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On the dynamics of $\mathbb{G}$-solenoids. Applications to Delone sets

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Abstract. A $\mathbb{G}$-solenoid is a laminated space whose leaves are copies of a single Lie group $\mathbb{G}$ and whose transversals are totally disconnected sets. It inherits a $\mathbb{G}$-action and can be considered as a dynamical system. Free $\mathbb{Z}^d$-actions on the Cantor set as well as a large class of tiling spaces possess such a structure of $\mathbb{G}$-solenoids. For a large class of Lie groups, we show that a $\mathbb{G}$-solenoid can be seen as a projective limit of branched manifolds modeled on $\mathbb{G}$. This allows us to give a topological description of the transverse invariant measures associated with a $\mathbb{G}$-solenoid in terms of a positive cone in the projective limit of the dim($\mathbb{G}$)-homology groups of these branched manifolds. In particular, we exhibit a simple criterion implying unique ergodicity. Particular attention is paid to the case when the Lie group $\mathbb{G}$ is the group of affine orientation-preserving isometries of the Euclidean space or its subgroup of translations.

1. $\mathbb{G}$-*solenoids*

Let $\mathbb{G}$ be a connected Lie group and M be a compact metric space. Assume that there exist a cover of M by open sets U_i called *boxes* and homeomorphisms called *charts* $h_i : U_i \to V_i \times T_i$ where the T_i are totally disconnected metric sets and the V_i are open subsets in $\mathbb{G}$. These open sets and homeomorphisms define an atlas of a $\mathbb{G}$-*solenoid* structure on M if the *transition maps* $h_{i,j} = h_j \circ h_i^{-1}$ read on their domains of definitions

$$h_{i,j}(v, t) = (g_{i,j} \cdot v, \tau_{i,j}(t)),$$

where $\tau_{i,j}$ is a continuous map and $g_{i,j}$ is an element in $\mathbb{G}$. Two atlases are *equivalent* if their union is again an atlas. A $\mathbb{G}$-*solenoid* is the data of a compact metric space M together with an equivalence class of atlases.

The fact that the transition maps are required to be very rigid implies the following properties.

(1) *Laminated structure*. We call a subset of the form $h_i^{-1}(V_i \times \{t\})$ a *slice* of a solenoid. The transition maps map slices onto slices. The *leaves* of M are the smallest connected subsets that contain all the slices they intersect. Each leaf is a differentiable manifold with dimension $\dim(\mathbb{G})$.

(2) $\mathbb{G}$-*action*. Since $h_{i,j}(v \cdot g, t) = (g_{i,j} \cdot (v \cdot g), \tau_{i,j}(t))$, it is possible to define on each leaf a $\mathbb{G}$-right action, at least for small elements in $\mathbb{G}$.

We make the assumption that each leaf is diffeomorphic to $\mathbb{G}$ *and that the* $\mathbb{G}$*-right action on* $\mathbb{G}$ *fits in the charts with the local* $\mathbb{G}$*-right action we have defined.* In this case, the free right action on $\mathbb{G}$ induces a free right $\mathbb{G}$-action on M whose orbits are the leaves of M.

(3) *Vertical germ*. We call a subset of the form $h_i^{-1}(\{x\} \times T_i)$ a *vertical* of a solenoid. The transition maps map verticals onto verticals. This allows us to define above each point in the $\mathbb{G}$-solenoid a local vertical (independently of the charts).

(4) *Locally constant return times*. Since the $\mathbb{G}$-action preserves the verticals, it follows that if a leaf L (the $\mathbb{G}$-orbit of some point) intersects two verticals V and V' at v and $v \cdot g$ where $g \in \mathbb{G}$ then, for any $\hat{v}$ in V close enough to v, $\hat{v} \cdot g$ is a point in V' close to v'.

A $\mathbb{G}$-solenoid M is *minimal* if each leaf of M is dense in M. It is *expansive* if the $\mathbb{G}$-action is expansive, i.e. there exists a positive constant $\epsilon(M)$ such that if two points on the same local vertical have their $\mathbb{G}$-orbits which remain $\epsilon(M)$-close to each other then these two points coincide. The constant $\epsilon(M)$ is called *the expansivity diameter* of the $\mathbb{G}$-solenoid.

We give below some standard examples.

1. The most naive example of a $\mathbb{G}$-solenoid is given by a compact, connected Lie group. In this case all the verticals are reduced to a single point.
2. More interesting is the case of the dynamical system given by the iteration of a homeomorphism ϕ on the Cantor set Σ with no periodic point. It is plain to see that the suspension $\Sigma \times \mathbb{R}/\phi$ inherits a structure of the $\mathbb{R}$-solenoid. On the one hand, the standard odometer is an example of a non-expansive $\mathbb{R}$-solenoid, on the other hand, the shift operation on a minimal non-periodic subset of $\{0, 1\}^{\mathbb{Z}}$ is an expansive $\mathbb{R}$-solenoid.
3. More generally, any free $\mathbb{Z}^d$-action on the Cantor set inherits a structure of a $\mathbb{R}^d$-solenoid.

This paper is organized as follows. Section 2 is devoted to providing a much broader set of examples of $\mathbb{G}$-solenoids. Let N be a Riemannian manifold on which a connected Lie group $\mathbb{G}$ has a transitive action by isometries and such that the stabilizer of a given point in N is a compact group K ($N \approx \mathbb{G}/K$). On this Riemannian manifold N, one can define Delone sets of finite $\mathbb{G}$-type. The $\mathbb{G}$-action on such a set X can be extended to a $\mathbb{G}$-action on the continuous hull $\Omega_{\mathbb{G}}(X)$. We prove that, when the Delone set is totally aperiodic, this dynamical system has a natural structure of an expansive $\mathbb{G}$-solenoid (Theorem 2.4). Conversely, we prove that for any such Lie group $\mathbb{G}$, an expansive and minimal $\mathbb{G}$-solenoid can be seen as a continuous hull of a totally aperiodic Delone set of finite $\mathbb{G}$-type (Theorem 2.5).

In §3, we focus our attention on the case when the Riemannian manifold $N \approx \mathbb{G}/K$ has a curvature nowhere positive, i.e. N is a $CAT(0)$ space. We develop a combinatorial approach of the structure of a $\mathbb{G}$-solenoid (Theorem 3.1). This approach can be interpreted as an extension to our general situation of the construction of infinite sequences of towers and Brateli diagrams that proved to be so powerful in the study of the $\mathbb{Z}$-action on the Cantor set.

This allows us to introduce, in §4, a special class of branched manifolds modeled on $\mathbb{G}$, that we call $\mathbb{G}$-branched manifolds, and to show that a $\mathbb{G}$-solenoid is a projective limit of $\mathbb{G}$-branched manifolds whose faces become arbitrarily large when going backward in the projective limit (Theorem 4.5). This construction gives us better and better approximants of M with bigger and bigger regions where the $\mathbb{G}$-action is defined. The known analogue of this description in the case of the $\mathbb{Z}$-action by a homeomorphism ϕ on the Cantor set Σ, is that the dynamical system (Σ, ϕ) can be seen as a projective limit of oriented graphs with all vertices but one with valence 2, and such that the length of the loops around the singular vertex goes to infinity (see for instance [**4, 11, 17**]). In the context of smooth dynamics, projective limits of branched manifolds were introduced by Williams in the 1960s [**18, 19**]. Our approach makes a bridge between these last two constructions.

This projective limit construction allows us to give, in §5, a topological description of the set of transverse invariant measures of a $\mathbb{G}$-solenoid. This set turns out to be the projective limit of cones in the $\dim \mathbb{G}$-homology groups of the $\mathbb{G}$-branched manifolds (Theorem 5.1). We show that if the number of faces of the branched manifolds is uniformly bounded, then this number is also a bound for the number of ergodic transverse invariant probability measures. Furthermore, we prove that if the linear maps projecting the homology groups one into the other in the projective limit are positive and bounded, then there exists a unique transverse invariant probability measure (Corollary 5.2).

Finally, §6 is devoted to a more specific analysis of the case when the Lie group is the group $\mathbb{E}^d$ of isometries in $\mathbb{R}^d$ or the group of translations $\mathbb{R}^d$.

Remark 1.1. Our definition of a $\mathbb{G}$-solenoid implies that the $\mathbb{G}$-action it wears is free. We could have worked out a similar analysis for the case when some leaves of the $\mathbb{G}$-solenoid are quotients of $\mathbb{G}$ by a finite group. This would have allowed us to also deal with Delone sets which are invariant under finite-order isometries (see Remark 2.2). We made this choice only for reasons of clarity. Actually, the whole construction of projective limits works in the same way as the topological interpretation of the transverse invariant measures.

Remark 1.2. For $\mathbb{R}^d$-actions, Anderson and Putnam [**1**] proved that the hull $\Omega_{\mathbb{R}^d}(Y)$ of a Delone set with finite $\mathbb{R}^d$-type obtained by a substitution rule is a projective limit of branched manifolds whose faces become arbitrarily large when going backward in the projective limit. This result has been extended by Bellissard *et al* [**2**] to the general case of a Delone set with finite $\mathbb{R}^d$-type. Gähler [**3**] announced recently a very simple proof of the fact that the hull is a projective limit of branched manifolds in the same context. However, in this proof, the sizes of the faces of the branched manifolds remain constant as we go backward in the projective limit and, thus, the number of faces goes to infinity. More recently, Sadun announced a construction similar to Gähler's in the case of a more general Lie group [**13**]. Finally the fact that the hull $\Omega_{\mathbb{G}}(Y)$ of a Delone set with finite $\mathbb{G}$-type Y has a laminated structure was observed by Ghys [**5**].

