Short course on definable groups: part I

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Outline

Structures

- A **structure** $\mathcal{M}$ is a non-empty set $M = \text{dom}(\mathcal{M})$ equipped with some functions, constants, and relations.
- Groups, rings, modules, ordered sets, boolean algebras, are examples of structures.
- Each structure has a language L, consisting of the "symbols" (or "names") of its functions, constants, and relations.
- The language of ordered rings consists of the symbols $\leq, +, \cdot, 0, 1$.
- Each symbols of L has a "type", speficying whether it has to be interpreted as a function, a relation or a constant, and its "arity" (numer of arguments).
- Given two structures M, N in the same language L, a morphisms from M to N is a function $f : M \to N$ which preserves the intepretation of the symbols of L.
- Sometimes it is convenient to consider many-sorted structures, with more than one domain and functions and relations between the various domains (for instance a valued field). For the moment we consider one-sorted structures.

Examples

Three of the most important structures in mathematics are:

1. $(\mathbb{Z}, +, \cdot)$, a "gödelian" structure;
2. $(\mathbb{C}, +, \cdot)$, a "stable" structure;
3. $(\mathbb{R}, <, +, \cdot)$, an "o-minimal" structure.

In the early foundational period logicians were mostly interested in gödelian structures, focusing on indecidability and incompleteness results.
The study of stable structures brought to light connections with algebraic geometry (e.g. [Hru96]).
O-minimal structures, have an order $<$ and a topology induced by the order. Real-algebraic and subanalytic geometry, as well as PL topology, fit into this context. They recently have found applications to number theory (e.g. [Wil04, PZ08, PW06]).
NIP structures encompass both the stable and the o-minimal structures. Keisler measure play a key role in their study.

Formulas

Given a language L, the L-formulas are expressions built up from:

1. the symbols of the language L (namely the names of the functions, constant and relations);
2. the equality sign;
3. variables and parenthesis;
4. the boolean connectives, and the quantifier $\forall x$ and $\exists x$

According to the following grammar:

- Term ::= variable | constant | function symbol applied to terms
- Formula ::= (Term = Term) | relation symbol applied to terms | (Formula $\wedge$Formula) | $\neg$Formula | $\forall x$Formula | etc.

Terms are generalizations of polynomials. Given a structure, they represent functions on the structure. Formulas represent statements about the structure and its elements. The variables range in the domain of the structure.

Definable sets

Given an L-structure $\mathcal{M}$, and an L-formula $\varphi(\bar{x})$ with free variables included in $\bar{x} = (x_1, \ldots, x_n)$, we write

$$\{\bar{a} \in M^n : \mathcal{M} \models \varphi(\bar{a})\},$$

for the set of n-tuples from M satisfying the formula (also denoted $\varphi(M)$).

- A $\emptyset$-definable set in $\mathcal{M}$ is a set of the form $\{\bar{a} \in M^n : \mathcal{M} \models \varphi(\bar{a})\}$ for some L-formula $\varphi(\bar{x})$;
- A definable set in $\mathcal{M}$ is a set of the form $\{\bar{a} \in M^n : \mathcal{M} \models \varphi(\bar{a}, \bar{b})\}$ for some L-formula $\varphi(\bar{x}, \bar{y})$ and parameters $\bar{b}$ from M.
- If the parameters $\bar{b}$ belong to a subset B of M we say that the set is B-definable (so "definable" means "M-definable"). We consider $\varphi(\bar{x}, \bar{b})$ as a formula with parameters from $\bar{b}$, or "over B".
- Formulas with no free variables are called sentences. They are either true or false in $\mathcal{M}$.

Example

In the structure $(\mathbb{N}; +, \cdot, 0, 1)$ the set P of primes is definable:

$$n \in P \iff \mathbb{N} \models \forall x, y(x \cdot y = n \to x = 1 \lor y = 1).$$

The factorial function is also definable in the same structure. Indeed, by Gödel's theorems, every computable function is definable in $(\mathbb{N}; +, \cdot, 0, 1)$, as well as many non-computable ones.

Adding constants to the language

1. In $(\mathbb{R}, +, \cdot)$ the non-negative elements are $\emptyset$-definable: $x \geq 0$ iff $\exists y(y^2 = x)$.
2. A circle of radious r is definable with parameter $r \in \mathbb{R}$ (by the formula $x^2 + y^2 = r^2$).
3. If r is real algebraic, the circle of radious r is $\emptyset$-definable. ✎
4. The positive elements are *not* definable in $(\mathbb{R}, +, 0)$. ✎

Given a subset $A \subseteq M$, we can turn A-definable subsets of M^n into $\emptyset$-definable sets by working in a bigger language $L(A) \supseteq L$ obtained by adding constants for the elements of A, and considering $\mathcal{M}$ as an $L(A)$-structure. Formally we should use a different notation, so we denote by $\mathcal{M}_A$, or $(\mathcal{M}, a)_{a \in A}$, the expansion of $\mathcal{M}$ to the bigger language.

Theories and elementary equivalence

- A *L-theory* T is a collection of L-sentences, called the axioms of T.
- By the compactness theorem, if every finite $T_0 \subseteq T$ has a model, then T has a model.
- We write $T \vdash \varphi$ if φ is true in all the models of T.
- Again by compactness, $T \vdash \varphi$ iff $T_0 \vdash \varphi$ for some finite $T_0 \subseteq T$.
- This has non-trivial consequences such as: if a sentence φ in the language of rings is true in all fields of characteristic zero, then it is true in all fields of sufficiently big finite characteristic.
- An L-theory T is complete if for every L-sentence φ, either $T \vdash \varphi$ or $T \vdash \neg\varphi$.
- The complete theory of $\mathcal{M}$, written $Th(\mathcal{M})$, is the set of all L-sentences true in $\mathcal{M}$.
- $\mathcal{M}$ and $\mathcal{N}$ are elementarily equivalent, written $\mathcal{M} \equiv \mathcal{N}$, if $Th(\mathcal{M}) = Th(\mathcal{N})$.

Morphisms

- Given two structures $\mathcal{M}, \mathcal{N}$ in the same language L, a morphism from $\mathcal{M}$ to $\mathcal{N}$ is a function $f : M \to N$ which preserves the intepretation of the symbols of L (e.g. the ring morphims $\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$).
- An isomorphism is an invertible morphism (whose inverse is a morphism).
- A morphism is an embedding if is an isomorphism towards its image.
- $\mathcal{M}$ is a substructure of $\mathcal{N}$ if $M \subseteq N$ and the inclusion map is an embedding. .
- An elementary embedding is an embedding $f : \mathcal{M} \to \mathcal{N}$ such that, for any L-formula $\varphi(x_1, \dots, x_n)$ and $a_1, \dots, a_n \in M$, $\mathcal{M} \models \varphi(a_1, \dots, a_n)$ iff $\mathcal{N} \models \varphi(a_1, \dots, a_n)$. Taking $n = 0$, this implies $Th(\mathcal{M}) = Th(\mathcal{N})$.
- $\mathcal{M}$ is an elementary substructure of $\mathcal{N}$, written $\mathcal{M} \preceq \mathcal{N}$, if $M \subseteq N$ and the inclusion map is an elementary embedding.

1. $\mathbb{Q}$ is a substructure of $\mathbb{R}$ but it is not an elementary substructure ✎.
2. A structure $\mathcal{M}$ can have a substructure $\mathcal{N} \subseteq \mathcal{M}$ isomorphic to itself (hence elementary equivalent) which is not an elementary substructure ✎.

Model completeness

- A theory is model complete if every embedding among models of T is an elementary embedding.
- Equivalently, every formula is equivalent in T to an existential formula.
- By Tarski's elimination of quantifiers, the complete theories of $(\mathbb{C}, +, \cdot, 0, 1)$ and $(\mathbb{R}, <, +, \cdot, 0, 1)$ are model complete.
- Model completeness allows to "transfer" first-order information (given by L-formulas with parameters) from one model to another. For instance if a system of polynomial equations with coefficients in $\mathbb{C}$ has a solution in some algebrically closed field $K \supset \mathbb{C}$, then it has a solution in $\mathbb{C}$.

Macintyre's article in [Bar77] contains more information and applications of model-completeness.

Types

Fix a L-structure $\mathcal{M}$ and a subset $A \subseteq M$. Let $p(\bar{x})$ be a collection of $L(A)$-formulas $\varphi(\bar{x})$ with free variables included in $\bar{x}$.

- We say that $p(\bar{x})$ is a type of $Th(\mathcal{M}_A)$ if it is finitely satisfiable in M, namely for every finite set $\{\varphi_1(\bar{x}), \dots, \varphi_n(\bar{x})\} \subseteq p(\bar{x})$ there is a tuple $\bar{b}$ from M such that $M \models \bigwedge_{i \leq n} \varphi_i(\bar{b})$.
- (More generally $p(\bar{x})$ is a type of a theory T if $p(\bar{x}) \cup T$ has a model.)
- For instance if $\mathcal{M} = (\mathbb{R}, <, +, \cdot)$ we may consider the type $p(x) = \{x > n \,:\, n \in \mathbb{N}\}$.
- By the compactness theorem, given a type $p(\bar{x})$ of $Th(\mathcal{M}_A)$ there is $\mathcal{N} \succeq \mathcal{M}$ and $\bar{b}$ in N such that $N \models p(\bar{b})$, namely $N \models \varphi(\bar{b})$ for all $\varphi(\bar{x}) \in p(\bar{x})$. We say that $\bar{b}$ realizes $p(\bar{x})$.
- For instance, there is an elementary extension $M \succeq \mathbb{R}$ with an element $a \in M$ bigger than any natural number. Its inverse $1/a$ is infinitesimal. We can re-interpret dy/dx in terms of infinitesimals as in Robinson's "non-standard analysis".

Complete types

- A type $p(\bar{x})$ of $Th(\mathcal{M}_A)$ is complete if for every $L(A)$-formula $\varphi(\bar{x})$, either $\varphi(\bar{x})$ or $\neg\varphi(\bar{x})$ belongs to $p(\bar{x})$.
- If $\bar{b}$ is a tuple from M, the type of $\bar{b}$ over A is the collection $tp_{\bar{x}}(\bar{b}/A)$ of all $L(A)$-formulas $\varphi(\bar{x})$ such that $\mathcal{M} \models \varphi(\bar{b})$. Clearly $tp_{\bar{x}}(\bar{b}/A)$ is a complete type.
- In $(\mathbb{R}, <, +)$ there are three types of elements over $\emptyset$: positive, negative, zero ✎.
- In $(\mathbb{R}, <, +, \cdot)$ every element has a different type over $\emptyset$ ✎.
- If $\bar{x} = (x_1, \ldots, x_n)$, the family of complete types $p(\bar{x})$ of $Th(\mathcal{M}_A)$ is denoted $S_{\bar{x}}(A)$ or $S_n(A)$ (We omit $\mathcal{M}$ from the notation since if $\mathcal{N} \succeq \mathcal{M}$, then $Th(\mathcal{M}_A) = Th(\mathcal{N}_{\mathcal{A}})$.)
- Each complete type of $Th(\mathcal{M}_A)$ is the type of some tuple in some elementary extension $\mathcal{N} \succeq \mathcal{M}$.

Types and ideals

Complete types can be see as a generalization of prime ideals.

- Given an algebraically closed field K, there is a bijective correspondence between the prime ideals of $K[\bar{x}]$ and the complete types $p(\bar{x}) \in S_{\bar{x}}(K)$. If $p(\bar{x}) \in S_{\bar{x}}(K)$, the prime ideal $I_p \in Spec(K[\bar{x}])$ associated to $p(\bar{x})$ consists of all the polynomials $f \in K[\bar{x}]$ such that the formula "$f(\bar{x}) = 0$" belongs to $p(\bar{x})$ ✎.
- The unique complete type containing all the formulas of the form $f(\bar{x}) \neq 0$ corresponds to the zero ideal.

