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Reidemeister–Turaev torsion of 3-dimensional Euler structures with simple boundary tangency and pseudo-Legendrian knots

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Abstract. We generalize Turaev's definition of torsion invariants of pairs (M, ξ), where M is a 3-dimensional manifold and ξ is an Euler structure on M (a non-singular vector field up to homotopy relative to ∂M and modifications supported in a ball contained in $\mathrm{Int}(M)$). Namely, we allow M to have arbitrary boundary and ξ to have simple (convex and/or concave) tangency circles to the boundary. We prove that Turaev's $H_1(M)$-equivariance formula holds also in our generalized context. Using branched standard spines to encode vector fields we show how to explicitly invert Turaev's reconstruction map from combinatorial to smooth Euler structures, thus making the computation of torsions a more effective one. Euler structures of the sort we consider naturally arise in the study of pseudo-Legendrian knots (*i.e.* knots transversal to a given vector field), and hence of Legendrian knots in contact 3-manifolds. We show that torsion, as an absolute invariant, contains a lifting to pseudo-Legendrian knots of the classical Alexander invariant. We also precisely analyze the information carried by torsion as a relative invariant of pseudo-Legendrian knots which are framed-isotopic.

0. Introduction

Reidemeister torsion is a classical yet very vital topic in 3-dimensional topology, and it was recently used in a variety of important developments. To mention a few, torsion is a fundamental ingredient of the Casson–Walker–Lescop invariants (see *e.g.* [8]). Relations have been pointed out between torsion and hyperbolic geometry [13]. Turaev's torsion of Euler structures [16] has been recognized by Turaev himself ([17, 18]) to have deep connections with the Seiberg–Witten invariants of Spin^c-structures on 3-manifolds, after the proof of Meng and Taubes [10] that a suitable combination of these invariants can be identified with the classical Milnor torsion.

0.1. Review of known results

Turaev's theory [16] actually exists in all dimensions. We quickly review it before proceeding. A *smooth Euler structure* ξ on a compact oriented manifold M, possibly

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with $\partial M = \emptyset$, is a non-singular vector field on M viewed up to arbitrary modifications supported in a closed ball contained in $\mathrm{Int}(M)$ and homotopy relative to ∂M. Orientability of M is not strictly necessary, but we find it convenient to assume it. Turaev only considers the case where the field is transversal to the boundary, so the boundary components are "monochromatic" (*black* if the field points outwards, and *white* if it points inwards). This implies the constraint that $\chi(M, W) = 0$, where W is the white portion of ∂M, but in [17] and [18] Turaev only focuses on the more specialized case where M is 3-dimensional and closed or bounded by black tori. In all dimensions, the set $\mathrm{Eul}^{\mathrm{s}}(M, W)$ of smooth Euler structures compatible with (M, W) is an affine space over $H_1(M; \mathbb{Z})$. The two main ingredients of Turaev's theory are as follows. First, he defines a certain set of 1-chains, called the space $\mathrm{Eul}^{\mathrm{c}}(M, W)$ of *combinatorial* Euler structures compatible with (M, W), he shows that this is again affine over $H_1(M; \mathbb{Z})$, and he describes an $H_1(M; \mathbb{Z})$-equivariant bijection $\Psi : \mathrm{Eul}^{\mathrm{c}}(M, W) \to \mathrm{Eul}^{\mathrm{s}}(M, W)$ called the *reconstruction map*. Second, for $\xi \in \mathrm{Eul}^{\mathrm{c}}(M, W)$ and for any representation φ of $\pi_1(M)$ into the units of a suitable ring Λ he defines a torsion invariant $\tau^{\varphi}(M, \xi)$, or more generally $\tau^{\varphi}(M, \xi, \mathfrak{h})$, as explained below in Sect. 2, with values in $K_1(\Lambda)/(\pm 1)$, where $K_1(\Lambda)$ denotes the Whitehead group of Λ. This invariant satisfies the $H_1(M; \mathbb{Z})$-equivariance formula

$$\tau^{\varphi}(M, \xi', \mathfrak{h}) = \tau^{\varphi}(M, \xi, \mathfrak{h}) \cdot \overline{\varphi}(\xi' - \xi) \tag{1}$$

where $\xi' - \xi \in H_1(M; \mathbb{Z})$ and $\overline{\varphi}$ is naturally induced by φ. In addition $\tau^{\varphi}(M, \xi)$ is by definition a lifting of the classical Reidemeister torsion (see [11]) $\tau^{\varphi}(M, W) \in K_1(\Lambda)/(\pm\varphi(\pi_1, (M)))$. For $\xi \in \mathrm{Eul}^{\mathrm{s}}(M, W)$ one defines $\tau^{\varphi}(M, \xi)$ as $\tau^{\varphi}(M, \Psi^{-1}(\xi))$, and the $H_1(M; \mathbb{Z})$-equivariance of the reconstruction map Ψ implies that formula (1) holds also for smooth structures. We emphasize that the definition of Ψ is based on an explicit geometric construction, but its bijectivity is only established through $H_1(M; \mathbb{Z})$-equivariance. This makes the definition of torsion for smooth structures somewhat implicit.

0.2. Aims of the paper

In [3] we have provided a combinatorial encoding of non-singular vector fields up to homotopy (also called "combings") in terms of branched standard spines, and the initial aim of this paper was to use this encoding in order to find a geometric description of the map Ψ^{-1}, and hence to turn the computation of Turaev's torsion into a more effective procedure. The use of branched standard spines naturally leads to considering Euler structures on 3-manifolds M (without restrictions on ∂M) with simple tangency circles to ∂M of *concave* type (see Fig. 1 below). This type of generalized Euler structure also arises in the study of a Legendrian knot K in a contact structure ξ, and more generally when one considers a "pseudo-Legendrian" pair (v, K), consisting of a knot K transversal to a non-singular vector field v. A pair (v, K) is viewed up to the natural relation of 'pseudo-Legendrian isotopy' (see Sect. 5), and, taking the restriction of v, it defines a concave Euler structure on the exterior $E(K)$ of K, with two parallel concave tangency lines on $\partial E(K)$ determined by the framing induced by v on K. On the other hand it turns out that, to define torsion, the natural objects to deal with are Euler structures with *convex* tangency circles. It is a fortunate fact, peculiar of dimension 3, that there is a canonical way to associate a convex field to any *simple* (*i.e.* mixed concave and convex) one. This allows to define torsion for all smooth simple Euler structures.

0.3. Summary – general theory

Let us now summarize the contents of this paper. In Sect. 1, extending Turaev's theory, we set the foundations of the theory of smooth and combinatorial Euler structures in the context of structures with simple tangency to the boundary. In particular we describe the obstruction to the existence of such a structure (Proposition 1.1) and we describe the reconstruction map Ψ (Theorem 1.4). This part follows the same scheme as [16] and relies on some technical results of Turaev. Our main contribution here is the proof that there exists a natural transformation of a simple structure into a convex one both at the smooth and at the combinatorial level, and that these transformations actually correspond to each other under the reconstruction map (Theorem 1.7). In Sect. 2 we introduce torsion and state the equivariance property. In Sect. 3 we show that if P is a branched standard spine and ξ is the Euler structure (with concave boundary tangency) carried by P, then P allows to explicitly find a representative of $\Psi^{-1}(\xi)$, namely the combinatorial counterpart of ξ (Theorem 3.7). In Sect. 4 we carry out a specific computation of torsion using the technology of Sect. 3.

0.4. Summary – pseudo-Legendrian knots

In Sects. 5 and 6 we concentrate on the application of the theory of torsion developed in Sects. 1 and 2 to pseudo-Legendrian knots. In Sect. 5 we analyze torsion as an *absolute* invariant, and we show in particular that, when the ambient manifold is a homology sphere, torsion contains (in a suitable sense) a lifting to pseudo-Legendrian knots of the classical Alexander invariant. In Sect. 6 we turn to the information carried by torsion as a *relative* invariant, and we show that it is capable of distinguishing knots which are framed isotopic but not pseudo-Legendrian-isotopic. A delicate point which emerges here is that, given pseudo-Legendrian pairs (v_0, K_0) and (v_1, K_1) such that (v_0, v_1) are homotopic to each other and (K_0, K_1) are framed-isotopic to each other, torsion does not provide in general a single-valued relative invariant of (v_0, K_0) and (v_1, K_1), because the action of a certain mapping class group (which depends on the framed isotopy class only) must be taken into account. This phenomenon is carefully described in Sect. 6, where we introduce and study the notion of 'good' framed knot, for which the action is trivial. We show that many knots are good (for instance, all knots in a homology sphere are good, and most knots with hyperbolic complement are good). In the special case of knots in a homology sphere we also prove that the relative torsion of two knots essentially coincides with the difference of their rotation numbers (Maslov indices), so torsion basically detects whether the knots are isotopic through pseudo-Legendrian immersions. In the more general case of a good knot in a manifold which may not be a homology sphere, we analyze the effect on torsion of the framed first Reidemeister move (which does not change the framing but locally changes the winding number by ± 2). Combining this analysis with the fact (proved in [4]) that framed isotopy is generated by pseudo-Legendrian isotopy and the framed first Reidemeister move, we are able to give an interpretation of torsion, as a relative invariant of knots, by means of a well-defined "relative winding number". For homology spheres the relation between the winding number and the Maslov index is clear, but we emphasize that the definition of relative winding number works in more general situations.

Our results on good knots and the relative winding number allow to single out several situations in which torsion actually can detect pseudo-Legendrian isotopy. For instance we can show the following: *consider pseudo-Legendrian knots* (v_0, K_0) *and* (v_1, K_1) *which are isotopic as framed knots. Assume that* K_0 *is good and that the meridian of* K_0 *has infinite order in* $H_1(E(K_0); \mathbb{Z})$. *Then the knots are pseudo-Legendrian-isotopic if and only if they have trivial relative torsion invariants.*

In Sects. 1 and 3 proofs which are long and require the introduction of ideas and techniques not used elsewhere are omitted. Section 7 contains all these proofs.

1. Euler structures

In this section we define smooth and combinatorial Euler structures and illustrate their correspondence. We fix once and for ever a compact oriented 3-manifold M, possibly with $\partial M = \emptyset$. Using the *Hauptvermutung*, we will always freely intermingle the differentiable, piecewise linear and topological viewpoints. Homeomorphisms will always respect orientations. All vector fields mentioned in this paper will be non-singular unless the contrary is explicitly stated, and they will be termed just *fields* for the sake of brevity.

1.1. Smooth and combinatorial Euler structures

We will call *boundary pattern* on M a partition $\mathcal{P} = (W, B, V, C)$ of ∂M where V and C are finite unions of disjoint circles, and $\partial W = \partial B = V \cup C$. In particular, W and B are interiors of compact surfaces embedded in ∂M. Even if $\mathcal{P}$ can actually be determined by less data, *e.g.* the pair (W, V), we will find it convenient to refer to $\mathcal{P}$ as a quadruple. Points of W, B, V and C will be called *white*, *black*, *convex* and *concave* respectively. We define the set of *smooth Euler structures* on M compatible with $\mathcal{P}$, denoted by $\mathrm{Eul}^{\mathrm{s}}(M, \mathcal{P})$, as the set of equivalence classes of fields on M which point inside M on W, point outside M on B, and have simple tangency to ∂M of *convex* type along V and *concave* type along C, as shown in a cross-section in Fig. 1. Two such fields are equivalent if they are obtained from each other by homotopy through fields of the same type and modifications supported into closed interior balls (namely, replacements of a field v by another one v' such that $v - v'$ vanishes outside a ball contained in $\mathrm{Int}(M)$). The following variation on the Hopf–Poincaré theorem is established in Sect. 7:

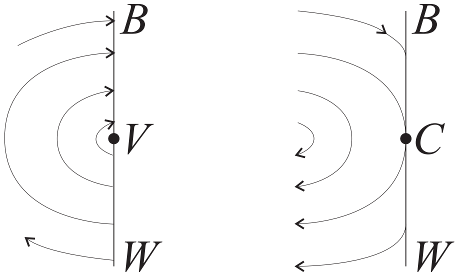

Fig. 1. Convex (left) and concave (right) tangency to the boundary

Proposition 1.1. $\mathrm{Eul}^{\mathrm{s}}(M,\mathcal{P})$ *is non-empty if and only if* $\chi(\overline{W})=\chi(M)$.

We remark here that $\chi(\overline{W})=\chi(W)$, $\chi(\overline{B})=\chi(B)$, $\chi(V)=\chi(C)=0$ and $\chi(W)+\chi(B)=\chi(\partial M)=2\chi(M)$, so there are various ways to rewrite the relation $\chi(\overline{W})=\chi(M)$, the most intrinsic of which is actually $\chi(M)-(\chi(\overline{W})-\chi(C))=0$ (see the discussion before Lemma 1.2 for the reason).

It is a standard fact of obstruction theory that, given fields v and v' compatible with a pattern $\mathcal{P}$ on ∂M, the first obstruction to v' being homotopic to v through fields compatible with $\mathcal{P}$ is given by a homology class $\alpha^{\mathrm{s}}(v,v')\in H_1(M;\mathbb{Z})$. The same theory also shows that v and v' represent the same Euler structure $\xi\in\mathrm{Eul}^{\mathrm{s}}(M,\mathcal{P})$ if and only if $\alpha^{\mathrm{s}}(v,v')=0$. It follows that if ξ and ξ' in $\mathrm{Eul}^{\mathrm{s}}(M,\mathcal{P})$ are represented by fields v and v' then $\alpha^{\mathrm{s}}(\xi,\xi')$ is unambiguously defined as $\alpha^{\mathrm{s}}(v,v')$. Moreover

$$\alpha^{\mathrm{s}}:\mathrm{Eul}^{\mathrm{s}}(M;\mathcal{P})\times\mathrm{Eul}^{\mathrm{s}}(M;\mathcal{P})\to H_1(M;\mathbb{Z})$$

gives to $\mathrm{Eul}^{\mathrm{s}}(M;\mathcal{P})$ the structure of an affine space over $H_1(M;\mathbb{Z})$. All these facts are carefully stated in Sect. 5 of [16] for the case where $C=V=\emptyset$ and both W and B are unions of tori, but they extend *verbatim* to our situation. See also the discussion in Sect. 6.2 of [3] for the case of closed manifolds.

A (finite) cellularization $\mathcal{C}$ of M is called *suited* to $\mathcal{P}$ if $V\cup C$ is a subcomplex, so W and B are unions of cells. Here and in the sequel by "cell" we will always mean an *open* one. Let such a $\mathcal{C}$ be given. For $\sigma\in\mathcal{C}$ define $\mathrm{ind}(\sigma)=(-1)^{\dim(\sigma)}$. We define $\mathrm{Eul}^{\mathrm{c}}(M,\mathcal{P})_{\mathcal{C}}$ as the set of equivalence classes of integer singular 1-chains z in M such that

$$\partial z=\sum_{\sigma\subset M\setminus(W\cup V)}\mathrm{ind}(\sigma)\cdot p_\sigma \tag{2}$$

where $p_\sigma\in\sigma$ for all σ. Two chains z and z' with $\partial z=\sum\mathrm{ind}(\sigma)\cdot p_\sigma$ and $\partial z'=\sum\mathrm{ind}(\sigma)\cdot p'_\sigma$ are defined to be equivalent if there exist $\delta_\sigma:([0,1],0,1)\to(\sigma,p_\sigma,p'_\sigma)$ such that

$$z-z'+\sum_{\sigma\subset M\setminus(W\cup V)}\mathrm{ind}(\sigma)\cdot\delta_\sigma$$

represents 0 in $H_1(M;\mathbb{Z})$. Elements of $\mathrm{Eul}^{\mathrm{c}}(M,\mathcal{P})_{\mathcal{C}}$ are called *combinatorial Euler structures* relative to $\mathcal{P}$ and $\mathcal{C}$, and their representatives are called *Euler* *chains*. The definition implies that, for $\xi, \xi' \in \mathrm{Eul}^{\mathrm{c}}(M, \mathcal{P})_{\mathcal{C}}$, their difference $\xi - \xi'$ can be defined as an element $\alpha^{\mathrm{c}}(\xi, \xi')$ of $H_1(M; \mathbb{Z})$. Since $\overline{W} = W \cup V \cup C$ and C is closed, the total algebraic number of points appearing in the right-hand side of (2) can be written as

$$\chi(M) - \chi(\overline{W} \setminus C) = \chi(M) - (\chi(\overline{W}) - \chi(C)).$$

Considering that $\chi(C) = 0$ and that M is connected, we easily get:

Lemma 1.2. $\mathrm{Eul}^{\mathrm{c}}(M, \mathcal{P})_{\mathcal{C}}$ *is non-empty if and only if* $\chi(\overline{W}) = \chi(M)$, *and in this case* α^{c} *turns it into an affine space over* $H_1(M; \mathbb{Z})$.

This discussion also explains why the most meaningful way to write the relation $\chi(\overline{W}) = \chi(M)$ is $\chi(M) - (\chi(\overline{W}) - \chi(C)) = 0$. From now on we will always assume that this relation holds. Turaev only considers the case where $V = C = \emptyset$, so $W = \overline{W}$ and $B = \overline{B}$, and our relation takes the usual form $\chi(M, W) = 0$. The following result was established by Turaev in Section 3 of [16] in his setting, but again the proof extends directly to our context, so we omit it. Only the first assertion is hard. We state the other two because we will use them.

Proposition 1.3. 1. *If* $\mathcal{C}'$ *is a subdivision of a cellularization* $\mathcal{C}$ *then there exists a canonical* $H_1(M; \mathbb{Z})$*-isomorphism* $\mathrm{Eul}^{\mathrm{c}}(M, \mathcal{P})_{\mathcal{C}} \to \mathrm{Eul}^{\mathrm{c}}(M, \mathcal{P})_{\mathcal{C}'}$. *In particular* $\mathrm{Eul}^{\mathrm{c}}(M; \mathbb{Z})$ *is canonically defined up to* $H_1(M; \mathbb{Z})$*-isomorphism independently of the cellularization.*
2. *If* $\mathcal{C}$ *is a cellularization of* M *suited to* $\mathcal{P}$ *and* $x_0 \in M$ *is an assigned point, any element of* $\mathrm{Eul}^{\mathrm{c}}(M, \mathcal{P})$ *can be represented, with respect to* $\mathcal{C}$, *as a sum* $\sum_{\sigma \subset M \setminus (W \cup V)} \mathrm{ind}(\sigma) \cdot \beta_\sigma$ *with* $\beta_\sigma : ([0, 1], 0, 1) \to (M, x_0, \sigma)$.
3. *If* $\mathcal{T}$ *is a triangulation of* M *suited to* $\mathcal{P}$, *any element of* $\mathrm{Eul}^{\mathrm{c}}(M, \mathcal{P})$ *can be represented, with respect to* $\mathcal{T}$, *as a simplicial* 1*-chain in the first barycentric subdivision of* $\mathcal{T}$.

A chain as in point 2 of this proposition will be later referred to as a (connected) *spider with head at* x_0. Our first main result, proved in Sect. 7, is the extension to the case under consideration of Turaev's correspondence between $\mathrm{Eul}^{\mathrm{c}}$ and $\mathrm{Eul}^{\mathrm{s}}$.

Theorem 1.4. *There exists a canonical* $H_1(M; \mathbb{Z})$*-equivariant isomorphism*

$$\Psi : \mathrm{Eul}^{\mathrm{c}}(M, \mathcal{P}) \to \mathrm{Eul}^{\mathrm{s}}(M, \mathcal{P}).$$

The definition of Ψ is based on an explicit geometric construction, but its bijectivity is only established through $H_1(M; \mathbb{Z})$-equivariance. As already mentioned in the introduction, this makes in general a very difficult task to determine the inverse of Ψ. One of the features of this paper is the description of Ψ^{-1} in terms of the encoding of fields by means of branched spines (Theorem 3.7). In this theorem we will actually describe Ψ^{-1} only in the special case where $\mathcal{P}$ is concave, but we will see in Remark 1.8 that there is an effective and geometric method to pass from a general simple structure to a concave structure.

In view of Theorem 1.4, when no confusion risks to arise, we shortly write $\mathrm{Eul}(M, \mathcal{P})$ for either $\mathrm{Eul}^{\mathrm{s}}(M, \mathcal{P})$ or $\mathrm{Eul}^{\mathrm{c}}(M, \mathcal{P})$, and α for the map giving the affine $H_1(M; \mathbb{Z})$-structure on this space.