2. *Delone sets*

Let N be a Riemannian manifold on which a connected Lie group $\mathbb{G}$ has a transitive action by isometries and such that the stabilizer of a given point 0 in N is a compact group K ($N \approx \mathbb{G}/K$). The standard projection $\pi : \mathbb{G} \to N$ is defined by $\pi(g) = g(0)$.

Consider two positive numbers r and R, a (r, R)*-Delone set* is a discrete subset X of N which satisfies the following two properties.

(i) *Uniformly discrete.* Each open ball with radius r in N contains at most one point in X.

(ii) *Relatively dense.* Each open ball with radius R contains at least one point in X.

When the constants r and R are not explicitly used, we will say *Delone set* for an (r, R)-Delone set. We refer the reader to [**9**] for a more detailed approach of the theory of Delone sets.

A *patch* of a Delone set is a finite subset of X. Two patches P_1 and P_2 of a Delone set X are of the same $\mathbb{G}$-type if there exists an element g in $\mathbb{G}$ such that $g \cdot P_1 = P_2$. *A Delone set of finite* $\mathbb{G}$*-type* is a Delone set such that for each positive number T, there exists only a finite number of $\mathbb{G}$-types of patches in X with diameters smaller than T. The Lie group $\mathbb{G}$ has a natural right action on the set of Delone sets of finite $\mathbb{G}$-type:

$$(g, X) \to g^{-1} \cdot X.$$

Furthermore, any metric on $\mathbb{G}$ induces a metric $\delta_{\mathbb{G}}$ on the orbit $\mathbb{G} \cdot X$ of a given Delone set X of finite $\mathbb{G}$-type defined as follows.

Let $B_\epsilon(0)$ stand for the open ball with radius ϵ and center 0 in $\mathcal{V}$. Consider two Delone sets $g_1^{-1} \cdot X$ and $g_2^{-1} X$ in $\mathbb{G} \cdot X$. Let A denote the set of $\epsilon \in]0, 1[$ such that there exists g and g' ϵ-close to the identity e in $\mathbb{G}$ such that $g^{-1} \cdot g_1^{-1} \cdot X \cap B_{1/\epsilon}(0) = g'^{-1} \cdot g_2^{-1} \cdot X \cap B_{1/\epsilon}(0)$, then

$$\delta_{\mathbb{G}}(g_1^{-1} \cdot X, g_2^{-1} \cdot X) = \inf A \quad \text{if } A \neq \emptyset$$
$$\delta_{\mathbb{G}}(g_1^{-1} \cdot X, g_2^{-1} X) = 1 \quad \text{if } A = \emptyset.$$

Hence, the diameter of $\mathbb{G} \cdot X$ is bounded by 1 and the $\mathbb{G}$-action on $\mathbb{G} \cdot X$ is continuous. The *continuous hull* $\Omega_{\mathbb{G}}(X)$ of a Delone set of finite $\mathbb{G}$-type X is the completion of the metric space $(\mathbb{G} \cdot X, \delta_{\mathbb{G}})$. It is straightforward to check (see for instance [**8**]) that $\Omega_{\mathbb{G}}(X)$ is a compact metric space and that any element in $\Omega_{\mathbb{G}}(X)$ is a Delone set whose $\mathbb{G}$-type of patches are those of X. Thus, the group $\mathbb{G}$ acts on $\Omega_{\mathbb{G}}(X)$ and the dynamical system $(\Omega_{\mathbb{G}}(X), \mathbb{G})$ possesses (by construction) a dense orbit (namely the orbit $X \cdot \mathbb{G}$).

A Delone set of finite $\mathbb{G}$-type X is *repetitive* if for any patch P in X there exists a radius $\rho(P) > 0$ such that any ball with radius $\rho(P)$ intersects X in a patch that contains a patch with the same $\mathbb{G}$-type as P. This last property can be interpreted in the dynamical system framework (see for instance [**8**]).

PROPOSITION 2.1. *The dynamical system* $(\Omega_{\mathbb{G}}(X), \mathbb{G})$ *is minimal (i.e. all its orbits are dense) iff the Delone set of finite type X is repetitive.*

A Delone set X is *totally aperiodic* if there exist no $g \neq e$ in $\mathbb{G}$ and no Y in $\Omega_{\mathbb{G}}(X)$ such that $g^{-1} \cdot Y = Y$.

Remark 2.2. It is not clear *a priori* that totally aperiodic Delone sets of finite $\mathbb{G}$-type exist for any $N \approx \mathbb{G}/K$. When the Lie group is the group of translations in $\mathbb{R}^2$, the Penrose Delone set is a standard example. When the Lie group is the group of direct isometries $\mathbb{E}^2$ acting on $\mathbb{R}^2$, the Penrose Delone set is no longer totally aperiodic since some elements in its hull have a five-fold symmetry. The Pinwheel Delone set [**12**] is an example of a repetitive Delone set of finite $\mathbb{G}$-type; no Delone set in its hull is fixed by an infinite-order element in $\mathbb{G}$, however there are elements with a two-fold symmetry. Very recently, Goodman-Strauss [**6, 7**], has shown a stronger result in this direction when the Lie group is $PSL(2, \mathbb{R})$ acting by isometries on the Poincaré disk. For Delone sets of $\mathbb{G}$-finite type associated with more sophisticated Lie groups we refer to Mozes [**10**].

Consider the subset $\Omega^0_{\mathbb{G}}(X)$ in $\Omega_{\mathbb{G}}(X)$ of the Delone sets Y such that 0 belongs to Y. The group $\mathbb{G}$ does not act on $\Omega^0_{\mathbb{G}}(X)$, but the compact subgroup K does. The *canonical transversal* of a Delone set X is the quotient space $\Omega^0_{\mathbb{G}}(X)/K$.

PROPOSITION 2.3. *The canonical transversal of a totally aperiodic Delone set of finite $\mathbb{G}$-type equipped with the quotient metric, is a totally disconnected metric space. Furthermore, if the Delone set is repetitive, it is a metric Cantor set.*

Proof. Two elements Y_1 and Y_2 in $\Omega_{\mathbb{G}}(X)$ are close to each other if, in a big ball surrounding 0, the two associated patches are images of each other by the action of an element in $\mathbb{G}$ close to e. If Y_1 and Y_2 are in $\Omega^0_{\mathbb{G}}(X)$, this implies that the two patches are images of each other by the action of an element in K close to e. Since X has finite $\mathbb{G}$-type, we have constructed in this way a basis of clopen neighborhoods of $\Omega^0_{\mathbb{G}}(X)/K$ which consequently is totally disconnected. The repetitivity directly yields the fact that $\Omega^0_{\mathbb{G}}(X)/K$ is perfect. □

Two Delone sets X and Y are $\mathbb{G}$*-equivalent (respectively* $\mathbb{G}$*-orbit equivalent)* if the two dynamical systems $(\Omega_{\mathbb{G}}(X), \mathbb{G})$ and $(\Omega_{\mathbb{G}}(Y), \mathbb{G})$ are conjugate (respectively orbit equivalent), i.e. there exists a homeomorphism $h : \Omega_{\mathbb{G}}(X) \to \Omega_{\mathbb{G}}(Y)$ that commutes with the $\mathbb{G}$-action (respectively that maps $\mathbb{G}$-orbits onto $\mathbb{G}$-orbits).

The following two theorems relate Delone sets and solenoids.

THEOREM 2.4. *Let N be a Riemannian manifold on which a connected Lie group $\mathbb{G}$ acts transitively by isometries and such that the stabilizer of a given point* 0 *in N is a compact group K and let X be a totally aperiodic Delone set of finite $\mathbb{G}$-type in N. Then, the continuous hull $\Omega_{\mathbb{G}}(X)$ inherits a structure of an expansive $\mathbb{G}$-solenoid and the $\mathbb{G}$-action on $\Omega_{\mathbb{G}}(X)$ coincides with the canonical $\mathbb{G}$-action of the $\mathbb{G}$-solenoid. This action is minimal if and only the Delone set is repetitive.*

Proof. Consider a totally aperiodic, repetitive Delone set X in $\mathbb{G}/K$ of finite $\mathbb{G}$-type and choose a section $s : \Omega^0_{\mathbb{G}}(X)/K \to \Omega^0_{\mathbb{G}}(X)$, such that the closer two points are in $\Omega^0_{\mathbb{G}}(X)/K$, the larger is the ball surrounding 0 in which their images exactly coincide. Any point X' in $\Omega_{\mathbb{G}}(X)$ reads $X' = g^{-1} \cdot X''$ where X'' is in $s(\Omega^0_{\mathbb{G}}(X)/K)$ and g is an element in $\mathbb{G}$. It follows that $\Omega_{\mathbb{G}}(X)$ can be covered by a finite union of sets $U_i = \phi_i(V_i \times T_i)$ where:

- T_i is a clopen set in $s(\Omega^0_{\mathbb{G}}(X)/K)$;
- V_i is an open set in $\mathbb{G}$;
- $\phi_i : V_i \times T_i \to \Omega_{\mathbb{G}}(X)$ is the map defined by $\phi_i(v, t) = v^{-1} \cdot t$.