The article of Marker in [MMP96] contains more information.

Saturation

Definition

If κ is an infinite cardinal, a structure $\mathcal{M}$ is κ-saturated if every type with $< \kappa$ parameters from M is realized in M. If κ is the cardinality of M we say that $\mathcal{M}$ is saturated.

- The field of real numbers is not saturated. Indeed the type containing all the formulas $x > n$ with $n \in \mathbb{N}$ is not realized in $\mathbb{R}$.
- The field of complex numbers is saturated.

Theorem

Given κ, every structure $\mathcal{M}$ has a κ-saturated elementary extension.

The question whether one can find a κ-saturated elementary extension of cardinality κ involves set-theoretic subtlelties (one needs the generalized continuum hypothesis or some stability assumptions). For our purposes it is harmless to ignore these difficulties and pretend that saturated extensions always exist.

Galois theoretic intepretation

A definable set X in $\mathcal{M}$ can be seen as a "functor" which associate to each $\mathcal{N} \succeq \mathcal{M}$ a definable set $X(N)$ in $\mathcal{N}$ (= the set defined by the same formula). We write X instead of $X(N)$ when $\mathcal{N}$ is clear from the context or irrelevant.

Fact

- *Two n-tuples $\bar{b}, \bar{c}$ of $\mathcal{M}$ have the same type over $A \subseteq M$ iff there is an elementary extension $\mathcal{N} \succeq \mathcal{M}$ and an automorphism of $\mathcal{N}$ fixing A pointwise and taking $\bar{b}$ to $\bar{c}$.*
- *Let $A \subseteq M$. Suppose that X is definable in $\mathcal{M}$. Then X is definable over A iff for every $\mathcal{N} \succeq \mathcal{M}$, $X(N)$ is setwise fixed by any automorphism of $\mathcal{N}$ fixing A pointwise. (We say that X is A-invariant.)*

- The set of positive elements is not definable in $(\mathbb{R}, +)$ since $x \mapsto -x$ is an automorphism of $(\mathbb{R}, +)$ which takes positive to negative elements.
- The set of even numbers is not definable in $(\mathbb{N}, 0, \mathrm{succ})$ ✎(hint: reason by contradiction and go to an elementary extension).

Tame and wild structures

Loosely speaking a structure is "tame" if its definable sets are not too complicated. A necessary condition is that the ring of integers is not definable.

Examples

1. Let $M = (\mathbb{C}, +, \cdot)$. A set $X \subseteq M^n$ is definable iff it is constructible namely a boolean combination of affine algebraic varieties.
2. Let $M = (\mathbb{R}, <, +, \cdot)$. A set $X \subseteq M^n$ is definable iff it is semialgebraic, namely a boolean combination of sets defined by polynomial inequalities $p(x_1, \ldots, x_n) \geq 0$.
3. Let $M = (\mathbb{R}, <, +, \cdot, \sin(x))$. The definable sets are very complicated, for instance one can define the Mandelbrot set or the Peano curve

In the last structure one can define the ring of integers: $n \in \mathbb{Z}$ iff $\sin(n\pi) = 0$. One can also find a definition without parameters ✎.

Strongly minimal structures

Definition

An structure M is minimal if every definable subset of M is finite or cofinite. We say that M is strongly miminal if all the structures elementary equivalent to M are minimal.

Example

By Tarski's elimination of quantifiers, the field $(\mathbb{C}, +, \cdot)$ is strongly-minimal.

There is a vast literature on strongly minimal structures and more generally on "stable" structures. Moreover, there are various monographs on stable groups of finite Morley rank [BN94]. In this lectures I will concentrate on groups definable in a different kind of structures, the "o-minimal" ones.

O-minimal structures

Definition

An ordered structure $M = (M, <, ...)$ is o-minimal if every definable subset of M is a finite union of points and intervals (a, b) with $a, b \in M \bigcup \{\pm\infty\}$.

Example

By Tarski's elimination of quantifiers, the ordered field $(\mathbb{R}, <, +, \cdot)$ is o-minimal (so the subset $\mathbb{Z}$ is not definable).

Remark

If M is o-minimal, any definable set $X \subseteq M$ has a sup in $M \cup \{+\infty\}$. Moreover X is either finite or it contains a non-trivial interval.

Examples

The following are o-minimal

- $(\mathbb{R}, <, +, \cdot)$;
- Any real closed field $\mathcal{M} = (M, <, +, \cdot)$;
- $(\mathbb{R}, <, +, \cdot, \exp)$ [Wil96];
- $(\mathbb{R}, <, +, \cdot, \exp, f)_{f \in an}$ where *an* is the collection of all the real analytic functions restricted to a compact box $[a, b]^n \subset \mathbb{R}^n$ [vdDM94, vdDM95]

$(\mathbb{Q}; <, +, \cdot)$ is ***not*** o-minimal (the integers are definable [Rob49]).

Definably complete exponential fields

Definition

A definably complete exponential field $\mathcal{M} = (M, <, +, \cdot, \exp)$ is an ordered field with a differentiable $\exp : M \to M$ satisfying $\exp(0) = 1$ and $\exp'(x) = \exp(x)$ and such that every definable set has a sup in $M \cup \{+\infty\}$.

Theorem (see [BS04, FS10, FS12, Hie11])

Every definably complete exponential field is o-minimal.

The theory of definably complete exponential fields is not known to be complete. If it were, the theory of $\mathbb{R}_{\exp}$ would be recursively axiomatizable, hence decidable (a major open problem of Tarski).

Topology

Definition

Let $\mathcal{M} = (M, <, \dots)$ be o-minimal. Put on M the topology generated by the open intervals (a, b) and on M^n the product topology.

When $M \neq \mathbb{R}$ this topology is rather bad:

- intervals $[a, b]$ are neither connected nor compact;
- M^2 can be homeomorphic to M.

However:

- intervals are definably connected: they cannot be written as the union of two definable non-empty open subsets X.
- there is no definable bijection from M^2 to M.

Piecewise monotonicity and uniform bounds

Assume $\mathcal{M}$ o-minimal.

Theorem ([PS86], see also [vdD98])

If $f : (a, b) \to M$ is definable, there are $a = a_0 < a_1 < \cdots < a_N = b$ such that, for every i, the restriction of f to (a_i, a_{i+1}) is constant, or strictly increasing and continuous, or strictly decreasing and continuous.

Theorem ([KPS86], see also [vdD98])

If $f : X \to Y$ is definable, there is $k \in \mathbb{N}$ such that all the fibers of f of cardinality $> k$ are infinite.

The existence of uniform bounds implies that every structure elementary equivalent to an o-minimal one is o-minimal ✎. For instance, together with $\mathbb{R}$, we get all the real closed fields.

Cells

- A cell in M is an open interval (possibly unbounded) or a point.
- A cell in M^{k+1} is either:
  - the graph of a definable continuous function $f: C \to M$, where $C \subset M^k$ is a cell,
  - or the region $(f, g)_C = \{(x, y) \in C \times M \mid fx < y < gx\}$ bounded by two such functions. (We allow $f = -\infty$ or $g = +\infty$.)
- Any cell C is definably homeomorphic to an open subset of M^d for some d and we define $\dim(C) = d$.

Cell decomposition theorem

Theorem ([KPS86], see also [vdD98])

- *Any definable subset of M^k can be partitioned into cells*
- *Given a definable function $f : X \to Y$ there is a cell decomposition of X (definable over the same parameters) such that f is continuous on every cell of the partition.*

Corollary

Every definable function $f : X \to Y$ is continuous almost everywhere.

Proof.

The union of the cells of X of maximal dimension is open and dense in X. ✎ □

Outline

Algebraic closure, definable closure

Definition

Given a structure $\mathcal{M}$ and $A \subseteq M$ we say $b \in \mathrm{acl}(A)$ if b belongs to a finite A-definable set X. If X has only one element, we say $b \in \mathrm{dcl}(A)$.

Examples

- In $(\mathbb{C}, +, \cdot)$ we have $\sqrt{-1} \in \mathrm{acl}(\emptyset)$ witnessed by the $\emptyset$-definable finite set $\{x : x^2 = -1\}$ (note that 1 is $\emptyset$-definable).
- In general, given $a \in \mathbb{C}$, we have $\sqrt{a} \in \mathrm{acl}(a)$.
- In $(\mathbb{R}, +, \cdot)$, $\sqrt{2} \in \mathrm{dcl}(\emptyset)$.
- In general, in any ordered structure, $\mathrm{acl} = \mathrm{dcl}$ ✏.
- In $\mathbb{R}$ or $\mathbb{C}$ the field-theoretic algebraic closure coincides with the model-theoretic one ✏(use Tarski's quantifier elimination).
- In any structure, $b \in \mathrm{dcl}(\bar{a})$ iff there is a $\emptyset$-definable (partial) function f such that $b = f(\bar{a})$ ✏.

Steinitz exchange property

Definition

A structure $\mathcal{M}$ has the exchange property if for all $a, b \in M$ and $A \subseteq M$

$$b \in \mathrm{acl}(Aa) \quad \& \quad b \notin \mathrm{acl}(A) \implies a \in \mathrm{acl}(Ab),$$

where $Aa := A \cup \{a\}$. A complete theory T is pregeometric if every model of T has the exchange property. We say that $\mathcal{M}$ is pregeometric if $Th(\mathcal{M})$ is pregeometric.

Examples

- $(\mathbb{R}, +, \cdot)$ and $(\mathbb{C}, +, \cdot)$ are pregeometric (in fact "geometric", see below).
- $(\mathbb{Z}, |)$ does not have the exchange property ($x|y$ means "x divides y").
  ✎ Hint: $3 \in \mathrm{acl}(15)$, but $15 \notin \mathrm{acl}(3)$ because every permutation of the primes induces an automorphism of $(\mathbb{Z}, |)$.

O-minimal $\implies$ exchange property

Theorem ([PS86, Theorem 4.1])

O-minimal structures have the exchange property (i.e. they are pregeometric).

Proof.

- Suppose $b \in \mathrm{acl}(Aa)$ and $b \notin \mathrm{acl}(A)$. We need to show $a \in \mathrm{acl}(Ab)$.
- In ordered structures, *acl* coincides with dcl.
- So there is an A-definable (partial) $f : M \to M$ such that $b = f(a)$.
- Since $b \notin \mathrm{acl}(A)$, b lies in the interior of an open A-definable interval I on which f is striclty monotone.
- So $a = g(b)$ where g is the inverse of f in that interval.
- Hence $a \in \mathrm{acl}(Ab)$.

$\square$

Dimension of types, transcendence degree

Let $\mathcal{M}$ be pregeometric, let $A \subseteq M$ and let $\bar{a}$ be a tuple from some elementary extension of M.

Definitions

- $\dim(\bar{a}/A)$ is the least cardinality of a subtuple $\bar{a'}$ of $\bar{a}$ such that $\bar{a} \subseteq \mathrm{acl}(A\bar{a'})$.
- $\dim(\bar{a}/A)$ depends only on the type $p(\bar{x})$ of $\bar{a}$ over A.
- Given $p \in S_{\bar{x}}(A)$, define $\dim(p) := \dim(\bar{a}/A)$ where $\bar{a}$ is a realization of $p(\bar{x})$ (in some $N \succeq M$).
- A set $I \subseteq M$ is independent (over A) if, $\forall b \in I$, b does not belong to the algebraic closure of $I \setminus \{b\}$ (union A).

If $\mathcal{M}$ is an algebraically closed field, $\dim(\bar{a}/A)$ coincides with the transcendence degree of $\bar{a}$ over the subfield generated by A.

