Fast Algorithms
for the
Removal of Non-Uniform Motion Blurs

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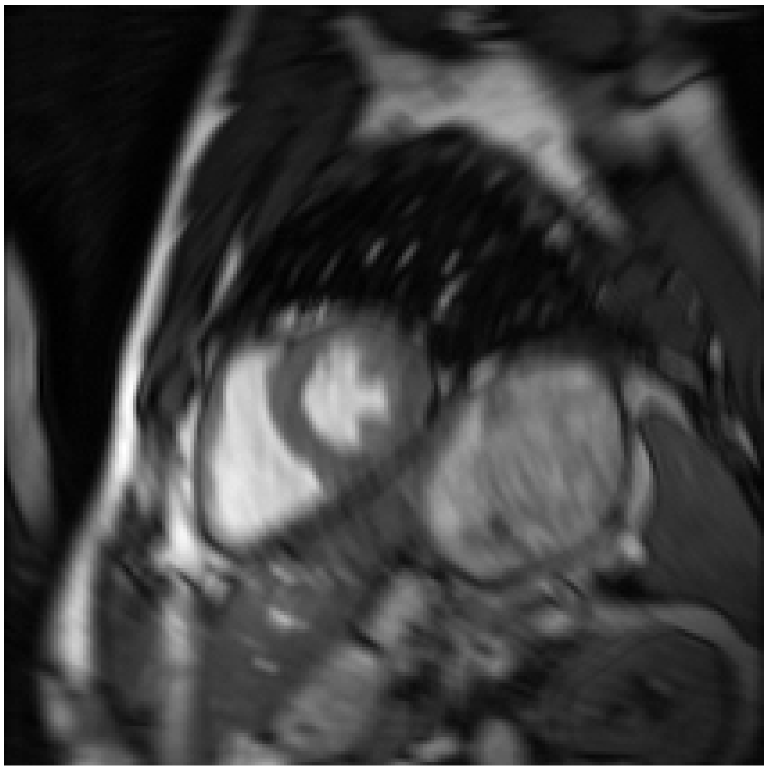
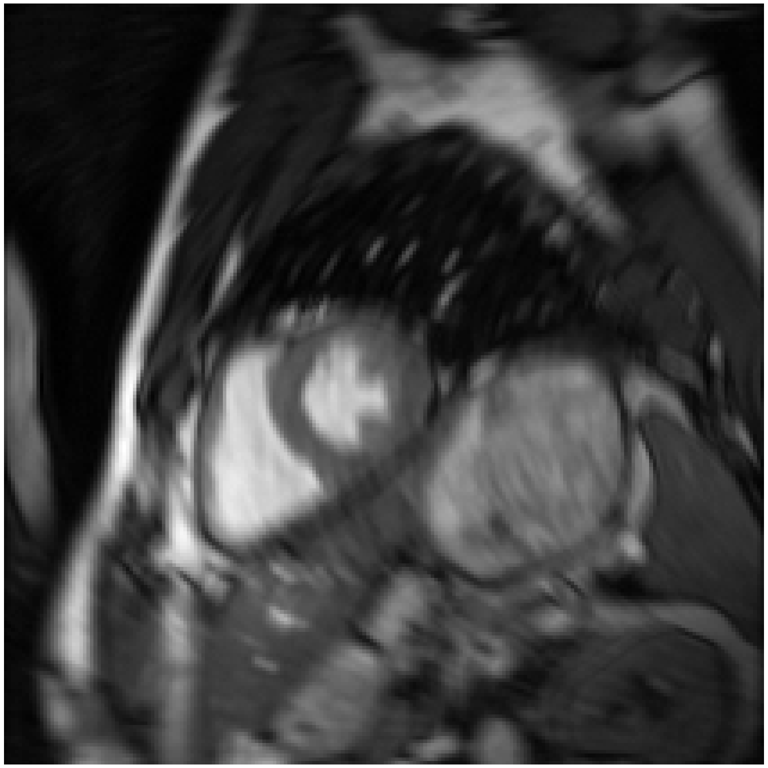
Motivating Example

Motion in MRI cardiac image

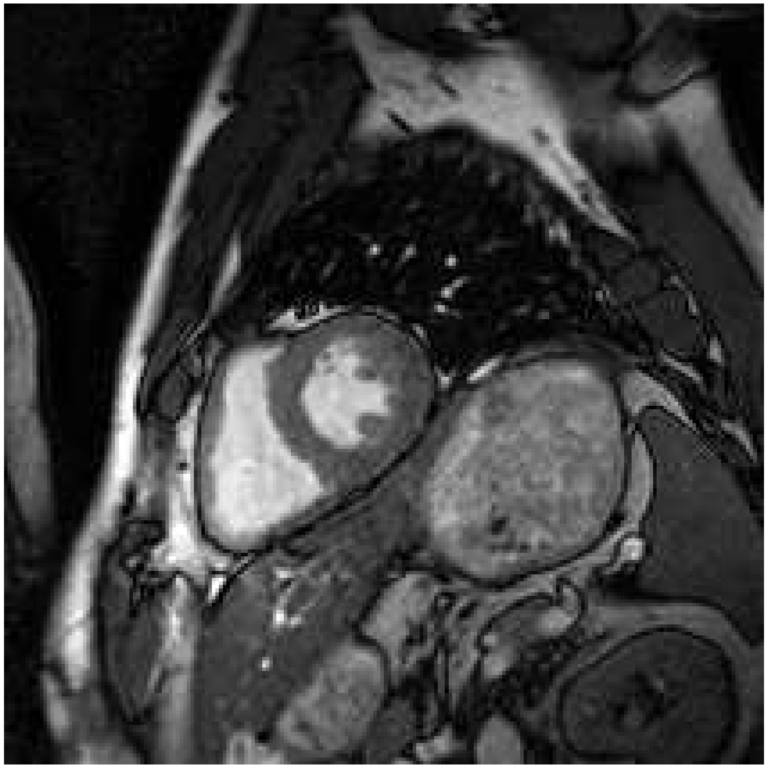


Motivating Example

Motion in MRI cardiac image



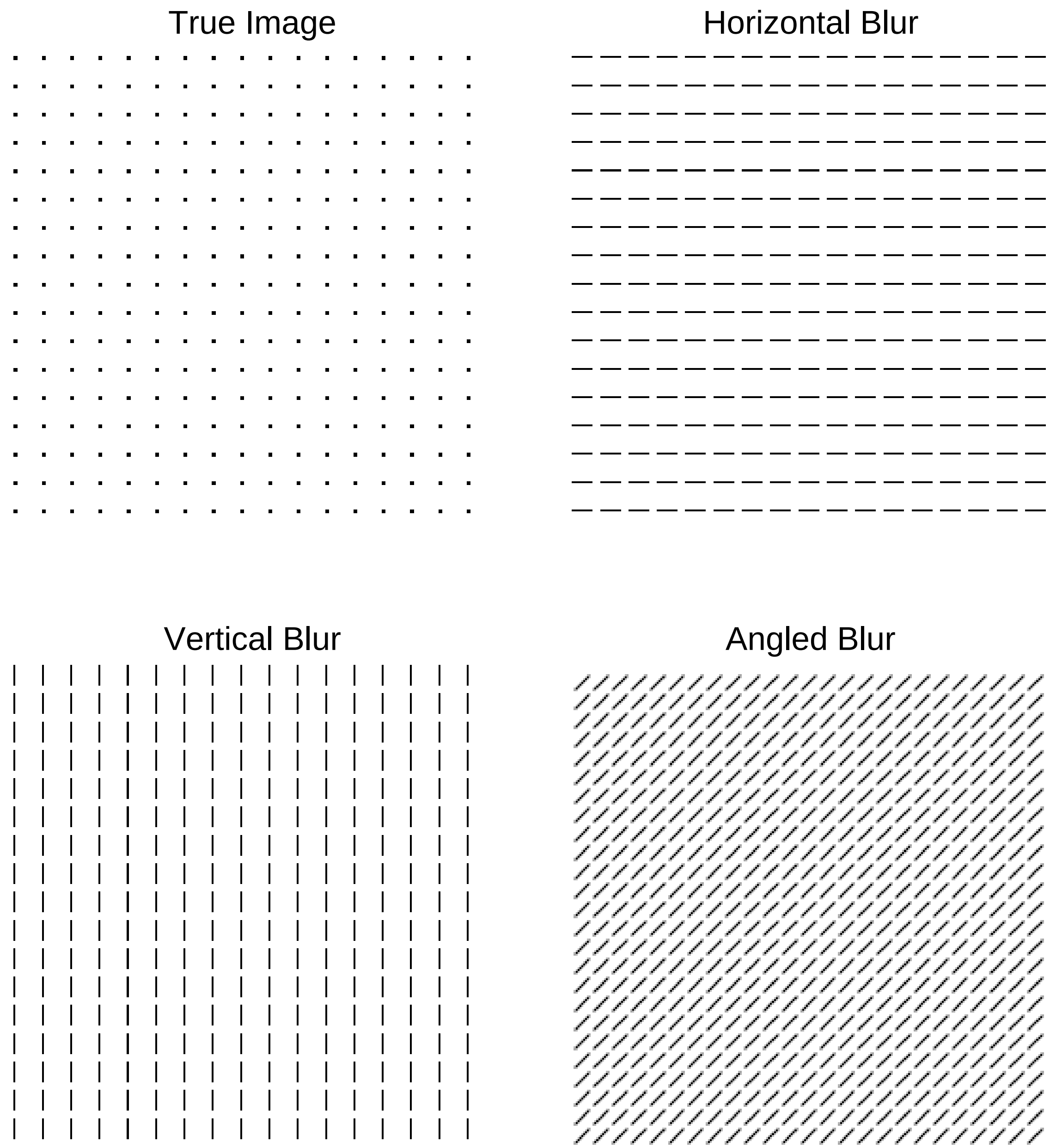
Restored MRI cardiac image



Outline

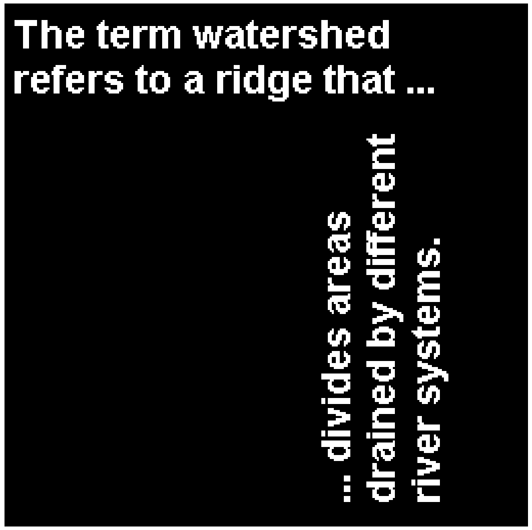
- Uniform motion blurs, matrix models
- Non-uniform motion blurs, matrix models
- Algorithms
- Difficulties