Since there exists a finite partition of $\Omega^0_{\mathbb{G}}(X)/K$ (and of any of its clopen sets) in clopen sets with arbitrary small maximal diameter, it is possible to choose this diameter small enough so that:

- the maps ϕ_i are homeomorphisms onto their images;
- whenever $Z \in U_i \cap U_j$, $Z = \phi_i(v,t) = \phi_j(v',t')$, the element $v \cdot v'^{-1}$ in $\mathbb{G}$ is independent of the choice of Z in $U_i \cap U_j$, we denote it by $g_{i,j}^{-1}$.

The transition maps read $(v',t') = (g_{i,j}v, s^{-1}(g_{i,j}^{-1} \cdot s(t)))$. It follows that the boxes U_i and charts $h_i = \phi_i^{-1} : U_i \to V_i \times T_i$ define a $\mathbb{G}$-solenoid structure on $\Omega_{\mathbb{G}}(X)$. By construction, the $\mathbb{G}$-action on this $\mathbb{G}$-solenoid coincides with the $\mathbb{G}$-action on $\Omega_{\mathbb{G}}(X)$. The $\mathbb{G}$-orbits of $\Omega_{\mathbb{G}}(X)$ are mapped to leaves of the $\mathbb{G}$-solenoid structure.

The $\mathbb{G}$-action on $\Omega_{\mathbb{G}}(X)$ is expansive. Indeed consider two Delone sets Y_1 and Y_2 in $s(\Omega^0_{\mathbb{G}}(X)/K)$ and within a distance ϵ small enough so that they coincide on a ball centered at 0 and that their restriction to this ball B is fixed by no element in $\mathbb{G} - \{e\}$. If $g^{-1} \cdot Y_1$ remains ϵ-close to $g^{-1} \cdot Y_2$ when g runs over a ball $B_{\mathbb{G}}$, then Y_1 and Y_2 coincide on $B_{\mathbb{G}}^{-1} \cdot B$. Thus, if the two $\mathbb{G}$-orbits remain ϵ-close, Y_1 and Y_2 coincide.

The equivalence between the minimality of these two dynamical systems is plain and is equivalent to the repetitivity of the Delone set X thanks to Proposition 2.1. □

Let $\pi : \mathbb{G} \to N \approx \mathbb{G}/K$ be the standard projection. We say that a finite collection of verticals V chosen in boxes of an atlas of a $\mathbb{G}$-solenoid M is *well distributed* if it satisfies the following properties:

(i) for any leaf L of the solenoid (identified with $\mathbb{G}$ by fixing the unity e in L), the projection $\pi(V \cap L)$ is a Delone set Y;

(ii) for any pair of points in $\tilde{y}_1$ and $\tilde{y}_2$ in $V \cap L$ such that there is $g_{1,2}$ in K satisfying $y_1^{-1}Y = g_{1,2}^{-1}\tilde{y}_2^{-1} \cdot Y$, then $\tilde{y}_1$ and $\tilde{y}_2$ are on a same vertical in V.

Since we have a lot of freedom in choosing the verticals, it is clear that for any given size ϵ there exists a well-distributed finite collection of verticals with diameter smaller than ϵ.

THEOREM 2.5. *Let N be a Riemannian manifold on which a connected Lie group $\mathbb{G}$ acts transitively by isometries and such that the stabilizer of a given point 0 in N is a compact group K and let M be a $\mathbb{G}$-solenoid. Then we have the following.*

(1) *For any finite well-distributed collection V of verticals chosen in boxes of an atlas of M, the intersection of V with any leaf L (identified with $\mathbb{G}$ by fixing the unity e in L) of M, defines by projection on $N \approx \mathbb{G}/K$, a Delone set Y of finite $\mathbb{G}$-type.*

(2) *If M is expansive and the diameter of the verticals is chosen small enough, Y is fixed by no element in $\mathbb{G}$ different from e.*

(3) *If, furthermore, M is minimal, the dynamical systems $(\Omega_{\mathbb{G}}(Y), \mathbb{G})$ and $(M, \mathbb{G})$ are conjugate and thus all the Delone sets constructed in this way are $\mathbb{G}$-equivalent, totally aperiodic and repetitive.*

Proof. (1) Let V be a finite union of verticals in the $\mathbb{G}$-solenoid and L be any leaf of the solenoid. This leaf L is diffeomorphic to $\mathbb{G}$ and we fix an origin e on L associated with the identity in $\mathbb{G}$. Consider the projection Y on N of the intersection $V \cap L$

$$Y = \pi(V \cap L).$$

Whenever V is well distributed, Y is a Delone set (i). Fix a number $\rho > 0$ and assume that $\tilde{y}_0$ is in $V \cap L$ and denote by $y_0 = \pi(\tilde{y}_0)$ its projection on N, the other intersection points in the ball with center y_0 and radius ρ being $y_1 = \pi(\tilde{y}_1), \ldots, y_n = \pi(\tilde{y}_n)$, where $\tilde{y}_1 = \tilde{y}_0 \cdot g_1, \ldots, \tilde{y}_n = \tilde{y}_0 \cdot g_n$. Consider a point $\tilde{z}_0$ in V close enough to $\tilde{y}_0$. Property (4) above insures that, for $i = 1, \ldots, n$, $\tilde{z}_0 \cdot g_i$ is in V. Thus, the set of points $z_0 = \pi(\tilde{z}_0), \ldots z_n = \pi(\tilde{z}_n)$ is an isometric copy of the set $y_0, \ldots, y_n$. By choosing a finite partition of V in clopen sets with small enough diameters, we get that there exists only a finite number of $\mathbb{G}$-type for patches with size smaller than ρ. Since this is true for all $\rho > 0$, Y is a Delone set of finite $\mathbb{G}$-type.

(2) Assume now that the diameter of the verticals in V is smaller than a given $\epsilon > 0$ and suppose that the Delone set Y is fixed by some element in $\mathbb{G}$ different from e. It follows that there exist $\tilde{y}_1$ and $\tilde{y}_2$ in $V \cap L$ and $g_{1,2}$ in K such that $y_1^{-1} Y = g_{1,2}^{-1} \tilde{y}_2^{-1} \cdot Y$. Since V is well distributed, we deduce from (ii) that $\tilde{y}_1$ and $\tilde{y}_2$ are on a same vertical in V and that $\tilde{y}_1' \cdot g$ is in a vertical in V if and only if $\tilde{y}_2' \cdot g$ is on the same vertical. It follows that when ϵ is sufficiently small, the $\mathbb{G}$ orbits of $\tilde{y}_1'$ and $\tilde{y}_2'$ remain $\epsilon(M)$ close and thus coincide.

(3) Thus, the dynamical systems $(\Omega_{\mathbb{G}}(Y), \mathbb{G})$ and $(M, \mathbb{G})$ are conjugate and, if M is minimal, Y is repetitive and totally aperiodic. □

3. *Box decompositions and tower systems*

From now on we restrict our attention to the case when the Riemannian manifold $N \approx \mathbb{G}/K$ is a $CAT(0)$ space.

3.1. *Box decomposition.*

A *box decomposition* of a $\mathbb{G}$-solenoid M is a finite collection of boxes $B_1, \ldots B_n$ such that any two boxes are disjoint and the closure of the union of all boxes covers the whole solenoid. A box decomposition is *polyhedral* if each box B_i reads in a chart $\pi^{-1}(U_i) \times T_i$ where U_i is an open convex geodesic polyhedron in $N \approx \mathbb{G}/K$. Since the intersection of two polyhedral boxes is a finite union of polyhedral boxes, it is simple to check that every $\mathbb{G}$-solenoid admits a polyhedral box decomposition. In general, box decompositions associated with a $\mathbb{G}$-solenoid M are not unique. We shall associate a canonical polyhedral box decomposition with any pair (M, V), where M is a $\mathbb{G}$-solenoid and V a finite collection of verticals in M. In particular, given a totally aperiodic, repetitive Delone set X of $\mathbb{G}$-finite type, there exists a canonical polyhedral box decomposition of its hull $\Omega_{\mathbb{G}}(X)$. We call this particular box decomposition, the *Voronoi box decomposition* and define it as follows.

Let V be a well-distributed finite union of verticals in a $\mathbb{G}$-solenoid M and Y be the projection on N of the intersection $V \cap L$, where L is any leaf of the solenoid identified with $\mathbb{G}$ by fixing the unity e in L. We know from Theorem 2.5 that Y is a Delone set of $\mathbb{G}$-finite type in N. Consider the Voronoi tiling associated with Y in L, i.e. the polyhedral tiling defined as follows.

For any y in Y the *Voronoi cell* V_y of y is the open set of points in N which are closer to y than to any other points in Y. The family $\{\overline{V_y};\ \ y \in Y\}$ defines a polyhedral tiling of N, in which the tiles are convex polyhedra that meet full face-to-full face. Again from Theorem 2.5, there exists a finite partition of V in clopen sets $C_1, \ldots, C_n$, such that for any pair of points $\tilde{y}_1$ and $\tilde{y}_2$ in a same C_i, $V_{\pi(\tilde{y}_2)}$ is obtained from $V_{\pi(\tilde{y}_1)}$ by the isometric action of an element in $\mathbb{G}$. The boxes of the Voronoi box decomposition read in charts $\pi^{-1}(V_y) \times C_i$ where y is in Y and $i = 1, \ldots, n$.