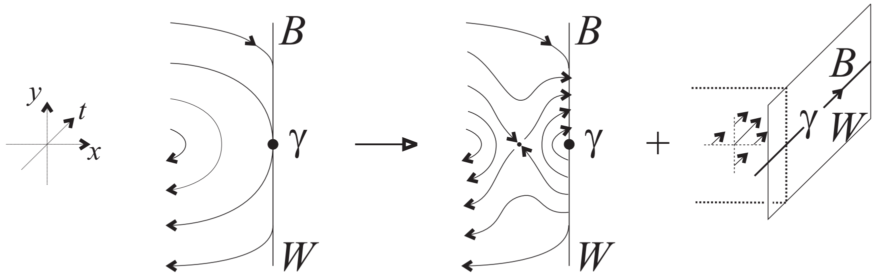

Fig. 2. Turning a concave tangency circle γ into a convex one: the apparent singularity in the cross-section is removed by adding a small bell-shaped field directed parallel to γ, *i.e.* orthogonal to the cross-section

1.2. Convex Euler structure associated to an arbitrary one

Let M and $\mathcal{P} = (W, B, V, C)$ be as in the definition of $\mathrm{Eul}(M, \mathcal{P})$. The pattern $\theta(\mathcal{P}) = (W, B, V \cup C, \emptyset)$ is a convex one canonically associated to $\mathcal{P}$. We define a map

$$\Theta^{\mathrm{s}} : \mathrm{Eul}^{\mathrm{s}}(M, \mathcal{P}) \to \mathrm{Eul}^{\mathrm{s}}(M, \theta(\mathcal{P}))$$

according to the following procedure:

1. For each component γ of C, we orient γ as a component of the boundary of B, which is oriented as a subset of the boundary of M;
2. Near γ we choose coordinates $[-1, 0]_x \times [-1, 1]_y \times S^1_t$ on M such that $\{0\} \times [-1, 1] \times S^1 \subset \partial M$, and $\gamma = \{0\} \times \{0\} \times S^1$ with orientation;
3. We choose a representative v of ξ such that each rectangle $[-1, 0]\times[-1, 1]\times\{t\}$ is a union of orbits of v, and therefore appears as in Fig. 2-left;
4. Within each rectangle $[-1, 0] \times [-1, 1] \times \{t\}$ we replace v by a singular field w having a saddle point at $(-1/2, 0, t)$ and a tangency of convex type to $\{0\} \times [-1, 1] \times \{t\}$ at $(0, 0, t)$, as in Fig. 2-centre; (of course it would be easy to write explicit analytic expressions, but we do not think this would be of much use);
5. We define $v'(x, y, t) = w(x, y, t) + f(x, y) \cdot (\partial/\partial t)$, where f is a bell-shaped function attaining its maximum 1 at $(-1/2, 0)$ and vanishing except very close to this point (see Fig. 2-right).

Lemma 1.5. Θ^{s} *is well-defined, $H_1(M; \mathbb{Z})$-equivariant, and bijective.*

Proof of Lemma 1.5. The first two properties are easy and imply the third property. The inverse of Θ^{s} may actually be described by a direct procedure very similar to the one used for Θ^{s}, but we will not use it. □

We define now a combinatorial version of Θ^{s}. Consider a cellularization $\mathcal{C}$ suited to $\mathcal{P}$, and denote by $\gamma_1, \ldots, \gamma_n$ the 1-cells contained in C. We choose the parameterizations $\gamma_j : (0, 1) \to C$ so that they respect the natural orientation of C already discussed above, and we extend the γ_j to $[0, 1]$, without changing notation. Now let z be an Euler chain relative to $\mathcal{P}$ such that the points of C appearing in

∂z are precisely the $\gamma_j(0)$'s (with positive sign), and the $\gamma_j(1/2)$'s (with negative sign). Then $z - \sum_{j=1}^n \gamma_j\big|_{[1/2,1]}$ is an Euler chain relative to $\theta(\mathcal{P})$. Setting

$$\Theta^{\mathrm{c}}([z]) = \left[z - \sum_{j=1}^{n} \gamma_j\big|_{[1/2,1]} \right]$$

we get a map $\Theta^{\mathrm{c}} : \mathrm{Eul}^{\mathrm{c}}(M,\mathcal{P}) \to \mathrm{Eul}^{\mathrm{c}}(M,\theta(\mathcal{P}))$.

Lemma 1.6. *Θ^{c} is well-defined, $H_1(M;\mathbb{Z})$-equivariant, and bijective.*

Proof of Lemma 1.6. Again, the first two properties are easy and imply the third one. □

In Sect. 7 we will see the following:

Theorem 1.7. *If Ψ is the reconstruction map of Theorem 1.4 then the following diagram is commutative:*

$$\begin{array}{ccc} \mathrm{Eul}^{\mathrm{c}}(M,\mathcal{P}) & \xrightarrow{\Theta^{\mathrm{c}}} & \mathrm{Eul}^{\mathrm{c}}(M,\theta(\mathcal{P})) \\ \Psi\downarrow & & \downarrow\Psi \\ \mathrm{Eul}^{\mathrm{s}}(M,\mathcal{P}) & \xrightarrow{\Theta^{\mathrm{s}}} & \mathrm{Eul}^{\mathrm{s}}(M,\theta(\mathcal{P})). \end{array}$$

Using this result we will sometimes just write $\Theta : \mathrm{Eul}(M,\mathcal{P}) \to \mathrm{Eul}(M,\theta(\mathcal{P}))$.

Remark 1.8. In the previous pages we have concentrated on the transformation of a simple field into a convex one, because we will later see that torsion is naturally defined only for convex fields. However one could easily provide an explicit procedure (very similar to that described in Fig. 2) to turn a simple field into a *concave* one. As already mentioned, this is one of the ingredients of a general geometric description of the map Ψ^{-1}. See Remark 3.9 for the complete recipe.

1.3. Pseudo-Legendrian knots

We spell out in this paragraph the fact, already mentioned in the introduction, that Euler structures with concave boundary pattern naturally arise in the study of pseudo-Legendrian knots. We actually refer here to the more general case of *links* (rather than knots), but later, when analyzing torsion, we will restrict to knots only. As above, fix a compact oriented manifold M and a boundary pattern $\mathcal{P}$ on M. The boundary of M may be empty or not. If v is a vector field on M and L is a link in $\mathrm{Int}(M)$, we defined L to be pseudo-Legendrian in (M,v) if v is transversal to L. We will also call (v,L) a pseudo-Legendrian pair. Having fixed $\mathcal{P}$, we will only consider fields v compatible with $\mathcal{P}$.

For a link L in M we consider a closed tubular neighbourhood $U(L)$ of L in M and we define $E(L)$ as the closure of the complement of $U(L)$. If F is a framing on L we extend the boundary pattern $\mathcal{P}$ previously fixed on M to a boundary pattern $\mathcal{P}(L^F)$ on $E(L)$, by splitting $\partial U(L)$ into a white and a black longitudinal annuli, the longitude being the one defined by the framing F. As a direct application of Proposition 1.1 one sees that $\mathrm{Eul}(E(L), \mathcal{P}(L^F))$ is non-empty (assuming $\mathrm{Eul}(M, \mathcal{P})$ to be non-empty).

A convenient way to think of $\mathcal{P}(L^F)$ is as follows. The framing F determines a transversal vector field along L. If we extend this field near L and choose $U(L)$ small enough then the pattern we see on $\partial U(L)$ is exactly as required. With this picture in mind, it is clear that if L is pseudo-Legendrian in (M, v), where v is compatible with $\mathcal{P}$, then the restriction of v to $E(L)$ defines an element

$$\xi(v, L) \in \mathrm{Eul}(E(L), \mathcal{P}(L^{(v)})).$$

Pseudo-Legendrian *knots* and their torsion invariants will be extensively studied in Sections 5 and 6 (see also [4] for related facts).

2. Torsion of an Euler structure

In this section we define torsion. We set up the usual algebraic environment [11] in which torsion can be defined, fixing a ring Λ with unit, with the property that if n and m are distinct positive integers then Λ^n and Λ^m are not isomorphic as Λ-modules. The Whitehead group $K_1(\Lambda)$ is defined as the Abelianization of $\mathrm{GL}_\infty(\Lambda)$, and $\overline{K}_1(\Lambda)$ is the quotient of $K_1(\Lambda)$ under the action of $-1 \in \mathrm{GL}_1(\Lambda) = \Lambda_*$.

We will directly define torsion only for a *convex* Euler structure, but the definition easily extends to any Euler structure ξ with simple boundary tangency, taking the torsion of the "convexified" structure $\Theta(\xi)$ discussed in Theorem 1.7. So, we fix a manifold M, a *convex* boundary pattern $\mathcal{P} = (W, B, V, \emptyset)$ on M, a cellularization $\mathcal{C}$ suited to $\mathcal{P}$ and a representation $\varphi : \pi_1(M) \to \Lambda_*$. We will denote by φ again the extension $\mathbb{Z}[\pi_1(M)] \to \Lambda$ (a ring homomorphism).

We consider now the universal cover $q : \tilde{M} \to M$ and the twisted chain complex $\mathrm{C}_*^\varphi(M, W \cup V)$, where $\mathrm{C}_i^\varphi(M, W \cup V)$ is defined as $\Lambda \otimes_\varphi \mathrm{C}_i^{\mathrm{cell}}(\tilde{M}, q^{-1}(W \cup V); \mathbb{Z})$, and the boundary operator is induced from the ordinary boundary. The homology of this complex is denoted by $H_*^\varphi(M, W \cup V)$ and called the φ-twisted homology. We assume that each $H_i^\varphi(M, W \cup V)$ is a free Λ-module and fix a basis $\mathfrak{h}_i$.

Remark 2.1. 1. To have a completely formal definition of $H_*^\varphi(M, W \cup V)$, one should fix from the beginning a basepoint $x_0 \in M$ for $\pi_1(M)$, and consider a pointed universal cover, because such a cover, and the action of $\pi_1(M)$ on it, are *canonically* defined only for pointed spaces.

2. To define $H_*^\varphi(M, W \cup V)$ we have used in an essential way the fact that $W \cup V = \overline{W}$ is closed, because the cellular theory of homology can only be employed in the relative case for pairs (X, Y) where X is a complex and Y is a subcomplex, so it is *closed* as a subset. Namely, if Y is merely a union of cells and $C_i^{\mathrm{cell}}(X, Y; \mathbb{Z})$ is defined as the $\mathbb{Z}$-module generated by i-cells lying in $X \setminus Y$, the boundary operator naturally defined on $C_*^{\mathrm{cell}}(X, Y; \mathbb{Z})$ does not turn it into an algebraic complex.
3. The Λ-module $\mathrm{C}_i^\varphi(M, W \cup V)$ is always a free one, and each $\mathbb{Z}[\pi_1(M)]$-basis of $\mathrm{C}_i^{\mathrm{cell}}(\tilde{M}, q^{-1}(W \cup V); \mathbb{Z})$ determines a Λ-basis of $\mathrm{C}_i^\varphi(M, W \cup V)$.

4. If we compose φ with the projection $\Lambda_* \to \overline{K}_1(\Lambda)$ we get a homomorphism of $\pi_1(M)$ into an *Abelian* group, so we get a homomorphism $\overline{\varphi} : H_1(M; \mathbb{Z}) \to \overline{K}_1(\Lambda)$.

Now let $\xi \in \mathrm{Eul}^{\mathrm{c}}(M, \mathcal{P})$ and choose a representative of ξ as in point 2 of Proposition 1.3, namely

$$\sum_{\sigma \in \mathcal{C},\ \sigma \subset M \setminus (W \cup V)} \mathrm{ind}(\sigma) \cdot \beta_\sigma$$

with $\beta_\sigma(0) = x_0$ for all σ, x_0 being a fixed point of M (such a representative was above called a spider with head at x_0). We choose $\tilde{x}_0 \in q^{-1}(x_0)$ and consider the liftings $\tilde{\beta}_\sigma$ which start at $\tilde{x}_0$. For $\sigma \subset M \setminus (W \cup V)$ we select its preimage $\tilde{\sigma}$ which contains $\tilde{\beta}_\sigma(1)$, and define $\mathfrak{g}(\xi)$ as the collection of all these $\tilde{\sigma}$. Arranging the i-dimensional elements of $\mathfrak{g}(\xi)$ in any order, by Remark 2.1(3) we get a Λ-basis $\mathfrak{g}_i(\xi)$ of $\mathrm{C}_i^{\varphi}(M, W \cup V)$. We consider a set $\tilde{\mathfrak{h}}_i$ of elements of $\mathrm{C}_i^{\varphi}(M, W \cup V)$ which project to the fixed basis $\mathfrak{h}_i$ of $H_i^{\varphi}(M, W \cup V)$.

Now note that, given a free Λ-module L and two finite bases $\mathfrak{b} = (b_k)$, $\mathfrak{b}' = (b'_k)$ of M, the assumption made on Λ guarantees that $\mathfrak{b}$ and $\mathfrak{b}'$ have the same number of elements, so there exists an invertible square matrix (λ_k^h) such that $b'_k = \sum_h \lambda_k^h b_h$. We will denote by $[\mathfrak{b}'/\mathfrak{b}]$ the image of (λ_k^h) in $K_1(\Lambda)$.

Proposition 2.2. *Choose a set* $\mathfrak{b}_i \subset \mathrm{C}_i^{\varphi}(M, W \cup V)$ *such that* $\partial\mathfrak{b}_i$ *is a* Λ*-basis of* $\partial(\mathrm{C}_i^{\varphi}(M, W \cup V))$. *Then* $(\partial\mathfrak{b}_{i+1}) \cdot \tilde{\mathfrak{h}}_i \cdot \mathfrak{b}_i$ *is a* Λ*-basis of* $\mathrm{C}_i^{\varphi}(M, W \cup V)$*, and*

$$\tau^{\varphi}(M, \mathcal{P}, \xi, \mathfrak{h}) = \pm \prod_{i=0}^{3} \left[\left((\partial\mathfrak{b}_{i+1}) \cdot \tilde{\mathfrak{h}}_i \cdot \mathfrak{b}_i \right) \Big/ \mathfrak{g}_i(\xi) \right]^{(-1)^{i+1}} \in \overline{K}_1(\Lambda)$$

is independent of all choices made. Moreover

$$\tau^{\varphi}(M, \mathcal{P}, \xi', \mathfrak{h}) = \tau^{\varphi}(M, \mathcal{P}, \xi, \mathfrak{h}) \cdot \overline{\varphi}(\alpha^{\mathrm{c}}(\xi', \xi)). \tag{3}$$

Proof of Proposition 2.2. The first assertion and independence of the $\mathfrak{b}_i$'s is purely algebraic and classical, see [11]. Now note that $\xi \in \mathrm{Eul}^{\mathrm{c}}(M, \mathcal{P})$ was used to select the bases $\mathfrak{g}_i(\xi)$. The $\mathfrak{g}_i(\xi)$'s are of course not uniquely determined themselves, but we can show that different choices lead to the same value of τ^{φ}.

First of all, the arbitrary ordering in the $\mathfrak{g}_i(\xi)$'s is inessential because torsion is only regarded up to sign. Second, consider the effect of choosing a different representative of ξ. This leads to a new family $\tilde{\sigma}'$ of cells. If $\tilde{\sigma}' = a(\sigma) \cdot \tilde{\sigma}$, with $a(\sigma) \in \pi_1(M)$, and $\overline{a}(\sigma)$ is the image of $a(\sigma)$ in $H_1(M; \mathbb{Z})$, we automatically have

$$\sum_{\sigma \subset M \setminus W \cup V} \mathrm{ind}(\sigma) \cdot \overline{a}(\sigma) = 0 \in H_1(M; \mathbb{Z}),$$

which allows to conclude that also the representative chosen is inessential. The choice of the lifting $\tilde{x}_0$ can be shown to be inessential either in the spirit of Remark 2.1(1), or by showing that a simultaneous a-translation of all $\tilde{\sigma}$, for $a \in \pi_1(M)$, multiplies the torsion by $\overline{\varphi}(a)^{\chi(M) - \chi(W \cup V)} = 1$.

Formula (3) is readily established by choosing representatives $\sum \mathrm{ind}(\sigma)\cdot\beta_\sigma$ and $\sum \mathrm{ind}(\sigma)\cdot\beta'_\sigma$ of ξ and ξ' such that $\beta'_\sigma = \beta_\sigma$ for all σ but one. $\square$

Since the above construction uses the cellularization $\mathcal{C}$ in a way which may appear to be essential, we add a subscript $\mathcal{C}$ to the torsion we have defined. The next result, which can be established following Turaev [16], shows that dependence on $\mathcal{C}$ is actually inessential.

Proposition 2.3. *Let $\mathcal{C}$ and $\mathcal{C}'$ be cellularizations suited to $\mathcal{P}$. Assume that $\mathcal{C}'$ subdivides $\mathcal{C}$, and consider the bijection $\mathcal{S}_{(\mathcal{C}',\mathcal{C})}:\mathrm{Eul}^{\mathrm{c}}(M,\mathcal{P})_{\mathcal{C}}\to\mathrm{Eul}^{\mathrm{c}}(M,\mathcal{P})_{\mathcal{C}'}$ of Proposition 1.3, and the canonical isomorphism $j_{(\mathcal{C}',\mathcal{C})}:H^{\varphi}_{*}(M,W\cup V)_{\mathcal{C}}\to H^{\varphi}_{*}(M,W\cup V)_{\mathcal{C}'}$. Then, with obvious meaning of symbols, we have:*

$$\tau^{\varphi}_{\mathcal{C}}(M,\mathcal{P},\xi,\mathfrak{h})=\tau^{\varphi}_{\mathcal{C}'}(M,\mathcal{P},\mathcal{S}_{(\mathcal{C}',\mathcal{C})}(\xi),j_{(\mathcal{C}',\mathcal{C})}(\mathfrak{h})).$$

It is maybe appropriate here to remark that the choice of a basis $\mathfrak{h}$ of $H^{\varphi}_{*}(M,W\cup V)$ and the definition of $\tau^{\varphi}(M,\mathcal{P},\xi,\mathfrak{h})$ implicitly assume a description of the universal cover of M, which is typically not doable in practical cases. However, if one starts from a representation of $\pi_1(M)$ into the units of a *commutative* ring Λ, *i.e.* a representation which factors through one of $H_1(M;\mathbb{Z})$, one can use from the very beginning the maximal Abelian rather than the universal cover, which makes computations more feasible.

Remark 2.4. Turaev [15] has shown that a homological orientation yields a sign-refinement of torsion, *i.e.* a lifting from $\overline{K}_1(\Lambda)$ to $K_1(\Lambda)$. This refinement extends with minor modifications to our setting of boundary tangency. This sign-refinement, in the closed and monochromatic case, is often an essential component of the theory (for instance, it is crucial for the relation with the 3-dimensional Seiberg–Witten invariants [17,18] and for the definition of the Casson invariant [8]), but we will not address it in the present paper.

2.1. *Computation of torsion via disconnected spiders*

In this subsection we show that to determine the family of lifted cells necessary to define torsion one can use representatives of Euler structures more general than the (connected) spiders used above. This is a technical point which we will use below to compute torsions using branched spines (Sect. 3).

We fix M, $\mathcal{P}$, $\mathcal{C}$ and φ as above, and $\xi\in\mathrm{Eul}^{\mathrm{c}}(M,\mathcal{P})$. Let $\mathfrak{g}(\xi)=\{\tilde\sigma\}$ be the family of liftings of the cells lying in $M\setminus(W\cup V)$ determined by a connected spider as explained above. Note that if $\mathfrak{g}'=\{\tilde\sigma'\}$ is any other family of liftings we have $\tilde\sigma'=a(\sigma)\cdot\tilde\sigma$ for some $a(\sigma)\in\pi_1(M)$, and we can define

$$h(\mathfrak{g}',\mathfrak{g}(\xi))=\sum_{\sigma\subset M\setminus(W\cup V)}\mathrm{ind}(\sigma)\cdot\overline{a}(\sigma)\in H_1(M;\mathbb{Z}).$$

Proposition 2.5. *Assume there exists a partition $\mathcal{C}_1 \sqcup \cdots \sqcup \mathcal{C}_k$ of the set of cells lying in $M \setminus (W \cup V)$, and let $\xi \in \mathrm{Eul}^{\mathrm{c}}(M, \mathcal{P})$ have a representative of the form*

$$z = \sum_{j=1}^{k} \left(\sum_{\sigma \in \mathcal{C}_j \setminus \{\sigma_j\}} \mathrm{ind}(\sigma) \cdot \gamma_\sigma^{(j)} \right)$$

where $\sigma_j \in \mathcal{C}_j$ and $\gamma_\sigma^{(j)} : ([0,1], 0, 1) \to (M, p_{\sigma_j}, p_\sigma)$. Choose any lifting $\tilde{p}_{\sigma_j}$ of p_{σ_j}, lift $\gamma_\sigma^{(j)}$ to $\tilde{\gamma}_\sigma^{(j)}$ starting from $\tilde{p}_{\sigma_j}$, let $\tilde{\sigma}'$ be the lifting of σ containing $\tilde{\gamma}_\sigma^{(j)}(1)$, and let $\mathfrak{g}'$ be the family of all these liftings. Then $h(\mathfrak{g}', \mathfrak{g}(\xi)) = 0 \in H_1(M; \mathbb{Z})$. In particular $\mathfrak{g}'$ can be used to compute $\tau^\varphi(M, \mathcal{P}, \xi, \mathfrak{h})$.