Uniform Motion Blur

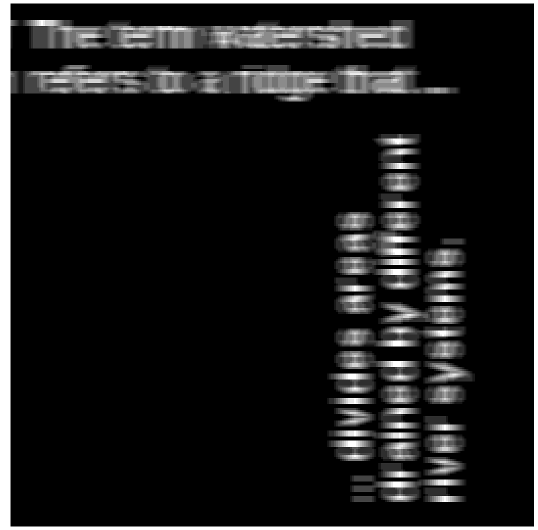


Uniform Motion Blur

True Image



Horizontal Blur



Vertical Blur

Angled Blur

Uniform Motion Blur

Linear model: $\mathbf{b} = A\mathbf{x} + \mathbf{n}$

The matrix A is structured:

Horizontal blur $\Rightarrow$ $A = T \otimes I$

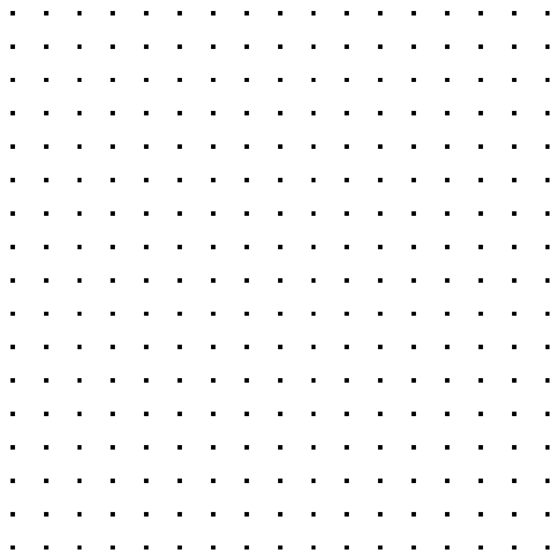
Vertical blur $\Rightarrow$ $A = I \otimes T$

$$\text{where} \quad T = \begin{bmatrix} \frac{1}{d} & & & & \\ \vdots & & & & \\ \frac{1}{d} & & \ddots & & \\ & \ddots & & & \\ & & \frac{1}{d} & \cdots & \frac{1}{d} \end{bmatrix}$$

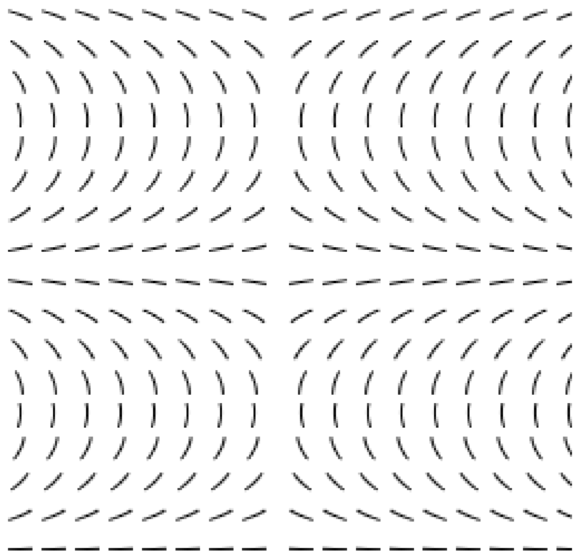
Angled blur $\Rightarrow$ $A =$ block Toeplitz, Toeplitz blocks