3.2. *Tower system.* The *vertical boundary* of a polyhedral box which reads in a chart $\pi^{-1}(P) \times T$, where P is a polyhedron in N and T a totally disconnected set, reads in the same chart $\pi^{-1}(\partial P) \times T$, where ∂P stands for the boundary of P. A polyhedral box decomposition $\mathcal{B}_2$ is *zoomed out* of a polyhedral box decomposition $\mathcal{B}_1$ if:

(1) for each point x in a box B_1 in $\mathcal{B}_1$ and in a box B_2 in $\mathcal{B}_2$, the vertical of x in B_2 is contained in the vertical of x in B_1;

(2) the vertical boundaries of the boxes of $\mathcal{B}_2$ are contained in the vertical boundaries of the boxes of $\mathcal{B}_1$;

(3) for each box B_2 in $\mathcal{B}_2$, there exists a box B_1 in $\mathcal{B}_1$ such that $B_1 \cap B_2 \neq \emptyset$ and the vertical boundary of B_1 does not intersect the vertical boundary of B_2.

(4) if a vertical in the vertical boundary of a box in $\mathcal{B}_1$ contains a point in a vertical boundary of a box in $\mathcal{B}_2$, then it contains the whole vertical†.

A *tower system* of a $\mathbb{G}$-solenoid is a sequence of polyhedral box decompositions $(\mathcal{B}_n)_{n\geq 1}$ such that for each $n \geq 1$, $\mathcal{B}_{n+1}$ is zoomed out of $\mathcal{B}_n$.

THEOREM 3.1. *Let N be a Riemannian manifold on which a connected Lie group $\mathbb{G}$ acts transitively by isometries and such that the stabilizer of a given point* 0 *in N is a compact group K and let M be a $\mathbb{G}$-solenoid. Then, M admits a tower system.*

Proof. Consider a triplet (M, V_1, x) where M is a $\mathbb{G}$-solenoid, V_1 is a well distributed finite collection of verticals and x a point in V_1.

Step 1. Fix $\delta_1 = 1$. The first box decomposition $\mathcal{B}_1$ is the Voronoi box decomposition associated with (M, V_1).

Step 2. Consider the box $B_{1,1}$ in $\mathcal{B}_1$ that contains x and choose in $B_{1,1} \cap V_1$ a clopen set V_2 which contains x and has a diameter smaller than δ_2. Consider the Voronoi box decomposition $\mathcal{B}'_2$ associated with (M, V_2). The box decomposition $\mathcal{B}'_2$ is not zoomed out of $\mathcal{B}_1$. Indeed, it satisfies point (1) of the definition and if δ_2 is chosen small enough it also satisfies points (3) and (4), but it certainly does not satisfy point (2). To construct a box decomposition which also satisfies point (2) we have to introduce a slight modification in the construction of the Voronoi box decomposition. More precisely, let $Y_2 = \pi(V_2 \cap L)$ where L is a leaf of M. Instead of considering the Voronoi tiling associated with Y_2 in L and for each point y in Y_2 its Voronoi cell V_y, we construct the best approximation possible‡ of the Voronoi tiling by changing each Voronoi cell V_y by a reunion $\tilde{V}_y$ of cells of the Voronoi tiling associated with the Delone set $Y_1 = \pi(V_1 \cap L)$. The new boxes of the decomposition are then constructed following the same rules.

† In substitution theory, this condition is called *forcing the border* [**8**]. In the study of Williams attractors it is called *flattening condition* [**19**].

‡ The mild ambiguity inherent with this construction will turn out to be irrelevant because of the $\mathbb{G}$-finite type hypothesis.

Step 3. We iterate this construction by considering the box $B_{1,2}$ in $\mathcal{B}_2$ that contains x and choose in $B_{1,2} \cap V_2$ a clopen set V_3 which contains x and has a diameter smaller than δ_3 etc. □

4. $\mathbb{G}$-*branched manifolds*

In this section we keep the same notation: N is a Riemannian manifold on which a connected Lie group $\mathbb{G}$ acts transitively by isometries and such that the stabilizer of a given point 0 in N is a compact group K, M is a $\mathbb{G}$-solenoid and $\pi : \mathbb{G} \to N \approx \mathbb{G}/K$ is the standard projection. Furthermore we set $\dim \mathbb{G} = g$ and $\dim K = k$.

4.1. *Local models.*

For $r > 0$ and x in N, we consider the open ball with radius r centered at x, $B_r(x)$. The $\mathbb{G}$-*ball of type 1* $B(x, r)$ is the open set $\pi^{-1}(B_r(x))$.

A polyhedral decomposition of $B_r(x)$ is a finite collection of polyhedral open cones centered at x, ($\mathcal{C} = \{C_1, \dots, C_n\}$) which are pairwise disjoint and whose closures cover the ball $B_r(x)$. For $p \geq 1$, a $\mathbb{G}$-*ball of type* p is defined in several steps.

- Consider in the ball $B_r(x)$ a finite collection of polyhedral decompositions $\mathcal{C}_1, \dots, \mathcal{C}_p$ such that if two cones in the same $\mathcal{C}_i$ have a $(g-k-1)$-face in common, then there exits $j \neq i$ such that one of the two cones also appears in $\mathcal{C}_j$ and the other one does not.
- Define in the disjoint union $\bigsqcup_{i=1}^{i=p} B_r(x)$ the equivalence relation: $(y, i) \sim (y', j)$ iff $\mathcal{C}_i$ and $\mathcal{C}_j$ share the same cone whose closure contains $y = y'$.
- In $\bigsqcup_{i=1}^{i=p} \pi^{-1}(B_r(x))$, consider the equivalence relation $\approx$ defined by $\tilde{y} \approx \tilde{y}'$ if and only if $\pi(\tilde{y}) \sim \pi(\tilde{y}')$.
- The $\mathbb{G}$-*ball of type* p, $B(x, r, \mathcal{C}_1, \dots, \mathcal{C}_p)$, is the set $\bigsqcup_{i=1}^{i=p} \pi^{-1}(B_r(x))/\approx$.

The $\mathbb{G}$-balls, as defined above, are said to be *centered at x and with radius r*. Note that the projection π induces a projection from $\bigsqcup_{i=1}^{i=p} \pi^{-1}(B_r(x))/\approx$ to $\bigsqcup_{i=1}^{i=p} B_r(x)/\sim$ which we still denote by π. Let $\pi_1 : \bigsqcup_{i=1}^{i=p} B_r(x) \to \bigsqcup_{i=1}^{i=p} B_r(x)/\sim$ be the canonical projection. This quotient space can be seen as a collection of p copies of the ball $B_r(x)$ glued to each other along polyhedral cones centered at x and is stratified as follows.

For $\tilde{y}$ in $\bigsqcup_{i=1}^{i=p} B_r(x)/\sim$ and y in $\pi_1^{-1}(\tilde{y})$, let $\nu(y)$ be the number of cones whose closure contains y and $\nu(\tilde{y})$ be the maximum of all $\nu(y)$ when y runs over $\pi_1^{-1}(\tilde{y})$. For l in $\{1, \dots, g-k\}$, $\mathcal{V}_l$ is the set of points $\tilde{y}$ in $\bigsqcup_{i=1}^{i=p} B_r(x)/\sim$ such that $\nu(\tilde{y}) = g-k-l+1$ and $\mathcal{V}_0$ is the set of points $\tilde{y}$ in $\bigsqcup_{i=1}^{i=p} B_r(x)/\sim$ such that $\nu(\tilde{y}) \geq g-k+1$. By denoting $cl(A)$ the closure of a set A, we have:

- $\bigsqcup_{i=1}^{i=p} B_r(x)/\sim\, = cl(\mathcal{V}_{g-k}) = \bigcup_{l=0}^{l=g-k} \mathcal{V}_l$;
- for each l in $\{1, \dots, g-k\}$, $cl(\mathcal{V}_l) = \mathcal{V}_l \cup \mathcal{V}_{l-1}$ and $\mathcal{V}_l \cap \mathcal{V}_{l-1} = \emptyset$;
- $cl(\mathcal{V}_0) = \mathcal{V}_0$ contains the point x.

This stratification yields a stratification of the $\mathbb{G}$-ball. The *singular locus* of the $\mathbb{G}$-ball with type p is the set $\pi^{-1}(cl(\mathcal{V}_{g-1}))$. For l in $\{0, \dots, g-k\}$, $\pi^{-1}(\mathcal{V}_l)$ is called the *$l+k$-stratum* of the $\mathbb{G}$-ball. Any $\mathbb{G}$-ball inherits a smooth structure, a K-right action, a local $\mathbb{G}$-right action defined outside the singular locus.

Let proj be the standard projection proj : $B(x, r, \mathcal{C}_1, \dots, \mathcal{C}_p) \to B(x, r)$. A $\mathbb{G}$-*sheet* in the $\mathbb{G}$-ball $B(x, r, \mathcal{C}_1, \dots, \mathcal{C}_p)$ is the image of $B(x, r)$ by a section of proj.