Properties

1. $\dim(\bar{a}/A)$ is the cardinality of any maximal independent (over A) subtuple of $\bar{a}$.
2. (monotonicity) If $A \subseteq B$, $\dim(\bar{a}/A) \geq \dim(\bar{a}/B)$;
3. (additivity) $\dim(\bar{a}\bar{b}/A) = \dim(\bar{a}/A\bar{b}) + \dim(\bar{b}/A)$;
4. (extension) If $p(\bar{x}) \in S_n(A)$ and $A \subseteq B$, then there is $p'(x) \in S_n(B)$ such that $p \subseteq p'$ and $\dim(p) = \dim(p')$.

Dimension of definable sets

Definition

Given a model M of a pregeometric theory and an A-definable set $X \subseteq M^n$, let

$$\dim(X) = \max\{\dim(a/A) : a \in X\}$$

where the tuple a may belong to an elementary extension $N \succeq M$ (i.e. $a \in X(N)$ for some $N \succeq M$).

1. $\dim(X)$ does not depend on the choice of A by the extension property ✎.
2. If M is ω-saturated, there is no need to go to an elementary extension.
3. If M is o-minimal, the definition of $\dim(X)$ agrees with the previous definition of dimension of a cell. In particular $\dim(X) \geq n$ iff there is a definable $f : X \to M^n$ whose image contains an open subset of M^n ✎.

Examples

Dimension of the circle

1. In $(\mathbb{R}, +, \cdot)$ the circle $x^2 + y^2 = 1$ has dimension 1 because you can find a point $(a, b) \in \mathbb{R}^2$ in the circle with a trancendental (so $\dim(ab/\emptyset) = 1$).
2. Consider the real algebraic numbers $\mathbb{R}^{alg} := \overline{\mathbb{Q}} \cap \mathbb{R}$. In this field there are no transcendental elements, but the circle has still dimension 1 (because you can find transcendental elements in elementary extensions).

Definition

Let X be a A-definable set. A point $a \in X$ is generic over A if $\dim(X) = \dim(a/A)$. (To find a generic point you may need to go to an elementary extension.)

Properties of dimension in pregeometric theories

In a model of a pregeometric theory T we have:

1. $\dim(X) = 0$ iff X is finite and non-empty. The empty set has dimension $-\infty$.
2. (Additivity) If $f : X \to Y$ is definable and all the fibers of f have constant dimension k, then $\dim(X) = \dim(Y) + k$.
3. (Monotonicity) $\dim(X \cup Y) = \max\{\dim(X), \dim(Y)\}$.

Geometric theories

We say that T is geometric if it is pregeometric and satisfies the following equivalent properties:

- (Definability of dimension) For every definable function $f : X \to Y$ and $k \in \mathbb{N}$, the set $\{y \in Y : \dim(f^{-1}(y)) = k\}$ is definable.
- (Uniform boundedness) for every L-formula $\varphi(x, \bar{y})$ there is $n \in \mathbb{N}$ such that, in every model, $\exists^{\infty} x \varphi(x, \bar{y}) \iff \exists^{\geq n} x \varphi(x, \bar{y})$, where $\exists^{\infty}$ means "there are infiniteley many".

A structure is geometric if its complete theory is geometric.

O-minimal structures are geometric (for instance real closed fields).
Strongly minimal structures are geometric (for instance algebraically closed fields).

Note that the theory of $(\mathbb{N}, <)$ does *not* have uniform bounds.

Outline

Definable groups

A definable group in $\mathcal{M}$ is a definable set $G \subseteq M^n$ with a definable group operation.

(Assume M has field operations)

- An algebraic subgroup of $GL(n, M)$, like for instance:

$$SO_2(M) = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} : a^2 + b^2 = 1 \right\}.$$

- An elliptic curve $y^2 = x^3 + ax + b$ in $\mathbb{P}^2(M)$ or $\mathbb{P}^2(M[\sqrt{-1}])$ (we can identify $\mathbb{P}^n(M)$ with a subset of M^{n+1} using charts).
- More generally, an abelian variety.
- Finally we observe that every compact real Lie group is definable in the o-minimal structure $\mathbb{R}_{an}$.

One-dimensional examples

Assume $M = (M, <, +, \cdot, \dots)$ expands a field. Besides $(M, +)$ and $(M^{\neq 0}, \cdot)$ we have the following one-dimensional examples of definable groups:

1. $SO(2, M)$.
2. $\mathbb{T} := [0, 1) \subseteq M$ with addition defined by

$$x + y \mod (1) = \begin{cases} x + y & \text{if } x + y < 1 \\ x + y - 1 & \text{if } x + y \geq 1 \end{cases}$$

3. For $a > 1$, the group $[1, a) \subseteq M$ with multiplication defined by

$$x \cdot y \mod (a) = \begin{cases} x \cdot y & \text{if } x \cdot y < a \\ x \cdot y / a & \text{if } x \cdot y \geq a \end{cases}$$

When $M = (\mathbb{R}, +, \cdot)$ these groups are isomorphic to $S^1 \cong \mathbb{R}/\mathbb{Z}$, but the isomorphism may not be definable in $L = \{+, \cdot\}$ (need $\sin, \cos, \exp$).

Elimination of imaginaries

Let M be a structure (one-sorted, for simplicity).

If G is a definable group in M and $H \triangleleft G$ is a definable subgroup, in general the quotient G/H is not definable in M (since its domain is not a subset of M^n).

One way to deal with this problem is to assume that $Th(M)$ has "elimination of imaginaries".

Definition

Given a complete theory T, we say that T has elimination of imaginaries if (in any model of T) for every definable equivalence relation E on a definable set X there is a definable set Y and a definable surjective function $f : X \to Y$ (over the same parameters) such that $xEy \iff f(x) = f(y)$. In this case we can identify X/E with Y and consider it as a definable object.

Imaginaries in o-minimal structures

- Assume $M = (M, <, +, \cdot, \dots)$ is an o-minimal expansion of a divisible group. Then $Th(M)$ is geometric and has elimination of imaginaries (because we can definably pick representatives equivalence classes, see [vdD98, p. 94]).
- Example: to pick a representative c from the interval (a, b), let $c := (a+b)/2$.
- Given a definable group G in M and a definable subgroup $H < G$ (not necessarily normal), we may then consider the coset space G/H as a definable set in M.
- By the addivitity of dimension,

$$\dim(G/H) = \dim(G) - \dim(H)$$

- Since the non-empty sets of dimension zero are exactly the finite sets, it follows that $[G : H]$ is finite $\iff \dim(H) = \dim(G)$.
- An arbitrary o-minimal structure M may not have elimination of imaginaries [Joh14].
- However any definable group G in M (with the induced structure from M) does have elimination of imaginaries [Edm03, Thm. 7.2],[EPR14], so G/H can always be considered as a definable set in M and the dimension formula remains valid.

M^{eq}

Given any structure M there is a related structure M^{eq} which eliminates imaginaries. Given a definable equivalence relation E on a definable set X the quotient X/E is definable in M^{eq} "by definition".

Definition of M^{eq}

- For each $\emptyset$-definable set X in M and each $\emptyset$-definable equivalence relation E on X, the structure M^{eq} has a sort S_E for elements ranging in X/E and a function symbol π_E for the projection $X \to X/E$.
- Taking for E the equality relation, we can identify the elements of the "home sort" M with the elements of sort $S_=$ of M^{eq}.
- The language L^{eq} of M^{eq} includes L and the various $\pi_E : S_= \to S_E$.

Intepreted structures

We say that a L'-structure N is interpretable in the L-structure M if N is **isomorphic** to a structure definable in M^{eq}.

Examples

1. We can interpret $(\mathbb{Z}, +)$ in $(\mathbb{N}, +)$ identifying an element of $\mathbb{Z}$ as a pair $(a, b) \in \mathbb{N} \times \mathbb{N}$ modulo the $\emptyset$-definable equivalence relation $(a, b)E(a', b') \iff a + b' = a' + b$. The idea is that $(a, b) \in \mathbb{N} \times \mathbb{N}$ represents $a - b \in \mathbb{Z}$.
2. Another example is the interpretation of the real numbers in the standard model of euclidean geometry (as axiomatized by Hilbert, say).
3. Finally, the whole of mathematics is interpretable in set theory!

If M is geometric, we can extend the dimension function $\dim(-)$ to M^{eq} [Gag05] (although M^{eq} is not geometric) so we can speak of the dimension of quotients X/E . If $Th(M)$ eliminates imaginaries there is no need to pass to M^{eq}.

The quantifiers "few" and "most"

In a geometric theory the following quantifiers are first order expressible:

- $(\text{Few } x \in X)\varphi(x) :\iff \dim(\{x \in X \ : \ \varphi(x)\}) < \dim(X)$;
- $(\text{Most } x \in X)\varphi(x) :\iff (\text{Few } x \in X)\neg\varphi(x)$.

Exercise ✐

$(\text{Most } x \in X)\varphi(x) \iff$ every generic point x of X satisfies $\varphi(x)$.

Large sets

Given $X \subseteq Y$ we say that X is large in Y if $\dim(Y \setminus X) < \dim(Y)$.

Theorem

Let M be a geometric structure and let $X \subseteq Y$ be M-definable sets. Suppose X is large in Y and Y defined over a model $M_0 \prec M$. Then $X(M) \cap Y(M_0) \neq \emptyset$.

Proof.

Suppose $Y(M) \subseteq M^n$ and argue by induction on n. Let $d = \dim(Y)$.

- If $d = 0$, then Y is finite and X coincides with Y, so assume $d > 0$.
- If $n = 1$ then $\dim(Y) = 1$ and $Y \setminus X$ is finite, so $X(M_0) \neq \emptyset$.
- Assume $n > 1$ and consider the projection $p : M^n \to M$.
- For most $m \in p(Y)$ the fiber $Y_m = Y \cap p^{-1}(m)$ must have dimension $d - 1$ and X_m must be large in Y_m.
- One of these m must lie in M_0, so the corresponding Y_m is defined over M_0 and we can apply the induction hypothesis.

$\square$

Covering a group by translates of a large set

Theorem ([Pil88])

Let G be a group definable in a geometric structure M and let X be a large definable subset of G. Then X is left-generic, namely finitely many left-translates of X cover G. Similarly for "right".

Proof.

Suppose G is defined over $M_0 \prec M$.

- By the previous result, every right tranlate Xg contains some $m \in G(M_0)$.
- Equivalently, every $g \in G$ is contained in a left-translate mX with $m \in G(M_0)$.
- By compactness finitely many left-translates mX cover G.

□

t-topology

Theorem ([Pil88])

Let $G \subseteq M^n$ be a definable group in an o-minimal structure M. Then G has a group topology, called the t-topology, which coincides with the topology induced by M^n on a large open subset V of G.

Proof.

- Let $Y \subseteq G \times G \times G$ be the set of points $(a, b, c) \in G \times G \times G$ such that $(x, y, z) \mapsto xyz \in G$ is continuous in a neighbourhood of (a, b, c). By o-minimality Y is large (and open) in G.
- Let V be the set of points $x \in G$ such that for most $(g_1, g_2) \in G \times G$ the triples (g_1, x, g_2) and $(g_1, g_1^{-1} x g_2^{-1}, g_2)$ belong to Y. Then V contains all generic points of G ✎, so it is large in G.
- Define $O \subseteq G$ to be t-open if for all $a, b \in G$ the subset $aOb \cap V$ is open in V.

$\square$

For a similar proof and the details see also [BM13, Lemma 9.7].

Subgroups of finite index

Theorem ([Pil88])

For groups $H < G$ definable in an o-minimal structure M, the following are equivalent:

1. $\dim(G) = \dim(H)$;
2. *H has finite index in G;*
3. *H is open in G (in the t-topology).*

Proof.