Non-Uniform Motion Blur

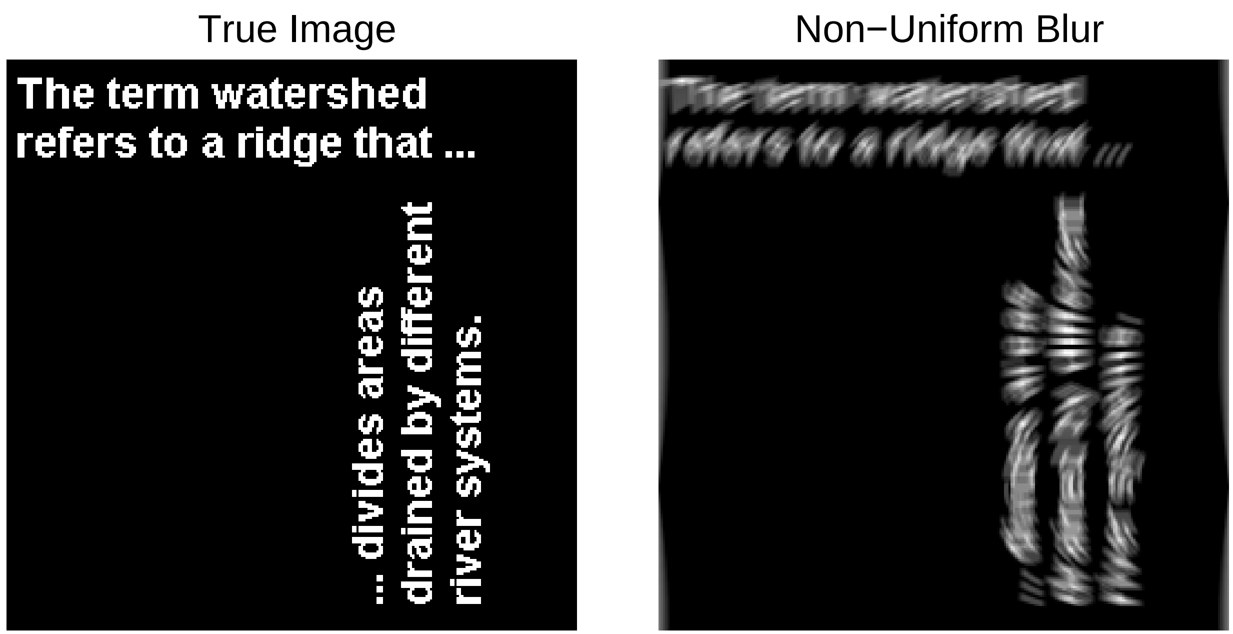
True Image



Non−Uniform Blur

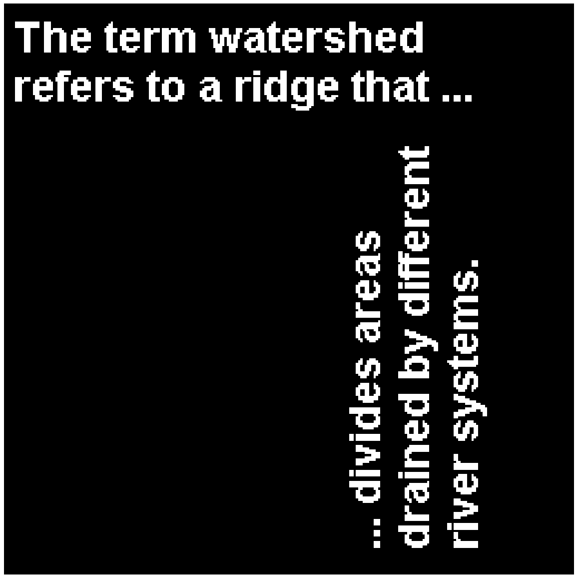


Non-Uniform Motion Blur



Non-Uniform Motion Blur

True Image



Non−Uniform Blur

We still have linear model

$$\mathbf{b} = A\mathbf{x} + \mathbf{n}$$

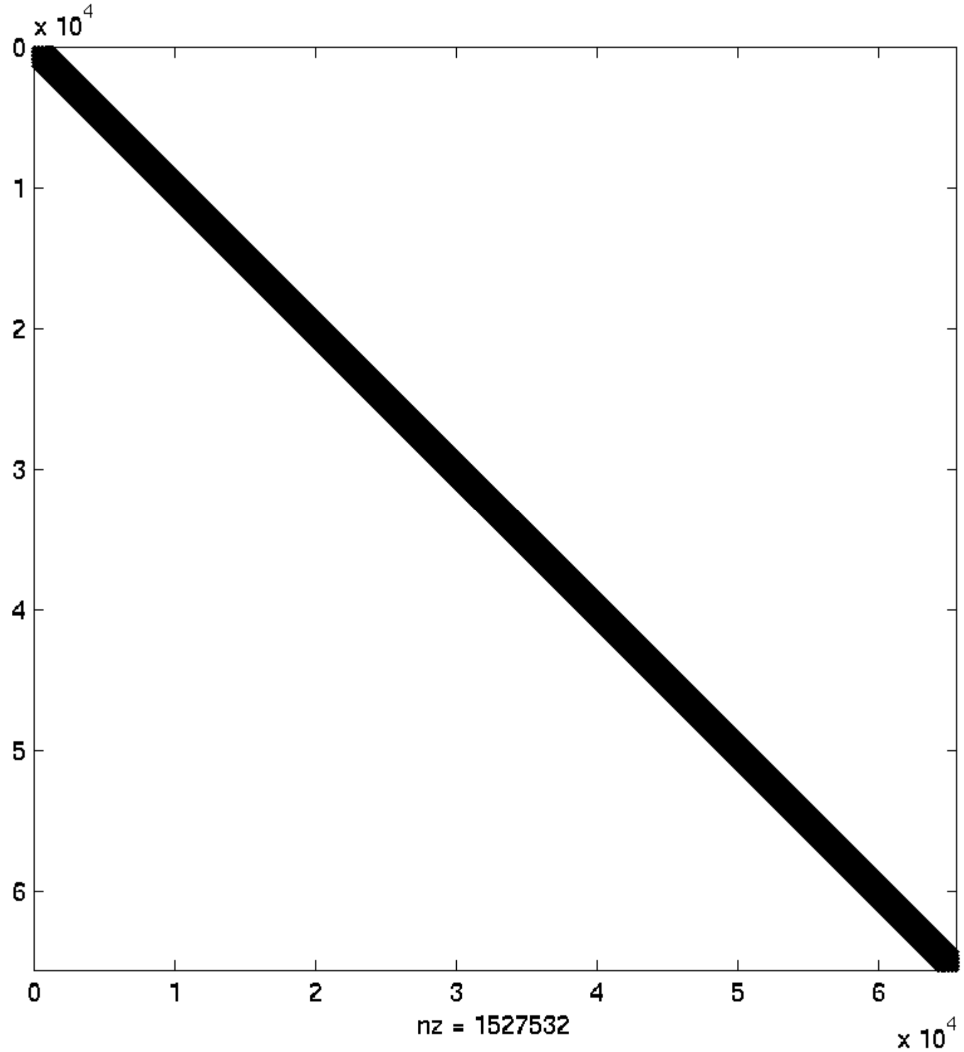
However, A has no obvious structure.

Non-Uniform Motion Blur

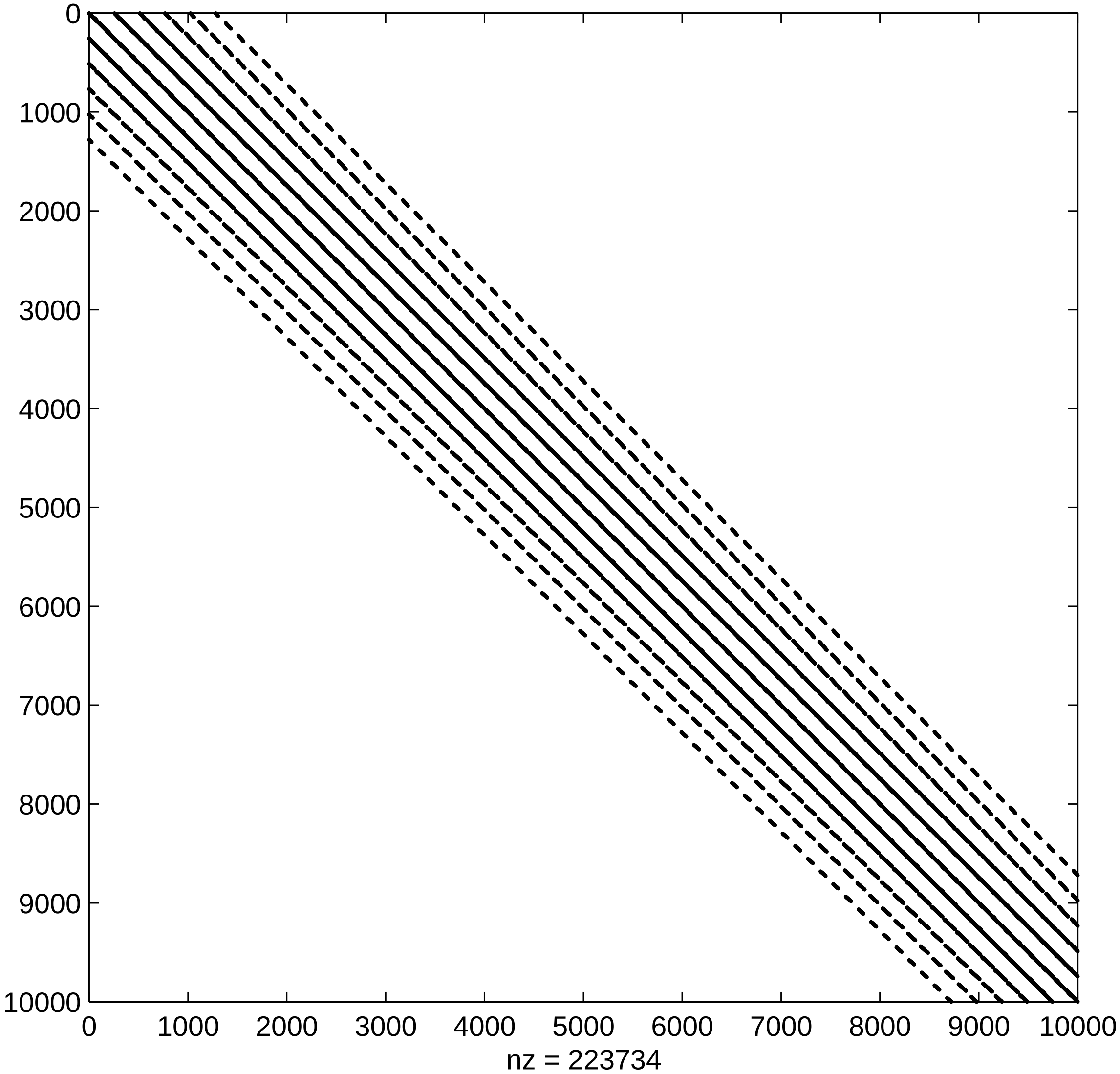
To explicitly construct A:

- Estimate direction and magnitude of motion at each pixel
- Construct corresponding column of A
- Use sparse data structure for A

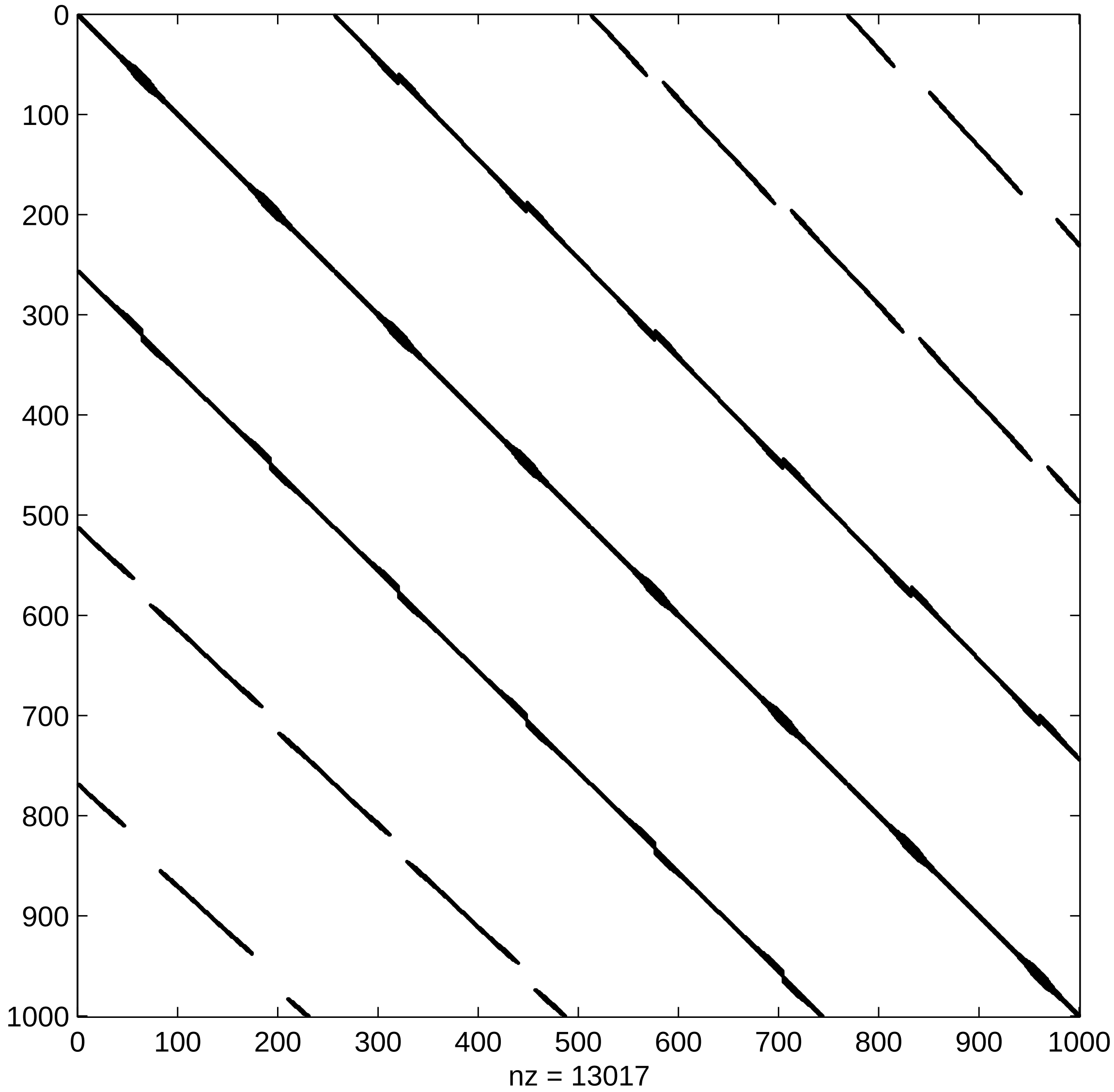
Sparsity Pattern of A



Sparsity Pattern of A



Sparsity Pattern of A



Non-Uniform Motion Blur

Problem: It may be difficult to estimate motion at *every* pixel

Non-Uniform Motion Blur

To approximate A:

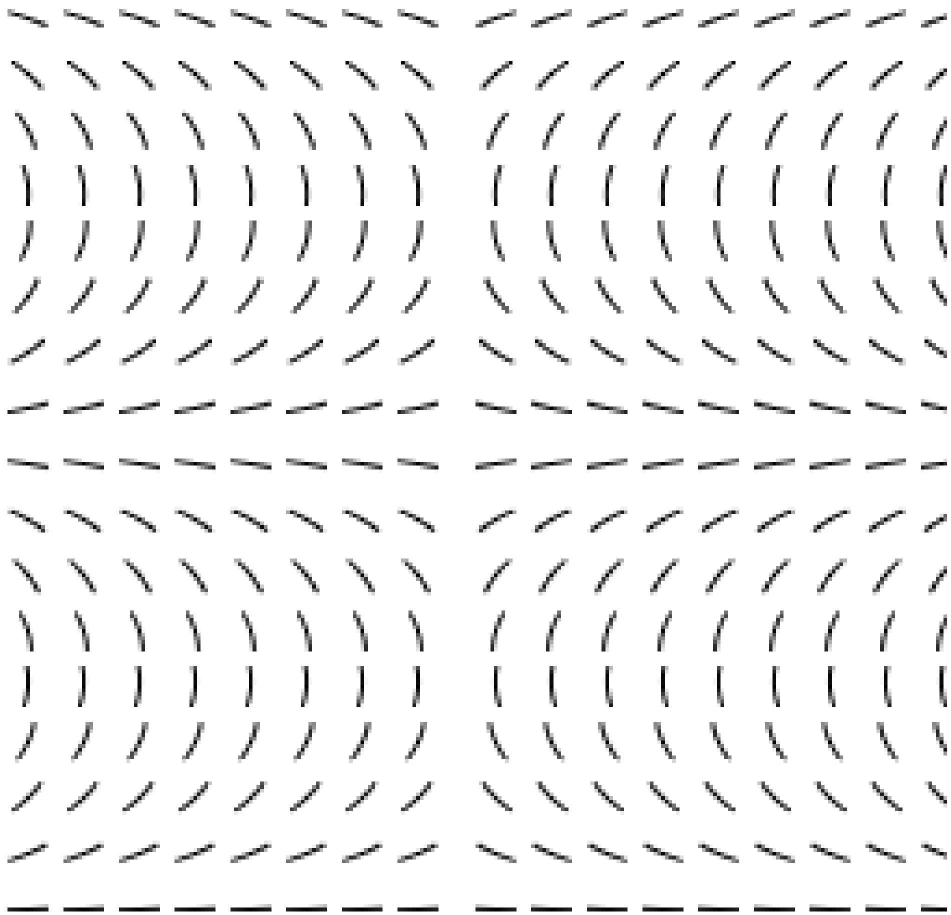
- Partition image into regions
- Assume motion is uniform in each region
- Estimate direction and magnitude of "uniform" motion in each region
- Use interpolation:

$$A \approx \sum I_k A_k$$

where A_k is defined by uniform motion in kth region and I_k is diagonal, with $\sum I_k = I$