The following lemma, whose proof is straightforward, describes the situation around the center of a $\mathbb{G}$-ball of type p.

LEMMA 4.1. *In a $\mathbb{G}$-ball with type p, there exists a neighborhood of each point y which is a $\mathbb{G}$-ball centered at y of type q where q is in $\{1, \dots, p\}$. Furthermore, if y is not in the singular locus, then $q = 1$.*

A map $\tau : B(x_2, r_2, \mathcal{C}_1^2, \dots, \mathcal{C}_{p_2}^2) \to B(x_1, r_1, \mathcal{C}_1^1, \dots, \mathcal{C}_{p_1}^1)$ is a $\mathbb{G}$-*local submersion* if:

- the map τ preserves the smooth structure i.e. it maps:
  - the singular locus into the singular locus;
  - the $\mathbb{G}$-sheets into $\mathbb{G}$-sheets;
- there exists g in $\mathbb{G}$ such that τ projects on the g action on $B(x_2, r_2)$, i.e. $\mathrm{proj} \circ \tau(\cdot) = g^{-1} \cdot \mathrm{proj}(\cdot)$.

A $\mathbb{G}$-local submersion is a $\mathbb{G}$-*local isometry* if it is one-to-one. It this case, the two $\mathbb{G}$-balls have the same radius and the same type and the inverse map is also a $\mathbb{G}$-local isometry.

4.2. *Global models.* Consider a compact metric space S and assume that there exists a cover of S by open sets U_i and chart homeomorphisms $h_i : U_i \to V_i$, where V_i is an open set in some $\mathbb{G}$-ball. These open sets and homeomorphisms define an atlas of a $\mathbb{G}$-*branched manifold* structure on S if the transition maps $h_{i,j} = h_j \circ h_i^{-1}$ satisfy the following property.

For each point y in $h_i(U_i \cap U_j)$, there exists a neighborhood B_y of y which is a $\mathbb{G}$-ball contained in $h_i(U_i \cap U_j)$ and a neighborhood $B_{h_{i,j}(y)}$ of $h_{i,j}(y)$ which is a $\mathbb{G}$-ball contained in $h_j(U_i \cap U_j)$, such that the restriction of $h_{i,j}$ to B_y is a local $\mathbb{G}$-isometry from B_y to $B_{h_{i,j}(y)}$.

Two atlases are *equivalent* if their union is again an atlas. A $\mathbb{G}$-*branched manifold* is the data of a metric compact space S together with an equivalence class of atlases $\mathcal{A}$.

LEMMA 4.2. *A $\mathbb{G}$-branched manifold is canonically equipped with a smooth structure, a stratification which is K-right invariant, and a local right $\mathbb{G}$-action defined outside of the singular locus.*

Proof. A smooth curve on a $\mathbb{G}$-branched manifold is a curve that, when read in a chart, remains in a $\mathbb{G}$-sheet and is mapped by proj onto a smooth curve in some $\mathbb{G}$.

The stratification $\mathcal{S}$ is defined as follows. For l in $\{0, \dots, g-k\}$, the *$l+k$-stratum of $\mathcal{S}$* is the set of points which, when read in a chart, have a neighborhood which is a $\mathbb{G}$-ball and belong to the $l+k$-stratum of this $\mathbb{G}$-ball. This property is independent of the chart and is K-right invariant. A *$l+k$-region* is a connected component of the $l+k$-stratum. The finite partition of S in $l+k$-regions, for l in $\{0, \dots, g-k\}$, is called the natural stratification of the $\mathbb{G}$-branched manifold. The union of all the $l+k$-regions for l in $\{0, \dots, g-k-1\}$ forms the *singular locus Sing(S) of S*. Outside of the singular locus, a neighborhood of a point is modeled on a neighborhood in $\mathbb{G}$, it is thus possible to define a local $\mathbb{G}$ action. □

Let S be a $\mathbb{G}$-branched manifold and x a point in S. The *injectivity radius of x* is the supremum of the radii of all $\mathbb{G}$-sheets centered at x. The *injectivity radius of S*, denoted $\mathrm{inj}(S)$, is the infimum over all x in S of the injectivity radius of x. The $\mathbb{G}$-*action radius* is the infimum over all connected components of $S \setminus Sing(S)$ of the supremum of the radius of a $\mathbb{G}$-ball embedded in the connected component, we denote it by $size(S)$.

Let S_1 and S_2 be two $\mathbb{G}$-branched manifolds. A continuous surjection $\tau : S_2 \to S_1$ is a $\mathbb{G}$-submersion if, for any y in S_2, there exist a neighborhood B_y of y which is a $\mathbb{G}$-ball (when read in a chart) and a neighborhood $B_{\tau(y)}$ of $\tau(y)$ which is a $\mathbb{G}$-ball (when read in a chart) such that the restriction of τ to B_y is a $\mathbb{G}$-local submersion from B_y to $B_{\tau(y)}$. It is a *strong $\mathbb{G}$-submersion* if $\tau(B_y)$ is in a $\mathbb{G}$-sheet of $B_{\tau(y)}$. The proof of the following lemma is straightforward.

LEMMA 4.3. *Let S_1 and S_2 be two $\mathbb{G}$-branched manifolds and $\tau : S_2 \to S_1$ a $\mathbb{G}$-submersion. Then $inj(S_2) \geq \mathrm{inj}(S_1)$ and $size(S_2) \geq size(S_1)$.*

4.3. *Projective limits.* For $n \geq 1$, let S_n be a sequence of $\mathbb{G}$-branched manifolds and $\tau_n : S_{n+1} \to S_n$ a sequence of strong $\mathbb{G}$-submersions. Let us recall that the elements of the *projective limit* $\lim_{\leftarrow}(S_n, \tau_n)$ consist of the elements $(x_1, x_2, \ldots, x_n, \ldots)$ in the product $\prod_{n\geq 1} S_n$ such that $\tau_n(x_{n+1}) = x_n$ for all $n \geq 1$. Since all the S_n are compact, $\prod_{n\geq 1} S_n$ equipped with the product topology is compact. The projective limit, being a closed subset of this product, is also compact. For every $i \geq 1$, we denote by p_i the natural continuous map $p_i : \lim_{\leftarrow}(S_n, \tau_n) \to S_i$.

PROPOSITION 4.4. *For $n \geq 1$, let S_n be a sequence of $\mathbb{G}$-branched manifolds and $\tau_n : S_{n+1} \to S_n$ a sequence of strong $\mathbb{G}$-submersions. Then, the projective limit $\lim_{\leftarrow}(S_n, \tau_n)$ is equipped with a natural local $\mathbb{G}$-action.*

Proof. As we have already observed (Lemma 4.2), there exists a local $\mathbb{G}$-action on each $\mathbb{G}$-branched manifold S_n outside of the singular locus. It is straightforward to observe that this action commutes with the $\mathbb{G}$-submersions τ_n. Consider now a point $(x_1, x_2, \ldots, x_n, \ldots)$ in the projective limit $\lim_{\leftarrow}(S_n, \tau_n)$. From Lemma 4.3, there exist a positive number $r > 0$ and a sequence of $\mathbb{G}$-sheets $\mathcal{F}_n$ in S_n centered at x_n with radius r, such that $\tau_n(\mathcal{F}_{n+1}) = \mathcal{F}_n$ for all $n \geq 1$. When the maps τ_n are $\mathbb{G}$-submersions, this sequence $\mathcal{S}_n$ is not necessarily unique. However, it is unique when the maps τ_n are required to be strong $\mathbb{G}$-submersions. For g in $\pi^{-1}(B_r(0)) \subset \mathbb{G}$ and for each $n \geq 1$, we define in a chart $x_n \cdot g = e_n(g)$, where $e_n : \pi^{-1}(B_r(0)) \to S_n$ is the isometric embedding defining $\mathcal{F}_n$. It is easy to check that $\tau_n(x_{n+1} \cdot g) = x_n \cdot g$ for all $n \geq 1$. □

4.4. *$\mathbb{G}$-solenoids and projective limits of $\mathbb{G}$-branched manifolds.* The following theorem relates $\mathbb{G}$-solenoids and projective limits of $\mathbb{G}$-branched manifolds.