Proof of Proposition 2.5. Note first that the coefficient of p_{σ_j} in ∂z is exactly

$$-\sum_{\sigma \in \mathcal{C}_j \setminus \{\sigma_j\}} \mathrm{ind}(\sigma).$$

On the other hand this coefficient must be equal to $\mathrm{ind}(\sigma_j)$. Summing up we deduce that $\sum_{\sigma \in \mathcal{C}_j} \mathrm{ind}(\sigma) = 0$.

Now choose $x_0 \in M$ and $\delta^{(j)} : ([0,1], 0, 1) \to (M, x_0, p_{\sigma_j})$. For $\sigma \in \mathcal{C}_j$ define

$$\beta_\sigma = \begin{cases} \delta^{(j)} & \text{if } \sigma = \sigma_j \\ \delta^{(j)} \cdot \gamma_\sigma^{(j)} & \text{otherwise,} \end{cases}$$

so that $\beta_\sigma : ([0,1], 0, 1) \to (M, x_0, p_\sigma)$, whence $w = \sum_{\sigma \subset M \setminus (W \cup V)} \beta_\sigma$ is an Euler chain. Moreover:

$$w - z = \sum_{j=1}^{k} \left(\sum_{\sigma \in \mathcal{C}_j} \mathrm{ind}(\sigma) \right) \cdot \delta^{(j)} = 0 \in H_1(M; \mathbb{Z}),$$

so $[w] = \xi$. Now choose $\tilde{x}_0$ over x_0, lift the $\delta^{(j)}$ and β_σ starting from $\tilde{x}_0$, and let $a^{(j)} \in \pi_1(M)$ be such that $\tilde{p}_{\sigma_j} = a^{(j)} \cdot \tilde{\delta}^{(j)}(1)$. Then

$$h(\mathfrak{g}', \mathfrak{g}(\xi)) = \sum_{j=1}^{k} \left(\sum_{\sigma \in \mathcal{C}_j} \mathrm{ind}(\sigma) \right) \cdot \overline{a}^{(j)} = 0 \in H_1(M; \mathbb{Z}),$$

and the proof is complete. □

The next result follows directly from the definition, but it is worth stating because it shows how torsions may be used to distinguish triples $(M, \mathcal{P}, \xi)$ from each other.

Proposition 2.6. *Let $f : M \to M'$ be a homeomorphism, consider $\xi \in \mathrm{Eul}(M, \mathcal{P})$, $\varphi : \pi_1(M) \to \Lambda_*$ and a Λ-basis $\mathfrak{h}$ of $H_*^\varphi(M, \overline{W})$. Then*

$$\tau^{\varphi \circ f_*^{-1}}(M', f_*(\mathcal{P}), f_*(\xi), f_*(\mathfrak{h})) = \tau^\varphi(M, \mathcal{P}, \xi, \mathfrak{h}).$$

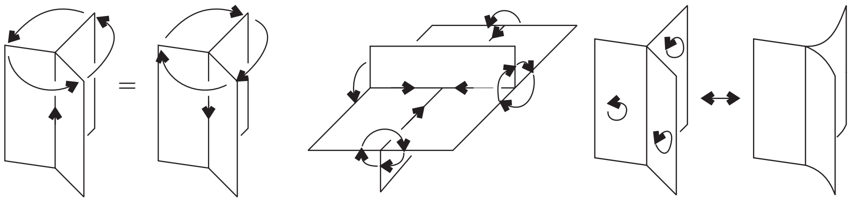

Fig. 3. Convention on screw-orientations, compatibility at vertices, and geometric interpretation of branching

3. Branched spines and inversion of the reconstruction map

In this section we show how to geometrically invert the reconstruction map Ψ, and how to compute torsions starting from an encoding of vector fields based on branched spines. Later in Sect. 4 we will provide an explicit example of computation. We first review the theory developed in [3]. See the beginning of Sect. 1 for our conventions on manifolds, maps, and fields. (We remind the reader in particular that our "fields" are non-singular by default.) In addition to the terminology introduced there, we will need the notion of *traversing* field on a manifold M, defined as a field whose orbits eventually intersect ∂M transversely in both directions (in other words, orbits are compact intervals).

3.1. Standard spines

A *simple* polyhedron P is a finite, connected, purely 2-dimensional polyhedron with singularity of stable nature (triple lines and points where six non-singular components meet; a regular neighbourhood of such a point is isomorphic to the cone over the 1-skeleton of a tetrahedron). Such a P is called *standard* if all the components of the natural stratification given by singularity are open cells. Depending on dimension, we will call the components *vertices*, *edges* and *regions*.

A *standard spine* of a 3-manifold M with $\partial M \neq \emptyset$ is a standard polyhedron P embedded in $\mathrm{Int}(M)$ so that M collapses onto P. Standard spines of oriented 3-manifolds are characterized among standard polyhedra by the property of carrying an *orientation*, defined (see Definition 2.1.1 in [3]) as a "screw-orientation" along the edges (as in the left-hand-side of Fig. 3), with the constraint that when the neighbourhood of a vertex is embedded in 3-space then the four initial portions of edge at the vertex should carry screw-orientations which are compatible in 3-space (as in the centre of Fig. 3). It is the starting point of the theory of standard spines that every oriented 3-manifold M with $\partial M \neq \emptyset$ has an oriented standard spine, and can be reconstructed (uniquely up to homeomorphism) from any of its oriented standard spines. See [5] for the non-oriented version of this result and [2] or Proposition 2.1.2 in [3] for the (slight) oriented refinement.

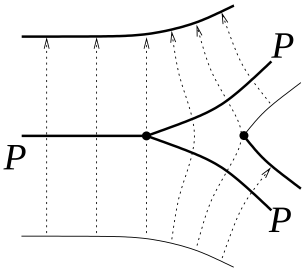

Fig. 4. Manifold and field associated to a branched spine.

3.2. Branched spines

A *branching* on a standard spine P of an orientable manifold M is a collection of one orientation for each region of P, such that no edge is induced the same orientation three times. As explained in [3, §. 3.1] and illustrated in Fig. 3-right, a branching can be used to consistently smoothen the singularity of P so to turn it into a branched surface, see [19] and also [6]. Namely, the embedding of P can be isotoped so that an oriented tangent plane is defined at each point, even along the topological singularity, and all the regions are smoothly immersed in P in an orientation-preserving way. A standard spine P with a certain branching will be called a *branched spine* of M. We will never use specific notations for the extra structures on P (*i.e.* the screw-orientation and the branching). These structures will be viewed as parts of P itself. The following result, proved as Theorem 4.1.9 in [3], is the starting point of our constructions.

Proposition 3.1. *To every branched spine P there corresponds a manifold $M(P)$ with non-empty boundary and a concave traversing field $v(P)$ on $M(P)$. The pair $(M(P), v(P))$ is well-defined up to diffeomorphism. Moreover an embedding i of P as a transversely oriented branched surface in* $\mathrm{Int}(M(P))$ *is defined, and it has the property that $v(P)$ is positively transversal to $i(P)$.*

The topological construction which underlies this proposition is actually quite simple, and it is illustrated in Fig. 4.

3.3. The encoding of combings via branched spines

Let v be a concave field on a manifold M. We denote by S^2_{triv} any sphere in ∂M which is split by the tangency line of v to ∂M into two discs. Now, notice that S^2_{triv} is also the boundary of the closed 3-ball with constant vertical field, denoted by B^3_{triv}. This shows that we can cap off every S^2_{triv} by attaching a copy of B^3_{triv}, getting a non-singular vector field on the resulting manifold. This vector field is however well-defined only up to homotopy.

We will denote by Comb the set of all pairs (M, v), where M is a compact oriented manifold and v is a concave field on M, viewed up to diffeomorphism of M and homotopy of v through concave fields. A class $[M, v] \in$ Comb is called a *combing* on the diffeomorphism class of the manifold M. Note that the boundary pattern on ∂M evolves isotopically during a homotopy of v, so a pair $(M, \mathcal{P})$, viewed up to diffeomorphism of M, can be associated to each $[M, v] \in$ Comb. In particular, Comb naturally splits as the disjoint union of subsets $\mathrm{Comb}(M, \mathcal{P})$, consisting of combings on M compatible with $\mathcal{P}$.

For a technical reason we actually rule out from Comb the set of those classes $[M, v]$ such that the corresponding boundary pattern contains components of the type S^2_{triv}. This is actually not a serious restriction, because each S^2_{triv} component can be capped off by a B^3_{triv} as explained above, and the result is well-defined up to homotopy. Note that we do accept pairs (M, v) with closed M, and pairs in which v has no tangency at all to ∂M.

Let us denote now by $\mathcal{B}$ the set of all branched spines P (up to orientation-preserving PL isomorphism) such that the boundary pattern $\mathcal{P}(P)$ of $v(P)$ on $M(P)$ contains only one S^2_{triv}. Such a P being given, we can cap off S^2_{triv} by attaching a copy of B^3_{triv}, getting $\widehat{M}(P)$ endowed with a concave field $\widehat{v}(P)$, and the pair $(\widehat{M}(P), \widehat{v}(P))$ gives rise to a well-defined element of Comb, which we denote by $\Phi(P)$. The following is proved in [4]:

Theorem 3.2. *The map* $\Phi : \mathcal{B} \to \mathrm{Comb}$ *is surjective.*

This theorem generalizes the main achievement of [3, Theorems 1.4.1 and 5.2.1], where it is proved in the special case of closed M. The assumption that ∂M contains no S^2_{triv} component has a purely technical nature, and has been inserted here only to make the statement simpler. The complete statement includes also the description of a finite set of local moves on branched spines generating the equivalence relation induced by Φ. We will not need the moves in this paper. The following geometric interpretation of the theorem may however be of some interest.

Remark 3.3. In general, the dynamics of a field, even a concave one, can be complicated, whereas the dynamics of a traversing field (in particular, B^3_{triv}) is simple. Theorem 3.2 means that for any (complicated) concave field there exists a sphere S^2 which splits the field into two (simple) pieces: a standard B^3_{triv} and a concave traversing field.

Back to the case of our fixed manifold M with boundary pattern $\mathcal{P}$, we note that we have a projection $\pi^{\mathrm{s}} : \mathrm{Comb}(M, \mathcal{P}) \to \mathrm{Eul}^{\mathrm{s}}(M, \mathcal{P})$. Our aim is now to define, using branched spines, another projection $\pi^{\mathrm{c}} : \mathrm{Comb}(M, \mathcal{P}) \to \mathrm{Eul}^{\mathrm{c}}(M, \mathcal{P})$ such that $\pi^{\mathrm{s}} = \Psi \circ \pi^{\mathrm{c}}$.

3.4. Spines and ideal triangulations

We remind the reader that an *ideal triangulation* of a manifold M with non-empty boundary is a partition $\mathcal{T}$ of $\mathrm{Int}(M)$ into open cells of dimensions 1, 2 and 3, induced by a triangulation $\mathcal{T}'$ of the space $Q(M)$, where:

1. $Q(M)$ is obtained from M by collapsing each component of ∂M to a point;

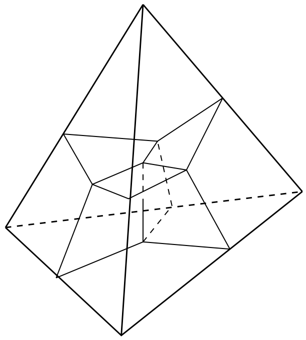

Fig. 5. Duality between standard spines and ideal triangulations

2. $\mathcal{T}'$ is a triangulation only in a loose sense, namely self-adjacencies and multiple adjacencies of tetrahedra are allowed;
3. The vertices of $\mathcal{T}'$ are precisely the points of $Q(M)$ which correspond to the collapsed components of ∂M.

It turns out (see for instance [9]) that there exists a natural bijection between standard spines and ideal triangulations of a 3-manifold. Given an ideal triangulation, the corresponding standard spine is just the 2-skeleton of the dual cellularization, as illustrated in Figure 5. The inverse of this correspondence will be denoted by $P \mapsto \mathcal{T}(P)$.

We can now give a dual interpretation, using $\mathcal{T}(P)$, of a branching on P. Since the ambient manifold is oriented, an orientation for a region of P is the same as an orientation for the dual edge of $\mathcal{T}(P)$, and it turns out that a collection of orientations on the edges of $\mathcal{T}(P)$ defines a branching if and only if on each tetrahedron of $\mathcal{T}(P)$ exactly one of the vertices is a sink and one is a source. Moreover, if P has a branching, the oriented edges of $\mathcal{T}(P)$ are precisely oriented orbits of $v(P)$, and the 2-faces are unions of such orbits.

Remark 3.4. It turns out that if P is a branched spine, not only the edges, but also the faces and the tetrahedra of $\mathcal{T}(P)$ have natural orientations. For tetrahedra, we just restrict the orientation of $M(P)$. For faces, we first note that the edges of P have a natural orientation (the prevailing orientation induced by the incident regions). Now, we orient a face of $\mathcal{T}(P)$ so that the algebraic intersection in $M(P)$ with the dual edge is positive.

3.5. Euler chain defined by a branched spine

We fix in this paragraph a standard spine P and consider its manifold $M = M(P)$. We start by noting that the ideal triangulation $\mathcal{T} = \mathcal{T}(P)$ defined by P can be interpreted as a realization of $\text{Int}(M)$ by face-pairings on a finite set of tetrahedra with vertices removed. If, instead of removing vertices, we remove open conic neighbourhoods of the vertices, thus getting *truncated* tetrahedra, after the face-pairings we obtain M itself. This shows that P determines a cellularization $\overline{\mathcal{T}} = \overline{\mathcal{T}}(P)$ of

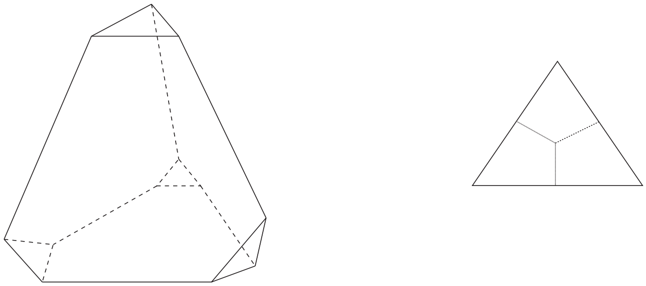

Fig. 6. Truncated tetrahedra and subdivision of the triangles on the boundary

M with vertices only on ∂M and 2-faces which are either triangles contained in ∂M or hexagons contained in $\mathrm{Int}(M)$, with edges contained alternatingly in ∂M and in $\mathrm{Int}(M)$.

Now assume that P is branched and that ∂M contains only one S^2_{triv} component, so $\widehat{M} = \widehat{M}(P)$ is defined, together with the concave boundary pattern $\widehat{\mathcal{P}} = \widehat{\mathcal{P}}(P) = (W, B, \emptyset, C)$ on $\widehat{M}$. Note that $\widehat{M}$ can be thought of as the space obtained from M by contracting S^2_{triv} to a point, so a projection $\pi : M \to \widehat{M}$ is defined, and $\pi(\overline{\mathcal{T}})$ is a cellularization of $\widehat{M}$. Next, we modify $\pi(\overline{\mathcal{T}})$ by subdividing each triangle on $\partial\widehat{M}$ into 3 "kites" (quadrilaterals having two "short" and two "long" edges) as shown in Fig. 6. We do this to get a cellularization suited to the boundary pattern: the tangency line C is now a union of short edges of kites. The result is a cellularization $\widehat{\mathcal{T}} = \widehat{\mathcal{T}}(P)$ of $\widehat{M}$. Note again that $\widehat{\mathcal{T}}$ on $\partial\widehat{M}$ consists of kites, with long edges coming from tetrahedra and short edges coming from subdivision. Note also that $\widehat{\mathcal{T}}$ has exactly one vertex x_0 in $\mathrm{Int}(\widehat{M})$, and that the cells contained in $\mathrm{Int}(\widehat{M})$, except x_0, are the duals to the cells of the natural cellularization $\mathcal{U} = \mathcal{U}(P)$ of P. For $u \in \mathcal{U}$ we denote by $\hat{u}$ its dual and by $p_u = p_{\hat{u}}$ the point where u and $\hat{u}$ intersect, called the *centre* of both.

We will now use the field $\widehat{v} = \widehat{v}(P)$ to construct a combinatorial Euler chain on $\widehat{M}$ with respect to $\widehat{\mathcal{T}}$. It is actually convenient to consider, instead of $\widehat{v}$, the field $\overline{v} = \pi(v)$, which coincides with $\widehat{v}$ except near x_0, where it has a (removable) singularity. For $u \in \mathcal{U}$ we denote by β_u the arc obtained by integrating $\overline{v}(P)$ in the positive direction, starting from p_u, until the boundary or the singularity is reached. We define:

$$s(P) = \sum_{u\in\mathcal{U}} \mathrm{ind}(u) \cdot \beta_u.$$

(We remind the reader that $\mathrm{ind}(u) = (-1)^{\dim(u)}$.) Our aim is now to use this chain $s(P)$ to define a combinatorial Euler structure, and then show that the smooth companion of this structure is indeed the structure represented by $\widehat{v}(P)$. Note however that $\partial s(P)$ contains, besides the centres of the cells in $\mathrm{Int}(M)$, only the centres of the cells of $\pi(\overline{\mathcal{T}})$ which lie *entirely* in B, but B is *not* a union of cells of $\pi(\overline{\mathcal{T}})$: this is precisely the reason why we have subdivided $\pi(\overline{\mathcal{T}})$ into $\widehat{\mathcal{T}}$. So we will need to add something to $s(P)$.

If p is a vertex of $\pi(\overline{\mathcal{T}})$ contained in B, we define its star $\mathrm{St}(p)$ as the sum of the straight segments going from p to the centres of all the kites containing p,

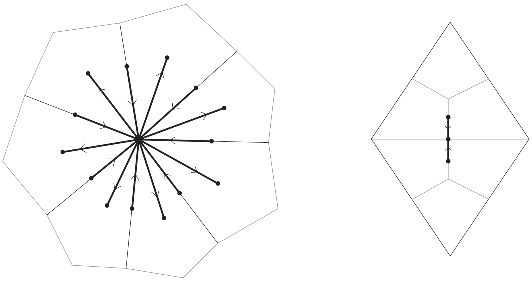

Fig. 7. The star $\mathrm{St}(p)$ centred at a vertex p contained in B and the bi-arrow $\mathrm{Ba}(\sigma)$ based at the midpoint of an edge σ contained in B

minus the sum of the straight segments going from p to the centres of all the long edges containing p. If σ is an edge of $\pi(\overline{\mathcal{T}})$ contained in B we define its bi-arrow $\mathrm{Ba}(\sigma)$ as the sum of the two straight segments going from the centre p_σ of σ to the centres of the two short kite-edges containing p_σ. A star and a bi-arrow are shown in Fig. 7. We define:

$$s'(P) = s(P) + \sum_{p \in \pi(\overline{\mathcal{T}})^{(0)},\ p \in B} \mathrm{St}(p) + \sum_{\sigma \in \pi(\widehat{\mathcal{T}})^{(1)},\ \sigma \subset B} \mathrm{Ba}(\sigma).$$

Lemma 3.5. *$s'(P)$ defines an element of* $\mathrm{Eul}^{\mathrm{c}}(\widehat{M}, \theta(\widehat{\mathcal{P}}))$.

Proof of Lemma 3.5. Recall that $\theta(\widehat{\mathcal{P}}) = (W, B, C, \emptyset)$, *i.e.* the concave line C is turned into a convex one. So by definition we have to show that $\partial s'(P)$ contains, with the right sign, the centres of all cells of $\widehat{\mathcal{T}}$ except those of $W \cup C$.

It will be convenient to analyze first the natural lifting of $s(P)$ to M, denoted by $\tilde{s}(P) = \sum_{u \in \mathcal{U}} \mathrm{ind}(u) \cdot \tilde{\beta}_u$ with obvious meaning of symbols. So

$$\partial \tilde{s}(P) = \sum_{u \in \mathcal{U}} -\mathrm{ind}(u) \cdot \tilde{\beta}_u(0) + \sum_{u \in \mathcal{U}} \mathrm{ind}(u) \cdot \tilde{\beta}_u(1). \tag{4}$$

Since the cellularization $\overline{\mathcal{T}}$ of M is dual to $\mathcal{U}$, the first half of (4) gives the centres of the cells contained in $\mathrm{Int}(M)$, with right sign. One easily sees that the second half gives exactly the centres of the cells (of $\overline{\mathcal{T}}$) contained in B, also with right sign.

When we project to $\widehat{M}$ and consider $\partial s(P)$, the first half of (4) again provides (with right sign) the centres of the all cells contained in $\mathrm{Int}(\widehat{M})$, except the special vertex x_0 obtained by collapsing S^2_{triv}. We can further split the points of the second half of (4) into those which lie on S^2_{triv} and those which do not. The points of the first type project to x_0, and the resulting coefficient of x_0 is $\chi(B \cap S^2_{\mathrm{triv}})$, but $B \cap S^2_{\mathrm{triv}}$ is an open 2-disc, so the coefficient is 1. (We are here using the very special property of dimension 2 that χ can be computed using a finite cellularization of an open manifold, because the boundary of the closure has $\chi = 0$.) The points of the second type faithfully project to $\widehat{M}$, giving the centres of the simplices contained in B of the triangulation $\pi(\overline{\mathcal{T}})\big|_{\partial \widehat{M}}$. However $\widehat{\mathcal{T}}$ on $\partial \widehat{M}$ is a subdivision of $\pi(\overline{\mathcal{T}})$, and this is the reason why we have added the stars and the bi-arrows to $s(P)$ getting

$s'(P)$. The following computation of the coefficients in $\partial s'(P)$ of the centres of the cells of $\widehat{\mathcal{T}}$ contained in B concludes the proof.