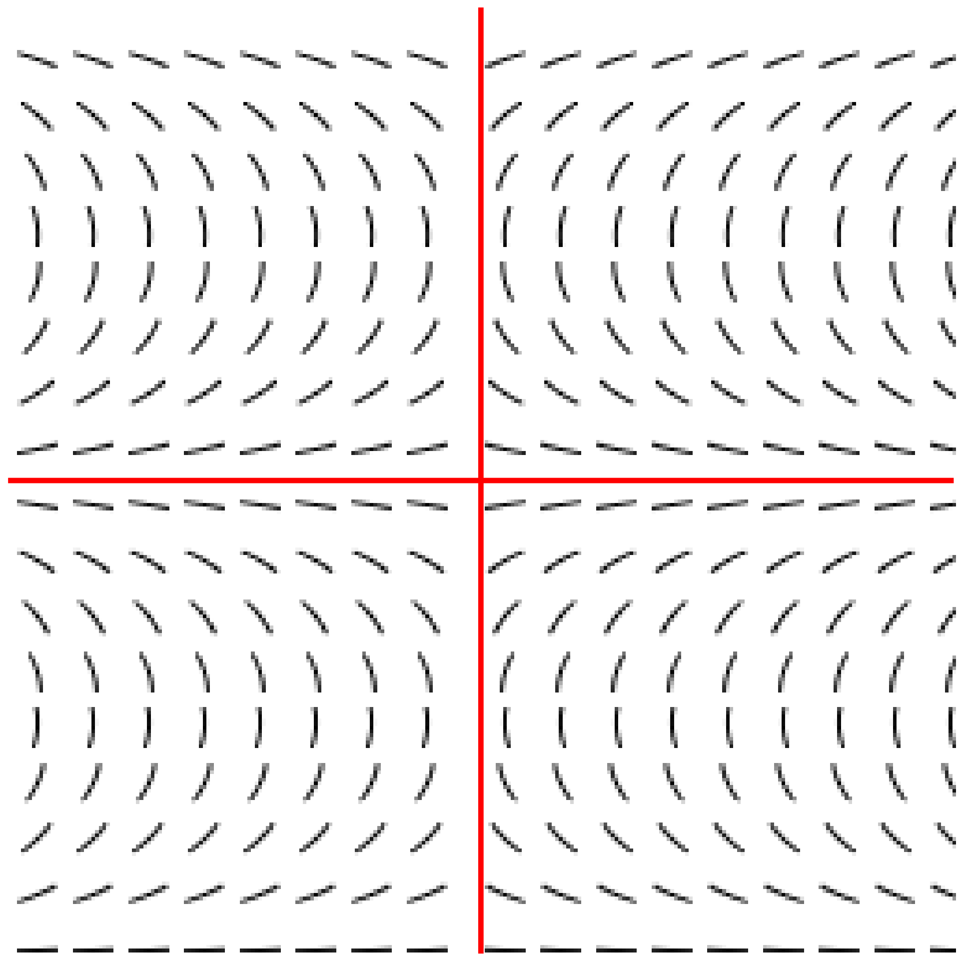
Region Partitioning

One region



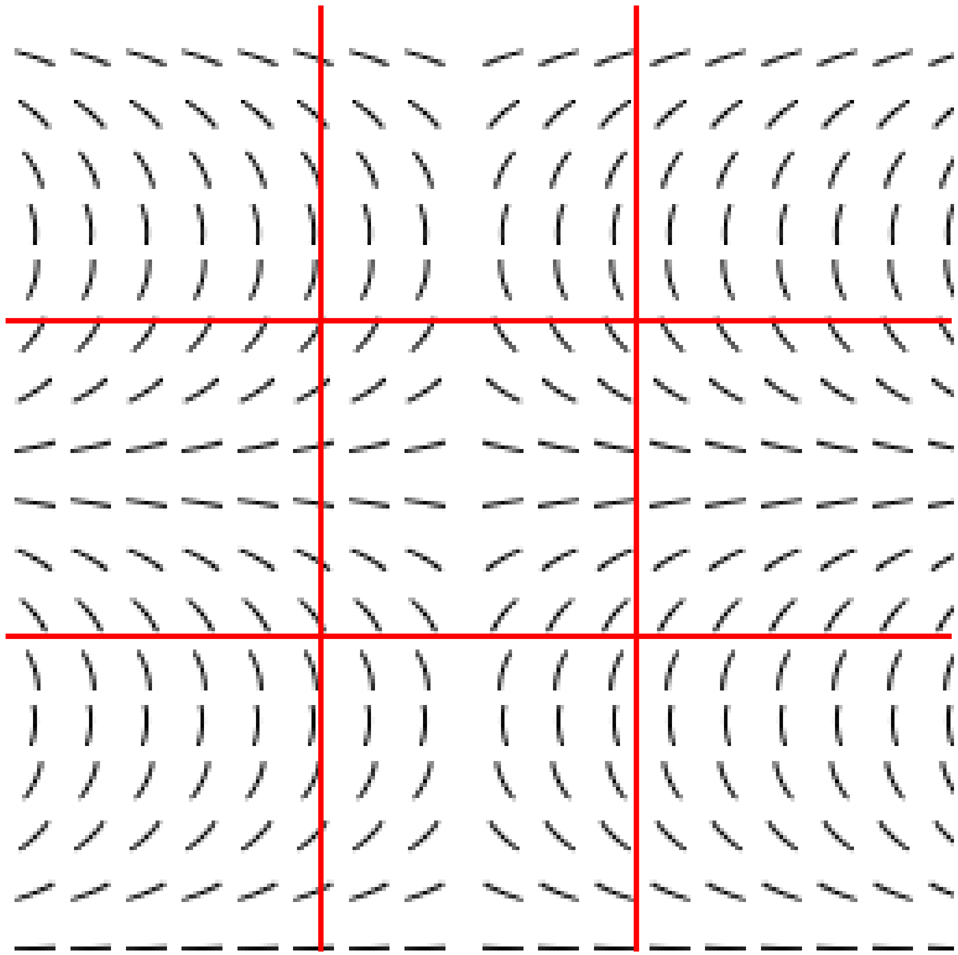
Region Partitioning

4 regions



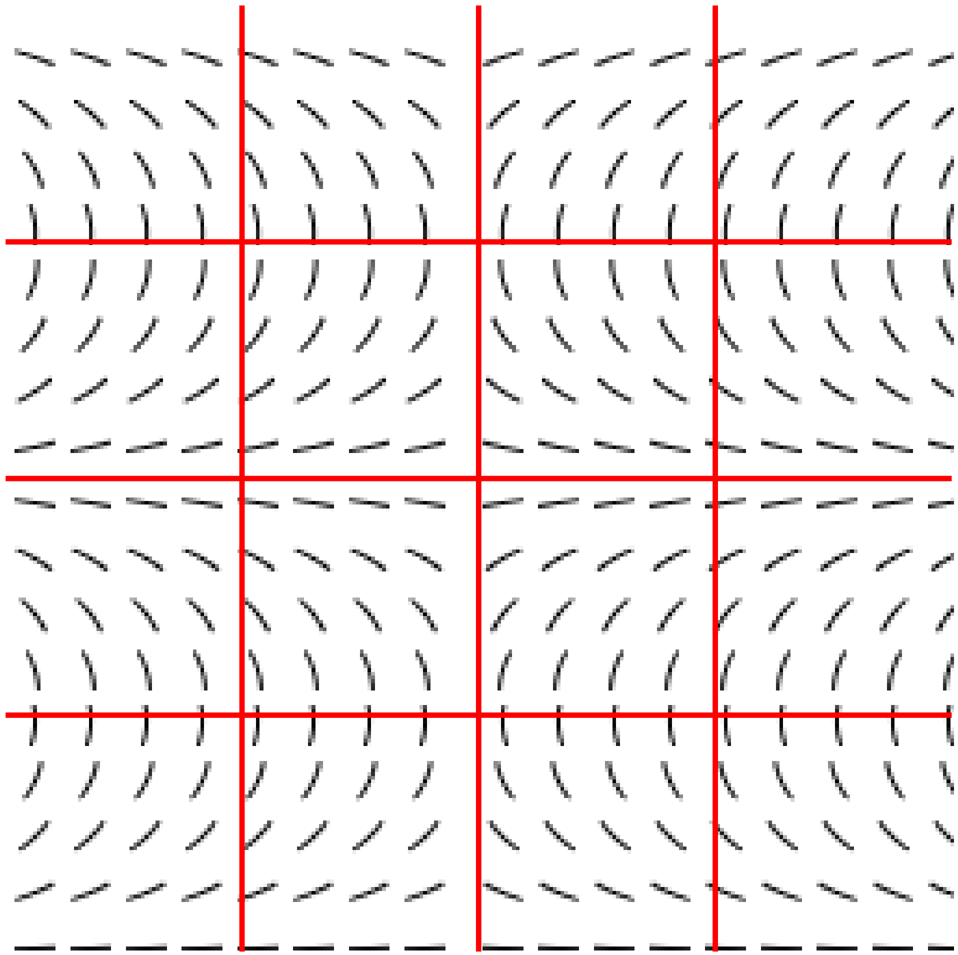
Region Partitioning

9 regions



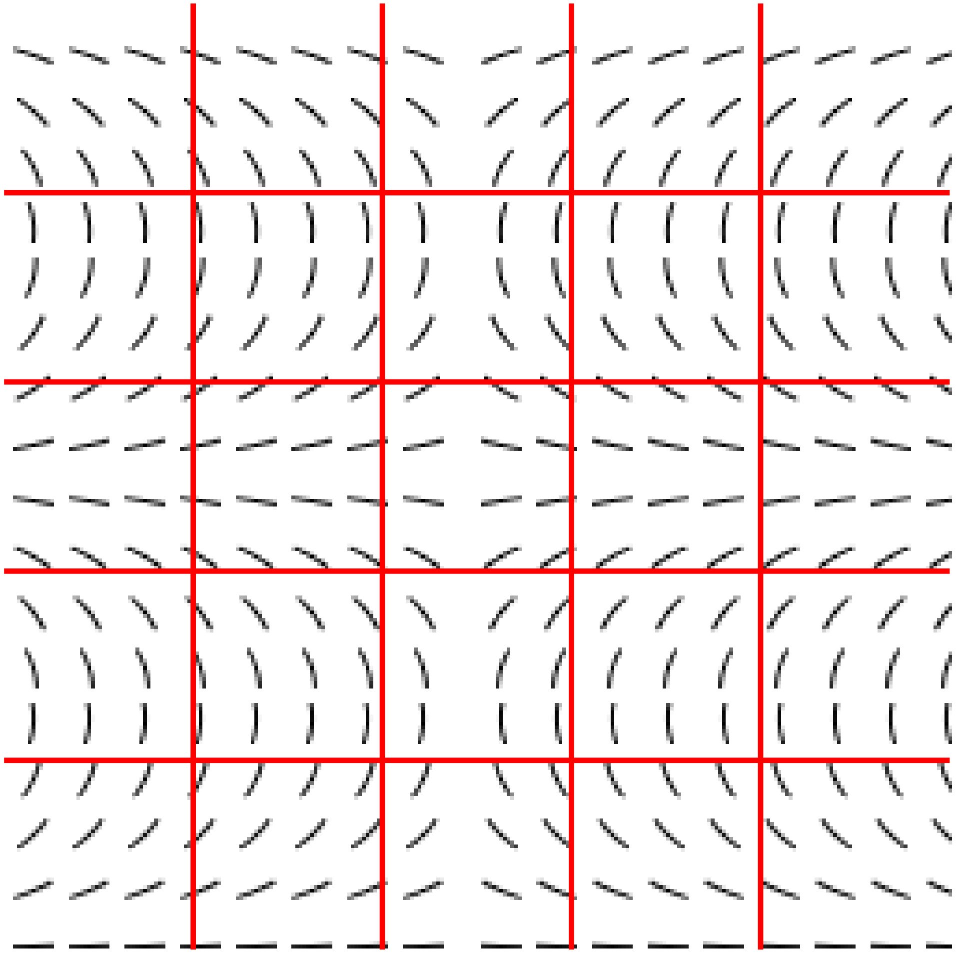
Region Partitioning

16 regions



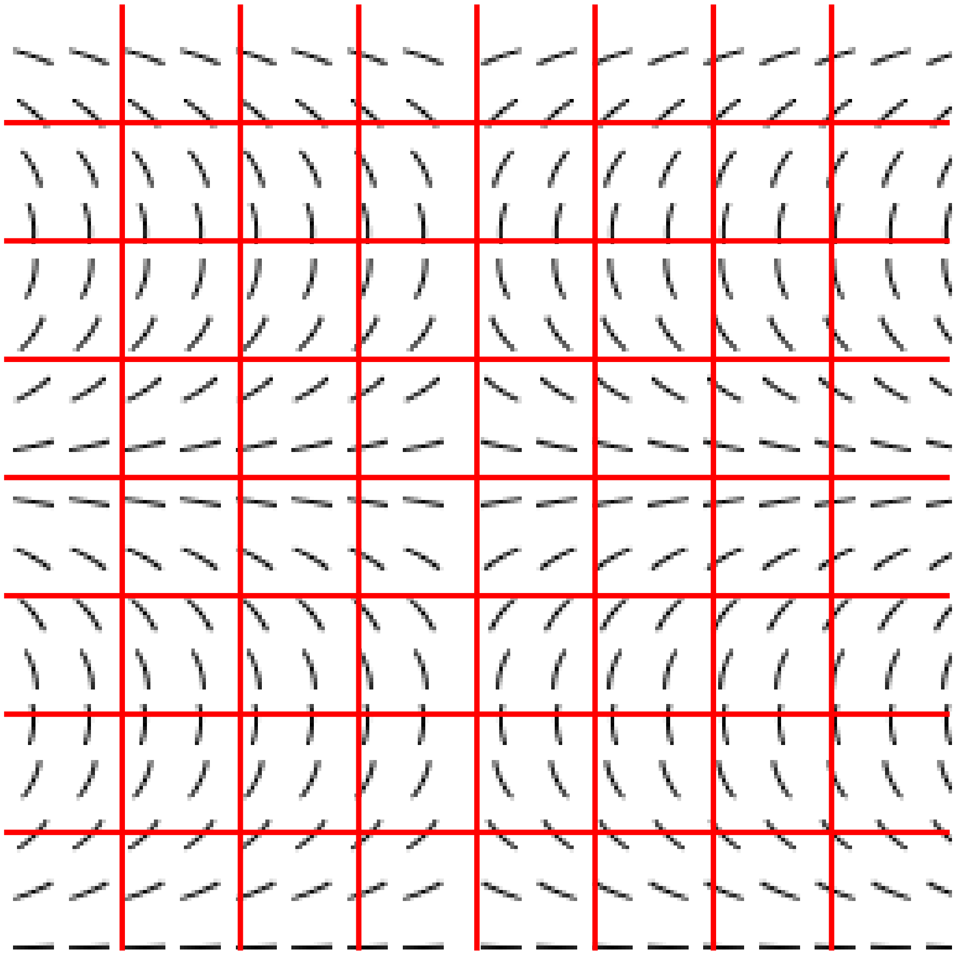
Region Partitioning

25 regions



Region Partitioning

64 regions



Deblurring Algorithms

Given A (or its approximation), need to solve

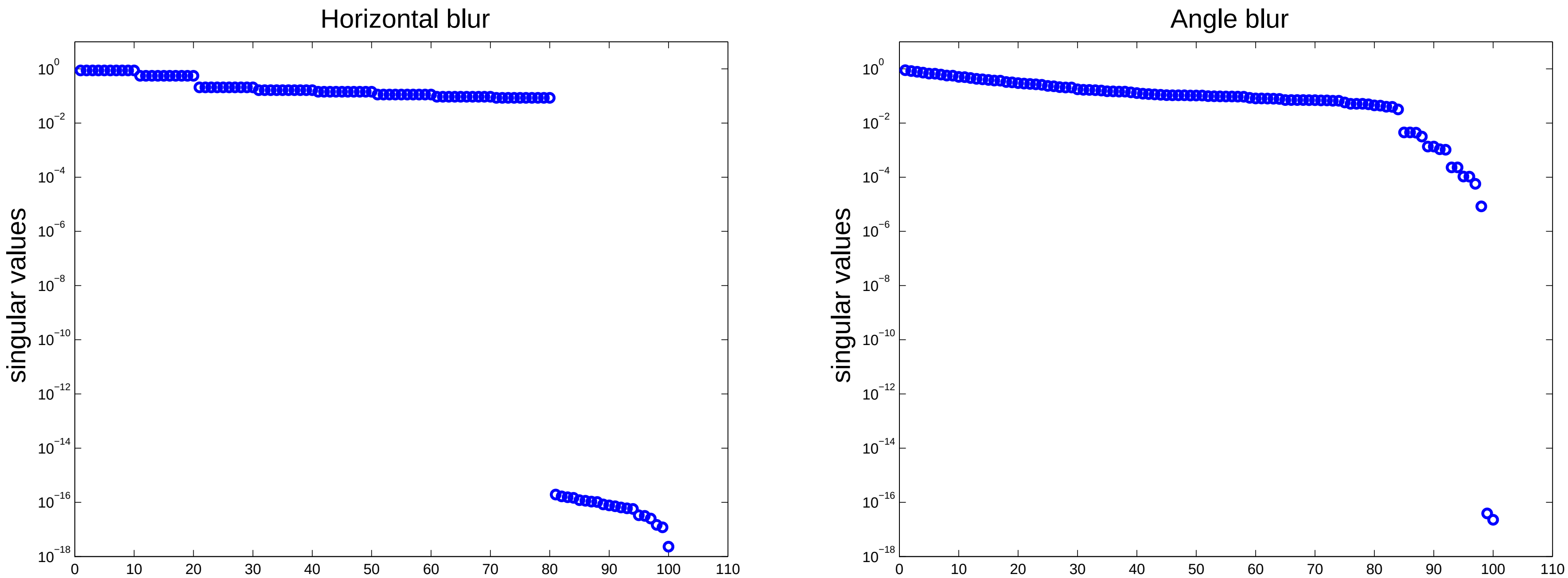
$$\mathbf{b} = A\mathbf{x} + \mathbf{n}$$

where

- A is ill-conditioned
  $\Rightarrow$ need regularization

- A is large
  $\Rightarrow$ usually need iterative method

Example Singular Value Distribution



Deblurring Algorithms

Possible regularization methods