THEOREM 4.5. *Let N be a Riemannian manifold on which a connected Lie group $\mathbb{G}$ acts transitively by isometries and such that the stabilizer of a given point* 0 *in N is a compact group K and let M be a $\mathbb{G}$-solenoid:*

(1) *there exist a sequence S_n of $\mathbb{G}$-branched manifolds and a sequence of strong $\mathbb{G}$-submersions $\tau_n : S_{n+1} \to S_n$ such that M is homeomorphic to the projective limit* $\lim_{\leftarrow}(S_n, \tau_n)$*;*

(2) *the local* $\mathbb{G}$*-action on* $\lim_{\leftarrow}(S_n, \tau_n)$ *extends to a* $\mathbb{G}$*-action and the homeomorphism realizes a conjugacy between the two dynamical systems* $(M, \mathbb{G})$ *and* $(\lim_{\leftarrow}(S_n, \tau_n), \mathbb{G})$*;*
(3) $\lim_{n\to+\infty} \mathrm{inj}(S_n) = +\infty$ *and* $\lim_{n\to+\infty} \mathit{size}(S_n) = +\infty$*;*
(4) *all the g-regions of* S_n *have the homotopy type of* K*;*
(5) *if* M *is minimal, then all the* S_n *are connected.*

Proof. Consider a polyhedral box decomposition $\mathcal{B}_1$ of M together with the equivalence relation $\simeq$ that identifies two points that are in the same vertical in the closure of a box of the decomposition. The whole construction we made in the previous section has been adapted so that the quotient space $M/\approx$ inherits a natural structure of $\mathbb{G}$-branched manifold S_1. Let us denote by $p_1 : M \to S_1$, the standard projection. From Theorem 3.1, any minimal $\mathbb{G}$-solenoid admits a tower system, i.e. a sequence of polyhedral box decompositions $(\mathcal{B}_n)_{n\geq 1}$ such that for each $n \geq 1$, $\mathcal{B}_{n+1}$ is zoomed out of $\mathcal{B}_n$. We associate with this sequence the corresponding sequence of $\mathbb{G}$-branched manifolds S_n and maps $p_n : \Omega_{\mathbb{G}}(X) \to S_n$. Remark that two points which are on a same vertical in a closed box in $\mathcal{B}_{n+1}$ are also on a same vertical of a box in $\mathcal{B}_n$. This defines a map $\tau_n : S_{n+1} \to S_n$. It is clear that τ_n is a $\mathbb{G}$-submersion and that $\tau_n \circ \pi_{n+1} = \pi_n$, for all $n \geq 1$. Furthermore, since each $\mathcal{B}_{n+1}$ is zoomed out of $\mathcal{B}_n$, we know that a vertical in the vertical boundary of a box in $\mathcal{B}_n$ contains or is disjoint from any vertical boundary of a box in $\mathcal{B}_{n+1}$. It follows that τ_n maps a small neighborhood of a point in the singular set of $\mathcal{S}_{n+1}$ into a $\mathbb{G}$-sheet of $\mathcal{S}_n$ and thus it is a strong $\mathbb{G}$-submersion.

At this point, we have constructed a sequence S_n of $\mathbb{G}$-branched manifolds and a sequence of strong $\mathbb{G}$-submersions $\tau_n : S_{n+1} \to S_n$ and a map $p : \Omega_{\mathbb{G}}(X) \to \lim_{\leftarrow}(S_n, \tau_n)$ defined by

$$p(x) = (p_1(x_1), \ldots, p_n(x_n), \ldots).$$

It is simple to check that p is a homeomorphism and that it maps the local $\mathbb{G}$-actions one onto another. Since the $\mathbb{G}$-action on M is global, so is the local action on $\lim_{\leftarrow}(S_n, \tau_n)$. Thus, the homeomorphism is a conjugacy between the two dynamical systems $(M, \mathbb{G})$ and $(\lim_{\leftarrow}(S_n, \tau_n), \mathbb{G})$.

Since each g-region in S_{n+1} is made of isometric copies of more than one region S_n, we immediately get that $\lim_{n\to+\infty} \mathrm{inj}(S_n) = +\infty$ and $\lim_{n\to+\infty} \mathit{size}(S_n) = +\infty$. The fact that the box decompositions $\mathcal{B}_n$ are polyhedral yields that the g-regions of $S_{\mathcal{B}_n}$ have the homotopy type of K. The fact that minimality implies connectedness is plain. Note that it is enough to have a dense $\mathbb{G}$-orbit to insure connectedness. □

5. *Transverse invariant measures of* $\mathbb{G}$*-solenoids*

5.1. *Transverse invariant measure and the Ruelle–Sullivan current.* Consider an atlas of a $\mathbb{G}$-solenoid M given by the charts $h_i : U_i \to V_i \times T_i$ where T_i are totally disconnected metric sets and V_i are open subsets in $\mathbb{G}$. A *finite transverse invariant measure* on M (see [**5**]) is the data of a finite positive measure on each set T_i in such a way that if B is a Borelian set in some T_i which is contained in the definition set of the transition map K_{ij}, then

$$\mu_i(B) = \mu_j(K_{ij}(B)).$$

It is clear that the data of a transverse invariant measure for a given atlas provides another invariant measure for any equivalent atlas and thus gives an invariant measure on each vertical. Thus, it makes sense to consider a transverse invariant measure μ^{t} of a $\mathbb{G}$-solenoid. The fact that the leaves of a lamination carry a structure of g-dimensional manifold, where g is the dimension of $\mathbb{G}$, allows us to consider differential forms on M. A k-differential form on M is the data of k-differential forms on the open sets V_i that are mapped onto each other by the differential of the transition maps g_{ij}. We denote by $A^k(M)$ the set of k-differential forms on M. A *foliated cycle* is a linear form from $A^g(M)$ to $\mathbb{R}$ which is positive on positive forms and vanishes on exact forms.

There exists a simple way to associate a transverse invariant measure with a foliated cycle. Consider a g-differential form ω in $A^g(M)$ and assume for the time being, that the support of ω is included in one of the U_i. In this case, the form can be seen as a form in $V_i \times T_i$. By integrating ω on the slices $V_i \times \{t\}$, we get a real valued map on T_i that we can integrate against the transverse measure μ_i to get a real number $\mathcal{C}_{\mu^{\mathrm{t}}}(\omega)$. When the support of ω is not in one of the U_i, we choose a partition of the unity $\{\phi_i\}_i$ associated with the cover of M by the open sets U_i and define

$$\mathcal{C}_{\mu^{\mathrm{t}}}(\omega) = \sum_i \mathcal{C}_{\mu^{\mathrm{t}}}(\phi_i \omega).$$

It is clear that we have defined in this way a linear form $\mathcal{C}_{\mu^{\mathrm{t}}} : A^g(M) \to \mathbb{R}$ which does not depend on the choice of the atlas in $\mathcal{L}$ and of the partition of the unity. It is also easy to check that this linear form is positive for positive forms. The fact that $\mathcal{C}_{\mu^{\mathrm{t}}}$ vanishes on closed forms is a simple consequence of the invariance property of the transverse measure. The foliated cycle $\mathcal{C}_{\mu^{\mathrm{t}}}$ is called *the Ruelle–Sullivan current* associated with the transverse invariant measure μ^{t}. It turns out that the existence of a foliated cycle implies the existence of a transverse invariant measure (see [**16**]) and thus both points of view, transverse invariant measure and foliated cycle, are equivalent. We denote $\mathcal{M}^{\mathrm{t}}(M)$ the convex set of transverse invariant measure on M.

5.2. *Transverse and $\mathbb{G}$-invariant measures.* In this section, we assume that the Lie group G is unimodular, i.e. it admits a right- and left-invariant measure $\lambda_{\mathbb{G}}$. Let us now use the fact a $\mathbb{G}$-solenoid carries a $\mathbb{G}$ action and assume that there exist a finite measure μ on M which is invariant for the $\mathbb{G}$-action. This invariant measure defines a transverse invariant measure. For any Borelian subset of a transverse set T_i

$$\mu_i(B) = \frac{\mu(h_i^{-1}(V_i \times B))}{\lambda_{\mathbb{G}}(V_i)}.$$

Conversely, consider a transverse invariant finite measure μ^{t} of the $\mathbb{G}$-solenoid M. Let $f : M \to \mathbb{R}$ be a continuous function and assume for the time being that the support of f is included in one of the U_i. In this case, the map $f \circ h_i^{-1}$ is defined on $V_i \times T_i$. By integrating $f \circ h_i^{-1}$ on the sheets $V_i \times \{t\}$ against the measure $\lambda_{\mathbb{G}}$ of $\mathbb{G}$, we get a real valued map on T_i that we can integrate against the transverse measure μ_i to get a real number $\int f \, d\mu$. When the support of f is not in one of the U_i, we choose a partition of the unity $\{\phi_i\}_i$

associated with the cover of M by the open sets U_i and define

$$\int f\, d\mu = \sum_i \int f\phi_i\, d\mu.$$

It is clear that we have defined in this way a finite measure on M which does not depend on the choice of the atlas in its equivalence class and of the partition of the unity and is invariant under the $\mathbb{G}$-action.

Thus, for a $\mathbb{G}$-solenoid M and when the Lie group is unimodular, the following three points of view are equivalent:

- a finite transverse invariant measure;
- a foliated cycle;
- a finite $\mathbb{G}$-invariant measure on M.

5.3. *Transverse invariant measures and homology.* Let us first recall a few elementary facts about the (cellular) homology of S. Consider the natural stratification of S, $\pi^{-1}(\mathcal{V}_0), \ldots, \pi^{-1}(\mathcal{V}_{g-k})$, where, for $l = 0, \ldots, g-k$, $\pi^{-1}(\mathcal{V}_l)$ is decomposed into a finite number a_{l+k} of $l+k$-regions that we orient and order in an arbitrary way but for the orientation of the g-regions which is the natural orientation induced by the $\mathbb{G}$-manifold structure. We refine this stratification by regions to get a cell decomposition and we orient each g-cell according to the orientation of the g-region that contains it. We denote by $C_i(S, \mathbb{R})$ the free $\mathbb{R}$-module which has as a (ordered) basis the set of ordered and oriented i-cells. By convention, for any oriented i-cell e, $-e = -1e$ and it consists of the same cell with the opposite orientation. Note that with the exception of the g-cells orientation, there are no canonical choices for the above orders and orientations, but these choices will be essentially immaterial in our discussion.