Since $\dim(G/H) = \dim(G) - \dim(H)$ we have $1 \iff 2$. Now, if H has finite index in G, then it has interior in the t-topology of G, and being a subgroup it is open in G. On the other hand if H has infinite index in G, then it has lower dimension, hence no interior. □

Definable subgroups are closed

Theorem ([Pil87, Prop. 2.7])

Let G be a definable group in an o-mimimal structure and $H < G$ a definable subgroup. Then H is closed in the t-topology.

Proof.

The closure $\overline{H}$ is definable group and H has full dimension in $\overline{H}$ (because in o-minimal structures $\dim(\overline{X} \setminus X) < \dim(X)$). So H is open in $\overline{H}$ and being a subgroup it is also closed in $\overline{H}$. $\square$

Example

The circle group $S^1 \cong \mathbb{R}/\mathbb{Z}$ is definable in $(\mathbb{R}, +, \cdot)$. There are dense sugroups of $S^1 \times S^1$ isomorphic to $\mathbb{R}$ (infinite spirals), but they are not closed, so they cannot be definable.

Descending chain condition on definable subgroups

Theorem ([Str94a, Thm. 2.6])

Let G be a definable group in an o-minimal structure M. Then G has finitely many definable subgroups H with $\dim(H) = \dim(G)$.

Proof.

Let $H < G$ with $\dim(H) = \dim(G)$. Then H is open in G, hence clopen. Let V be a large open subset of G where the t-topology coincides with the o-minimal topology. Decompose V into cells. Since H is clopen in the t-topology and cells are definably connected, every cell is contained in H or disjoint from H. So there are $\leq 2^k$ choices for $H \cap V$, where k is the number of cells of V. Now observe that $H = \overline{H} = \overline{H \cap V}$. □

Corollary (DCC on definable subgroups)

G is a definable group in an o-minimal structure, then G has no infinite descending chains of definable subgroup.

By contrast, $(\mathbb{Z}, +)$ does not have the DCC.

Connected component G^0

1. Given a definable group G in a structure M, we say that G is connected if it has no subgroups of finite index.
2. The connected component G^0 of G is the intersection of all definable subgroups of finite index.
3. In the o-minimal case there is a smallest such subgroup (by the DCC), so G^0 is definable.
4. It can be shown that G^0 coincides with the definable path-connected component in the t-topology.

Divisibility

Proposition ([Pil88])

If G is divisible, then G is definably connected.

Proof.

Consider the connected component $G^0 \lhd G$ and the morphism $G \to G/G^0$. Since G/G^0 is finite and divisible it must be trivial. $\square$

Definability of centralizers

- Another consequence of the DCC is that the intersection $\bigcap_{i \in I} H_i$ of a family of definable subgroups of a definable group G coincides with the intersection of a finite subfamily, and therefore it is definable.
- In particular, if $A \subseteq G(M)$ is a set of parameters (not necessarily definable), then the centralizer

$$C_G(A) = \{g \in G : (\forall a \in A)(ga = ag)\}$$

  is definable (because by the DCC it must coincide with the centralizer of a finite subset of A).
- Moreover, there is a smallest definable sugroup $\langle A \rangle$ containing A.

Existence of infinite abelian subgroups

Using the above tools one can prove the following:

Theorem ([Pil87, Prop. 5.6])

Let G be an infinite group definable in an o-minimal structure. Then G has an infinite definable abelian subgroup. Indeed, any infinite connected subgroup $H < G$ of minimal dimension is abelian.

Outline

Euler characteristic

O-minimal Euler characteristic:

Let $X \subseteq M^k$ be a definable set and consider a partition of X into cells. Define

$$E(X) := \Sigma_i (-1)^i \ \#\text{cells of dimension } i$$

(= the number of even dimensional cells minus the number of odd dimensinal cells).

For X closed and bounded, $E(X)$ is the o-minimal analogue of the classical Euler characteristic χ.
When X is not closed and bounded there are differences:

1. Classically $\chi((0,1)) = \chi([0,1]) = 1$ (because both spaces are contractible).
2. In the o-minimal case $E((0,1)) = -1$, while $E([0,1]) = E(pt) = 1$.

Properties of E(X)

Properties

1. $E(X) = \#X$ if X is finite,
2. $E(X \cup Y) = E(X) + E(Y)$ if the union is disjoint,
3. $E(X \times Y) = E(X) \cdot E(Y)$,
4. If $f : X \to Y$ is definable and $E(f^{-1}(y)) = m$ for each $y \in Y$, then $E(X) = E(\bigcup_{y \in Y} f^{-1}(y)) = E(Y) \cdot m$.
5. If $f : X \to Y$ is a definable bijection $E(X) = E(Y)$.

In 4 and 5 we do not require f to be continuous !

Computation of $E(S^1)$

We can compute $E(S^1)$ in two ways:

1. Write it as a union of two 0-cells and two 1-cells:

$E(S^1) = 1 + 1 + (-1) + (-1) = 0.$

2. Consider the fibers of $p : S^1 \to [0, 1]$:

the fibers over 0 and 1 are single points, the fibers over $(0, 1)$ consist of two points. So $E(S^1) = 1 + 1 + E((0, 1)) \cdot 2 = 0.$

Euler characteristic of groups

Theorem

If $H < G$ are definable groups, then $E(H)$ divides $E(G)$.

Proof.

$E(G) = E(H) \cdot E(G/H)$ (by definable choice quotients are definable). $\square$

Corollary

If $E(G) = \pm 1$, then G has no elements of finite order.

Example

No semialgebraic group structure on $\mathbb{R}$ or $\mathbb{R}^2$ can have torsion (because $E(\mathbb{R}) = -1$ and $E(\mathbb{R}^2) = 1$).

Elements of order p

Theorem ([Str94a])

Let G be a definable group and p a prime number. If p divides $E(X)$, then G has an element of order p.

Proof.

- Let $S := \{(a_1, \dots, a_p) \mid a_i \in G, \prod_i a_i = 1\}$;
- S is in (definable) bijection with G^{p-1};
- $p|E(G)$, so $E(S) = E(G^{p-1}) = E(G)^{p-1} \equiv 0 \mod p$;
- $\mathbb{Z}/p\mathbb{Z}$ acts on S by cyclic permutations;
- write $S = S_1 \sqcup S_p$ where:
- S_p is the union of the orbits of size p, so $E(S_p) \equiv 0 \mod p$;
- S_1 is the union of the orbits of size 1 and is in bijection with $G[p] := \{x \in G : x^p = 1\}$;
- Thus $0 \equiv E(S) \equiv E(S_1) + E(S_p) \equiv E(S_1) \equiv E(G[p]) \mod p$;
- Therefore $0 \equiv 1 + E(\text{elements of order p}) \mod p$.

$\square$

Groups with $E(G) = 0$

Corollary

If $E(G) = 0$, then G has elements of every prime order.

Thus for instance if the underlying set of G is a circle, then G has elements of every prime order.

p-groups

Theorem ([Str94a, 2.17, 2.21])

Let p be prime and p^k divide $E(G)$. Then:

1. *G has a subgroup of order p^k.*
2. *If $E(G) \neq 0$ there is a maximal such k and all the subgroups of order p^k are conjugated.*

Groups of bounded exponent

Theorem ([Str94a, 5.7])

Let G be a definable abelian group and $G[n] := \{x \in G : x^n = 1\}$. Then $G[n]$ is finite.

So for instance $(\mathbb{Z}/2\mathbb{Z})^{(\omega)}$ cannot be isomorphic to a definable group.

Proof.

Since $H := G[n]$ contains no elements of order $> n$, $E(H) \neq 0$. Write $E(H) = \pm 1 \cdot \prod_{i=1}^{k} p_i^{a_i}$ with p_i prime. Let F_i be a subgroup of H of order $p_i^{a_i}$. Then $E(\oplus_i F_i) = \prod_i p_i^{a_i}$ and therefore $E(H/\oplus_i F_i) = \pm 1$. It follows that $H/\oplus_i F_i$ has no elements of finite order. But H is torsion. So $H = \oplus_i F_i$. $\square$

By a reduction to the abelian case one can prove:

Theorem ([Str94a, 6.1])

Any definable group of bounded exponent is finite.

Torsion free groups

Theorem

A definable group G is torsion free if and only if $E(G) = \pm 1$.

Proof.

$E(G) \neq \pm 1 \iff E(G)$ is divisible by a prime $p \iff G$ contains an element of order p (for some p). $\square$

Outline

Definable compactness

A subset X of M^n is closed and bounded iff every definable curve $f : (0, \varepsilon) \to X$ has a limit in X (with the induced topology from M^n). This suggests the following:

Definition [PS99]

G is **definably compact** if every definable curve $f : (0, \varepsilon) \to G$ has a limit in G in the t-topology.

Note that when M has field operations, replacing G with a definably isomorphic copy, we can assume that the t-topology coincides with the induced topology from M^n (Robson's embedding theorem [vdD98]).

Question [PS99]

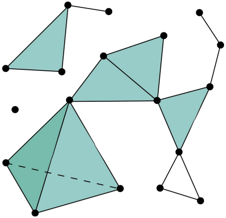
Let G be definably compact. Does G have torsion elements?

The efforts to answer the question led to the introduction of tools from algebraic topology in the o-minimal context. We shall prove that if G is definably compact and infinite, then $E(G) = 0$, and therefore, by [Str94a], G has torsion (indeed it has elements of every prime order).

Simplicial complexes

We modify the classical definition by allowing "open" simplexes.

- Given an **ordered field** M, an (open) n-simplex in M^k, with vertices $p_0, \dots, p_n \in M^k$, is the set of all M-linear combinations $\sum_i x_i p_i \in M^k$ with $0 < x_i < 1$ in M and $\sum x_i = 1$.



- A simplicial complex in M^k is a finite collection K of simplexes in M^k such that for for all σ_1, σ_2 in K either $cl(\sigma_1) \cap cl(\sigma_2)$ is empty or it is equal to $cl(\tau)$ for some common face τ of σ_1 and σ_2.
- Let $|K| \subseteq M^k$ be the union of all the simplexes of K.

We say that K is closed, if whenever it contains a simplex, it contains all its faces.

Triangulation theorem

Let $\mathcal{M} = (M, <, +, \cdot, \dots)$ be an o-minimal expansion of an ordered field (necessarily real closed).

Theorem [vdD98]

Every $\mathcal{M}$-definable set $X \subseteq M^n$ can be triangulated, namely there is a finite simplicial complex K and a definable homeomorphism $f : |K|(M) \to X$.

To deal with the case when X is not closed, we must allow "open simplexes", namely simplexes without some of the faces.

Definable homotopy

Consider an o-minimal structure M and fix two points "0" and "1" in M with $0 < 1$.

Definition

Two definable functions f_0, f_1 from X to Y are definably homotopic if there is a definable continuous map $F : [0, 1] \times X \to Y$ such that $f_0(x) = F(0, x)$ and $f_1(x) = F(1, x)$.

We say that X and Y are definably homotopy equivalent if if there are continuous maps $f : X \to Y$ and $g : Y \to X$ such that $g \circ f$ is definably homotopic to id_X and $f \circ g$ is definably homotopic to id_Y

Example

The figure "8" is (definably) homotopy equivalent to $\mathbb{R}^2$ minus two points.

O-minimal fundamental group

Work in an o-minimal expansion of an ordered field. In analogy with the classical case we define:

Fundamental group

Let X be a definable set with a fixed base point $x_0 \in X$. The o-minimal fundamental group $\pi_1(X, x_0)$ is the group of definable loops modulo definable homotopies, where a definable loop is a definable continuous maps $\gamma : [0,1] \to X$ with $\gamma(0) = \gamma(1) = x_0$. The group operation is concatenation of loops.

