is a natural number. It is immediate to realize that

$$N''(z) = \left(\begin{array}{c|c} N(z) & 0 \\ \hline 0 & N'(z) \end{array}\right).$$

Since $N(1)$ (respectively, $N'(1)$, $N''(1)$) is a Seifert matrix for K (respectively, K', $K + K'$), this readily implies the well-known:

Lemma 9.1. *We have*

$$\Delta(K_1 + K_2 + \cdots + K_h)(t) = \prod_{j=1}^{h} \Delta(K_j)(t),$$

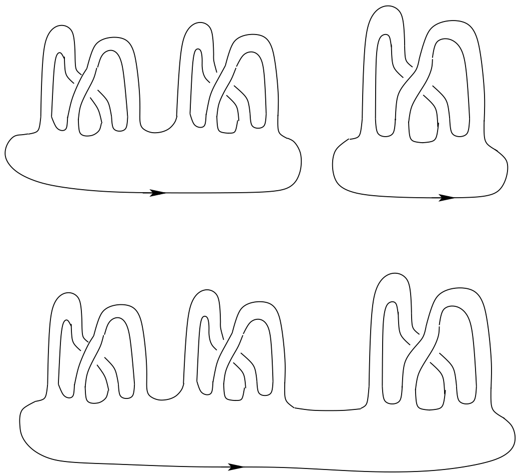

Fig. 15. On the top: two special diagrams $\mathcal{D}, \mathcal{D}'$ of the unknot K_0. On the bottom: the special diagram for $K_0 + K_0 = K_0$ obtained by "summing" the special diagrams on the top.

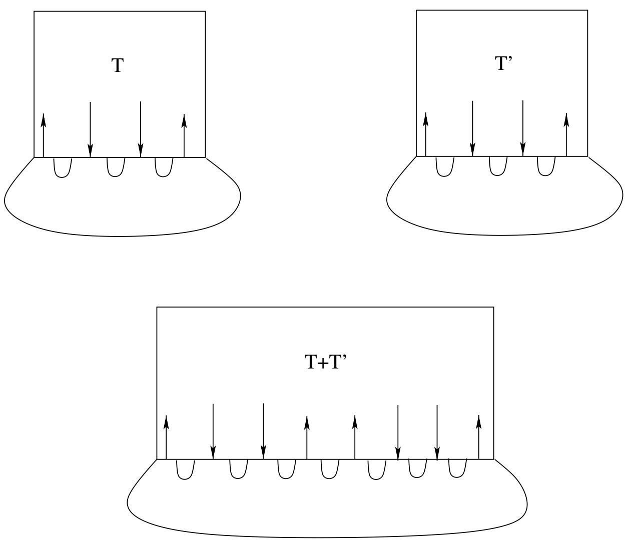

Fig. 16. If T and T' are the primary tangles of special diagrams of K and K', then $T + T'$ is the primary tangle of a special diagram of $K + K'$.

whence

$$\mathcal{A}(K_1 + K_2 + \cdots + K_h) = \sum_{j=1}^{h} \mathcal{A}(K_j).$$

Let us now fix an odd prime p and an irreducible element $h(t) \in \Lambda_p$ of positive breadth. Since $N(z, p, h(t))$ (respectively, $N'(z, p, h(t))$, $N''(z, p, h(t))$) is obtained from $N(z)$ (respectively, $N'(z)$, $N''(z)$) just by projecting the coefficients onto $\mathbb{F}(p, h(t))$, from Lemma 6.1 we deduce the following lemma.

Lemma 9.2. *For every* $\mathbb{X} \in \mathcal{Q}_{\mathcal{F}}$, $z \in \mathbb{Z}_{t_{\mathbb{X}}}$, *we have*

$$a_{\mathbb{X}}(K_1 + K_2 + \cdots + K_h, z) = \sum_{i=1}^{h} a_{\mathbb{X}}(K_j, z).$$

Therefore,

$$\mathcal{A}_{\mathcal{Q}}(K_1 + K_2 + \cdots + K_h, z) \leq \sum_{j=1}^{h} \mathcal{A}_{\mathcal{Q}}(K_j).$$

We observe that the equality $\mathcal{A}_{\mathcal{Q}}(K_1 + K_2 + \cdots + K_h, z) = \sum_{j=1}^{h} \mathcal{A}_{\mathcal{Q}}(K_j)$ does not hold in general. The equality holds if a single quandle $\mathbb{X} \in \mathcal{Q}_{\mathcal{F}}$ exists which realizes all the $\mathcal{A}_{\mathcal{Q}}(K_i)$'s with respect to the same cycle.

We are now ready to prove Proposition 1.9, which we recall here for the convenience of the reader.

Proposition 9.3. *Let us fix* $g \geq 1$. *Then, for every* r_1, r_2 *such that* $1 \leq r_1 \leq r_2 \leq 2r_1 \leq 2g$, *there exist knots* K_1 *and* K_2 *such that the following conditions hold*:

$$g(K_1) = g(K_2) = g, \quad \Delta(K_1) = \Delta(K_2) \quad (\text{whence } \mathcal{A}(K_1) = \mathcal{A}(K_2)),$$

while

$$\mathcal{A}_{\mathcal{Q}}(K_1) = r_1, \quad \mathcal{A}_{\mathcal{Q}}(K_2) = r_2.$$

Moreover, we can require that both $\mathcal{A}_{\mathcal{Q}}(K_1)$ *and* $\mathcal{A}_{\mathcal{Q}}(K_2)$ *are realized by means of some dihedral quandle with cycle* $\overline{z} = 1$.