0. Cells of dimension 0 are listed as follows:
   (a) Centres of triangles of $\pi(\overline{\mathcal{T}})$, which receive coefficient $+1$ from $\partial s(P)$;
   (b) Midpoints of edges of $\pi(\overline{\mathcal{T}})$, which receive coefficient -1 from $\partial s(P)$ and $+2$ from the bi-arrows they determine;
   (c) Vertices of $\pi(\overline{\mathcal{T}})$, which receive $+1$ from $\partial s(P)$ and (algebraically) 0 from the star they determine;
1. Cells of dimension 1 are:
   (a) Short edges of kites, whose midpoints receive -1 from the bi-arrows;
   (b) Long edges of kites, whose midpoints receive -1 from the stars;
2. Cells of dimension 2 are kites, and their centres receive $+1$ from the stars. □

Now we denote by $\gamma_j : (0, 1) \to C$, for $j = 1, \ldots, n$, orientation-preserving parameterizations of the 1-cells of $\widehat{\mathcal{T}}$ contained in C, and we extend the γ_j to $[0, 1]$, without changing notation. We define

$$s''(P) = s'(P) + \sum_{j=1}^{n} \gamma_j\big|_{[1/2,1]}.$$

Lemma 3.6. *$s''(P)$ defines an element of* $\mathrm{Eul}^{\mathrm{c}}(\widehat{M}, \widehat{\mathcal{P}})$*, and*

$$[s'(P)] = \Theta^{\mathrm{c}}([s''(P)]) \in \mathrm{Eul}^{\mathrm{c}}(\widehat{M}, \theta(\widehat{\mathcal{P}})).$$

Proof of Lemma 3.6. At the level of representatives, the second assertion is a direct consequence of the definition of Θ^{c}, and it implies the first assertion. □

We defer to Sect. 7 the proof of the next result, which shows that the map $P \mapsto [s''(P)] \in \mathrm{Eul}^{\mathrm{c}}(\widehat{M}, \widehat{\mathcal{P}})$ allows, using branched spines, to explicitly find the inverse of the reconstruction map Ψ of Theorem 1.4.

Theorem 3.7. $\Psi([s''(P)]) = [\widehat{v}(P)] \in \mathrm{Eul}^{\mathrm{s}}(\widehat{M}, \widehat{\mathcal{P}})$.

Recall now that we have defined torsions directly only for convex patterns, and we have extended the definition to concave patterns via the map Θ. As a consequence of Lemma 3.6 and Theorem 3.7, and by direct inspection of $s'(P)$, we have the following result which summarizes our investigations on the relation between spines, Euler structures, and torsion:

Theorem 3.8. *If P is a branched spine which represents a manifold $\widehat{M}$ with concave boundary pattern $\widehat{\mathcal{P}} = (W, B, \emptyset, C)$ in the sense of Theorem 3.2, then for any representation $\varphi : \pi_1(M) \to \Lambda_*$ and any Λ-basis $\mathfrak{h}$ of $H_*^{\varphi}(\widehat{M}, W \cup C)$, the torsion $\tau^{\varphi}(\widehat{M}, \widehat{\mathcal{P}}, [\widehat{v}(M)], \mathfrak{h})$ can be computed using (in the sense of Proposition 2.5) the lifting to the universal cover of $\widehat{M}$ of the chain $s'(P)$ defined above. In particular, $s'(P)$ can be used directly, without replacing it by a connected spider.*

In the next section we will illustrate an explicit example of torsion computation carried out using the last assertion of the previous result.

Remark 3.9. We can now be more explicit about how to compute the torsion of an Euler structure ξ represented by a given smooth vector field $\widehat{w}$ on $\widehat{M}$. The first step is to turn $\widehat{w}$ into a concave field $\widehat{v}$, in the spirit of Remark 1.8. Now we need a branched spine P of $\widehat{v}$, and we only need to note that the proof of Theorem 3.2 given in [4] has a constructive nature. The construction of $s'(P)$ starting from P is of course an effective one, and the previous theorem shows that $s'(P)$ allows to compute the torsion of the original ξ.

We mention here that in [1] we have developed a different extension of Turaev's theory of torsion, by considering combed manifolds with completely arbitrary configurations on the boundary. The torsion defined in [1] coincides with Turaev's one (and hence with that considered in the present paper) only when the manifold is closed. One point worth remarking is that in [1] the proof that torsion is well-defined and H_1-equivariant is completely self-contained, *i.e.* it does not depend on Turaev's sophisticated results on subdivisions of cellularizations. Instead, it is based on a combinatorial analysis of the elementary catastrophes associated to the moves of the calculus of branched spines.

4. An example of torsion computation

As an example of application of Theorem 3.8, we are going to work out in this section a specific computation of torsion. The example is simple enough to be treated by hand, but the method we describe extends to the general case. The present section may be skipped by the readers mainly interested in applications of torsion to pseudo-Legendrian knots.

4.1. Boundary operators

To actually apply Theorem 3.8 in order to compute torsion starting from a branched spine P, besides describing the universal (or maximal Abelian) cover of $\widehat{M}=\widehat{M}(P)$ and determining the preferred liftings of the cells in $\widehat{M}\setminus(W\cup C)$, one needs to compute the boundary operators in the twisted chain complex $\mathrm{C}_*^{\varphi}(M,W\cup C)$. These operators are of course twisted liftings of the corresponding operators in the cellular chain complex of $(\widehat{M},W\cup C)$, with respect to $\widehat{\mathcal{T}}$. We briefly describe here the form of the latter operators. Recall first that $\widehat{\mathcal{T}}$ consists of a special vertex x_0, the kites (with their vertices and edges) on $\widehat{M}$, and the duals of the cells of P. On $\partial\widehat{M}$ the situation is easily described, so we consider the internal cells.

1. If R is a region of P, the ends of its dual edge $\hat{R}$ are either x_0 or vertices of $\partial\widehat{M}$ contained only in long edges of kites.
2. If e is an edge of P then $\partial\hat{e}$ is given by $\hat{R}_1+\hat{R}_2-\hat{R}_0$ plus 3 long edges of kites, where r_0,r_1,r_2 are the regions incident to e, numbered so that r_1 and r_2 induce on e the same orientation. Here r_0,r_1,r_2 need not be different from each other. The 3 long edges of kites must be given an appropriate sign, and some of them may actually be collapsed to the point x_0. Note that we have only 3 kite-edges, out of the 6 which geometrically appear on $\partial\hat{e}$, because the other 3 are white.

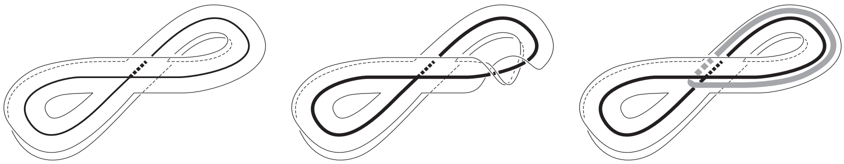

Fig. 8. The abalone, and a C^1 knot on it

3. If v is a vertex of P then $\partial\hat{v}$ is given by $\hat{e}_1 + \hat{e}_2 - \hat{e}_3 - \hat{e}_4$ plus 6 kites, where e_1, e_2 are the edges which (with respect to the natural orientation) are leaving v, and e_3, e_4 are those which are reaching it. Again, there could be repetitions in the e_i's. The kites all have coefficient $+1$, and again some of them may actually be collapsed to x_0. As above, we have only 6 kites because the other 6 are white.

4.2. *Simplified cellularization*

To define the cellularization $\widehat{\mathcal{T}}(P)$ associated to a spine we have decided to subdivide all the triangles of $\pi(\overline{\mathcal{T}})$ on $\partial\widehat{M}$ into 3 kites, but when doing actual computations this is not necessary and impractical. The only triangles which we really need to subdivide are those intersected by C, because we need the cellularization to be suited to the pattern. If we consider the 4 triangles corresponding to the ends of a certain tetrahedron, and in each of them we count the number of black kites and the number of white kites, we get respectively $(3, 0)$, $(2, 1)$, $(1, 2)$, $(0, 3)$. So, the first and last triangles do not have to be subdivided, and the other two can be subdivided using one segment only. Summing up, for each vertex of P we only need to add two segments on the boundary. Before projecting $M(P)$ to $\widehat{M}(P)$ one sees that the number of cells, with respect to $\overline{\mathcal{T}}(P)$, is increased in all dimensions 0, 1 and 2 by twice the number of vertices of P. When projecting to $\widehat{M}(P)$ the cells lying in S^2_{triv} get collapsed to x_0.

4.3. *The example*

Figure 8 shows a neighbourhood of the singular set of the so-called abalone, a branched standard spine of S^3, which we denote by A. Note that A has one vertex, two edges and two regions. The figure on the left is easier to understand, but it does not represent the genuine embedding of A in S^3, which is instead shown in the centre (hint: compute linking numbers). On the right we show (using the easier picture) a knot K on A. Of course K is transversal to the field v carried by A, so (v, K) is a pseudo-Legendrian pair (see the end of Sect. 1). Moreover, using the genuine picture of A, one sees that K is actually trivial in S^3, and its framing is $+1$. So the knot exterior $E(K)$ is a solid torus, with an induced Euler structure ξ, and the white annulus $W \subset \partial E(K)$ is a longitudinal one. Let us now take the representation $\varphi : \pi_1(E(K)) \to Q(\mathbb{Z}[t^{\pm 1}])$ which maps the generator to t (here, as usual, Q stands for the field of fractions). It is not hard to see that

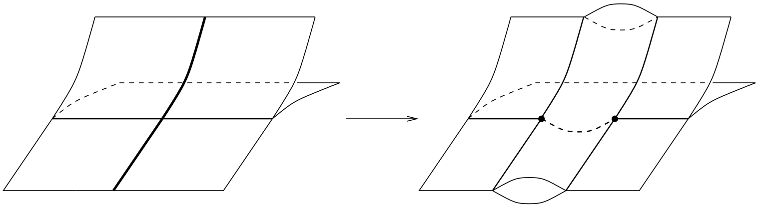

Fig. 9. How to dig a tunnel in a spine

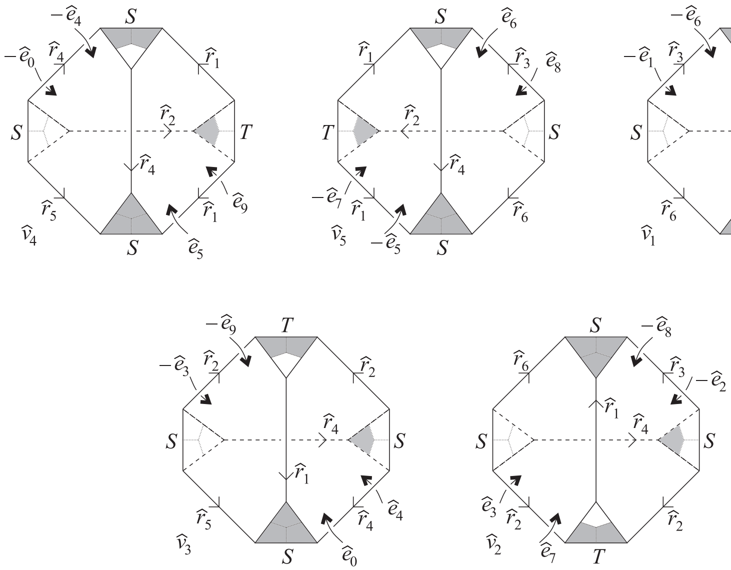

Fig. 10. Truncated ideal triangulation of the knot exterior

$H_*^{\varphi}(E(K), \overline{W}) = 0$, so we can compute $\tau^{\varphi}(E(K), \xi)$. We describe the method to be followed, skipping several details and all explicit formulae.

4.4. Spine of a knot complement

We first note that a branched standard spine P of $E(K)$ can be easily constructed by digging a tunnel through A along K, as suggested by Fig. 9. By construction the field carried by P on $E(K)$ is precisely the restriction to $E(K)$ of the original field v carried by the abalone on S^3. Now P is easily recognized to have 5 vertices (denoted $v_1, \ldots, v_5$), 10 edges (denoted $e_0, \ldots, e_9$) and 6 regions (denoted $r_1, \ldots r_6$). Figure 10 shows the truncated ideal triangulation dual to P. In the figure the hat denotes duality as usual. We have written $-\hat{e}_i$ instead of $\hat{e}_i$ when $\hat{e}_i$ lies on $\hat{v}_j$ but the natural orientation of $\hat{e}_i$ is not induced by the orientation of $\hat{v}_j$. The letters S and T refer to the boundary sphere and torus respectively (S should actually be collapsed to one point x_0, but the picture is easier to understand before collapse).

Recall that the algebraic complex of which we must compute the torsion has one generator for each cell in the cellularization of $E(K)$ arising from P, excluding the white cells and the tangency circles on the boundary. From Fig. 10 we can see how many such cells there will be in each dimension, namely 3 in dimension 0 (x_0 and two vertices on T), 14 in dimension 1 (the $\hat{r}_i$'s and 8 edges on T), 16 in dimension 2 (the $\hat{e}_i$'s and the 6 black kites on T) and 5 in dimension 3 (the $\hat{v}_i$'s). We can also easily describe the 1-chain $s'(P)$ which will be used to find the preferred cell liftings: besides the orbits of the field there are only one star and one bi-arrow; the support of $s'(P)$ has 3 connected components (one spider with 19 legs and head at x_0, the star union the second half of $\hat{r}_2$, and the bi-arrow union a segment contained in $\hat{e}_3$).

To actually determine the preferred liftings we need an effective description of the lifting of the cellularization to the universal cover $\tilde{E}(K) \to E(K)$. Since $\pi_1(E(K)) = \mathbb{Z}$, each cell c will have liftings $\tilde{c}^{(n)}$ for $n \in \mathbb{Z}$, where $\tilde{c}^{(n)}$ is the n-th translate of $\tilde{c}^{(0)}$. The choice of $\tilde{c}^{(0)}$ itself is of course arbitrary, but to understand the cover we must express the $\partial\tilde{c}^{(0)}$'s in terms of the other $\tilde{d}^{(n)}$'s. To do this we start with a lifting $\tilde{x}_0$ of the basepoint x_0 and we lift the other cells one after each other, taking into account the relations in $\pi_1(E(K))$ and making sure that the union of cells already lifted is always connected. When a cell c is reached for the first time, its lifting is chosen arbitrarily and declared to be $\tilde{c}^{(0)}$, but its boundary will involve in general $\tilde{d}^{(n)}$'s with $n \neq 0$. Once the lifted cellularization is known, it is a simple matter to determine preferred cell liftings: since the support of $s'(P)$ consists of 3 spiders, we only need to choose liftings of the 3 heads and then lift the legs.

Carrying out the computations we have explicitly found the algebraic complex with coefficients in $Q(\mathbb{Z}[t^{\pm 1}])$, and the preferred generators of the 4 moduli appearing. Then, using Maple, we have checked that indeed the complex is acyclic, and we have computed its torsion as follows:

$$\tau^{\varphi}(E(K), \xi) = \pm t^{-1}. \tag{5}$$

5. Torsion of pseudo-Legendrian knots and the Alexander invariant

In this section and in the next one we apply the general theory we have developed to the study of pseudo-Legendrian (and hence of Legendrian) knots. We fix a compact oriented manifold M and a boundary pattern $\mathcal{P}$ on M. The boundary of M may be empty or not. Recall that if v is a vector field on M and K is a knot in $\mathrm{Int}(M)$, we have defined K to be pseudo-Legendrian in (M, v) if v is transversal to K. We will also call (v, K) a pseudo-Legendrian pair. Having fixed $\mathcal{P}$, we will only consider fields v compatible with $\mathcal{P}$. Some of the results we will establish hold also for links, but we will stick to knots for the sake of simplicity. First, we need to spell out the equivalence relation on pseudo-Legendrian pairs which we consider.

Let v_0, v_1 be compatible with $\mathcal{P}$ and let K_0, K_1 be pseudo-Legendrian in (M, v_0) and (M, v_1) respectively. We define (v_0, K_0) to be *pseudo-Legendrian-isotopic* to (v_1, K_1) if there exist a homotopy $(v_t)_{t\in[0,1]}$ through fields compatible with $\mathcal{P}$ and an isotopy $(K_t)_{t\in[0,1]}$ such that K_t is transversal to v_t for all t. If $v_0 = v_1$

then K_0 and K_1 are called *strongly* pseudo-Legendrian-isotopic if the homotopy (v_t) can be chosen to be constant.

Remark 5.1. Of course strong pseudo-Legendrian isotopy implies pseudo- Legendrian isotopy. The latter relation is the natural one to consider on pseudo-Legendrian pairs (v, K), while the former is natural for pseudo-Legendrian knots in a fixed (M, v). A classical Legendrian isotopy of a Legendrian knot in a contact structure ξ is a strong pseudo-Legendrian isotopy with respect to any vector field v orthogonal to ξ. See [4] for further discussion on these notions.

Before proceeding recall that if K is pseudo-Legendrian in (M, v) then v turns K into a framed knot, which we denote by $K^{(v)}$, and the framed-isotopy class of $K^{(v)}$ is of course invariant under pseudo-Legendrian isotopy. We also know that, given a boundary pattern $\mathcal{P}$ on M, we have a well-defined pattern $\mathcal{P}(K^{(v)})$ on $E(K)$. Moreover, if v is compatible with $\mathcal{P}$, then the restriction of v to $E(K)$ defines an element

$$\xi(v, K) \in \mathrm{Eul}(E(K), \mathcal{P}(K^{(v)})),$$

so the theory of torsion applies. In the rest of this section we will discuss torsion as an *absolute* invariant of (v, K), showing in particular that in a homology sphere it lifts the classical Alexander invariant of K. The relation between torsion and the Alexander invariant is however more complicated than in Turaev's situation ([15] and [16]), because here two different algebraic complexes will be involved at the same time. In the next section we will discuss the extent to which torsion can be employed as a relative invariant, *i.e.* as an obstruction to pseudo-Legendrian isotopy of pairs (v_0, K_0) and (v_1, K_1).

For the sake of simplicity we only consider, in the present section and in the next one, representations of the fundamental group obtained from representations of the first homology group.

5.1. *Turaev's lifting of Milnor torsion*

Let us first recall again in what sense Turaev's torsion lifts the classical one. Let M be a manifold which is closed or bounded by tori, and take a representation $\varphi : H_1(M; \mathbb{Z}) \to \Lambda_*$, where Λ is as usual. The classical theory [11] allows to define an invariant

$$\tau^{\varphi}(M) \in K_1(\Lambda) \Big/ (\pm\varphi(H_1(M; \mathbb{Z}))),$$

usually stipulated to be 1 if the φ-twisted homology of M does not vanish, *i.e.,* using the above notation, if the complex $C_*^{\varphi}(M) = \Lambda \otimes_{\varphi} C_*^{\mathrm{cell}}(\tilde{M}; \mathbb{Z})$ is not acyclic, where $\tilde{M} \to M$ is the maximal Abelian cover. When ξ is an Euler structure on M with monochromatic boundary components, Turaev [16] shows that his torsion $\tau^{\varphi}(M, \xi) \in \overline{K}_1(\Lambda)$ is a lifting of $\tau^{\varphi}(M)$ with respect to the obvious projection of $K_1(\Lambda)/ \pm \varphi(H_1(M; \mathbb{Z})$ onto $\overline{K}_1(\Lambda)$.

In the special case where Λ is the field of fractions obtained from the group ring of $H_1(M; \mathbb{Z})$ modulo torsion, and $\varphi : H_1(M; \mathbb{Z}) \to \Lambda$ is the natural projection, the invariant $\tau^{\varphi}(M)$ is called Milnor torsion, and its sign-refinement provided by Turaev in [15] has been shown to be equivalent to the classical Alexander invariant. So Turaev's torsion for Euler structures contains a lifting of the Alexander invariant. We will discuss in the rest of this subsection the extent to what the same holds when the Euler structure arises from a pseudo-Legendrian knot. What we will say applies to any allowed representation $\varphi : H_1(M; \mathbb{Z}) \to \Lambda$, but we keep in mind that the relation with the Alexander invariant emerges for a special choice of φ and Λ. Since we will also drop the condition that the involved complexes be acyclic, we note that torsion is only defined when the resulting homology is free. This is not true in general, but it is for instance when Λ is a field.