- Truncated singular value decomposition (TSVD)

$$A = U\Sigma V^T \quad \Rightarrow \quad \mathbf{x}_{\text{tsvd}} = \sum_{i=1}^{k} \frac{\mathbf{u}_i^T \mathbf{b}}{\sigma_i} \mathbf{v}_i$$

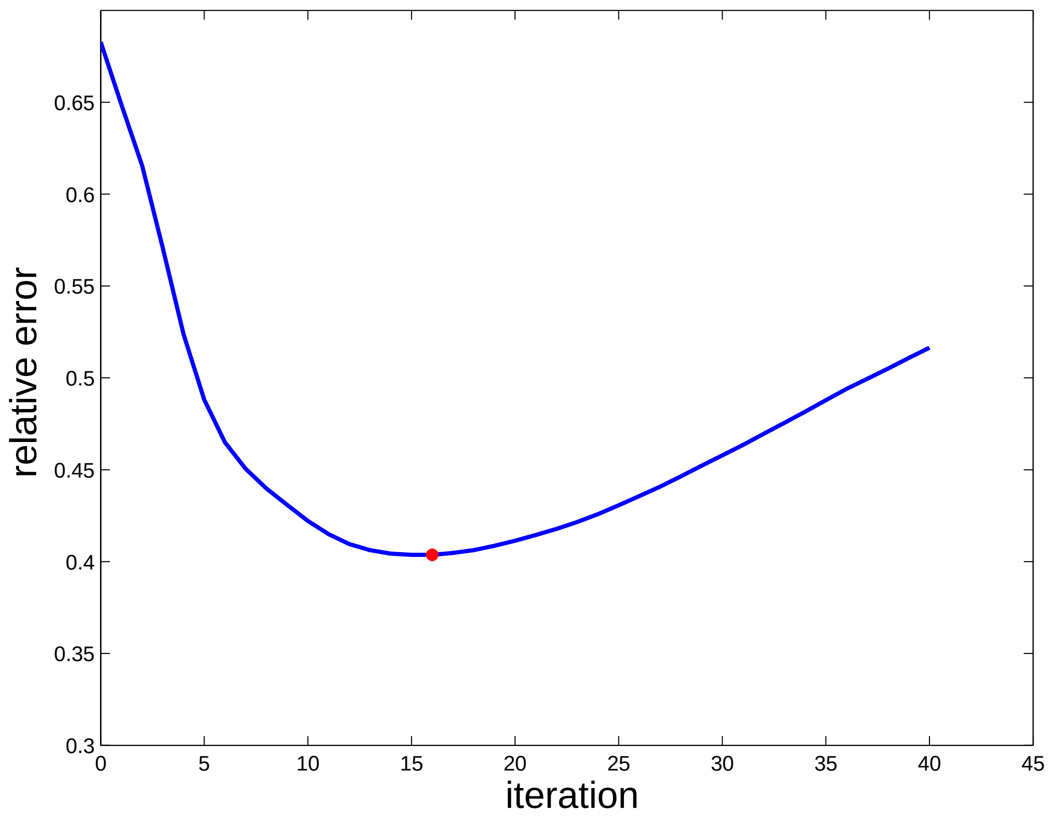
  (can use efficiently for horizontal or vertical blurs)

- Tikhonov:

$$\min \left\{ ||\mathbf{b} - A\mathbf{x}||^2 + \mu^2 ||L\mathbf{x}||^2 \right\}$$

- Iterative: Terminate iterations early
  (e.g., conjugate gradients)

Example of iterative regularization ...



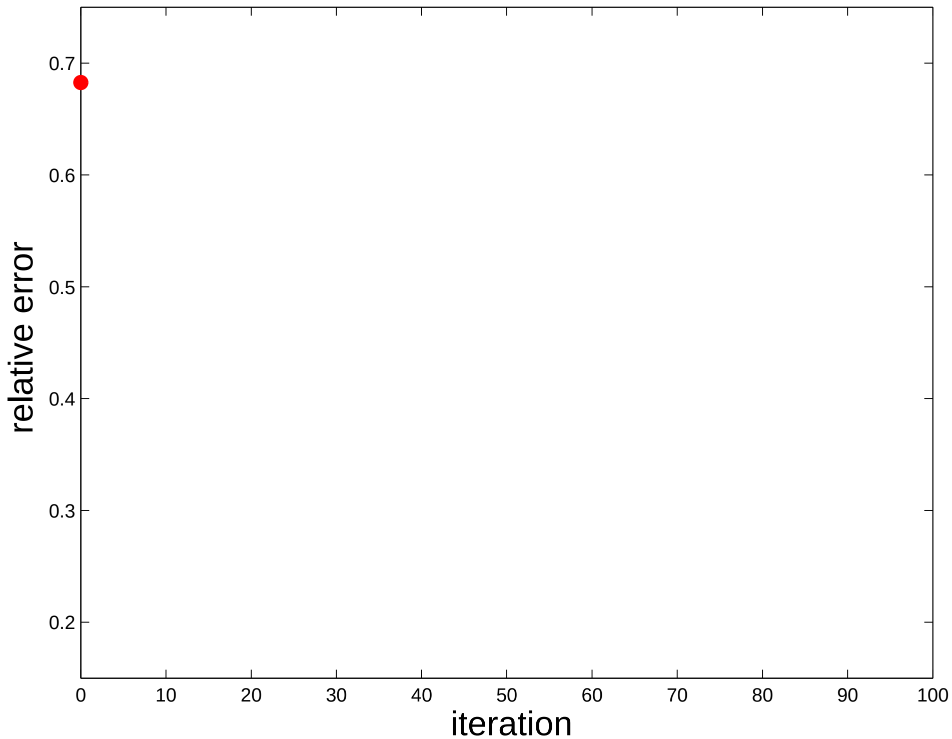
Numerical Examples

First consider text data:

True Image | Non−Uniform Blur

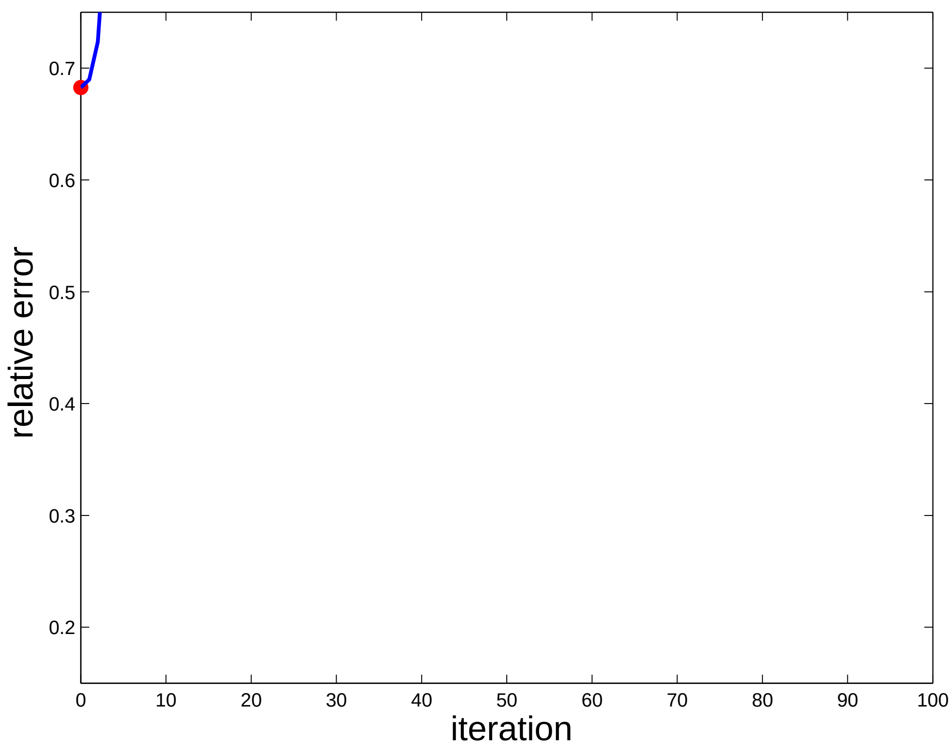
- Use CGLS, iterative regularization.
- For A,
  - Approximate motion on regions
  - Use motion information at every pixel