If X is definably connected, $\pi_1(X, x_0)$ does not depend on the choice of the base point, so we can write $\pi_1(X)$.

Example

Let S^1 be the circle. Then $\pi_1(S^1) \cong \mathbb{Z}$, where $[\gamma] \mapsto n$ if γ winds n times around the circle in the clockwise direction.

Properties

Work in an o-minimal expansion $\mathcal{M} = (M, <, +, \cdot, \dots)$ of an ordered field.

Theorem [BO02]

1. Every definable continuous map $f : X \to Y$ induces a group homomorphism $\pi_1(f) : \pi_1(X) \to \pi_1(Y)$.
2. Definably homotopic maps induce the same group homomorphism.
3. $\pi_1(X)$ is invariant under elementary extension $N \succeq M$, i.e. $\pi_1(X(N)) = \pi_1(X(M))$.
4. $\pi_1(X)$ is invariant under o-minimal expansions of the language. In particular, if X is semialgebric (definable in $L' = \{<, +, \cdot\} \subseteq L$), it suffices to consider semialgebraic loops.
5. $\pi_1(X)$ is finitely generated.

IDEA: triangulate $X \approx |K|(M)$ and show, using an o-minimal version of van Kampen theorem, that $\pi_1(X) \cong \pi_1(K) \cong \pi_1(|K|(\mathbb{R}))$ can be computed simplicially.

WARNING. The following classical argument fails: given $\varepsilon > 0$ there is a subdivision of the given triangulation where all simplexes have diameter $< \varepsilon$.

Reason: ε can be infinitesimal.

Homology groups

- One can define an o-minimal version of the singular homology groups $H_i(X)$ and prove their invariance under elementary extensions and o-minimal expansions of the language [EW08, BO03, BEO07].
- One adapts the classical definition by working with definable singular simplexes $\sigma : |\Delta|(M) \to X(M)$ in the given o-minimal structure M (expanding a field).
- Given a finite simplicial complex K (with vertices in $\mathbb{Q}$, say) we have

$$H_i(|K|(M)) = H_i(|K|(\mathbb{R})).$$

- By the triangulation theorem it follows that, for every definable set X, the group $H_i(X)$ is finitely generated.

Lefschetz fixed point theorem

Using the properties of the o-minimal homology functors H_i one can prove:

Theorem [BO03, EW09]

Let K be a finite **closed** simplicial complex of Euler characteristic different from zero and let $X = |K|$. Suppose that $X(M)$ is an orientable definable manifold. Let $f : X \to X$ be a definable continuous map definably homotopic to the identity. Then f has a fixed point.

The corresponding classical result holds without the assumption that X is orientable (in fact one does not even need the fact that X is a manifold). For our applications to definable groups the above version will suffice.

Example

We prove the classical result that every every element of $SO_3(\mathbb{R})$ corresponds to a rotation around some axis (under the natural action of $SO_3(\mathbb{R})$ on $\mathbb{R}^3$). Indeed an element of $SO_3(\mathbb{R})$ induces a self-map $f : S^2 \to S^2$ on the unit 2-sphere in $\mathbb{R}^3$ and since $E(S^2) \neq 0$ there is a fixed point $x = f(x) \in S^2$. Clearly f fixes the axis from the origin $0 \in \mathbb{R}^3$ to x.

Existence of torsion elements

Corollary (Edmundo, see survey [Ote08])

If G is infinite and definably compact, then $E(G) = 0$, so G has torsion.

We present the proof in [BO03].

Proof.

Since $E(G) = E(G/G^0)E(G^0)$ we can assume G connected (hence definably path-connected in the t-topology). It follows that if $1_G \neq g \in G$ the map $x \mapsto gx$ is definably homotopic to the identity. This map has no fixed points, so by the o-minimal Lefschetz fixed point theorem $E(G) = 0$ (using the fact that G is an orientable definable manfifold). $\square$

For simplicity we implicitly assumed that the t-topology coincides and the ambient topology, but if M expands a field we can reduce to this case by Robson's embedding theorem [vdD98].

Outline

0-groups and Strzebonksi tori

By the above results $E(G) = 0$ iff G has torsion elements of arbitrarily high order.

Definition

A zero-group is a definable group G such that $E(G/H) = 0$ for each proper definable subgroup H of G (not necessarily normal).
A Strzebonski torus is a zero-group G such that all its connected definable subgroups are 0-groups.

Theorem ([Str94a, 5.17])

Any zero-group is abelian and connected (i.e. has no definable subgroups of finite index).

A fundamental result in the theory of Lie groups is that every compact Lie group is covered by the conjugates of a maximal torus. Our next goal is to study an o-minimal version of this result.

A 0-group which is not a Strzebonski torus

Example ([Str94a, Ex. 5.3])

Let $M = (\mathbb{R}, +, \cdot)$ and define addition in $G = \mathbb{R} \times [1, e)$ by

$$(x, u) + (y, w) = \begin{cases} (x + y,\ u \cdot w & \text{if } u \cdot w < e \\ (x + y + 1,\ u \cdot w/e & \text{if } u \cdot w \geq e \end{cases}$$

Then G is a 0-group containing a connected subgroup which is not a 0-group. Indeed $\mathbb{R}$ is a subgroup of G and $E(\mathbb{R}) = -1$. (Note that $[1, e)$ is *not* a sugroup of G.)

The notion of zero-group is not very "robust", namely it is not invariant under expansions of the language (if we add exp the above group is not a zero group). By contrast, we shall see that the notion of Strzebonski torus is robust.

(Lack of) one-dimensional subgroups

Despite the similarites with Lie groups, a definable group may lack one-dimensional definable subgroups. In particular, unlike classical tori, a Strzebonski torus may not have one-dimensional definable subgroups [PS99] (examples could be abelian varieties). In the non-compact case things behave better:

Theorem ([PS99])

If G is not definably compact, it contains a torsion free definable subgroup.

Using this, we can give a "robust" topological characterization of Strzebonski tori.

Characterization of Strzebonski tori

Theorem (see [Ber08])

The following are equivalent

1. G is a Strzebonski torus.
2. G is abelian, connected, definably compact.

Proof.

Assume 2. By the Lefschetz fixed point theorem if G is a definably compact infinite group, then $E(G) = 0$. Let $H < G$ be a proper definable subgroup. Since G is abelian and connected, H is normal. To obtain 1 it suffices to observe that definable compactness is preserved under taking subgroups and quotient groups. . Assume 1. Then G is connected (if not $E(G/G^0) = \#G/G^0$ is finite non-zero) and by [Str94a, 5.17] it is also abelian. If it were not definably compact, it would contain a torsion free definable subgroup L [PS99]. But such subgroups have $E(L) = \pm 1$, contradicting the assumption. □

Maximal tori

Theorem ([Str94a, Cor. 5.19, Thm. 2.14])

Maximal Strzebonski tori of a definable group are conjugated. If $H < G$ is maximal Strzebonki torus, $E(G/H) \neq 0$.

Example

The maximal tori of $SO(3, \mathbb{R})$ (the group of rotations of $\mathbb{R}^3$) are the one-dimensional subgroups fixing an axis of rotation.

Union of the maximal tori

The following result was proved independently in [Edm05, Ber08]. I present the proof in [Ber08].

Theorem

If G is definably compact and connected, G is the union of its maximal tori.

Proof.

Let $g \in G$ and let $H < G$ be a maximal torus. Consider the map $L_g : G/H \to G/H$, $xH \mapsto gxH$. Since G is connected L_g is definably homotopic to the identity. By Lefschetz there is a fixed point $xH = gxH \in G/H$. But then $gx \in xH$ and $g \in xHx^{-1}$. □

Corollary

If G is definably compact, then G is divisible.

Proof.

By a reduction to the abelian case using the fact that G is the union of its maximal tori (see [Ote09] for a different proof). □

Outline

Counting the torsion points

Let G be Strzebonski torus (i.e. G is definably compact, abelian, definably connected.
We want to study the structure of the k-torsion subgroup $G[k]$.

- When $M = \mathbb{R}$, there is an (analytic) isomorphism $G(\mathbb{R}) \cong (\mathbb{R}/\mathbb{Z})^n$, so $G[k] \cong (\mathbb{Z}/k\mathbb{Z})^n$.
- If $\dim(G) = 1$, then $G[k] = \mathbb{Z}/k\mathbb{Z}$ [Raz91].
- If $\dim(G) > 1$, G may not have 1 dimensional definable subgroups [PS99], so we cannot reduce to the one-dimensional case.

The strategy in [EO04] is the following:

1. It can be shown that $x \mapsto kx$ is a covering map $G \to G$.
2. From the theory of covering spaces we have $G[k] \cong \pi_1(G)/k\pi_1(G)$.
3. So we must study $\pi_1(G)$.

Fundamental group of a Strzebonski torus

Theorem ([EO04])

Let G be a Strzebonski torus of dimension n. Then $\pi_1(G) \cong \mathbb{Z}^n$, and therefore $G[k] = (\mathbb{Z}/k\mathbb{Z})^n$.

Proof.

- We know that $\pi_1(G)$ is finitely generated [BO02, Cor. 2.10]
- Since G is abelian, the map $p_k : G \to G$, $x \mapsto kx$, is a homomorphisms.
- It is also a definable covering map, so it induces an injective homomorphism $p_{k*} : \pi_1(G) \to \pi_1(G)$, given by $[\gamma] \mapsto k[\gamma]$.
- Since this holds for every k, $\pi_1(G)$ is torsion free.
- Being also abelian and finitely generated, $\pi_1(G) \cong \mathbb{Z}^s$ for some s.
- The proof of $s = n$ is more difficult: it uses the study of $H^*(G; \mathbb{Q})$ as a graded Hopf algebra.

$\square$

Proof of s=n

1. We have $\pi_1(G) \cong H_1(G;\mathbb{Z}) \cong \mathbb{Z}^s$.
2. We can write $H^1(G,\mathbb{Q}) = \mathbb{Q}y_1 + \ldots + \mathbb{Q}y_s$ (direct sum).
3. Let $p_2 : G \to G$, $x \mapsto 2x$.
4. $H^*(G,\mathbb{Q}) = \Lambda[y_1,\ldots,y_s,\ldots,y_r]$ with $r \geq s$ and y_i "primitive", so that $p_2^*(y_i) = 2y_i$.
5. $\omega_G := \prod_{i\leq r} y_i$ is a generator of the top-cohomology $H^n(G;\mathbb{Q})$.
6. We have $p_2^*(\omega_G) = 2^r\omega_G$, hence $\deg(p_2) = 2^r$.
7. On the other hand $\deg(p_2)$ is bounded by $|\ker(p_2)| = |G[2]|$.
8. $G[2] \cong \pi_1(G)/2\pi_1(G) \cong (\mathbb{Z}/2\mathbb{Z})^s$.
9. So $r \leq s$. Hence $s = r = n$.

Outline

Higher homotopy groups

- O-minimal versions of the higher homotopy groups $\pi_n(X)$ are studied in [BO09, BMO10]
- A definable subgroup $H < G$ determines a definable fibration $G \to G/H$ and gives rise to a long exact sequence

$$\ldots \to \pi_{n+1}(G/H) \to \pi_n(H) \to \pi_n(G) \to \pi_n(G/H) \to \pi_{n-1}(H) \to \ldots$$

- The higher homotopy groups of a definable set need not be finitely generated.
- For instance let $X = S^1 \wedge S^2$. This is a circle with a 2-sphere tangent to it, and its universal cover $\widetilde{X}$ is a line with infinitely many 2-spheres tangent to it. It follows that $\pi_2(X) = \pi_2(\widetilde{X}) = \mathbb{Z}^{(\omega)}$ is not finitely generated (however it is finitely generated as a $\mathbb{Z}[\pi_1(X)]$-module).
- However we shall see that when G is a definable group, $\pi_n(G)$ is finitely generated.