5.2. *Torsion of a knot complement*

Let us restrict to the case of a closed manifold M, and let us consider a pseudo-Legendrian pair (v, K) in M and a representation $\varphi : H_1(E(K); \mathbb{Z}) \to \Lambda$ as usual. We would like to interpret the torsion of the Euler structure $\xi(v, K)$ on $E(K)$ with respect to φ as a lifting of $\tau^{\varphi}(E(K))$, but a difficulty immediately emerges, because the algebraic complexes used to compute these torsions do not coincide.

To be more specific, let us first spell out how the torsion of $\xi(v, K)$ is defined. Let $\mathcal{P}(K^{(v)}) = (B, W, \emptyset, C)$ be the boundary pattern defined on $E(K)$. Then we define $\tau^{\varphi}(M, v, K, \mathfrak{h})$ as $\tau^{\varphi}(M, \theta(\mathcal{P}(K^{(v)})), \Theta(\xi(v, K)), \mathfrak{h})$. More specifically, $\tau^{\varphi}(M, v, K, \mathfrak{h})$ is the torsion of the complex $C_*^{\varphi}(E(K), \overline{W})$, where W is the (open) white annulus on $\partial E(K)$, as above the maximal Abelian cover of $E(K)$ is used to define the complex, the preferred cell lifting is obtained using an Euler chain for the convexified structure $\Theta(\xi(v, K))$, and $\mathfrak{h}$ is a basis of the twisted homology of $E(K)$ relative to $\overline{W}$.

Now, $\tau^{\varphi}(E(K))$ is the torsion of $C_*^{\varphi}(E(K))$, and this complex can be radically different from the previous one. For instance, when M is a homology sphere, the absolute complex is always acyclic, while the complex relative to $\overline{W}$, which depends only on the framed knot $K^{(v)}$, in general is not. We will see how to overcome this difficulty using the fundamental multiplicativity properties of torsion.

5.3. *How to turn a torus into black*

We will describe in this paragraph two explicit methods for modifying $\xi(v, K)$ to an Euler structure $\beta(v, K)$ such that $\partial E(K)$ becomes monochromatic black. These methods are respectively a geometric and an algebraic one. The fact that they actually lead to the same result is true but not very important, so we will omit the proof. Both methods involve the choice of an orientation of K. The first method is explained in a cross-section in Fig. 11. The cross-section is transversal to K, and the apparent singularity of the modified field is removed by summing a field parallel to K and supported near the singularity (*cf.* Fig. 2, where a similar method was used).

To describe the algebraic construction of β, recall that if z is a 1-chain representing $\xi(v, K)$ then ∂z contains, with the appropriate sign, the centres of all cells

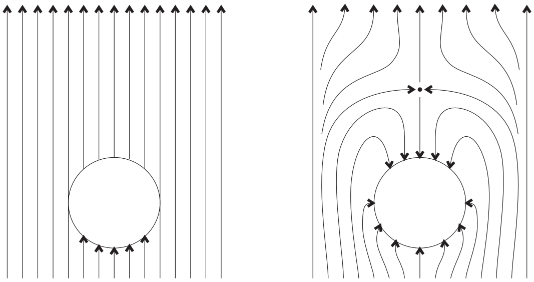

Fig. 11. Black field on a knot complement

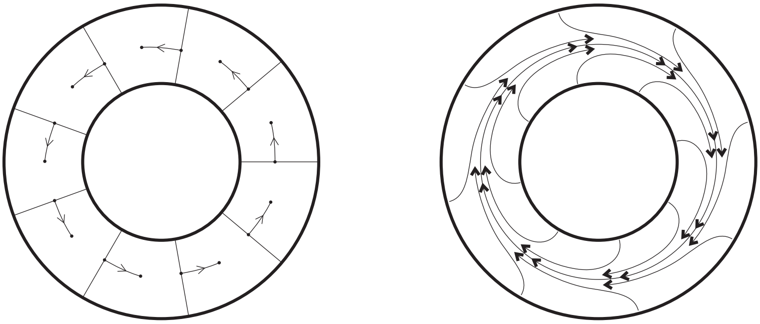

Fig. 12. A 1-chain on the annulus W

in $E(K) \setminus W$. Knowing the subdivision rule for Euler chains (Proposition 1.3) we can also assume that the cellularization on W has a particularly simple shape. We assume it consists of rectangles as in Fig. 12 (left), where we also show a 1-chain z_W having the property that ∂z_W contains the centres of all cells in W. We can now define $\beta(v, K)$ as the Euler structure carried by $z + z_W$. The boundary of $E(K)$ is completely black with respect to this structure, because $\partial(z + z_W)$ contains the centres of all cells of $E(K)$.

One easily sees from both our descriptions of β that it is canonically defined and H_1-equivariant. Since we will need these properties, we spell out their meaning, starting from an oriented framed knot K^F rather than a pseudo-Legendrian knot. Let $(W, B, \emptyset, C)$ be the concave boundary pattern determined by F on $E(K)$: then $\beta : \mathrm{Eul}(E(K), (W, B, \emptyset, C)) \to \mathrm{Eul}(E(K), (\emptyset, \partial E(K), \emptyset, \emptyset))$ is well-defined (depending on K^F only) and $H_1(E(K); \mathbb{Z})$-equivariant.

Remark 5.2. If $-K$ denotes the same knot K with reversed orientation then

$$\alpha(\beta(v, K), \beta(v, -K)) = [\lambda] \in H_1(E(K); \mathbb{Z})$$

where λ is the longitude on $\partial E(K)$ determined by the framing $K^{(v)}$.

A geometric interpretation of the chain r_W entering in the second description of β is possible and used below. We have mentioned that a theory of Euler structures exists in all dimensions. While the case $n \geq 4$ requires some technicalities, the reader can easily work out the case $n = 2$ using the case $n = 3$ treated in the present paper. And one readily sees that z_W is just an Euler chain of the inward-pointing Reeb field r_W on $\overline{W}$ shown in Fig. 12 (right). Moreover r_W can be canonically turned into an outward-pointing field $\Theta(r_W)$, which of course is the outward-pointing Reeb field (but the core spins in the opposite direction). So a torsion $\tau^{\psi}(\overline{W}, \Theta(r_W))$ can be computed (possibly with a basis of the twisted homology added to the data).

5.4. *Knot torsion as a lifting of Milnor's torsion*

Let as above (v, K) be pseudo-Legendrian and let $\varphi : H_1(E(K); \mathbb{Z}) \to \Lambda$ be a representation. If $i : \overline{W} \to E(K)$ is the inclusion, we set $\varphi_W = \varphi \circ i_*$. Considering the twisted homology of the pair $(E(K), \overline{W})$ we get an exact sequence

$$\mathcal{H} = \Big(\cdots \longrightarrow H_i^{\varphi_W}(\overline{W}) \longrightarrow H_i^{\varphi}(E(K)) \longrightarrow H_i^{\varphi}(E(K), \overline{W}) \\ \longrightarrow H_{i-1}^{\varphi_W}(\overline{W}) \longrightarrow \cdots \Big).$$

We choose bases $\mathfrak{h}$, $\mathfrak{h}'$, and $\mathfrak{h}''$ respectively for $H_*^{\varphi}(E(K); \overline{W})$, $H_*^{\varphi}(E(K))$ and $H_*^{\varphi_W}(\overline{W})$, so we can compute $\tau^{\varphi}(M, v, K, \mathfrak{h})$, $\tau^{\varphi}(E(K), \beta(v, K), \mathfrak{h}')$ and $\tau^{\varphi_W}(\overline{W}, \Theta(r_W), \mathfrak{h}'')$. In addition we can compute $\tau(\mathcal{H}, \mathfrak{h}, \mathfrak{h}', \mathfrak{h}'')$. The following result is a refinement of Theorem 3.2 in [11], and a proof can be given imitating the argument given in [17] (where a special case of the result is established).

Proposition 5.3. *The following equality holds:*

$$\tau^{\varphi}(E(K), \beta(v, K), \mathfrak{h}') \\ = \tau^{\varphi}(M, v, K, \mathfrak{h}) \cdot \tau^{\varphi_W}(\overline{W}, \Theta(r_W), \mathfrak{h}'') \cdot \tau(\mathcal{H}, \mathfrak{h}, \mathfrak{h}', \mathfrak{h}''). \quad (6)$$

The following remarks and corollary of the previous proposition eventually explain in what sense our torsion can be viewed as a lifting of the classical torsion (in particular, Milnor torsion and the Alexander invariant).

Remark 5.4. In equation (6) the term $\tau^{\varphi}(E(K), \beta(v, K), \mathfrak{h}')$ is one of Turaev's torsion, so it is indeed a lifting of the classical torsion. The term $\tau^{\varphi}(M, v, K, \mathfrak{h})$ is the torsion for pseudo-Legendrian knots introduced in this paper, while $\tau^{\varphi_W}(\overline{W}, \Theta(r_W), \mathfrak{h}'')$ and $\tau(\mathcal{H}, \mathfrak{h}, \mathfrak{h}', \mathfrak{h}'')$ can be viewed as normalizing terms. One can for instance choose homology bases so that $\tau(\mathcal{H}, \mathfrak{h}, \mathfrak{h}', \mathfrak{h}'') = 1$, and note that $\tau^{\varphi_W}(\overline{W}, \Theta(r_W), \mathfrak{h}'')$ depends only on the framed knot $K^{(v)}$, not on the Euler structure.

Remark 5.5. The factor $\tau^{\varphi_W}(\overline{W}, \Theta(r_W), \mathfrak{h}'')$ may be understood quite easily. Denoting by 1 the generator of $H_1(\overline{W}; \mathbb{Z})$, the result only depends on $\varphi_W(1)$. If $\varphi_W(1) = 1$ then the φ_W-twisted homology of $\overline{W}$ is not twisted at all, so it is non-zero and free, and we can choose $\mathfrak{h}''$ so that $\tau^{\varphi_W}(\overline{W}, \Theta(r_W), \mathfrak{h}'') = 1$. On the contrary, if $\varphi_W(1)-1$ is invertible, then the φ_W-twisted homology is zero, and $\tau^{\varphi_W}(\overline{W}, \Theta(r_W))$ is computed to be $(\varphi_W(1) - 1)^{-1}$. In the intermediate cases where $\varphi_W(1) - 1$ is neither zero nor a unit, which can only occur when Λ is not a field, $\tau^{\varphi_W}(\overline{W}, \Theta(r_W))$ is likely not to be defined.

We can further specialize the understanding of $\tau^{\varphi_W}(\overline{W}, \Theta(r_W), \mathfrak{h}'')$ when M is a homology sphere and $\varphi : H_1(E(K); \mathbb{Z}) \to \Lambda$ is the representation which gives the Milnor torsion. In this case we recall that $\Lambda = Q(\mathbb{Z}[t^{\pm 1}])$, the generator of $H_1(E(K); \mathbb{Z})$ is mapped to t by φ, and $H_*^{\varphi}(E(K)) = 0$. It follows that the generator of $H_1(\overline{W}; \mathbb{Z})$ is mapped to t^n by φ_W, where $n \in \mathbb{Z}$ is the framing of K. So the complex $C_*^{\varphi_W}(\overline{W})$ is only acyclic when $n \neq 0$. If indeed $n \neq 0$ then the torsion of $C_*^{\varphi_W}(\overline{W})$ is computed to be $(t^n - 1)^{-1}$, and from the exact sequence we deduce that $H_*^{\varphi}(E(K), \overline{W}) = 0$. If instead $n = 0$ then $H_*^{\varphi_W}(\overline{W})$ is non-zero, and canonically isomorphic to $H_*^{\varphi}(E(K), \overline{W})$. We deduce the following result which summarizes the relations between our torsion and the Alexander invariant:

Corollary 5.6. *Let (v, K) be a pseudo-Legendrian pair in a homology sphere M, and let $n \in \mathbb{Z}$ be the framing on K defined by v.*

- *If $n \neq 0$ then*

$$\tau^{\varphi}(E(K), \beta(v, K)) = \tau^{\varphi}(M, v, K) \cdot (t^n - 1)^{-1};$$

- *If $n = 0$ and we choose the same basis $\mathfrak{h}$ for $H_*^{\varphi_W}(\overline{W})$ and $H_*^{\varphi}(E(K), \overline{W})$ under their natural isomorphism, then*

$$\tau^{\varphi}(E(K), \beta(v, K)) = \tau^{\varphi}(M, v, K, \mathfrak{h}) \cdot \tau^{\varphi_W}(\overline{W}, \Theta(r_W), \mathfrak{h}).$$

6. Torsion as a relative invariant of knots

We study in this section how torsion can be employed to distinguish pseudo-Legendrian knots from each other. We first show that as a relative invariant torsion is only well-defined as a multi-valued function, the ambiguity being given by the action of a suitable group. Then we concentrate on the knots (called 'good' below) for which this action is trivial, and we interpret the relative information carried by torsion as a relative winding numbers.

6.1. Group action on Euler structures

Consider a knot K and a self-diffeomorphism f of $E(K)$ which is the identity near $\partial E(K)$. Then f extends to a self-diffeomorphism $\widehat{f}$ of M, where $\widehat{f}\big|_{U(K)} = \mathrm{id}_{U(K)}$. We define $G(K)$ as the group of all such f's with the property that $\widehat{f}$ is isotopic to the identity on M. Elements of $G(K)$ are regarded up to isotopy relative to $\partial E(K)$. If F is a framing on K then the pull-forward of vector fields induces an action of $G(K)$ on $\mathrm{Eul}(E(K), \mathcal{P}(K^{(v)}))$. We will now see that an obstruction to pseudo-Legendrian isotopy can be expressed in terms of this group action.

Let (v_0, K_0) and (v_1, K_1) be pseudo-Legendrian pairs in M, and assume that $K_0^{(v_0)}$ is framed-isotopic to $K_1^{(v_1)}$ under a diffeomorphism f relative to ∂M. Using the restriction of f and the pull-back of vector fields we get a bijection

$$f^* : \mathrm{Eul}(E(K_1), \mathcal{P}(K_1^{(v_1)})) \to \mathrm{Eul}(E(K_0), \mathcal{P}(K_0^{(v_0)})).$$

Proposition 6.1. *Under the current assumptions, if* (v_0, K_0) *and* (v_1, K_1) *are pseudo-Legendrian-isotopic to each other then* $f^*(\xi(v_1, K_1))$ *belongs to the* $G(K_0)$*-orbit of* $\xi(v_0, K_0)$ *in* $\mathrm{Eul}(E(K_0), \mathcal{P}(K_0^{(v_0)}))$.

Proof of Proposition 6.1. By assumption K_0, K_1 and v_0, v_1 embed in continuous families $(K_t)_{t\in[0,1]}$ and $(v_t)_{t\in[0,1]}$, where v_t is transversal to K_t for all t. Now $(K_t^{(v_t)})_{t\in[0,1]}$ is a framed-isotopy, so there exists a continuous family $(g_t)_{t\in[0,1]}$ of diffeomorphisms of M fixed on ∂M and such that $g_0 = \mathrm{id}_M$ and $g_t(K_0^{(v_0)}) = K_t^{(v_t)}$. So we get a map

$$[0, 1] \ni t \mapsto \alpha(\xi(v_0, K_0), g_t^*(\xi(v_t, K_t))) \in H_1(E(K_0); \mathbb{Z}).$$

Since $H_1(E(K_0); \mathbb{Z})$ is discrete and the map is continuous, we deduce that the map is identically 0. So $g_1^*(\xi(v_1, K_1)) = \xi(v_0, K_0)$. Now

$$f^*(\xi(v_1, K_1)) = (f^* \circ (g_1)_* \circ g_1^*)(\xi(v_1, K_1)) = (f^{-1} \circ g_1)_*(\xi(v_0, K_0))$$

and the conclusion follows because $f^{-1} \circ g_1$ defines an element of $G(K_0)$. □

The group $G(K)$ is in general rather difficult to understand (see [7]), so we introduce a special terminology for the case where its action can be neglected. We will say that a framed knot K^F is *good* if $G(K)$ acts trivially on $\mathrm{Eul}(E(K), \mathcal{P}(K^F))$. If K^F is good for all framings F, we will say that K itself is good. The following are easy examples of good knots:

- M is S^3 and K is the trivial knot;
- M is a lens space $L(p, q)$ and K is the core of one of the handlebodies of a genus-one Heegaard splitting of M.

The reason is that in both cases $E(K)$ is a solid torus, and we know that an automorphism of the solid torus which is the identity on the boundary is isotopic to the identity relatively to the boundary, so $G(K)$ is trivial. The next three results show that on one hand $G(K)$ is very seldom trivial, but on the other hand many knots are good. We will give proofs in the sequel, after introducing some extra notation. In the statements, by "$E(K)$ is hyperbolic" we mean "$\mathrm{Int}(E(K))$ is complete, finite-volume hyperbolic".

Proposition 6.2. *If* M *is closed and* $E(K)$ *is hyperbolic then* $G(K)$ *is non-trivial.*

Theorem 6.3. *If* M *is closed,* $E(K)$ *is hyperbolic and either* $\mathrm{Out}(\pi_1(E(K)))$ *is trivial or* $H_1(E(K); \mathbb{Z})$ *is torsion-free then* K *is good.*

Theorem 6.4. *If* M *is a homology sphere then every knot in* M *is good.*

The next result, which follows directly from Proposition 6.1, the definition of goodness, and Proposition 2.6, shows that for good knots torsion can be used as an obstruction to pseudo-Legendrian isotopy (and hence to strong pseudo-Legendrian isotopy).

Proposition 6.5. *Let* (v_0, K_0) *and* (v_1, K_1) *be pseudo-Legendrian pairs in* M*, and assume that* $K_0^{(v_0)}$ *is framed-isotopic to* $K_1^{(v_1)}$ *under a diffeomorphism* f *relative to* ∂M*. Suppose that* $K_0^{(v_0)}$ *is good, and that for some representation* $\varphi : \pi_1(E(K_0)) \to \Lambda$ *and some* Λ*-basis* $\mathfrak{h}$ *of* $H_*^{\varphi}(E(K_0), \overline{W(\mathcal{P}(K_0^{(v_0)}))})$ *we have*

$$\tau^{\varphi}(E(K_0), \mathcal{P}(K_0^{(v_0)}), \xi(v_0, K_0), \mathfrak{h}) \neq \tau^{\varphi \circ f_*^{-1}}(E(K_1), \mathcal{P}(K_1^{(v_1)}), \xi(v_1, K_1), f_*(\mathfrak{h})). \quad (7)$$

Then (v_0, K_0) *and* (v_1, K_1) *are not pseudo-Legendrian-isotopic.*

Remark 6.6. 1. The right-hand side of equation (7) actually equals

$$\tau^{\varphi}(E(K_0), \mathcal{P}(K_0^{(v_0)}), f^*(\xi(v_1, K_1)), \mathfrak{h}) = \overline{\varphi}(\alpha(v_0, f^*(v_1)) \cdot \tau^{\varphi}(E(K_0), \mathcal{P}(K_0^{(v_0)}), \xi(v_0, K_0), \mathfrak{h}).$$

This shows that the most torsion can capture as a relative invariant of (v_0, K_0) and (v_1, K_1) is $\alpha(v_0, f^*(v_1))$. We will show below that in some cases torsion indeed allows to determine $\alpha(v_0, f^*(v_1))$ completely.
2. By definition of goodness the homology class $\alpha(v_0, f^*(v_1))$ just considered is actually independent of f. We will denote it by $\alpha((v_0, K_0), (v_1, K_1))$.
3. For non-good knots the relative invariant is an orbit of the action of $G(K_0)$. So an obstruction in terms of torsion could be given also for non-good knots, but the statement would become awkward, and we have refrained from giving it.
4. If equation (7) holds for some basis $\mathfrak{h}$ then it holds for any basis.

To conclude this paragraph we note that using the technology of Turaev [16], one can actually see that the action on Euler structures of an automorphism is invariant under *homotopy* (not only isotopy) relative to the boundary. We will not use this fact.

6.2. Good knots

We introduce now some notation needed for the proofs of Proposition 6.2 and Theorem 6.3 (for Theorem 6.4 we will use a different approach, see below). Recall that $(M, \mathcal{P})$ is fixed for the whole section. We temporarily fix also a framed knot K^F in M, a regular neighbourhood U of K, and we denote by T the boundary torus of U. On T we consider 1-periodic coordinates (x, y) such that $x \mapsto (x, 0)$ is a meridian of U and $y \mapsto (0, y)$ is a longitude compatible with F. We denote a collar of T in $E(K)$ by V and parametrize V as $T \times [0, 1]$, where $T = T \times \{0\}$. We consider on $[0, 1]$ a coordinate s. For $p, q \in \mathbb{Z}$ we define automorphisms $\mathcal{D}_{(p,q)}$ of $E(K)$ as follows. Each $\mathcal{D}_{(p,q)}$ is supported in V, and on V, using the coordinates just described, it is given by

$$\mathcal{D}_{(p,q)}(x, y, s) = (x + p \cdot s, y + q \cdot s, s).$$

We will call such a map a *Dehn twist*. It is easy to verify that the extension of $\mathcal{D}_{(p,q)}$ to M is isotopic to the identity of M. Note that $\mathcal{D}_{(p,q)}$ is actually not smooth on $T \times \{1\}$, but we can consider some smoothing and identify $\mathcal{D}_{(p,q)}$ to an element of $G(K)$, because the equivalence class is independent of the smoothing.