Numerical Examples



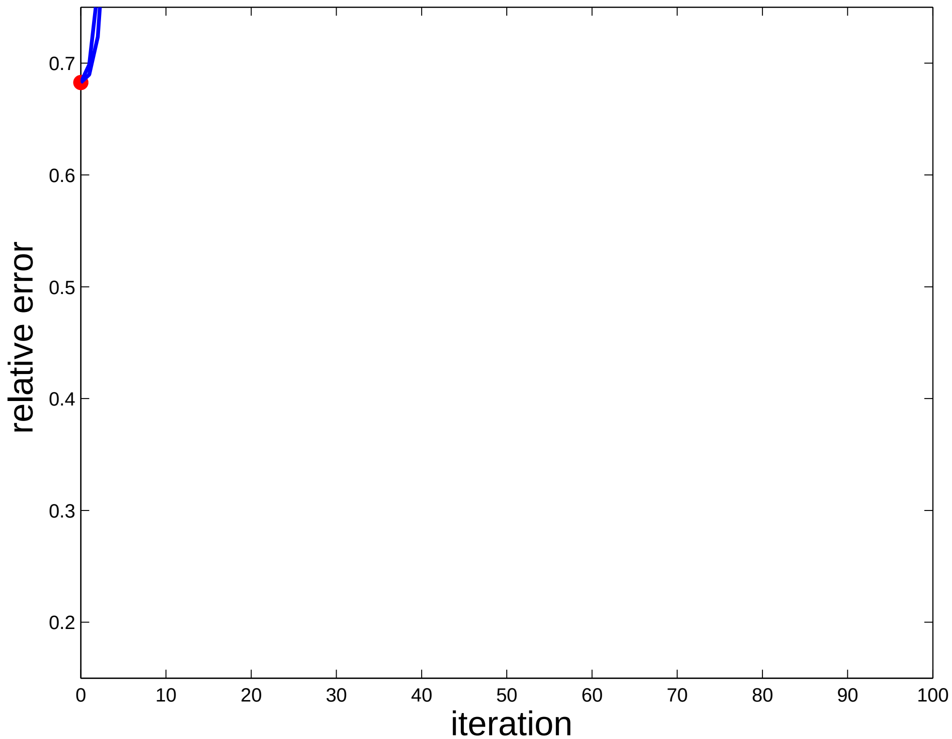
Blurred image

Numerical Examples



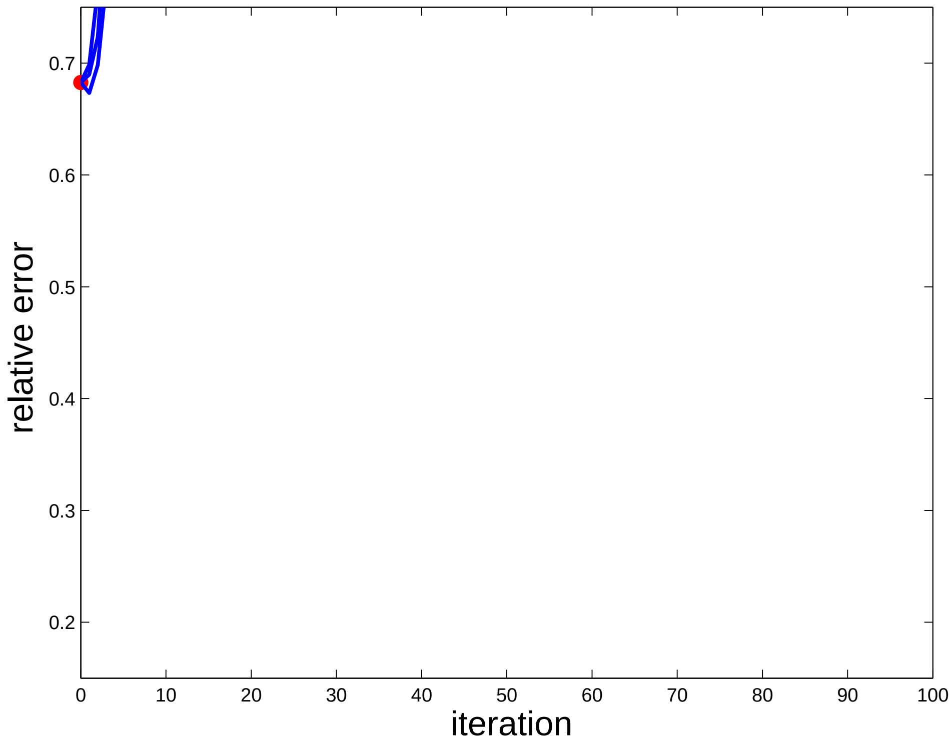
1 region, 0 iterations

Numerical Examples



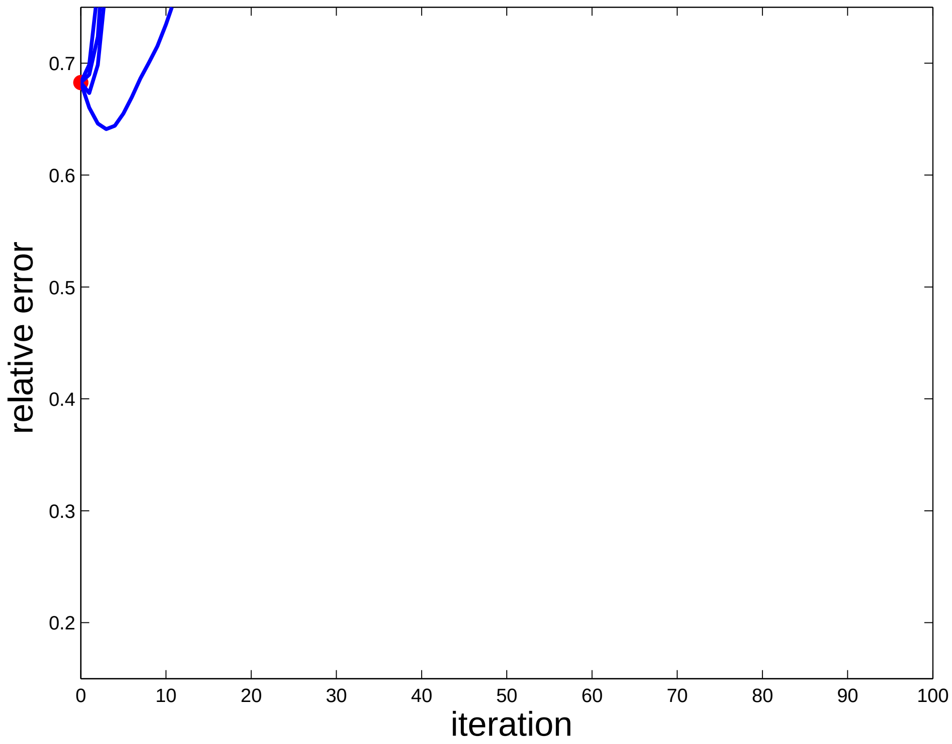
4 region, 0 iterations

Numerical Examples

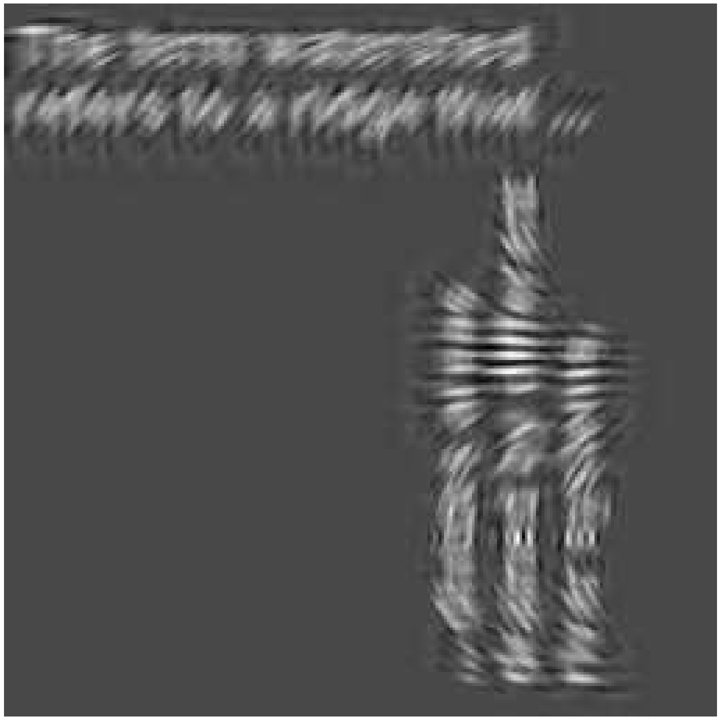


16 region, 1 iterations