We define the linear *boundary operator*

$$\partial_{i+1} : C_{i+1}(S, \mathbb{R}) \to C_i(S, \mathbb{R})$$

which assigns to any $i+1$-region, the sum of the i-regions that are in its closure considered with a positive (respectively negative) sign if the induced orientation fits (respectively does not fit) with the orientation chosen for these i-regions. It is clear that $\partial_i \circ \partial_{i+1} = 0$.

The space $Z_i(S, \mathbb{R}) = \operatorname{Ker} \partial_i$ is called the space of i-cycles of S and the space $B_i(S, \mathbb{R}) = \partial_{i+1}(C_{i+1}(S, \mathbb{R}))$ is called the space of i-boundaries of S. In fact, $B_i(S, \mathbb{R}) \subset Z_i(S, \mathbb{R})$ and $H_i(S, \mathbb{R}) = Z_i(S, \mathbb{R})/B_i(S, \mathbb{R})$ is the ith homology group of S. Note that $H_g(S, \mathbb{R}) = Z_g(S, \mathbb{R})$.

A standard result of algebraic topology ensures that (up to $\mathbb{R}$-module isomorphism) $H_i(S, \mathbb{R})$ is a topological invariant of S that coincides with the ith singular homology of S (see for example [**15**]).

It is important to observe that a g-chain z is a g-cycle (i.e. $\partial_d(z) = 0$) if and only if:

- two g-cells which are in a same region appear in z with the same coefficient;
- the coefficients of z satisfy the 'switching rules' (or Kirchoff-like laws). This means that along every $g-1$-region e of S the sum of the weights on the germs of g-cells along e on one side equal the sum of the weights of the germs of g-cells on the other side.

Identifying the free $\mathbb{R}$-module which has as a (ordered) basis the set of ordered and oriented g-regions with of $\mathbb{R}^{a_g}$, the homology group $H_g(S, \mathbb{R})$ is then the subspace of $\mathbb{R}^{a_g}$ defined by the switching rules.

We denote by $H_g^+(S, \mathbb{R})$ the intersection of $H_g(S, \mathbb{R})$ with the positive cone in $\mathbb{R}^{a_g}$.

THEOREM 5.1. *Let M be a $\mathbb{G}$-solenoid and S_n, $n \geq 1$, be a sequence of $\mathbb{G}$-branched manifolds and $\tau_n : S_{n+1} \to S_n$ a sequence of strong $\mathbb{G}$-submersions such that the two dynamical systems $(M, \mathbb{G})$ and $(\lim_{\leftarrow}(S_n, \tau_n), \mathbb{G})$ are conjugate. Then $\mathcal{M}^{\mathrm{t}}(M)$ is isomorphic to $\lim_{\leftarrow}(H_g^+(S_n, \mathbb{R}), \tau_{n,\star})$ where $\tau_{n,\star} : H_g(S_{n+1}, \mathbb{R}) \to H_g(S_n, \mathbb{R})$ is the map induced by τ_n on homology.*

Proof. Let us choose one of these branched manifolds S_n together with the natural projection $p_n : M \to S_n$. Let $R_{1,n}, \ldots, R_{p(n),n}$ be the ordered sequence of g-regions of S_n equipped with the natural orientation and choose a g-region $R_{i,n}$ of S_n. The set $p_n^{-1} R_{i,n}$ reads in some chart $V_{i,n} \times T_{i,n}$. A transverse measure μ^{t} associates a weight $w(R_{i,n}) = \mu^{\mathrm{t}}(T_{i,n})$ with $R_{i,n}$ and thus defines an element $i_n(\mu_t)$ in $C_g(S_n, \mathbb{R})$. The fact that the transverse measure is invariant implies that the switching rules are satisfied, thus $i(\mu_t)$ is in $H_g(S_n, \mathbb{R})$. Since a transverse invariant measure associates a positive weight with each region, we conclude that $i_n(\mu_t)$ is in $H_g^+(S_n, \mathbb{R})$.

Let $R_{1,n+1}, \ldots, R_{p(n+1),n+1}$ be the ordered sequence of regions of S_{n+1} equipped with the natural orientation and $\mathbb{R}^{p(n+1)}$ the associated $\mathbb{R}$-module. By identifying $H_g(S_{n+1}, \mathbb{R})$ with the subspace of $\mathbb{R}^{p(n+1)}$ defined by the switching rules and $H_g(S_n, \mathbb{R})$ with the subspace of $\mathbb{R}^{p(n)}$ also defined by the switching rules, we assign to the linear map

$$\tau_{n,*} : H_g(S_{n+1}, \mathbb{R}) \to H_g(S_n, \mathbb{R})$$

a $n(p) \times n(p+1)$ matrix A_n with integer non-negative coefficients. The coefficient $a_{i,j,n}$ of the ith line and the jth column is exactly the number of pre-images in $R_{j,n+1}$ of a point in $R_{i,n}$. Thus, we have the relations

$$w(R_{i,n}) = \sum_{j=1}^{j=p(n+1)} a_{i,j,n} w(R_{j,n+1})$$

for all $i = 1, \ldots, p(n)$ and all $j = 1, \ldots p(n+1)$, which exactly means that $\tau_{n,*} \circ i_n = i_{n+1}$. Thus any finite, transverse, invariant measure can be seen as an element in $\lim_{\leftarrow}(H_g^+(S_n, \mathbb{R}), \tau_{n,\star})$.

Conversely, let $T_{1,1}, \ldots T_{p(1),1}$ be the vertical associated with each g-region of the first $\mathbb{G}$-branched manifold S_1. Since the $T_{i,1}$ are totally disconnected metric spaces, they can be covered by a partition in clopen sets with arbitrarily small diameters. Such a partition $\mathcal{P}$ is *finer* than another partition $\mathcal{P}'$ if the defining clopen sets of the first are included in the clopen sets of the second. Consider a sequence of partitions $\mathcal{P}_n$, $n \geq 0$ of $T_{i,1}$ such that, for all $n \geq 0$, $\mathcal{P}_{n+1}$ is finer than $\mathcal{P}_n$ and the diameter of the defining clopen sets of $\mathcal{P}_n$ goes to zero as n goes to $+\infty$. A finite measure on $T_{i,1}$ is given by the countable data of non-negative numbers associated with each defining clopen set of each partition $\mathcal{P}_n$ which satisfy the obvious additivity relation. An element $y = (y_1, \ldots, y_n, \ldots)$ in $\lim_{\leftarrow}(H_g^+(S_n, \mathbb{R}), \tau_{n,\star})$ provides us with such a sequence of partitions $\mathcal{P}_n$ defined as follows:

- let $\hat{\pi}_i : p^{-1}(R_{i,1}) \to T_{i,1}$, the chart map composed with the projection;
- the clopen sets of $\mathcal{P}_n$ are the $\hat{\pi}_i(p_n^{-1}(R_{j,n}) \cap R_{i,1})$ for $j = 1, \ldots, p(n)$.

The fact that all y_n satisfy the switching rules implies that the transverse measure is invariant. □

COROLLARY 5.2.

- *If the dimension of $H_g(S_n, \mathbb{R})$ is uniformly bounded by N, then $\mathcal{M}^{\mathrm{t}}(M)$ contains at most N ergodic measures up to a global rescaling.*
- *If, furthermore, the coefficients of all the matrices A_n are uniformly bounded and positive, then $\mathcal{M}^{\mathrm{t}}(M)$ is reduced to a single direction, i.e. the transverse dynamics are uniquely ergodic.*

Proof. The proof is standard and can be found in [**4**] in a quite similar situation in the particular case when $g = 1$. To prove the first statement, we may assume that the dimension of $H_g(S_n, \mathbb{R})$ is constant and equal to N. The set $\mathcal{M}^{\mathrm{t}}(M)$ is a convex set and its extremal directions coincide with the set of ergodic measures. Since $\mathcal{M}^{\mathrm{t}}(M)$ is isomorphic to $\lim_{\leftarrow}(H_g^+(S_n, \mathbb{R}), \tau_{n,\star})$ the convex cone $\mathcal{M}^{\mathrm{t}}(M)$ is the intersection of the convex nested sets

$$\mathcal{M}^{\mathrm{t}}(M) = \bigcap_{n \geq 0} W_n$$

where

$$W_n = \tau_{1,\star} \circ \cdots \circ \tau_{n-1,\star}(H_g^+(S_n, \mathbb{R})).$$

Since each convex cone W_n possesses at most N extremal lines, the limit set $\mathcal{M}^{\mathrm{t}}(M)$ also possesses at most N extremal points and thus at most N ergodic probability measures.

In order to prove the second statement, consider two points x and y in the positive cone of $\mathbb{R}^N$. Let T be the largest line segment containing x and y and contained in the positive cone of $\mathbb{R}^N$. We recall that the hyperbolic distance between x and y is given by

$$\mathrm{Hyp}(x, y) = -\ln \frac{(m + l)(m + r)}{l \cdot r},$$

where m is the length of the line segment $[x, y]$ and l and r are the length of the connected components of $T \setminus [x, y]$. Positive matrices contract the hyperbolic distance in the positive cone of $\mathbb{R}^N$. Since the matrices corresponding to the maps $\tau_{\star n}$ are uniformly bounded in sizes and entries, this contraction is uniform. Because of this uniform contraction, the set $\mathcal{M}^{\mathrm{t}}(M)$ is one dimensional.