$\pi_m(G)$ is finitely generated

Definition

A path connected space is simple if its fundamental group acts trivially on all homotopy groups.

Fact

1. *A path connected H-space is simple [Spa66, Ch. 7, Thm. 3.9].*
2. *(Serre 1953) If X is a simple space and $H_m(X)$ is finitely generated for all m, then $\pi_m(X)$ is finitely generated for all m [Whi78, Ch. 13, Cor. 7.14].*

Using this and some homotopy transfer results in [BO09] we obtain:

Theorem ([BMO10, Thm. 3.2])

Let G be a definable group. Then $\pi_n(G)$ is finitely generated for all $n \in \mathbb{N}$.

Higher homotopy of Strzebonksi tori

Theorem ([BMO10])

Let G be a Strzebonski torus. Then $\pi_m(G) = 0$ for all $m > 1$.

The proof for real Lie tori does not apply because it depends on factorization into one-dimensional sugroups.

Proof.

- The morphism $p_k : G \to G$, $x \mapsto kx$, is a covering map, so it induces an injective endomorphism of $\pi_m(G)$ given by multiplication by k.
- Since $m > 1$ this is actually an automorphism of $\pi_m(G)$ [BO09, Cor. 4.11],[Hat02, Prop. 4.1].
- Since this holds for all k, we deduce that $\pi_m(G)$ is divisible.
- Since it is also abelian and finitely generated, it must be zero.

□

Homotopy type of a Strzebonski torus

Theorem ([BMO10])

Let G be a Strzebonski torus of dimension n. Then G is definably homotopy equivalent to $\mathbb{T}^n$ (a product of n circles).

Proof.

- By [EO04], $\pi_1(G) \cong \mathbb{Z}^n$.
- Consider the map $f : \mathbb{T}^n \to G$ sending $(t_1, \ldots, t_n) \in [0,1)^n$ to $\gamma_1(t_1) + \ldots + \gamma_n(t_n)$ where $[\gamma_1], \ldots, [\gamma_n]$ are free generators of $\pi_1(G)$.
- Then clearly $f_* : \pi_1(\mathbb{T}^n) \cong \pi_1(G)$.
- Since $\pi_m(G) = 0$ for $m > 1$, f induces an isomorphism on all the π_m's.
- By the o-minimal version of Whitehead's theorem ([BO09]) f is a definable homotopy equivalence.

$\square$

Topology of Strzebonski tori

Let G be a Strzebonski torus of dimension n. We have seen that G is definably homotopy equivalent to $\mathbb{T}^n$. The natural conjecture is that it is actually definably homeomorphic to $\mathbb{T}^n$.

Theorem ([Str94b])

Let G be a Strzebonski torus. Assume $\dim(G) = 1$. Then G, with the t-topology, is definably homeomorphic to the circle S^1.

We shall prove:

Theorem ([BB12])

Let G be a Strzebonski torus of dimension $n \neq 4$. Then G is definably ***homemorphic*** *(not isomorphic) to a product of n circles $S^1(M)$ in the given o-minimal structure M.*

A crucial ingredient of the proof is Shiota's o-minimal Hauptvermutung.

Homotopies are robust, homeomorphisms are not

Let M be an o-minimal expansion of a field and let K, L be finite simplicial complexes. We want to compare their realizations in M with the realizations in $\mathbb{R}$ (we can assume K, L have vertices in $\mathbb{Q}$, say).

Theorem ([BO09, Thm. 3.1])

The following are equivalent:

1. $|K|(\mathbb{R})$ *and* $|L|(\mathbb{R})$ *are homotopy equivalent;*
2. $|K|(\mathbb{R})$ *and* $|L|(\mathbb{R})$ *are semialgebraically homotopy equivalent;*
3. $|K|(M)$ *and* $|L|(M)$ *are definably (or semialgebraically) homotopy equivalent.*

So when speaking of homotopy equivalence, it does not matter the category we are working in. By contrast we shall see that $|K|(\mathbb{R})$ and $|L|(\mathbb{R})$ can be homeomorphic without beeing definably homeomorphic in any o-minimal structure. This is connected to the failure of the "Hauptvermutung".

Hauptvermutung

In the early 1900s the main conjecture of combinatorial topology was the following:

Hauptvermutung

If two compact polyedra $|K|(\mathbb{R})$ and $|L|(\mathbb{R})$ are homeomorphic, then they are PL-homeomorphic. Equivalently: the simplicial complexes K and L have isomorphic subdivisions.

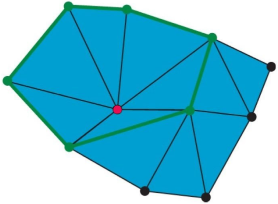
The bad news is that the Hauptvermutung is false (Milnor 1961).
The good news is that it is true in the o-minimal category:

Theorem (O-minimal Hauptvermutung: [Shi97, Shi13])

If two closed complexes $|K|(M)$ and $|L|(M)$ are definably homeomorphism in an o-minimal structure, then they are PL-homeomorphic.

PL-manifolds

If a polyhedron $|K|$ has the link of each vertex homeomorphic to a sphere S^{m-1}, then $|K|$ is a topological manifold (see [Thu97, Prop. 3.2.5]).



The converse holds in dimension ≤ 3 but it is not true in general. What is true is the following: if $|K|$ is a topological manifold, then $|K|$ has simply connected links (but they need not be topological manifolds!).

Fact: $|K|$ is a PL-manifold if and only if the link of each simplex is PL-homeomorphic to the standard PL sphere of the appropriate dimension.

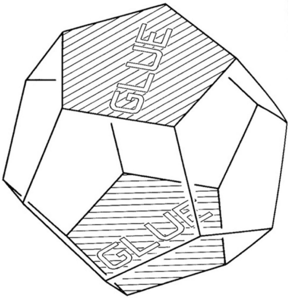
Theorem ([BB12, Fact. 3.3])

Let M be an o-minimal expansion of a field.

- *If $|K|(M)$ is a definable manifold, then $|K|(\mathbb{R})$ is a PL-manifold (and vice versa).*
- *A definable group in M (with the t-topology) has a triangulation which is a PL-manifold.*

Digression: Poincaré dodecahedral space

We given an example of two subsets of $\mathbb{R}^n$ which which are definable in $(\mathbb{R}, +, \cdot)$ and homeomorphic, but the homeomorphism cannot be defined in any o-minimal structure (see [BO03],[BEO07]).



The Poincaré space is a 3-dimensional topological manifold obtained by gluing the opposite faces of the solid dodecahedron after a clokwise rotation of $2\pi/10$.

Its double suspension $\Sigma\Sigma P$ is a polyhedron $|K|$ homeomorphic to a 5-dimensional sphere S^5, but it is not a PL-manifold (see [Thu97, p. 192]). Hence, under the standard triangulation of S^5, the homeomorphism $\Sigma\Sigma P \cong S^5$ is not PL (so by the o-minimal Hauptvermutung it is not definable in any o-minimal structure).

Semialgebraic Strebonski tori

Theorem ([BB12])

Let G be a semialgebraic Strzebonski torus of dimension n. Then G is semialgebraically homeomorphic to $\mathbb{T}^n(M)$.

Proof.

1. Since G is semialgebraic, by model completness we can reduce to the case when G is defined over the real algebraic numbers and consider its real points $G(\mathbb{R})$.
2. The t-topology and Hilbert's 5th, give $G(\mathbb{R})$ the structure of an abelian real Lie group, so there is an analytic isomorphism of Lie groups $h : G(\mathbb{R}) \to \mathbb{T}^n(\mathbb{R})$.
3. Since $G(\mathbb{R})$ is compact, h is definable in the o-minimal structure $\mathbb{R}_{an}$.
4. By the o-minimal Hauptvermutung there is a semialgebraic homeomorphism $f : G(\mathbb{R}) \to \mathbb{T}^n(\mathbb{R})$.
5. By model completeness we can take another f defined over the real algebraic numbers, so we get a semialgebraic homeomorphism $f(M) : G(M) \to \mathbb{T}^n(M)$.

□

Next goal: from homotopy equivalence to homeomorphism

Let G be a Strzebonski torus of dimension n definable in some o-minimal structure. We have seen that if G is semialgebraic, then G is definably homeomorphic to $\mathbb{T}^n$. In the general case we have only shown that G is definably homotopy equivalent to $\mathbb{T}^n$. Our next goal is to obtain a definable homeomophism, but we will succeed only when $\dim(G) \neq 4$. Note that we can always assume that the domain of G is semialgebraic (by the triangulation theorem), however the difficulty is that group operation may not be semialgebraic.

Around Borel's conjecture

Let X be a closed PL-manifold homotopy equivalent to the n-torus $\mathbb{T}^n(\mathbb{R})$. Is X homeomorphic to $\mathbb{T}^n(\mathbb{R})$? This is connected to Borel's conjecture, which is false in general, but we have the following weaker statement:

Theorem ([BB12, 1.3])

Let X be a (closed) PL-manifold of dimension $n \neq 4$ homotopy equivalent to $\mathbb{T}^n(\mathbb{R})$. Then there is a finite PL-covering $f : \mathbb{T}^n(\mathbb{R}) \to X$.

Proof.

For $n \geq 5$, see [HW69]. The case $n = 3$ follows from results in [KS77] plus the positive solution of Poincaré's conjecture. For $n \leq 2$ X is already *PL*-homeomorphic to $\mathbb{T}^n(\mathbb{R})$. □

Corollary ([BB12])

If G is a Strzebonski torus, there is a semialgebraic finite cover

$$f : \mathbb{T}^n(M) \to G(M)$$

(as spaces, not as groups).

Reduction to the semialgebraic case

Theorem ([BB12])

Let $\mathbb{G} = (G, \cdot)$ be a Strzebonski torus with $\dim(G) \neq 4$ *and semialgebraic domain G. Then there is a new group operation $\circ$ making $(G, \circ)$ into a semialgebraic Strzebonski torus.*

Proof.

- As $\dim(G) \neq 4$, there is a semialgebraic finite cover $f : \mathbb{T}^n(M) \to G(M)$.
- There is a group operation $*$ on $\mathbb{T}^n$ (possibly not semialgebraic) making f into a group homeomorphism with finite kernel.
- Let $T := (\mathbb{T}^n, *)$. Since T is abelian and connected, $\ker(f) < T[m]$ for some m and $T/T[m] \cong T$.
- So we get a definable group homomorphism $\mathbb{G} \cong T/ker(f) \to T/T[m] \cong T$.
- By the Hauptvermung and "good reduction" [EJP10] we can modify it to get a semialgebraic finite cover $h : G(M) \to \mathbb{T}^n(M)$ (but only as spaces).
- The standard semialgebraic group operation on $\mathbb{T}^n$ (addition mod 1) can now be lifted to a semialgebraic group operation on G making it into a semialgebraic Strzebonski torus.

Topology of Strebonski tori: conclusion

From the above results we get:

Theorem ([BB12])

Let G be a Strzebonski torus of dimension $n \neq 4$. Then G is definably homeomorphic to $\mathbb{T}^n(M)$.

Outline

Cherlin-Zilber algebraicity conjecture

The Cherlin-Zilber conjecture says that if G is a simple group of finite Morley rank, then G is isomorphic to an algebraic group over an algebraically closed field K interpretable in G [Zil77, Che79].
The conjecture is true if G is interpretable in an o-minimal structure thanks to part (1) of the following:

Theorem ([PPS00b, Thm. 1.1])

Let $\mathbb{G} = (G, \cdot)$ be an infinite definably simple (non-abelian) group. Then there is a real closed field R such that one of the following holds:

1. *(Stable case) $\mathbb{G}$ and the field $R[\sqrt{-1}]$ are bi-interpretable and G is $\mathbb{G}$-definably isomorphic to a linear algebraic group defined over $R[\sqrt{-1}]$.*
2. *(Unstable case) $\mathbb{G}$ and the field R are bi-interpretable and G is $\mathbb{G}$-definably isomorphic to the connected component of an algebraic group defined over R.*

Interpreting a field

Theorem ([PS00, Thm. 4.3])

Let $\mathbb{G} = (G, \cdot)$ be a group intepretable in an o-minimal structure M. Then $\mathbb{G}$ interprets an infinite field if and only if $\mathbb{G}$ is not abelian-by-finite (i.e. has not abelian subgroups of finite index).