Proof of Proposition 6.2. We show that $\mathcal{D}_{(p,q)}$ is non-trivial in $G(K)$ for all $(p,q) \neq (0,0)$. Fix the basepoint $a_0 = (0,0) \in T$ for the fundamental groups of T and $E(K)$. Then $\mathcal{D}_{(p,q)}$ acts on $\pi_1(E(K), a_0)$ as the conjugation by $i_*(p,q)$, where $i : T \to E(K)$ is the inclusion and $(p,q) \in \mathbb{Z} \times \mathbb{Z} = \pi_1(T, a_0)$. If $\mathcal{D}_{(p,q)}$ is trivial in $G(K)$, *i.e.* it is isotopic to the identity relatively to $\partial E(K)$, in particular it acts trivially on $\pi_1(E(K), a_0)$. This implies that $i_*(p,q)$ is in the centre of $\pi_1(E(K), a_0)$. Now it follows from hyperbolicity that this centre is trivial and i_* is injective, whence the conclusion. $\square$

The proof of Theorem 6.3 will rely on properties of hyperbolic manifolds and on the following fact, which we consider to be quite remarkable (note that the 2-dimensional analogue, which may be stated quite easily, is false). Remark that the result applies in particular to Dehn twists.

Proposition 6.7. *If* $[f] \in G(K)$ *and* f *is supported in the collar* V *of* ∂U *then* $[f]$ *acts trivially on* $\mathrm{Eul}(E(K), \mathcal{P}(K^F))$.

Proof of Proposition 6.7. Consider a vector field v on $E(K)$ compatible with $\mathcal{P}(K^F)$. Since v and $f_*(v)$ differ only on V, their difference belongs to the image of $H_1(V;\mathbb{Z})$ in $H_1(E(K);\mathbb{Z})$. So we may as well assume that $E(K) = V$, *i.e.* M is the solid torus $U \cup V$.

By contradiction, let $\xi \in \mathrm{Eul}(V, \mathcal{P}(K^F))$ be such that $\alpha(\xi, (\mathcal{D}_{(p,q)})_*(\xi))$ is non-zero in $H_1(V;\mathbb{Z})$, so it is given by $k \cdot [\gamma]$ for some $k \in \mathbb{Z} \setminus \{0\}$ and some simple closed curve γ on $T \times \{1\} \subset \partial V$. Let us now take another simple closed curve δ on $T \times \{1\}$ which intersects γ transversely at one point. Let us define N as the manifold obtained by attaching the solid torus to V along $T \times \{1\}$, in such a way that the meridian of the solid torus gets identified with δ. Note that N is again a solid torus and that the homology class of γ in $H_1(N;\mathbb{Z}) \cong \mathbb{Z}$ is a generator. Now we can apply Proposition 1.1 to extend ξ to an Euler structure ξ_N on N. Moreover we can extend f to an automorphism g of N which is the identity on $\partial N = T \times \{0\}$. Now by construction $\alpha(\xi_N, g_*(\xi_N))$ equals $k \cdot [\gamma]$ in $H_1(N;\mathbb{Z}) \cong \mathbb{Z}$, so it is non-zero. But g is isotopic to the identity of N relatively to the boundary of N, so we have a contradiction. $\square$

For the proof of Theorem 6.3 we will also need the following easy fact.

Lemma 6.8. *Let* f *be an automorphism of* M *relative to* ∂M, *and consider the induced automorphisms of* $H_1(M;\mathbb{Z})$ *and* $\mathrm{Eul}(M, \mathcal{P})$, *both denoted by* f_*. *Then:*

$$\alpha(f_*(\xi_0), f_*(\xi_1)) = f_*(\alpha(\xi_0, \xi_1)), \qquad \forall \xi_0, \xi_1 \in \mathrm{Eul}(M, \mathcal{P}).$$

Proof of Theorem 6.3. Consider $[f] \in G(K)$. It follows from the work of Johansson (see [7]) that, under the assumption that $E(K)$ is hyperbolic, the group generated by Dehn twists has finite index in the mapping class group of $E(K)$ relative to the boundary. More precisely, the quotient group can be identified to a subgroup of $\mathrm{Out}(\pi_1(E(K))$, which is finite as a consequence of Mostow's rigidity. If $\mathrm{Out}(\pi_1(E(K))$ is trivial then $[f]$ is equivalent to a Dehn twist, so f acts trivially on $\mathrm{Eul}(E(K),\mathcal{P}(K^F))$ by Proposition 6.7.

We are left to deal with the case where $H_1(E(K);\mathbb{Z})$ is torsion-free. By Johansson's result, there exists an integer n such that f^n acts trivially on $\mathrm{Eul}(E(K),\mathcal{P}(K^F))$. Consider now $\xi\in\mathrm{Eul}(E(K),\mathcal{P}(K^F))$, and set $\alpha=\alpha(\xi,f_*(\xi))$. We must show that $\alpha=0$. We denote by $\widehat{\alpha}$ the image of α in $H_1(M;\mathbb{Z})$, and by $\widehat{f}$ the extension of f to M. Since $\widehat{f}$ is isotopic to the identity, we have $\widehat{f}_*(\widehat{\alpha})=\widehat{\alpha}$. If we take an oriented 1-manifold a representing α and disjoint from $\partial U(K)$, this means that there exists an oriented surface Σ in M such that $\partial\Sigma=a\cup(-f(a))$. Up to isotopy we can assume that Σ intersects $\partial U(K)$ transversely in a union of circles. This shows that $f_*(\alpha)=\alpha+k\cdot\mu$, where μ is the meridian of T. Note that $f_*(\mu)=\mu$, so for all integers m we have $f_*^m(\alpha)=\alpha+m\cdot k\cdot\mu$. Now, using Lemma 6.8, we have:

$$\begin{aligned}0&=\alpha(\xi,f_*^n(\xi)=\sum_{m=0}^{n-1}\alpha(f_*^m(\xi),f_*^{m+1}(\xi))\\&=\sum_{m=0}^{n-1}f_*^m(\alpha(\xi,f_*(\xi)))=\sum_{m=0}^{n-1}f_*^m(\alpha)=\sum_{m=0}^{n-1}(\alpha+m\cdot k\cdot\mu)\\&=n\cdot\alpha+\frac{n(n-1)}{2}\cdot k\cdot\mu.\end{aligned}$$

This shows that $2\cdot\alpha+(n-1)\cdot k\cdot\mu$ is a torsion element of $H_1(E(K);\mathbb{Z})$, so it is null by assumption. So $(1-n)\cdot k\cdot\mu=2\cdot\alpha$. If we apply f_* to both sides of this equality we get $(1-n)\cdot k\cdot f_*(\mu)=2\cdot f_*(\alpha)$. Using the equality again and the relations $f_*(\mu)=\mu$ and $f_*(\alpha)=\alpha+k\cdot\mu$ we get

$$(1-n)\cdot k\cdot\mu=2\cdot\alpha+2\cdot k\cdot\mu=(1-n)\cdot k\cdot\mu+2\cdot k\cdot\mu.$$

Therefore $k\cdot\mu$ is a torsion element, and hence null. But $2\cdot\alpha=(1-n)\cdot k\cdot\mu$, so also α is null. $\square$

6.3. Rotation number, and goodness of knots in homology spheres

We will show in this paragraph that in a homology sphere the rotation number of a pseudo-Legendrian knot can be (defined and) expressed in terms of an Euler structure on its exterior. This will lead us to a simple interpretation of torsion as a relative invariant of knots, and it will allow us to show that in a homology sphere all knots are good (Theorem 6.4).

To begin, we note that the notion of rotation number, classically defined in the contact case, actually extends to the situation we are considering. Since we will need this definition, we recall it. Let M be a homology sphere, let v be a field on M and let K be an oriented pseudo-Legendrian knot in (M,v). Take a plane field η transversal to v and tangent to K, and a Seifert surface S for K. Up to isotopy of

S we can assume that η is tangent to S only at isolated points. Then $\mathrm{rot}_v(K)$ is the sum of a contribution for each of these tangency points p. Define $\mathrm{o}(p)$ to be $+1$ if $\eta_p = T_pS$ and -1 if $\eta_p = -T_pS$. If $p \in \partial S = K$ then p contributes just with $\mathrm{o}(p)$. If $p \in \mathrm{Int}(S)$ we can consider near p a section of $\eta \cap TS$ which vanishes at p only, and denote by $\mathrm{i}(p)$ its index. Then p contributes to $\mathrm{rot}_v(K)$ with $\mathrm{o}(p) \cdot \mathrm{i}(p)$.

It is quite easy to see that the resulting number is indeed independent from η and S. Moreover $\mathrm{rot}_v(K)$ is unchanged under homotopies of v relative to K, and local modifications away from K, so we can actually define $\mathrm{rot}_\xi(K)$ where $\xi = \xi(v, K) \in \mathrm{Eul}(E(K), \mathcal{P}(K^{(v)}))$.

Proposition 6.9. *Let M be a homology sphere, let v be a field on M and let K_0 and K_1 be oriented pseudo-Legendrian knots in (M, v). Assume that there exists a framed-isotopy f which maps $K_1^{(v)}$ to $K_0^{(v)}$. Identify $H_1(E(K_0); \mathbb{Z})$ to $\mathbb{Z}$ using a meridian. Then:*

$$\mathrm{rot}_v(K_1) = \mathrm{rot}_v(K_0) + 2\alpha(f_*(\xi(v, K_1)), \xi(v, K_0)).$$

Proof of Proposition 6.9. Let $K := K_0$, $v_0 := v$ and $v_1 := f_*(v)$. Note that v_0 and v_1 coincide along K. Of course $\mathrm{rot}_v(K_1) = \mathrm{rot}_{v_1}(K)$. We are left to show that

$$\mathrm{rot}_{v_1}(K) = \mathrm{rot}_{v_0}(K) + 2\alpha(\xi(v_1, K)), \xi(v_0, K)).$$

We can now homotope v_0 and v_1 away from K until they differ only in the neighbourhood $W(L)$ of an oriented link L, and within this neighbourhood they differ exactly by a "Pontrjagin move", as defined for instance in [3]. Namely, v_0 runs parallel to L in $W(L)$, while v_1 runs opposite to L on L and has non-positive radial component on $W(L)$ (see below for a picture). Note that L represents $\alpha(\xi(v_1, K)), \xi(v_0, K))$.

Let us choose now a Seifert surface S for K and a Riemannian metric on M, and define $\eta_i = v_i^\perp$, for $i = 0, 1$. Since $\eta_0\big|_K = \eta_1\big|_K$, the contributions along K to $\mathrm{rot}_{v_0}(K)$ and $\mathrm{rot}_{v_1}(K)$ are the same. Up to isotoping S we may assume that L is transversal but never orthogonal to S. At the points where η_0 is tangent to S also η_1 is tangent to S, and the contributions are the same. So $\mathrm{rot}_{v_1}(K) - \mathrm{rot}_{v_0}(K)$ is given by the sum of the contributions of the tangency points of η_1 to S within $W(L)$. We will show that each point of $L \cap S$ gives rise to exactly two tangency points, which both contribute with $+1$ or -1 according to the sign of the intersection of L and S at the point. This will show that $\mathrm{rot}_{v_1}(K) - \mathrm{rot}_{v_0}(K)$ is twice the algebraic intersection of L and S. This algebraic intersection is exactly the value of $[L] = \alpha(\xi(v_1, K)), \xi(v_0, K))$ as a multiple of $[m]$, so the local analysis at $L \cap S$ will imply the desired conclusion.

For the sake of simplicity we only examine a positive intersection point of L and S. This is done in a cross-section in Fig. 13, which shows the local effect of the Pontrjagin move. Both the fields have a rotational symmetry around L, suggested in the figure. The two tangency points which arise with the move are a positive focus (on the right) and a negative saddle (on the left), so the local contribution is indeed $+2$, and the proof is complete. $\square$

Remark 6.10. The definition of rotation number and Proposition 6.9 easily extend to the case of manifolds which are not homology spheres, by restricting to homologically trivial knots and choosing a relative homology class in the exterior.

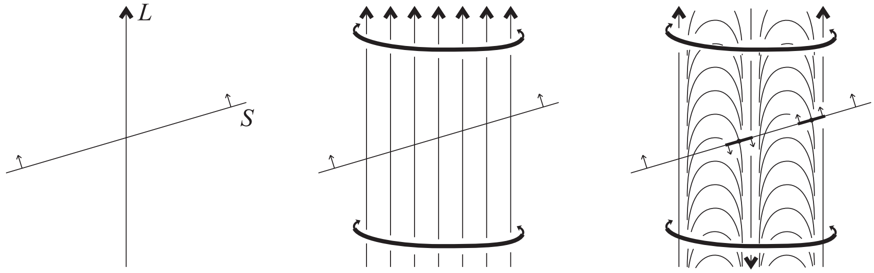

Fig. 13. Effect of the Pontrjagin move

We can now prove that in a homology sphere all knots are good.

Proof of Theorem 6.4. Consider $[f] \in G(K)$, a framing F on K and $\xi \in \mathrm{Eul}(E(K), \mathcal{P}(K^F))$. We must show that $f_*(\xi) = \xi$. Let $\xi = [v]$ and denote by $\widehat{v}$ the obvious extension of v to M. As above, let $\widehat{f}$ be the extension of f to M. During the proof of Proposition 6.9 we have shown that

$$\mathrm{rot}_{\widehat{f}_*(\widehat{v})}(K) - \mathrm{rot}_{\widehat{v}}(K) = 2\alpha(f_*(v), v).$$

But $\mathrm{rot}_{\widehat{f}_*(\widehat{v})}(K)$ is actually equal to $\mathrm{rot}_{\widehat{v}}(K)$, because $\widehat{f}$ is the identity near K. Therefore $f_*(v)$ and v differ by a torsion element of $H_1(E(K);\mathbb{Z}) \cong \mathbb{Z}$, so they are equal. By definition $f_*(\xi) = [f_*(v)]$ and $\xi = [v]$, and the proof is complete. □

Theorems 6.3 and 6.4 provide a partial answer to the problem of determining which knots are good. The general problem does not appear to be straight-forward, and we leave it for further investigation. We will only show below an example of knot which is not good.

6.4. Curls and the winding number

We show in this paragraph the relation between the relative invariant $\alpha((v_0, K_0), (v_1, K_1))$ of two pseudo-Legendrian knots (when this invariant is well-defined) and a relative analogue of the winding number (the invariant which allows to distinguish framed-isotopic planar link diagrams which are not equivalent under the second and third of Reidemeister's moves, see [14]).

Consider the local modification of pseudo-Legendrian pairs which is shown in Fig. 14. Here we consider a field v on a manifold M and a portion of M on which v can be identified to the vertical field in $\mathbb{R}^3$; we consider oriented knots K_0 and $K_{\pm 1}$ which are transversal to v and differ only within the chosen portion of M, as shown in the figure. We say that the two pseudo-Legendrian knots differ for a positive or a negative double curl. We state now a result proved in [4].

Proposition 6.11. *Let (v_0, K_0) and (v_1, K_1) be pseudo-Legendrian in M, assume that v_0 and v_1 are homotopic fields, and that $K_0^{(v_0)}$ and $K_1^{(v_1)}$ are isotopic as framed knots. Then (v_0, K_0) and (v_1, K_1) become pseudo-Legendrian-isotopic up to addition of a finite number of positive or negative double curls.*

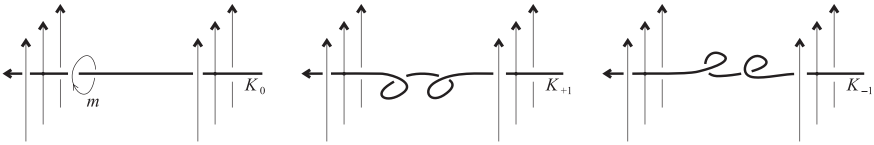

Fig. 14. Knots which differ for a positive or a negative double curl

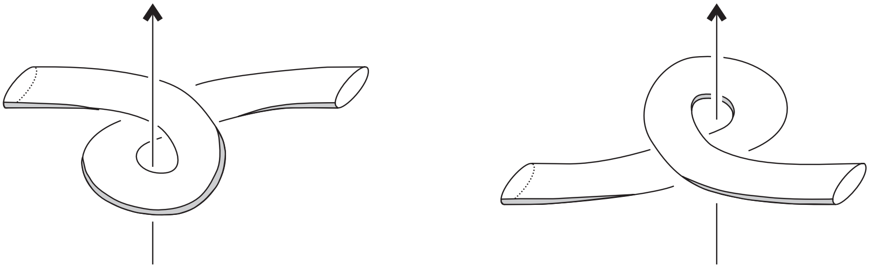

Fig. 15. Differently curled tubes in the vertical field.

We show now the effect on $\alpha((v_0, K_0), (v_1, K_1))$ of a double curl.

Proposition 6.12. *With the notations of Fig. 14 choose the positive meridian m of K_0, as also shown in the figure. Let f be an isotopy which maps $K_{\pm 1}^{(v)}$ to $K_0^{(v)}$ and is supported in a tubular neighbourhood of K_0. Then:*

$$\alpha(\xi(v, K_0), f_*(\xi(v, K_{\pm 1}))) = \pm[m] \in H_1(E(K_0); \mathbb{Z}).$$

Proof of Proposition 6.12. Let us first note that $\alpha(\xi(v, K_0), f_*(\xi(v, K_{+1})))$, which we must show to be $+[m]$, is independent of f by Proposition 6.7. Note also that this comparison class can be factorized through the inclusion of a collar of $\partial E(K)$ in M, and on this collar certainly it is an integer multiple of $[m]$, say $k \cdot [m]$. Moreover k is independent of the ambient manifold (M, v) and of the knot K. By symmetry, we will also have that $\alpha(\xi(v, K_0), f_*(\xi(v, K_{-1}))) = -k \cdot [m]$. So we can take M to be S^3. In particular, $[m]$ has infinite order. Using either the classical machinery of obstruction theory or the techniques developed in [4], one can see that there exists another pseudo-Legendrian knot K' in (S^3, v), framed isotopic to K, such that $\alpha(\xi(v, K), \xi(v, K')) = [m]$, where by simplicity we are omitting the framed-isotopies necessary to compare these classes. Using Proposition 6.11 we know that, up to pseudo-Legendrian isotopy, K' differs from K only for a finite number of transformations of the form $K \mapsto K_1$ or $K \mapsto K_{-1}$. This shows that $[m]$ is a multiple of $k \cdot [m]$, so $k = \pm 1$.

To check that actually $k = +1$, instead of comparing a "straight" knot with one with two curls, we compare two knots with one curl, chosen so that the framing is the same but the (local) winding number is different. This is of course equivalent. The two knots are shown in Fig. 15 as thick tubes, together with one specific orbit of the field they are immersed in. The resulting pattern on the boundary of the tubes is also outlined. To compare the curls we isotope the tubes to the same straight tube, and we show how the boundary patterns and the orbits of the field are transformed

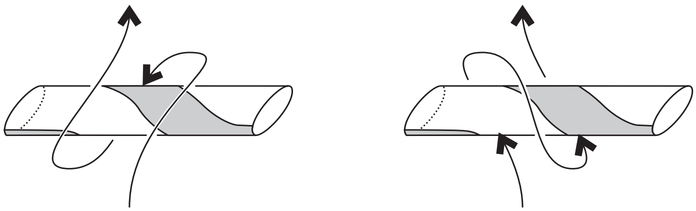

Fig. 16. Straightened curls

under this isotopy. This is done in Fig. 16. Also from this very partial picture it is quite evident that the resulting fields wind in opposite directions around the tube, in accordance with the sign rule stated in the proposition. □

Remark 6.13. In the above proof we have referred to Proposition 6.11, but with a little care one could actually use only Trace's [14] well-known version of this proposition for *planar* knot diagrams. For instance, one could choose K and K' to be represented by diagrams on the "smaller" disc of the abalone (the branched spine of S^3 used in the example of Sect. 4).

Proposition 6.11 shows that framed-isotopic non-pseudo-Legendrian-isotopic knots differ at most by double curls. The next result will imply that, under certain additional assumptions, the converse holds, *i.e* framed-isotopic knots which differ by double curls are not pseudo-Legendrian-isotopic.