Remark 5.3. Note that existence of a transverse invariant measure for a $\mathbb{G}$-solenoid M implies, when writing M as an inverse limit of $\mathbb{G}$-branched manifolds

$$M = \lim_{\leftarrow}(S_n, \tau_n),$$

that $H_g^+(S_n, \mathbb{R}) \neq 0$ for each $n \geq 1$.

Remark 5.4. As we have seen, considering a $\mathbb{G}$-solenoid as a projective limit of $\mathbb{G}$-branched manifolds allows us to give a topological characterization of the invariant measures. It turns out that in the case of an $\mathbb{R}$-solenoid, i.e. a $\mathbb{Z}$-action free action on the Cantor set, the description in terms of projective limit of graphs yields a topological characterization of some other dynamical invariants (topological entropy, semi-conjugacy to rotations, etc.) [**4**]. It is likely that a similar approach works in the general case of $\mathbb{G}$-solenoids. Furthermore, projective limits of graphs can be used to construct $\mathbb{R}$-solenoids with some nice exotic properties (for example, minimal, uniquely ergodic and with infinite topological entropy or minimal with many ergodic invariant probability measures) and a simple way to construct these examples is to use Toeplitz systems (branches with equal lengths). At least in the case of $\mathbb{R}^d$-solenoids, similar constructions can be done by considering $\mathbb{R}^d$-branched manifolds made of tori with dimension d glued along tori with dimension $d-1$.

6. *More specific analyses*

6.1. *$\mathbb{E}^d$-solenoids.* The Lie group $\mathbb{E}^d$ of affine orientation-preserving isometries of the Euclidean space $\mathbb{R}^d$ is the semi-direct product $SO(d) \rtimes \mathbb{R}^d$. Thus, in addition to the $\mathbb{E}^d$-action on a $\mathbb{E}^d$-solenoid M, there also exists a $\mathbb{R}^d$-(sub)-action. Choose a point x in M, we denote by $\omega_{\mathbb{E}^d,\mathbb{R}^d}(x)$ the closure of the orbit $x + \mathbb{R}^d$ in M.

PROPOSITION 6.1. *For each x in a minimal $\mathbb{E}^d$-solenoid M, the $\mathbb{R}^d$-invariant subset $\omega_{\mathbb{E}^d,\mathbb{R}^d}(x)$ is $\mathbb{R}^d$-minimal, i.e. all its $\mathbb{R}^d$-orbits are dense in $\omega_{\mathbb{E}^d,\mathbb{R}^d}(x)$.*

Proof. Let g be an element in $\mathbb{E}^d$ and x a point in M. Since $(x + \mathbb{R}^d) \cdot g = x \cdot g + \mathbb{R}^d$, we have $\omega_{\mathbb{E}^d,\mathbb{R}^d}(x \cdot g) = \omega_{\mathbb{E}^d,\mathbb{R}^d}(x) \cdot g$. Any $\mathbb{R}^d$-action on a compact set possesses a minimal invariant subset, thus there exists x_0 in M such that $\omega_{\mathbb{E}^d,\mathbb{R}^d}(x_0)$ is $\mathbb{R}^d$-minimal. It follows that for any g in $\mathbb{E}^d$, $\omega_{\mathbb{E}^d,\mathbb{R}^d}(x_0 \cdot g)$ is either equal to or completely disjoint from $\omega_{\mathbb{E}^d,\mathbb{R}^d}(x_0)$, but in both cases it is also $\mathbb{R}^d$-minimal. Since M is minimal for the $\mathbb{E}^d$-action, any point x in M, belongs to a $\omega_{\mathbb{E}^d,\mathbb{R}^d}(x_0 \cdot g)$ for some g in $\mathbb{E}^d$ and thus $\omega_{\mathbb{E}^d,\mathbb{R}^d}(x)$ is also $\mathbb{R}^d$-minimal. □

Let K_y be the stabilizer of $\omega_{\mathbb{E}^d,\mathbb{R}^d}(y)$ in $\mathbb{E}^d$, i.e. the closed subgroup of K in $SO(d)$ such that $\omega_{\mathbb{E}^d,\mathbb{R}^d}(y) \cdot K = \omega_{\mathbb{E}^d,\mathbb{R}^d}(y)$. The stabilizers associated with two different points in M are conjugate. The proof of the following proposition is immediate.

PROPOSITION 6.2. *Let M be a minimal $\mathbb{E}^d$-solenoid. Then, for any y in M, the following assertions are equivalent:*

(i) $\omega_{\mathbb{E}^d,\mathbb{R}^d}(y) = \Omega_{\mathbb{E}^d}(y)$;

(ii) $K_y = SO(d)$.

Furthermore, if one of these properties is satisfied for some y in M, then it is also satisfied for any other y' in M.

A minimal, $\mathbb{E}^d$-solenoid which satisfies one of the properties in Proposition 6.2 is called *isotropic*.

Consider now a projective limit of $\mathbb{E}^2$-branched manifolds associated with a free minimal $\mathbb{E}^2$-solenoid $M = \lim_{\leftarrow}(S_n, \tau_n)$. The $\mathbb{E}^2$-action on the hull M decomposes itself into two parts when projected on the $\mathbb{E}^2$-branched manifold S_n. On the one hand, the $SO(2)$-action on M projects by p_n on the $SO(2)$ action on the $\mathbb{E}^2$-branched manifold S_n. On the other hand, there is of course no $\mathbb{R}^2$-action on S_n, but for any y in M, $\omega_{\mathbb{E}^2,\mathbb{R}^2}(y)$ projects on a closed subset of S_n which is K_y invariant. The $\mathbb{E}^2$-branched manifold S_n is foliated by the projection of the orbits $y + \mathbb{R}^2$ for y in M. In the particular case when M is isotropic, all the projected $\mathbb{R}^2$-orbits are dense.

6.2. *$\mathbb{R}^d$-solenoids.* In the particular case when the group $\mathbb{G} = \mathbb{R}^d$ we have the following result.

PROPOSITION 6.3. *Any minimal $\mathbb{R}^d$-solenoid admits a rectangular box decomposition.*

Proof. Consider rectangular boxes that read in the charts of an atlas, $R \times T$, where R is a rectangle with faces parallel to one of the unit d-cubes generated by the canonical basis of $\mathbb{R}^d$ and T is a totally disconnected set. We call such rectangular boxes well-oriented rectangular boxes. The intersection of two well-oriented rectangular boxes is a finite union of well-oriented rectangular boxes and that the closure of the union of two well-oriented rectangular boxes is also the closure of the union of a finite collection of pairwise disjoint well-oriented rectangular boxes. It follows that there exists a finite collection rectangular boxes $R_1 \times T_1 \times, \ldots, R_n \times T_n$ pairwise disjoint and whose closures cover the whole solenoid. □

This yields the following result.

THEOREM 6.4. *Any minimal $\mathbb{R}^d$-solenoid is orbit equivalent to an $\mathbb{R}^d$-solenoid which fibers on the d-torus.*

Proof. Consider a minimal $\mathbb{R}^d$-solenoid M. From Proposition 6.3, we can associate with M a rectangular box decomposition $\mathcal{B}$ which, by projection along the vertical direction in the boxes, gives a $\mathbb{R}^d$-branched manifold $S_{\mathcal{B}}$ whose d-regions are d-dimensional rectangles. Consider the (1)-stratum of the stratification of $S_{\mathcal{B}}$. It is made with a finite number of one-dimensional edges parallel to the directions of the vectors of the canonical orthonormal basis and a finite number of vertices. Assume that all the sizes of these edges are commensurable, i.e. there exists a positive real τ such that the $\mathbb{R}^d$-branched manifold S can be filled with d-cubes with size τ that intersect on their boundary, full faces to full faces. This induces a projection $\pi : S \to \mathbb{T}^d$ where $\mathbb{T}^d$ is the d-torus $\mathbb{R}^d/\tau\mathbb{Z}^d$ which commutes with translations. In the general case, the sizes of the rectangles are not commensurable. However, we can perform a change of the metric g in the leaves that satisfy the following properties:

- at each point in $S_{\mathcal{B}}$, the metric g preserves the orthogonality of basis corresponding in charts to the canonical basis of $\mathbb{R}^d$;
- in the chart associated with each rectangular box of the box decomposition, the metric g is independent of the choice of the point on the same vertical;
- the size of the rectangles are commensurable when computed with the metric g.

This new metric defines a homeomorphism between the $\mathbb{R}^d$-solenoid $S_{\mathcal{B}}$ and a $\mathbb{R}^d$-solenoid which possesses a box decomposition in commensurable rectangles and thus fibers on a d-torus. □

As a direct consequence, we recover a result obtained by Sadun and Williams [**14**].

COROLLARY 6.5. *Let X be a totally aperiodic, repetitive, Delone set of $\mathbb{R}^d$-finite type, then there exists a totally aperiodic, repetitive, Delone set of $\mathbb{R}^d$-finite type Y which is $\mathbb{R}^d$-orbit equivalent to X and is contained in the lattice $\mathbb{Z}^d$.*

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