Example

Let $M = (M, +, \cdot)$ be a real closed field. Consider the group $(G, \circ)$ of affine transformations $(a, b) : x \mapsto a + bx$ from M to M. The composition is given by $(a, b) \circ (a', b') = (a + ba', bb')$. Clearly $(G, \circ)$ is intepretable in $(M, +, \cdot)$.

Proposition

$(M, +, \cdot)$ is interpretable in $(G, \circ)$.

Proof.

We have $(a, 1) \circ (a', 1) = (a + a', 1)$, so the subgroup $A \lhd G$ of all elements of the form $(a, 1) : x \mapsto x + a$ is isomorphic to $(M, +)$. Since $A = C_G(A)$, we have that A is definable in $(G, \circ)$ (definability of centralizers). The elements $(0, b) : x \mapsto bx$ form a sugroup $T < G$ isomorphic to K^*, which is definable since $T = C_G(T)$. We have $(b, 1) = (1, 1)^{(0,b)}$ and $(1, 1)^{(0,b)} \circ (1, 1)^{(0,b')} = (1, 1)^{(0,bb')} = (bb', 1)$, so the operation $(b, 1) * (b', 1) = (bb', 1)$ is definable in $(G, \circ)$ and makes $(M, +, \cdot) \cong (A, \circ, *)$ interpretable in $(G, \circ)$. □

Lie algebras

Theorem ([PPS00a])

Let G be a definably simple (non-abelian) group definable in an o-minimal structure M expanding a field $\mathcal{R} = (R, <, +, \cdot)$. Then G is definably isomorphic to a group definable in $\mathcal{R}$.

Proof.

We can put a differential structure on G (as for the t-topology) and define the notion of two definable curves in G being "tangent" at $e \in G$. An equivalence class of curves modulo tangency is a tangent vector. The class of all tangent vectors is the tangent space $T_e(G) \cong R^n$, $n = \dim(G)$. Conjugation by $g \in G$ is an automorphism of G and its differential at e is the adjoint map $Ad_g : T_e(G) \to T_e(G)$. The differential at e of the map $g \mapsto Ad_g \in GL(T_e(G))$ is a liner map $ad : T_e(G) \to End(T_e(G))$. For $\xi, \zeta \in T_e(G)$ let $[\xi, \zeta] := ad(\xi)(\zeta) \in T_e(G)$. Then $[-, -]$ makes $T_e(G)$ into a Lie algebra $\mathfrak{g}$ and $g \mapsto Ad_g$ is an isomorphism from G to the connected component H^0 of the linear algebraic group $H = Aut(\mathfrak{g}) < GL(T_e(G)) \cong GL(n, R)$. □

Almost direct products

Definition

Given a group G and two subgroups A and B of G. We say that G is the almost direct product of A and B if $G = AB$ and the function $\mu : A \times B \to G$ sending (a, b) to ab is a surjective group homomorphism with a finite kernel. This implies $ab = ba$ for all $a \in A, b \in B$ and $\Gamma := A \cap B$ is a finite (hence central) subgroup of G ✐. In this situation we write $G = A \times_\Gamma B$ and note that $\ker(\mu) = \{(c, c^{-1}) \,:\, c \in \Gamma\}$.

The derived subgroup [G,G]

In general the derived subgroup $[G, G]$ of a definable group is not definable [Con09, BJO12]. However in the definably compact case $[G, G]$ is definable and we have:

Theorem ([HPP11, Thm. 6.4])

If G is definably compact and definably connected, then $[G, G]$ is definable (and semisimple) and there is a morphism $Z^0(G) \times [G, G] \to G$ with finite kernel $\Gamma < Z(G)$, namely we can write G as an almost direct product $G = Z^0(G) \times_\Gamma [G, G]$.

This reduces many questions on definably compact groups to the abelian and semisimple cases.

Semisimple case

The study of semisimple definable groups can be reduced to the study of groups defined in the real field $(\mathbb{R}, +, \cdot)$. This depends on the fact that any o-minimal expansion of a field contains an isomorphic copy of the field $\mathbb{R}^{alg}$ of the real algebraic numbers, and any definably connected semisimple definable group G is definably isomorphic to a semialgebraic group defined over the real algebraic numbers $\mathbb{R}^{alg}$.

References

A definably simple group is isomorphic to a group defined over $\mathbb{R}^{alg}$ [PPS00a, Thm. 4.1],[PPS02, Proof of Thm. 5.1]. A semisimple centreless definable group G is a finite product of definably simple groups [PPS00a]. General semisimple groups are also isomorphic to groups defined over $\mathbb{R}^{alg}$ by "very good reduction" [EJP10, Cor. 1.3],[HPP11, thm. 4.4].

Outline

Pillay's conjectures: introduction

Let G be a definable group G in an o-minimal structure M. We have seen that G has a natural topology, the t-topology, making it into a "Lie group over M". If M is sufficiently saturated, we shall prove:

There is a type-definable subgroup G^{00} of G, called "infinitesimal subgroup", such that G/G^{00}, with the "logic topology", is a real Lie group.

The intuition is the "moding out the infinitesimals" we are left with the reals, but we need the appropriate notion of "infinitesimal relative to G". We introduce below the necessary definitions.

Bounded equivalence relations

Definition

An A-invariant equivalence relation $E \subseteq X \times X$ on a definable set X is bounded if there is a cardinal κ such that, in any model, E has $\leq \kappa$ equivalence classes.

One can in fact take $\kappa = |L| + |A|$. If E is a bounded and M is sufficiently saturated, then given $N \succeq M$ every $a \in X(N)$ is equivalent to some $a' \in X(M)$, so the natural map $X(M)/E(M) \to X(N)/E(N)$ is a bijection, namely X/E does not depend on the model.

Logic topology

Definition

Given a definable set X in a sufficiently saturated structure $\mathcal{U}$ and a type-definable equivalence relation E on X of bounded index, the logic topology on X/E is defined as follows. A subset C of X/E is closed if and only if its preimage in X is type-definable. Equivalently, a subset O of X/E is open if and only if its preimage is $\bigvee$-definable.

Proposition ([Pil04, Lemma 2.5])

X/E, with the logic topology, is a compact (Hausdorff) topological space. ✎

Example: the standard part map

Work in a real closed field $M \succeq \mathbb{R}$. Let $X = [0,1]$ and let $E(x,y) \iff |x-y| < 1/n$ for all $n \in \mathbb{N}$. Then:

Proposition

E is a type definable definable equivalence relation on $X = [0,1]$, and X/E, with the logic topology, is homeomorphic to $[0,1](\mathbb{R})$, with the euclidean topology.

Proof.

The standard part map $\mathrm{st} : [0,1] \to [0,1](\mathbb{R})$ is such that for every closed $C \subseteq [0,1](\mathbb{R})$ the preimage $\mathrm{st}^{-1}(C)$ is type-definable. $\square$

The infinitesimal subgroup G^{00}

Definition

Let G be a definable group in a sufficiently saturated structure $\mathfrak{U}$. Given a small model M (or a small set of parameters), we denote G^{00}_M the intersection of all type-definable over M subgroups of bounded index. Note:

1. G^{00}_M is of bounded index and if $M \preceq N$, $G^{00}_N \subseteq G^{00}_M$;
2. if G^{00}_M does not depend on M, we call it G^{00} and say that G^{00} exists;
3. G^{00} (when it exists) is the smallest type-definable subgroup of bounded index;
4. G^{00} (when it exists) is definable without parameters: $G^{00} = G^{00}_{\emptyset} = G^{00}_M$ for all M.

We want to study G/G^{00}_M as a compact group with the logic topology.

The logic topology does not coincide with the quotient topology. Indeed G^{00}_M is open in the t-topology of G [Pil04], so with the quotient topology G/G^{00}_M is discrete.

NIP theories

Definition

A formula $\phi(x, y)$ shatters $\{a_i : i \in I\}$ if for every $J \subseteq I$ there is b_J (in the monster model) such that $\models \phi(a_i, b_J)$ if and only if $i \in J$. A formula $\phi(x, y)$ is NIP if it does not shatter an infinite set. A theory T is NIP if every formula in T is NIP.

NIP theories include the o-minimal and the stable theories. The reason we mention NIP theories is that G^{00} always exists in that context.

Pillay's conjectures

Let G be a definable group in a sufficently saturated structure $\mathfrak{U}$ and let $T = Th(\mathfrak{U})$.

Theorem

We have:

1. G/G^{00}_M*, with the logic topology, is a compact topological group [Pil04].*
2. *If T is NIP, G^{00} exists [She08].*
3. *If T is o-minimal G^{00} exists and G/G^{00} is a real Lie group [BOPP05].*
4. *If moreover G is definably compact,* $\dim(G) = \dim(G/G^{00})$ *[HPP08].*

3+4 are known as "Pillay's conjectures" [Pil04] (now theorems).

Circle group

We illustrate Pillay's conjectures in the one-dimensional case [Pil04]. Work in a sufficiently saturated o-minimal structure M.

Proposition ([Pil04, p. 156, Case II])

Let G be a definably compact definably connected abelian one-dimensional definable group. Then:

1. *the t-topology on G is induced by a circular ordering;*
2. *G^{00} is the largest arc of the circle containing $e \in G$ and not containing any torsion element;*
3. *$G/G^{00} \cong \mathbb{R}/\mathbb{Z}$.*

For instance G can be $[0, 1)$ with addition modulo 1, or $G = SO(2, M)$.

Algebraic groups

Let $\mathcal{U} \succeq \mathbb{R}$ and let $G(\mathcal{U}) < GL(n, \mathcal{U})$ be an algebraic linear group defined over $\mathbb{R}$. There are two groups that we can associate to G:

$$G(\mathbb{R}) \leftarrow G(\mathcal{U}) \rightarrow G/G^{00}.$$

A natural question is whether $G(\mathbb{R}) \cong G/G^{00}$. Since G/G^{00} is compact, a necessary condition is that $G(\mathbb{R})$ is compact, which amounts to say that G is definably compact.

Example

1. Let $G(M) = SL(2, M)$. In this case $G(\mathbb{R})$ is not compact, and $G/G^{00} \not\cong G(\mathbb{R})$. In fact it can be shown that $G/G^{00} = \{1\}$.
2. Let $G(M) = SO(3, M)$. In this case $G(\mathbb{R})$ is compact, and $G/G^{00} \cong G(\mathbb{R})$.

When $G(\mathcal{U})$ is not defined over $\mathbb{R}$ (for instance an elliptic curve with non-standard parameters), $G(\mathbb{R})$ is not defined, but we shall see that G/G^{00} is nevertheless a real Lie group.