Proposition 6.14. *Let (v, K_0) be a pseudo-Legendrian pair in M, and denote by $[m] \in H_1(E(K_0); \mathbb{Z})$ the homology class of the meridian of $U(K_0)$. Assume either that $K_0^{(v)}$ is good and $[m] \neq 0$ or that $E(K_0)$ is hyperbolic and $[m]$ has infinite order. Let K_{+1} be a knot obtained from K_0 as in Fig. 14. Then (v, K_0) and (v, K_{+1}) are not pseudo-Legendrian-isotopic.*

Proof of Proposition 6.14. By contradiction, using Propositions 6.1 and 6.12, we would get elements ξ_0, ξ_1 of $\mathrm{Eul}(E(K_0), \mathcal{P}(K_0^{(v)})$ such that $\alpha(\xi_0, \xi_1) = [m]$ and $\xi_1 = f_*(\xi_0)$ for some $[f] \in G(K_0)$. If $K_0^{(v)}$ is good and $[m] \neq 0$ this is a contradiction. Assume now that $E(K_0)$ is hyperbolic and $[m]$ has infinite order. Since $f_*([m]) = [m]$, using Lemma 6.8 we easily see that $\alpha(\xi_0, f_*^k(\xi_0)) = k \cdot [m]$ for all k. Proposition 6.7 and the result of Johansson already used in the proof of Theorem 6.3 now imply that f^k acts trivially on $\mathrm{Eul}(E(K_0), \mathcal{P}(K_0^{(v)})$ for some k, whence the contradiction. □

Back to the situation considered in Proposition 6.11 of pseudo-Legendrian pairs (v_0, K_0) and (v_1, K_1) with v_0 homotopic to v_1 and $K_0^{(v_0)}$ framed-isotopic to $K_1^{(v_1)}$, one would be tempted to define an invariant $\mathrm{w}((v_0, K_0), (v_1, K_1)) \in \mathbb{Z}$ as the algebraic number of double curls which one has to add to (v_0, K_0) to make it pseudo-Legendrian-isotopic to (v_1, K_1). However to define this algebraic number we need a (coherent) orientation on K_0 and K_1, and the sign changes if we change orientation. As a second attempt one could then try to define as an invariant the product $\mathrm{w}((v_0, K_0), (v_1, K_1)) \cdot [m_0] \in H_1(E(K_0); \mathbb{Z})$, where m_0 is the meridian of K_0. This product is now independent of the orientation, but again it is not well-defined in general. However Propositions 6.11 and 6.12 easily imply the following results:

Corollary 6.15. *If $K_0^{(v_0)}$ is good, so $\alpha((v_0, K_0), (v_1, K_1))$ is well-defined, then*

$$\alpha((v_0, K_0), (v_1, K_1)) = \mathrm{w}((v_0, K_0), (v_1, K_1)) \cdot [m_0] \in H_1(E(K_0); \mathbb{Z}).$$

In particular, the invariant on the right-hand side is also well-defined.

Corollary 6.16. *If $K_0^{(v_0)}$ is good and $[m_0]$ has infinite order in $H_1(E(K_0); \mathbb{Z})$ then $\mathrm{w}((v_0, K_0), (v_1, K_1)) \in \mathbb{Z}$ is a well-defined relative invariant of oriented pseudo-Legendrian pairs, which we call the* relative winding number.

Remark 6.17. Let M be a homology sphere with a field v, and let K_0 and $K_{\pm 1}$ be related as in Fig. 14. Then, using Propositions 6.9 and 6.12, we deduce that

$$\mathrm{rot}_v(K_{\pm 1}) - \mathrm{rot}_v(K_0) = \pm 2. \tag{8}$$

On the other hand one could prove formula (8) directly for $M = S^3$ and deduce an alternative proof of Proposition 6.12 using Proposition 6.9 only.

The next proposition implies, in particular, the result stated at the end of the introduction.

Proposition 6.18. *Under the assumptions of Proposition 6.11, assume that $K_0^{(v_0)}$ is good and that $[m_0]$ has infinite order in $H_1(E(K_0); \mathbb{Z})$. The following facts are pairwise equivalent:*

1. *The relative winding number of (v_0, K_0) and (v_1, K_1) vanishes;*
2. *All relative torsion invariants of (v_0, K_0) and (v_1, K_1) are trivial;*
3. *(v_0, K_0) and (v_1, K_1) are pseudo-Legendrian-isotopic.*

Proof of Proposition 6.18. Equivalence of (6.18) and (6.18) comes from the previous discussion and from the fact that a positive double curl and a negative double curl cancel via pseudo-Legendrian isotopy. To show that (6.18) and (6.18) are equivalent we only need to consider torsion with respect to a representation $\varphi : H_1(E(K_0); \mathbb{Z}) \to \Lambda$ such that $\varphi([m_0])$ has infinite order. □

Corollary 6.19. *Under the assumptions of Proposition 6.11, assume that M is a homology sphere. Then the facts (1), (2), and (3) of Proposition 6.18 are also equivalent to the following:*

4. *(v_0, K_0) and (v_1, K_1) have the same rotation number.*

Proof of Corollary 6.19. Equivalence of (6.18) and (6.19) comes from the previous discussion and Proposition 6.12. □

Since in a homology sphere two pseudo-Legendrian knots which are homotopic through pseudo-Legendrian immersions certainly have the same Maslov index, the previous corollary seems to suggest that all torsion can capture in a homology sphere is the homotopy class through immersions. We believe that it would be interesting to check if also for a general manifold M, under the assumptions of Corollary 6.15, homotopy through pseudo-Legendrian immersions implies $\mathrm{w}((v_0, K_0), (v_1, K_1)) \cdot [m_0] = 0$. We conclude by informing the reader that in [4] we have discussed the extent to which the category of pseudo-Legendrian knots can be represented by the category of genuine Legendrian knots in overtwisted contact structures.

6.5. *Non-good knots*

As another application of Proposition 6.12, we can show that there exist knots which are not good. Consider $S^2 \times [0, 1]$ with vector field parallel to the $[0, 1]$ factor. Let K_0 be the equator of $S^2 \times \{1/2\}$, and let K_1 be obtained from K_0 by the modification described in Fig. 14. Using Proposition 6.12, if we choose a framed-isotopy g of $K_1^{(v)}$ onto $K_0^{(v)}$ supported in $U(K_0)$, we have

$$\alpha(\xi(v, K_0), (g\big|_{E(K_1)})_*(\xi(v, K_1))) = [m],$$

where $[m]$ is a generator of $H_1(E(K_0); \mathbb{Z}) \cong \mathbb{Z}$. On the other hand, K_1 is strongly pseudo-Legendrian-isotopic to K_0 in (M, v) (the winding number only exists on $\mathbb{R}^2$, not on S^2). So there exists an isotopy h of $K_1^{(v)}$ onto $K_0^{(v)}$ through links transversal to v, and we have

$$\alpha(\xi(v, K_0), (h\big|_{E(K_1)})_*(\xi(v, K_1))) = 0.$$

This implies that $(h \circ g^{-1})\big|_{E(K_0)}$ acts non-trivially on $\xi(v, K_0) \in \mathrm{Eul}(E(K_0), \mathcal{P}(K_0^{(v)}))$.

7. Main proofs

In this section we provide the proofs which we have omitted in Sects. 1 and 3. We will always refer to the statements for notation.

Proof of Proposition 1.1. Let us first recall the classical Hopf–Poincaré theorem, according to which if v is a vector field with isolated singularities on a manifold M, v is transverse to ∂M and points outwards M (*i.e.* ∂M is black), then the sum of the indices of all singularities is $\chi(M)$. For the proof of this fact, and for the definitions of the notions involved, we address the reader to [12]. Assume now that v has isolated singularities and on ∂M it is compatible with a pattern $\mathcal{P} = (W, B, V, C)$. We claim that if $\mathcal{C}$ is a cellularization of M suited to $\mathcal{P}$ we have:

$$\sum_{x \in \mathrm{Sing}(v)} \mathrm{ind}_x(v) = \chi(M) - \sum_{\sigma \in \mathcal{C},\ \sigma \subset W \cup V} \mathrm{ind}(\sigma). \tag{9}$$

This formula is enough to prove the statement: if a non-singular field v compatible with $\mathcal{P}$ exists then the left-hand side of (9) vanishes, and the right-hand side of (9)

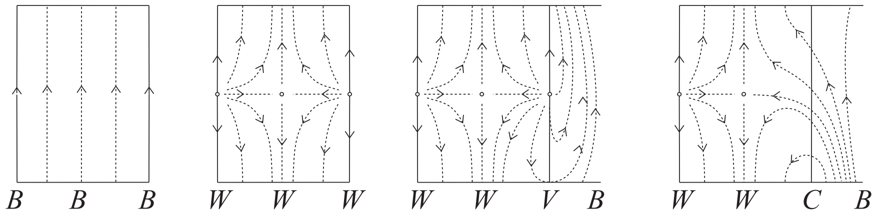

Fig. 17. Extension of the field to the collared manifold: dimension 2

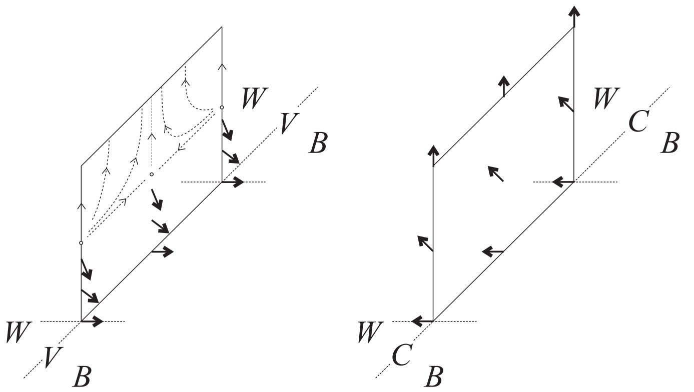

Fig. 18. Extension of the field to $\sigma \times [0, 1]$ for $\sigma \subset V$ and for $\sigma \subset C$.

equals the obstruction of the statement. On the other hand, if the obstruction vanishes, then one can first consider a singular field compatible with $\mathcal{P}$, then group up the singularities in a ball, and remove them.

To prove (9) we consider the manifold M' obtained by attaching a collar $\partial M \times [0, 1]$ to M along $\partial M = \partial M \times \{0\}$. Of course $M' \cong M$. We will now extend v to a field v' on M' with the property that v' points outwards on $\partial M'$, and in $\partial M \times (0, 1)$ the field v' has exactly one singularity for each cell $\sigma \subset W \cup V$, with index $\mathrm{ind}(\sigma)$. An application of the classical Hopf–Poincaré theorem then implies the conclusion. The construction of v' is done cell by cell. We first show how the construction goes in dimension 2, see Fig. 17.

For the 3-dimensional case, we choose a cellularization $\mathcal{C}$ of special type. Namely, we require that $\mathcal{C}\big|_{\partial M}$ on a neighbourhood of $C \cup V$ consists of rectangles, and each rectangle has exactly one edge on $V \cup C$. We describe now the extension of v' first on $\sigma \times [0, 1]$ for $\sigma \in \mathcal{C}\big|_{\partial M}$ and $\dim(\sigma) \leq 1$. When σ is not contained in $C \cup V$ the rules are exactly the same as in the 2-dimensional case, see Fig. 17. When $\sigma \subset C \cup V$ the rules are given in Fig. 18. Concerning the rule when σ is an edge contained in V, note that v' is only tangent to $\sigma \times [0, 1]$ on $\sigma \times [1/2, 1]$, and Fig. 18-left shows how it is constructed. To give a precise rule on $\sigma \times [0, 1/2]$ we choose local coordinates $[-1, 1]_x \times [-1, 1]_y$ on ∂M such that $[-1, 0) \times [-1, 1] \subset W$, $(0, 1] \times [-1, 1] \subset B$ and $\sigma = \{0\} \times [-1, 1] \subset V$. Then

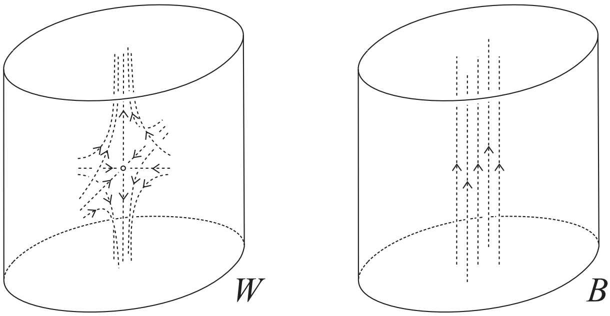

Fig. 19. Extension of the field near $p_\sigma \times [0, 1]$ for $\dim(\sigma) = 2$

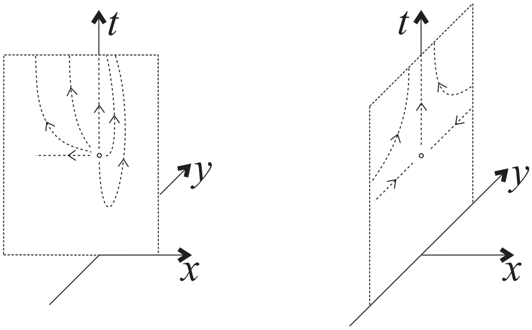

Fig. 20. Computation of the index of a singularity

we define:

$$v'(0, y, t) = \cos(\pi \cdot t) \cdot \frac{\partial}{\partial x} + \sin(-\pi \cdot y) \cdot \frac{\partial}{\partial y} + \sin(-2\pi \cdot t) \cdot \frac{\partial}{\partial t}.$$

For $\dim(\sigma) = 2$ we note that v' is already defined on $(\partial\sigma) \times [0, 1]$. We next define v' near $p_\sigma \times [0, 1]$, where p_σ is the centre of σ, as shown in Fig. 19. Now in $\sigma \times [0, 1]$, at all heights $t \in [0, 1]$, we extend v' radially from p_σ to $\partial\sigma$, using convex combinations. The fact that such a radial extension is indeed possible without introducing further singularities is a direct consequence of the previous choices, and the precise way the extension is made is actually immaterial.

The verification that indices of singularities of v' are as required is now a routine matter. We only do this in the hardest case, namely at $p_\sigma \times \{1/2\}$ for $\sigma \subset V$ and $\dim(\sigma) = 1$. Using the coordinates already introduced above and Figs. 18 and 19, we see that v' is a positive multiple of $\partial/\partial t$ near $(0, 0, 1/2)$ only at points of the form $(0, 0, 1/2 + t)$ for small $t > 0$. Moreover the field v' can be written as $v'(x, y, 1/2 + t) = (x, -y, t)$ for $t > 0$ (compare with the cross-sections shown in Fig. 20). Taking the normalized field $v'/\|v'\|$ and $(0, 0, 1)$ as a regular value, we readily see that the index is -1, as required. $\square$

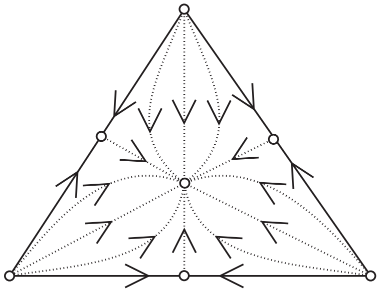

Fig. 21. The fundamental singular field $w_{\mathcal{S}}$ on a 2-simplex

Proof of Theorem 1.4. Our proof follows the scheme given by Turaev in [16], with some technical simplifications and some extra difficulties due to the tangency circles. We first recall that it is possible to associate to any smooth triangulation $\mathcal{S}$ of a manifold N a singular vector field $w_{\mathcal{S}}$ on N called the *fundamental* field of $\mathcal{S}$. This field has the property of having one singularity of index $\mathrm{ind}(\sigma) = (-1)^{\dim(\sigma)}$ at the barycentre of each simplex σ of $\mathcal{S}$. Qualitatively $w_{\mathcal{S}}$ can be defined by the requirements that: (1) each simplex is a union of orbits; (2) the singularities are exactly the barycentres of the simplices; (3) barycentres of higher dimensional simplices are more attractive that those of lower dimensional simplices. More precisely, each orbit (asymptotically) goes from a barycentre p_σ to a barycentre $p_{\sigma'}$, where $\sigma \subset \sigma'$. See Fig. 21 for a description of $w_{\mathcal{S}}$ on a 2-simplex of $\mathcal{S}$ and [16], page 642, for an explicit formula in barycentric coordinates.

Let us consider now a triangulation $\mathcal{T}$ of M, and let us choose a representative z of the given $\xi \in \mathrm{Eul}^{\mathrm{c}}(M, \mathcal{P})$ as in Proposition 1.3(3). We consider now the manifold M' obtained by attaching $\partial M \times [0, \infty)$ to M along $\partial M = \partial M \times \{0\}$. Note that $M' \cong \mathrm{Int}(M)$. Moreover $\mathcal{T}$ extends to a "triangulation" $\mathcal{T}'$ of M', where on $M \times [0, \infty)$ we have simplices with exactly one ideal vertex, obtained by taking cones over the simplices in ∂M and then removing the cone vertex. Even if $\mathcal{T}'$ is not strictly speaking a triangulation, the construction of $w_{\mathcal{T}'}$ makes sense, because the missing vertex at infinity would be a repulsive singularity anyway. We arrange things in such a way that if $\sigma \subset \partial M$ then the singularity in $\sigma \times (0, \infty)$ is at height 1, so it is $p_\sigma \times \{1\}$.

We will define now a smooth function $h : \partial M \to (0, \infty)$ and set $M_h = M \cup \{(x, t) \in \partial M \times [0, \infty) : t \le h(x)\}$, in such a way that $w_{\mathcal{T}'}$ is non-singular on ∂M_h, and, modulo the natural homeomorphism $M \cong M_h$, it induces on ∂M_h the desired boundary pattern $\mathcal{P}$. Later we will show how to use z to remove the singularities of $w_{\mathcal{T}'}$ on M_h.

To define the function h we consider a (very thin) left half-collar L of V on ∂M and a right half-collar R of C. Here "left" and "right" refer to the natural orientations of ∂M and of V and C. Note that $L \subset B$ and $R \subset W$. Now we set $h\big|_{B \setminus L} \equiv 1/2$, and $h\big|_{W \setminus R} \equiv 2$. Figures 22 and 23 respectively show that away from $V \cup C$ indeed the pattern of $w_{\mathcal{T}'}$ on ∂M_h is as required. Now we identify L to $V \times [-1, 0]$ and R to $C \times [0, 1]$, and we define $h(x, s) = f(s)$ for $(x, s) \in V \times [-1, 0]$ and

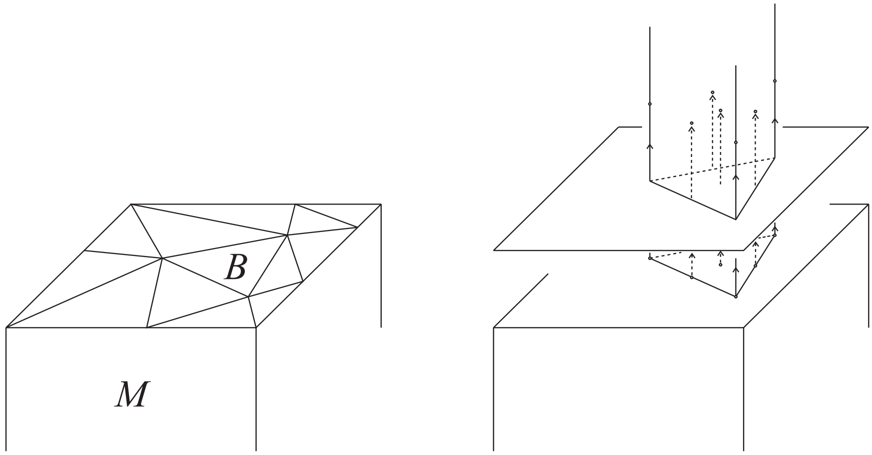

Fig. 22. Where $h = 1/2$ the field points outwards

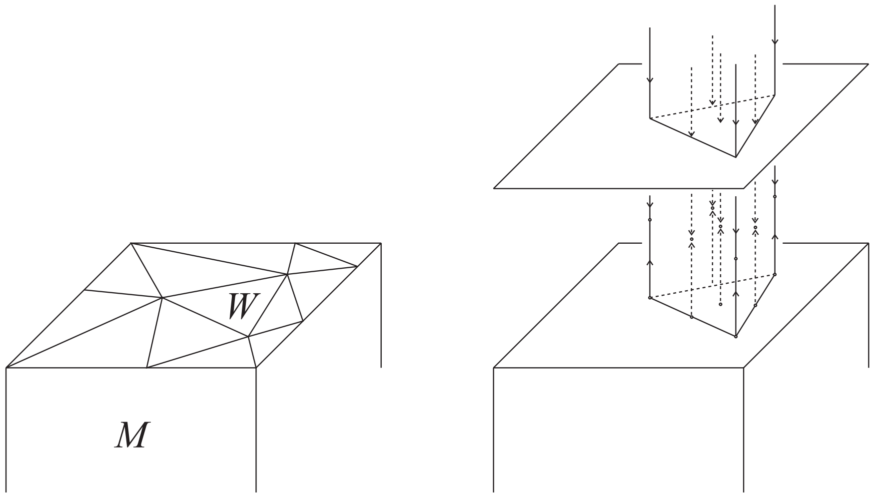

Fig. 23. Where $h = 2$ the field points inwards

$h(x, s) = f(s-1)$ for $(x, s) \in C \times [0, 1]$, where $f : [-1, 0] \to [1/2, 2]$ is a smooth monotonic function with all the derivatives vanishing at -1 and 0. Instead of describing f explicitly we picture it and show that also near $V \cup C$ the pattern is as required. This is done near V and C respectively in Figg. 24 and 25. In both pictures we have only considered a special configuration for the triangulation on ∂M, and we have refrained from picturing the orbits of the field in the 3-dimensional figure. Instead, we have separately shown the orbits on the vertical simplices on which the value of h changes.