Proof. Let K, K' be the genus-1 knots provided by Corollary 8.15, and let K'' be a genus-1 knot with trivial Alexander polynomial (see Proposition 8.16). Then we may define K_1 as the sum of r_1 copies of K' and $g - r_1$ copies of K'', and K_2 as the sum of $2r_1 - r_2$ copies of K', $r_2 - r_1$ copies of K and $g - r_1$ copies of K''. The additivity of the genus gives that $g(K_1) = g(K_2) = g$, and Lemma 9.1 readily implies that $\Delta(K_1)(t) = \Delta(K_2)(t)$. Moreover, by Lemma 9.2 we have that $\mathcal{A}_{\mathcal{Q}}(K_1) \leq r_1$ and $\mathcal{A}_{\mathcal{Q}}(K_2) \leq r_2$. However, Corollary 8.15 ensures that there exists a dihedral quandle $\mathbb{X}$ such that $\mathcal{A}_{\mathcal{Q}}(K) = a_{\mathbb{X}}(K, 1) = 2$ and $\mathcal{A}_{\mathcal{Q}}(K') = a_{\mathbb{X}}(K', 1) = 1$, so by Lemma 9.2 $a_{\mathbb{X}}(K_1, 1) = r_1$ and $a_{\mathbb{X}}(K_2, 1) = r_2$, whence the conclusion. □

9.1. *The case of links*

Let $L = K_1 \cup \cdots \cup K_h$ be a split link, where K_i is a knot for every $i = 1, \ldots, h$. Let also $\mathcal{P}_M$ be the maximal partition of L, let $\overline{z}\colon \mathcal{P}_M \to \mathbb{N}$ be a $\mathcal{P}_M$-cycle and set

$z_i = \overline{z}(K_i)$. The following lemma is an immediate consequence of the definition of quandle coloring.

Lemma 9.4. *For every quandle $\mathbb{X} \in \mathcal{Q}_{\mathcal{F}}$ we have*

$$c_{\mathbb{X}}(L, \mathcal{P}_M, \overline{z}) = \prod_{i=1}^{h} c_{\mathbb{X}}(K_i, z_i),$$

so

$$a_{\mathbb{X}}(L, \mathcal{P}_M, \overline{z}) = \left(\sum_{i=1}^{h} a_{\mathbb{X}}(K_i, z_i)\right) + h - 1,$$

and

$$\mathcal{A}_{\mathcal{Q}}(L, \mathcal{P}_M) \leq \left(\sum_{i=1}^{h} \mathcal{A}_{\mathcal{Q}}(K_i)\right) + h - 1.$$

Moreover, if $h \geq 2$ then $\Delta(L)(t) = 0$, so $\mathcal{A}(L) = 0$.

Just as in Lemma 9.2, the equality $\mathcal{A}_{\mathcal{Q}}(L, \mathcal{P}_M) \leq (\sum_{i=1}^{h} \mathcal{A}_{\mathcal{Q}}(K_i)) + h - 1$ does not hold in general.

Let now K_0 be a genus-1 knot such that $\mathcal{A}_{\mathcal{Q}}(K_0) = 2$ (see Sec. 8 for examples of such knots), and let L_h be the split link having h components, each isotopic to K_0. The following result implies Proposition 1.8.

Proposition 9.5. *We have $\mathcal{A}_{\mathcal{Q}}(L_h) = 3h - 1$.*

Proof. We have $a_{\mathbb{X}}(K_0, 1) = 2$ for some quandle $\mathbb{X} \in \mathcal{Q}_{\mathcal{F}}$, so Lemma 9.4 implies that

$$\mathcal{A}_{\mathcal{Q}}(L_h) \geq a_{\mathbb{X}}(L, \mathcal{P}_m, \overline{1}) = a_{\mathbb{X}}(L, \mathcal{P}_M, \overline{1}) = 3h - 1. \qquad \square$$

Remark 9.6. Strictly speaking, the equality $\mathcal{A}_{\mathcal{Q}}(L_h) = 3h - 1$ does not provide a sharp bound on $g(L_h)$, since Corollary 1.3 states that $\mathcal{A}_{\mathcal{Q}}(L) \leq 2g(L) + 2k - 2$ for every k-component link. This inequality provides the bound $2g(L_h) \geq h + 1$, which is not sharp since of course $g(L_h) = h$. However, it is immediate to see that if L is a split link, then $g(L) = g(L, \mathcal{P}_M)$. The inequalities

$$3h - 1 = \mathcal{A}_{\mathcal{Q}}(L_h) \leq \mathcal{A}_{\mathcal{Q}}(L_h, \mathcal{P}_M) \leq 2g(L_h, \mathcal{P}_M) + h - 1 = 2g(L_h) + h - 1$$

imply now $g(L_h) \geq h$.

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