A stable example

When the structure is NIP but not o-minimal, G/G^{00} is a compact group, but in general not a Lie group. For instance let $T = Th(\mathbb{Z}, +)$ (a stable theory) and let $\mathfrak{U} \succeq \mathbb{Z}$ be a sufficiently saturated model of T. Then:

- $\mathfrak{U}^{00} = \bigcap_{n \in \mathbb{N}} n\mathfrak{U}$ ✎.
- $\mathfrak{U}/\mathfrak{U}^{00} \cong \widehat{\mathbb{Z}} := \varprojlim_n (\mathbb{Z}/n\mathbb{Z})$ ✎

Outline

The abelian case

Our next goal is to prove Pillay's conjectures in the abelian case, namely:

Theorem

Let G be a definably connected abelian group in an o-minimal structure. Then:

1. *G^{00} exists and G/G^{00}, with the logic topology, is Lie isomorphic to a real Lie group, namely $G/G^{00} \cong (\mathbb{R}/\mathbb{Z})^m$ for some $m \leq \dim(G)$ [BOPP05];*
2. *If G is definably compact, then $m = \dim(G)$ [HPP08].*

We need to prove some facts about G^{00}_M (lastly we show $G^{00} = G^{00}_M$).

How do we prove that a group is a Lie group?

Fact

1. A compact connected locally connected separable abelian group is isomorphic to a torus of possibly infinite dimension [HM98, Thm. 8.36].
2. Any compact group is the limit of a strict projective system (projective system with surjective maps) of compact Lie groups [HM98, Cor. 2.36-2.43].
3. Cor: A compact group Γ is a Lie group iff it has the descending chain condition on closed subgroups.

To prove Pillay's conjectures we need to deal with connectedness and the DCC.

Proposition ([Pil04, Prop. 2.12])

If G has the DCC on type-definable subgroups of bounded index, then G^{00} exists and G/G^{00} is a compact Lie group.

Proof.

Let $\Gamma = G/G^{00}$. Then Γ has the DCC on closed subgroups. □

G/G^{00}_M is connected

Proposition

Let $\pi : G \to G/G^{00}_M$ be the natural map and let $V \subseteq G$ be a definably connected set. Then $\pi(V) \subseteq G/G^{00}_M$ is connected. In particular G/G^{00}_M is connected.

Proof. For a contradiction there are two non-empty disjoint closed sets Z_1, Z_2 with $\pi(V) = Z_1 \cup Z_2$. The sets $V_1 = V \cap \pi^{-1}(Z_1)$ and $V_2 = V \cap \pi^{-1}(Z_2)$ are type-definable and disjoint and since their union $V = V_1 \cup V_2$ is definable, they are definable. But they are also open in G (since G^{00}_M is open), hence they disconnect V.

Proposition

Let X be a definable subset of G containing gG^{00}_M. Then $\pi(g)$ is in the interior of $\pi(X)$.

Proof: π sends definable sets to closed sets, so $\pi(X^{\complement})$ is closed, and since it does not contain $\pi(g)$, its complement is an open nbd of $\pi(g)$ contained in $\pi(X)$.

G/G^{00}_M is locally connected

We say that a type-definable set is connected if it cannot be splitted in two non-empty relatively definable open subsets.

Proposition ([BOPP05])

1. *A type-definable set has at most $2^{|T|}$ type-definable connected components.*
2. *G^{00}_M is connected.*
3. *G/G^{00}_M is locally connected.*

Proof: (1) Left to the reader, see [BOPP05, Thm. 2.3]. ✎
(2) If not, the connected component of G^{00}_M contradicts the minimality of G^{00}_M.
(3) Let U be an open neighbourhood of $\pi(g) \in G$. It suffices to find a connected neighbourhood C of $\pi(g)$, not necessarily open, contained in U. Note that $\pi^{-1}(U)$ is $\bigvee$-definable and contains gG^{00}_M. By compactness there is a definable subset X of $\pi^{-1}(U)$ containing gG^{00}_M, which can be taken to be connected since G^{00}_M is connected. Then $C := \pi(X)$ is the desired connected nbd of $\pi(g)$.

G^{00}_M is divisible

Proposition

G^{00}_M is divisible (G abelian).

Proof.

Since G is abelian, $G[n]$ is finite [Str94a], hence $g \mapsto ng$ has finite kernel and its image nG has finite index in G. But G is connected, so $nG = G$ and G is divisible. Now nG^{00}_M has bounded index in $nG = G$, so $G^{00}_M \subseteq nG^{00}_M$. Hence $G^{00}_M = nG^{00}_M$ and G^{00}_M is divisible. □

Corollary

The n-torsion of G/G^{00}_M is bounded by the n-torsion of G, hence it is finite.

DCC on type-definable subgroups

Proposition ([BOPP05])

Let G be abelian, definably connected. Then G has the DCC on type-definable subgroups of bounded index.

Proof.

If not we can find a counterexample $(H_n : n \in \mathbb{N})$ to the DCC in a **countable** sublanguage L_0 with each H_i type-definable over a countable L_0-substructure N. Now G/G^{00}_N is compact connected **separable** abelian, therefore it is a possibly infinite product of circle groups. Since G^{00}_N is divisible, the n-torsion of G/G^{00}_M is bounded by the n-torsion of G, hence it is finite. Thus G/G^{00}_N is finite product of circle groups. This is absurd since G/G^{00}_N contains the infinite descending chain of closed subgroups H_i/G^{00}_N. □

G^{00} exists and G/G^{00} is a torus

Theorem ([BOPP05])

Let G be a definably connected abelian group in an o-minimal structure. Then G^{00} exists and G/G^{00}, with the logic topology, is Lie isomorphic to a real Lie group, namely $G/G^{00} \cong (\mathbb{R}/\mathbb{Z})^m$ for some $m \leq \dim(G)$.

Proof.

By the DCC on type definable subgroups of bounded index, G^{00} exists and G/G^{00} is a compact group with the DCC on closed subgroups, hence a real Lie group. Being abelian and connected, it is a torus. Since G^{00} is divisible, the n-torsion of G/G^{00} is bounded by that of G, so $\dim(G/G^{00}) \leq \dim(G)$ (the former is the dimension as a Lie group, the latter is the o-minimal dimension). □

Torsion free subgroups and G^{00}

Proposition

Let G be a definable abelian group and suppose $H < G$ is a type-definable torsion free subgroup of bounded index. Then $H = G^{00}$.

Proof.

If not H/G^{00} is a non-trivial abelian Lie group (being a closed subgroup of G/G^{00}) which is compact and torsion free (as H is torsion free and G^{00} is divisible), a contradiction. □

The proof of the following result needs the theory of "generic sets" and is postponed.

Theorem ([HPP08])

If G is abelian, G^{00} is torsion free.

Dimension of G/G^{00}. Abelian case

Theorem ([HPP08])

Let G be a definably connected abelian group in an o-minimal structure. Then $\dim(G) = \dim(G/G^{00})$.

Proof.

1. If G is abelian, G^{00} is torsion free [HPP08];
2. since it is also divisible, G and G/G^{00} have the same k-torsion;
3. $\Gamma := G/G^{00}$ is a torus, so $\Gamma[k] \cong (\mathbb{Z}/k\mathbb{Z})^{\dim(\Gamma)}$;
4. By [EO04], $G[k] \cong \pi_1(G)/k\pi_1(G) \cong (\mathbb{Z}/k\mathbb{Z})^{\dim(G)}$;
5. Thus $\dim(G) = \dim(G/G^{00})$.

Note that in the non-compact case G/G^{00} could be the trivial group 0 (e.g. when $G = (M, +)$).

$\square$

Outline

$G(\mathbb{R}) \cong G/G^{00}$ when $G(\mathbb{R})$ is defined and compact

Proposition (See [Pil04, Prop. 3.6])

Let $\mathfrak{U} \succeq (\mathbb{R}, +, \cdot, \dots)$ be an o-minimal saturated structure. Assume $G(\mathfrak{U})$ is defined over $\mathbb{R}$ and $G(\mathbb{R})$ is a compact Lie group. Then $G/G^{00} \cong G(\mathbb{R})$ as topological group, so Pillay's conjecture holds.

Proof.

We claim $G^{00} = \ker(\mathrm{st})$ where $\mathrm{st} : G(\mathfrak{U}) \to G(\mathbb{R})$ is the standard part map. This yields an isomorphism $G/G^{00} \cong G(\mathbb{R})$. It is an isomorphism of **topological** groups, because a subset X of $G(\mathbb{R})$ is closed (in the euclidean topology) iff $\mathrm{st}^{-1}(X)$ is type-definable. For the claim, let $T \subseteq G(\mathbb{R})$ be a maximal torus of $G(\mathbb{R})$ and let $T' \supseteq T$ be the smallest definable subgroup of $G(\mathbb{R})$ containing T. Then T' is abelian ([Str94a, Lemma 4.2]) and connected. By maximality $T = T'$, so T is definable. Since $\ker(\mathrm{st}) \cap T$ is torsion free, it coincides with T^{00}. Let $H < G$ be a type definable subgroup of bounded index. It suffices to show $\ker(\mathrm{st}) < H$. Now $H \cap T$ has bounded index in T, so $T^{00} < H \cap T$. But $T^{00} = \ker(\mathrm{st}) \cap T$ because the latter is torsion free. Thus $\ker(st) \cap T < H$. The conjugates of $T(\mathbb{R})$ cover $G(\mathbb{R})$, so the conjugates of T cover G. It follows that $\ker(\mathrm{st}) < H$. □

Pillay's conjectures: semisimple case

A definable group G is definably simple if it is not abelian and has no definable normal subgroups; G is semisimple if it has no infinite abelian normal subgroups [PPS00a, Def. 2.33].

Theorem

Let G be a definably simple group in an o-minimal structure.

1. *G interprets a field and it is definably isomorphic, via the adjoint representation to a linear group G_1 definable over the real algebraic numbers [PPS00a, 4.1],[PPS02, Proof of Thm. 51].*
2. *In the definably compact case, $G/G^{00} \cong G_1/G_1^{00} \cong G_1(\mathbb{R})$, so $\dim(G) = \dim(G/G^{00})$.*
3. *If G is not definably compact, then it is abstractly simple [PPS02, 6.3], thus $G/G^{00} = 1$.*

In particular Pillay's conjectures holds in the simple case [Pil04]. A semisimple centreless definable group G is a finite product of definably simple groups [PPS00a, Thm. 4.1], so Pillay's conjectures hold for G as well. The general semisimple case can be handled by "very good reduction" [EJP10, Cor. 1.3], [EJP07, Prop. 3.2], but it also follows by the arguments below.

DCC on type definable subgroups. General case

Proposition ([BOPP05, Lemma 1.10])

If $N \lhd G$ and G/N have the DCC on type-definable subgroups of bounded index, so does G.

Corollary ([BOPP05])

Given a definable group G in an o-minimal structure, G has the DCC on type-definable subgroups of bounded index (so G^{00} exists and G/G^{00} is a compact real Lie group).

Proof.

If G is not semisimple it has an infinite normal abelian subgroup $N \lhd G$. Since N is abelian the DCC holds for N. By induction on dimension it holds for G/N, hence for G. □

Dimension of G/G^{00}. General case

Proposition

If G is a definably compact group and H is a definable subgroup of G, then $H^{00} = G^{00} \cap H$ [HPP08, Thm. 8.1],[Ber07, Thm. 4.4].

Corollary ([Ber07, Thm. 5.2])

The functor $G \mapsto G/G^{00}$ preserves exact sequences.

In particular, if $N \lhd G$ and $H = G/N$, then $\dim(G/G^{00}) = \dim(H/H^{00}) + \dim(N/N^{00})$.

Corollary

If G is definably compact, $\dim(G/G^{00}) = \dim(G)$.

Proof.

The abelian and semisimple centreless case have already been proved. For the general use the fact that if G not semisimple then it has an infinite abelian normal subgroup N and by induction on dimension the result holds for G/N (alternatively use the finite cover $Z(G)^0 \times [G, G] \to G$ [HPP11, Thm. 6.4]). $\square$

What's next

We have proved Pillay's conjectures, but we took for granted that G^{00} is torsion free. I am going to expand the slides to include a proof of this result and also of "compact domination". They are essentially ready, but I need to put them in order.

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