The conclusion is now exactly as in Turaev's argument (Section 6.6 of [16]), so we only give a sketch. The chosen representative z of $\xi \in \mathrm{Eul}^{\mathrm{c}}(M, \mathcal{P})$ can be described as an integer linear combination of orbits of $w_{\mathcal{T}'}$, which we can describe

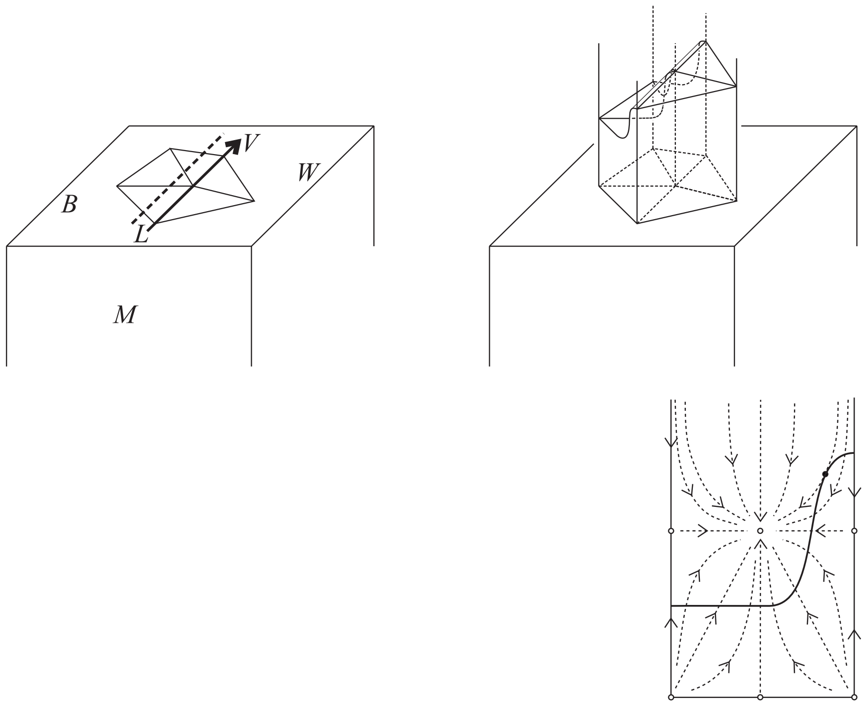

Fig. 24. On V the field has convex tangency

as segments $[p_\sigma, p_{\sigma'}]$ with $\sigma \subset \sigma'$. Now we consider the chain

$$z' = z - \sum_{\sigma \subset W \cup V} \mathrm{ind}(\sigma) \cdot p_\sigma \times [0,1]. \tag{10}$$

By definition of h we have that z' is a 1-chain in M_h, and $\partial z'$ consists exactly of the singularities of $w_{\mathcal{T}'}$ contained in M_h, each with its index. For each segment s which appears in z' we first modify $w_{\mathcal{T}'}$ to a field which is "constant" on a tube T around s, and then we modify the field again within T, in a way which depends on the coefficient of s in z'. The resulting field has the same singularities as $w_{\mathcal{T}'}$, but one checks that these singularities can be removed by a further modification supported within small balls centred at the singular points. We define $\Psi(\xi)$ to be the class in $\mathrm{Eul}^{\mathrm{s}}(M, \mathcal{P})$ of this final field. Turaev's proof that Ψ is indeed well-defined and $H_1(M; \mathbb{Z})$-equivariant applies without essential modifications. □

Remark 7.1. In the previous proof we have defined Ψ using triangulations, in order to apply directly Turaev's technical results (in particular, invariance under subdivision). However the geometric construction makes sense also for cellularizations $\mathcal{C}$ more general than triangulations, the key point being the possibility of defining a field $w_{\mathcal{C}}$ satisfying the same properties as the field defined for triangulations. This is certainly true, for instance, for cellularizations $\mathcal{C}$ of M induced by realizations of

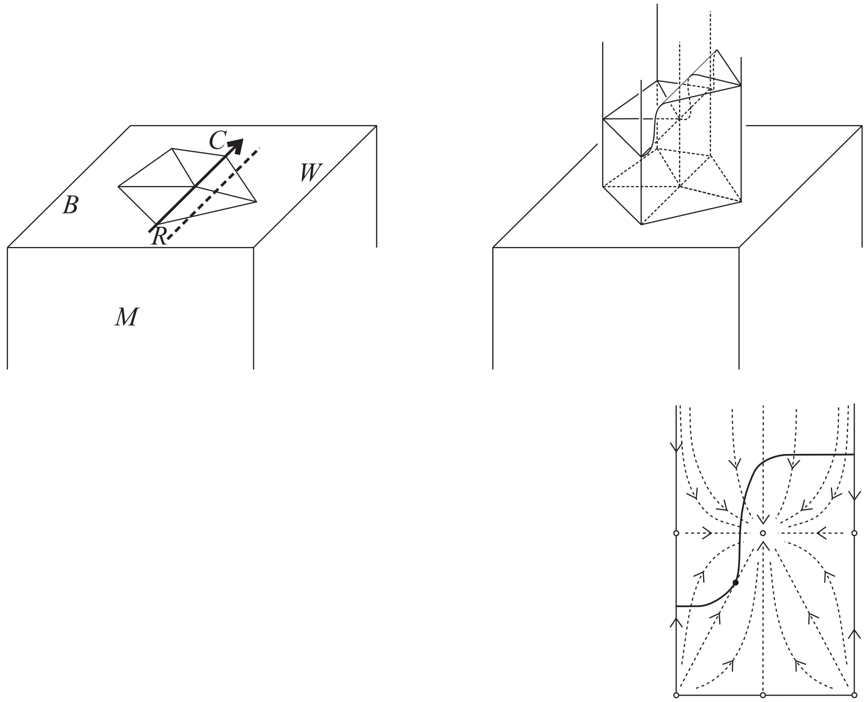

Fig. 25. On C the field has concave tangency

M by face-pairings on a finite number of polyhedra, assuming that the projection of each polyhedron to M is smooth.

Proof of Theorem 1.7. For the reader's convenience, we first outline the scheme of the proof:

1. By identifying M to a collared copy of itself, we choose a representative z of the given $\xi \in \mathrm{Eul}^{\mathrm{c}}(M, \mathcal{P})$ such that the extra terms added to define $\Theta^{\mathrm{c}}(\xi)$ cancel with terms already appearing in z.
2. We apply Remark 7.1 and choose a cellularization of M in which it is particularly easy to construct $\Psi(\xi)$ and $\Psi(\Theta^{\mathrm{c}}(\xi))$ using the representatives obtained above, and to show that $\Theta^{\mathrm{s}}(\Psi(\xi)) = \Psi(\Theta^{\mathrm{c}}(\xi))$.

We consider a cellularization $\mathcal{C}$ of M satisfying the same assumptions on ∂M as those considered in the proof of Proposition 1.1, namely $C \cup V$ is surrounded on both sides by a row of rectangular tiles. We denote by $\gamma_1, \ldots, \gamma_n$ the arcs in C, oriented as C.

Let us consider a representative z relative to $\mathcal{C}$ of the given $\xi \in \mathrm{Eul}^{\mathrm{c}}(M, \mathcal{P})$. We construct a new copy M_1 of M by attaching $\partial M \times [-1, 0]$ to M along $\partial M = \partial M \times \{-1\}$, and we extend $\mathcal{C}$ to $\mathcal{C}_1$ by taking the product cellularization on $\partial M \times [-1, 0]$.

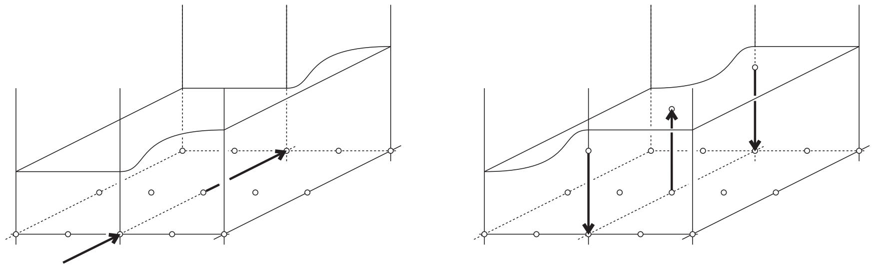

Fig. 26. Local difference near C between z' (left) and z'_θ (right)

We define a new chain as

$$z_1 = z + \sum_{\sigma \subset B} \mathrm{ind}(\sigma) \cdot p_\sigma \times [-1/2, 0] - \sum_{\sigma \subset W \cup V} \mathrm{ind}(\sigma) \cdot p_\sigma \times [-1, -1/2] + \sum_{j=1}^{n} \Big(\gamma_j\big|_{[1/2,1]} \times \{-1/2\} - \gamma_j\big|_{[1/2,1]} \times \{0\} \Big).$$

Note that z_1 is an Euler chain in M_1 with respect to $\mathcal{C}_1$. Consider the natural homeomorphism $f : M \to M_1$ and the class

$$a = \alpha^{\mathrm{c}}(f_*(\xi), [z_1]) \in H_1(M_1; \mathbb{Z})$$

which may or not be zero. Since the inclusion of M into M_1 is an isomorphism at the H_1-level, a can be represented by a 1-chain in M, so z_1 can be replaced by a new Euler chain z_2 such that $[z_2] = f_*(\xi)$ and z_2 differs from z_1 only on M.

Renaming M_1 by M and z_2 by z we have found a representative z of ξ such that $z = z_\theta + \sum_{j=1}^{n} \gamma_j\big|_{[1/2,1]}$, where z_θ is a sum of simplices contained in $B \cup \mathrm{Int}\, M$. Note that of course $\Theta^{\mathrm{c}}(\xi) = [z_\theta]$. To conclude the proof we will now apply the reconstruction map using z and z_θ, thus getting $\Psi(\xi)$ and $\Psi(\Theta^{\mathrm{c}}(\xi))$, and then we will analyze the smooth convexification $\Theta^{\mathrm{s}}(\Psi(\xi))$ to show that it actually coincides with $\Psi(\Theta^{\mathrm{c}}(\xi))$. By construction $\Theta^{\mathrm{s}}(\Psi(\xi))$ and $\Psi(\Theta^{\mathrm{c}}(\xi))$ only differ near C, so we concentrate on one component of C and show that the desired equality holds near it.

To understand $\Psi(\xi)$ and $\Psi(\Theta^{\mathrm{c}}(\xi))$ we follow the steps of the proof of Theorem 1.4 applied to z and z_θ respectively. The first step consists in choosing the height function h (respectively, h_θ) and replacing the chain z (respectively, z_θ) by a chain z' (respectively, z'_θ) as in formula (10). This is done in Fig. 26 where only the difference between the chains is shown.

The next step is to modify the fundamental field $w_{\mathcal{C}}$ of the cellularization on a neighbourhood of the support of z' and z'_θ, to get representatives of $\Psi(\xi)$ and $\Psi(\Theta^{\mathrm{c}}(\xi))$ on the modified versions of M called M_h and M_{h_θ} respectively. This is done in Figs. 27 and 28 respectively, where, for the sake of simplicity, the field is only shown on $C \times [0, \infty)$, where the essential modification takes place.

To conclude our description of $\Psi(\xi)$ and $\Psi(\Theta^{\mathrm{c}}(\xi))$ we must now bring the modified manifolds M_h and M_{h_θ} back to the original M. This is done in Figs. 29

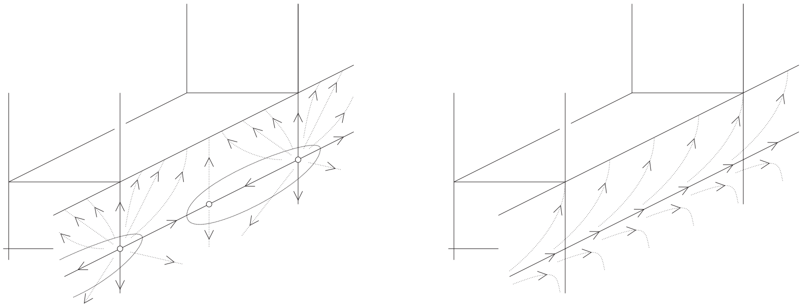

Fig. 27. Construction of $\Psi(\xi)$ on $C \times [0, \infty)$. On the left we show $w_{\mathcal{C}}$ and the zones where it must be modified, on the right we show the desingularized field.

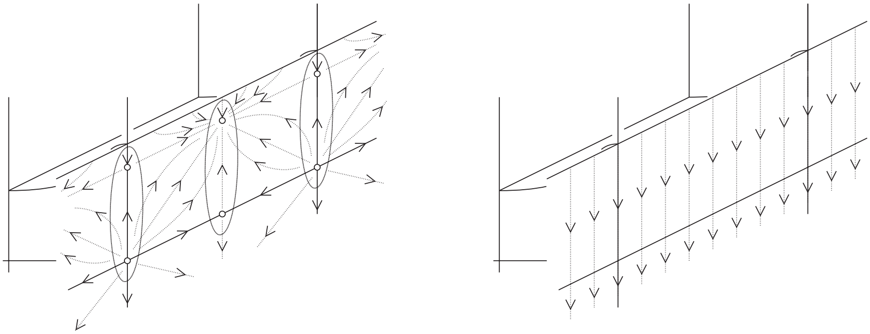

Fig. 28. Construction of $\Psi(\Theta^{\mathrm{c}}(\xi))$ on $C \times [0, \infty)$

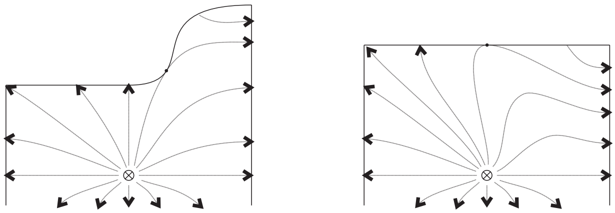

Fig. 29. A cross-section of a representative of $\Psi(\xi)$.

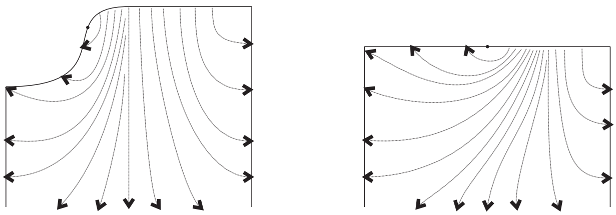

Fig. 30. A cross-section of a representative of $\Psi(\Theta^{\mathrm{c}}(\xi))$.

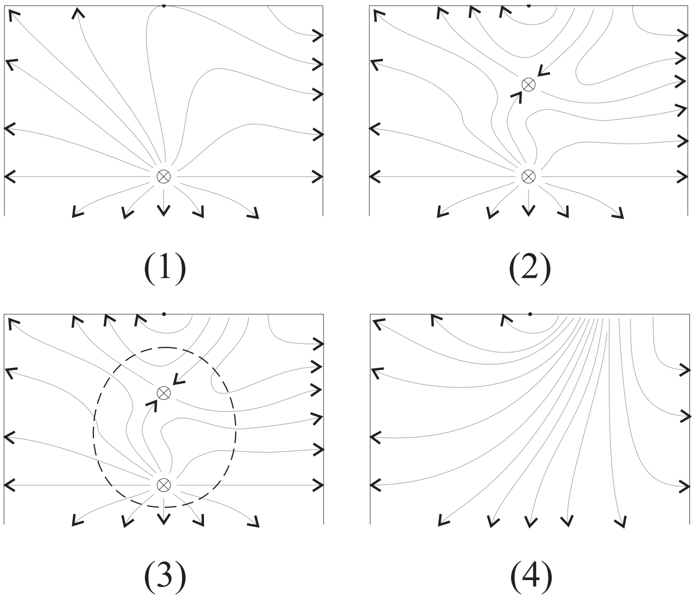

Fig. 31. Smooth convexification and homotopy with the smoothening of the combinatorial convexification

and 30 respectively, where a cross-section transversal to C is shown; in Fig. 30 the field is parallel to the cross-section, *i.e.* its C-component vanishes, and the same holds in Fig. 29 except near the apparent singularity, where the field has a positive C-component.

Now that $\Psi(\xi)$ and $\Psi(\Theta^{c}(\xi))$ have been described completely, we can construct $\Theta^{s}(\Psi(\xi))$ and show it equals $\Psi(\Theta^{c}(\xi))$. This is done in Fig. 31, which shows: (1) The representative of $\Psi(\xi)$ described above; (2) The representative of $\Theta^{s}(\Psi(\xi))$ constructed as in Fig. 2; (3) An alternative representative of $\Theta^{s}(\Psi(\xi))$, obtained by adding a positive C-component to the field in the whole region encircled by a thick dashed line; (4) The representative of $\Psi(\Theta^{c}(\xi))$ obtained above. The fields shown in (3) and (4) are nowhere opposite to each other, so they are homotopic, and the proof is complete. $\square$

Proof of Theorem 3.7. We fix P and set $s'' = s''(P), \widehat{v} = \widehat{v}(P)$. Using Remark 7.1 we see that the construction of $\Psi([s''])$ explained in the proof of Theorem 1.4 can be directly applied to the cellularization $\widehat{\mathcal{T}} = \widehat{\mathcal{T}}(P)$ of $\widehat{M}$. Recall that this construction requires identifying $\widehat{M}$ to a collared copy of itself, and extending s'' to a chain s''' whose boundary consists precisely of the singularities of the fundamental field $w = w_{\widehat{\mathcal{T}}}$ of the cellularization $\widehat{\mathcal{T}}$. (Here s'' plays the role of z in the proof of Theorem 1.4, and s''' plays the role of z'.) A representative of $\Psi([s''])$ is then obtained by applying to w a certain desingularization procedure. This desingularization is supported in a neighborhood of s''', and one can easily check that each connected component of the support of s''' is actually contractible, so its regular neighbourhood is a

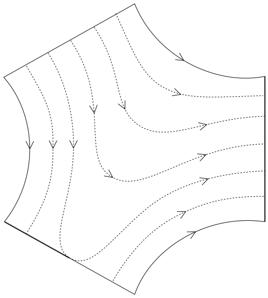

Fig. 32. The field $\widehat{v}$ on a hexagon

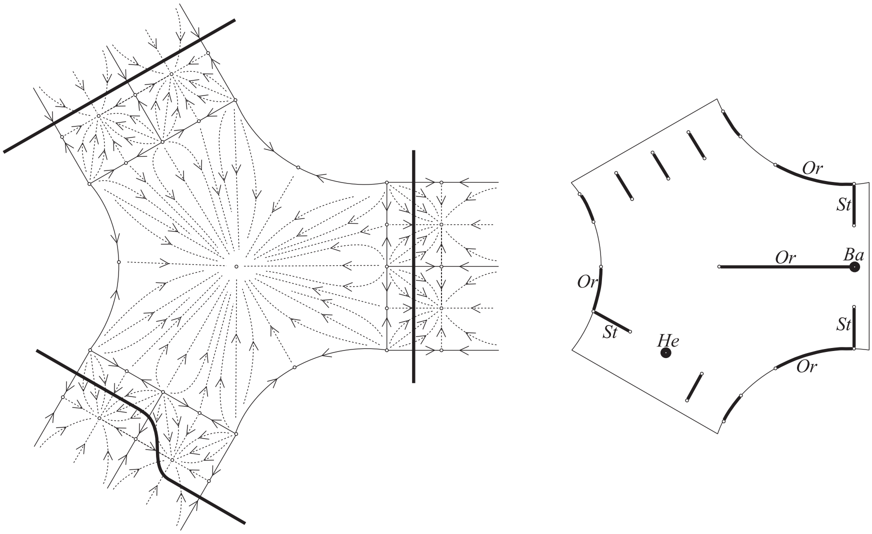

Fig. 33. The field w and the trace of S on a hexagon

ball. Since we already know that such a desingularization is indeed possible, and by definition an Euler structure is unaffected by a modification within a ball, the conclusion is readily deduced from the following claim: *the set of points where w is antipodal to $\widehat{v}$ is contained in the support of s'''*. We will show our claim neglecting the contraction of S^2_{triv} which maps M onto $\widehat{M}$. (The desired result actually holds at the level of M, and it easily implies the result for $\widehat{M}$.)

To prove the claim, we denote the support of s''' by S and note that the cells dual to those of P are unions of orbits of both w and $\widehat{v}$. Therefore we can analyze cells separately. We do this explicitly only for 2-dimensional cells, leaving to the reader the other cases. In Fig. 32 we describe $\widehat{v}$. In the left-hand side of Fig. 33 we describe w on the collared hexagon. In the right-hand side of the same figure we only show the singularities of w on the hexagon brought back to its original position, and the intersection of S with the hexagon. In this figure the 7 short segments come from $s''' - s''$; the other bits of S have been labeled by 'Or', 'St', 'Ba' or 'He' to indicate that they come from orbits of $\widehat{v}$, stars, bi-arrows or half-edges. Since indeed w and $\widehat{v}$ are only antipodal on S, the proof is complete. □

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