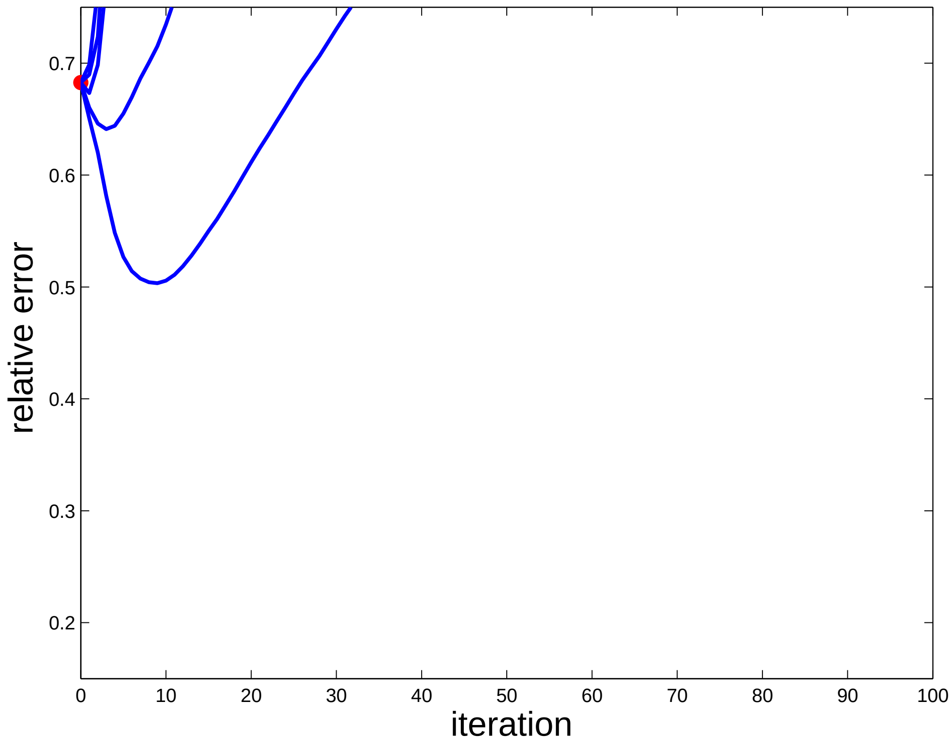
Numerical Examples



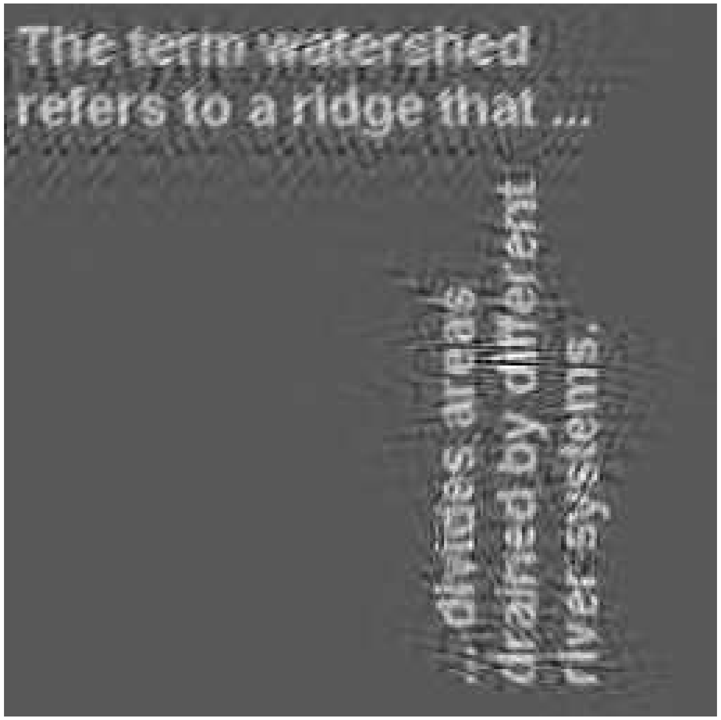
64 region, 3 iterations



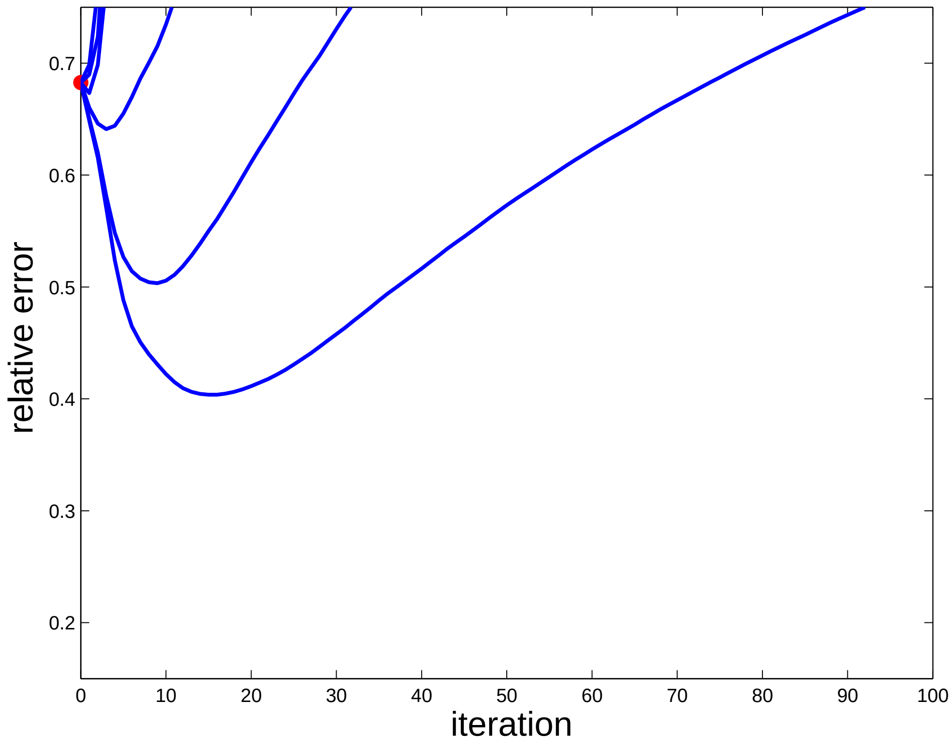
Numerical Examples



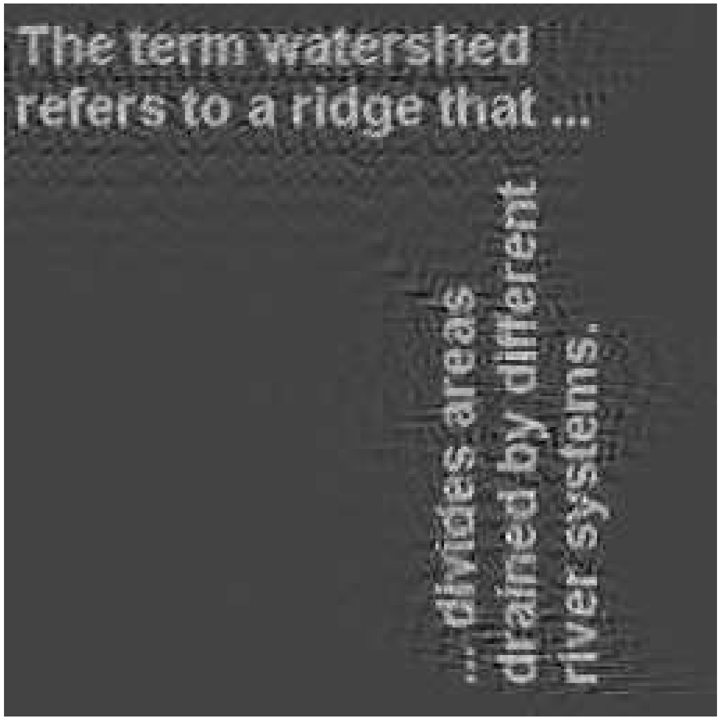
256 region, 9 iterations



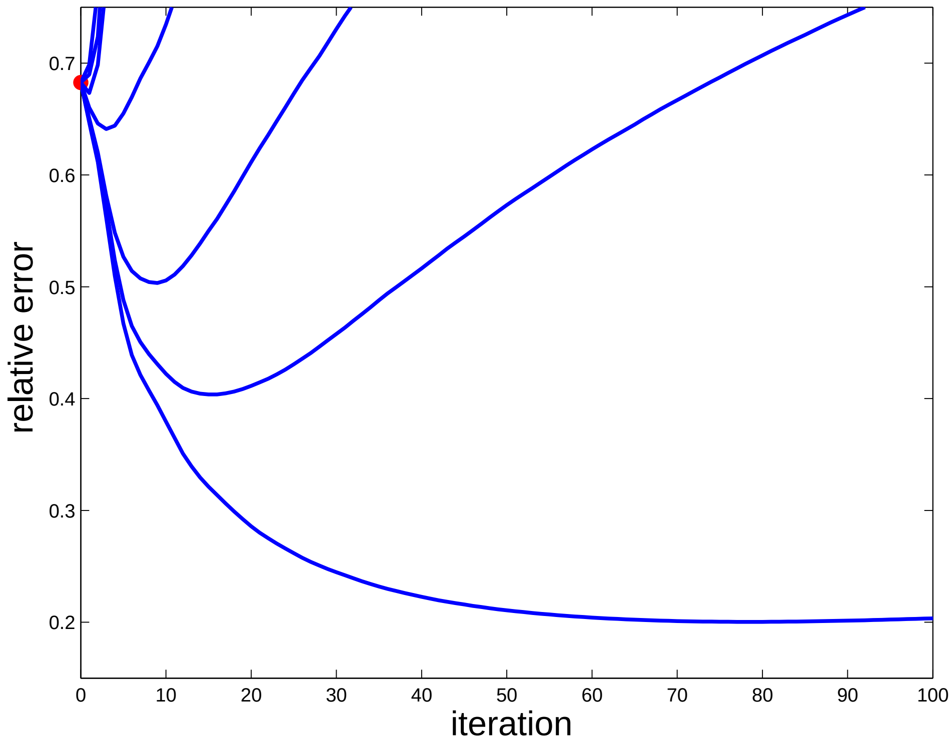
Numerical Examples



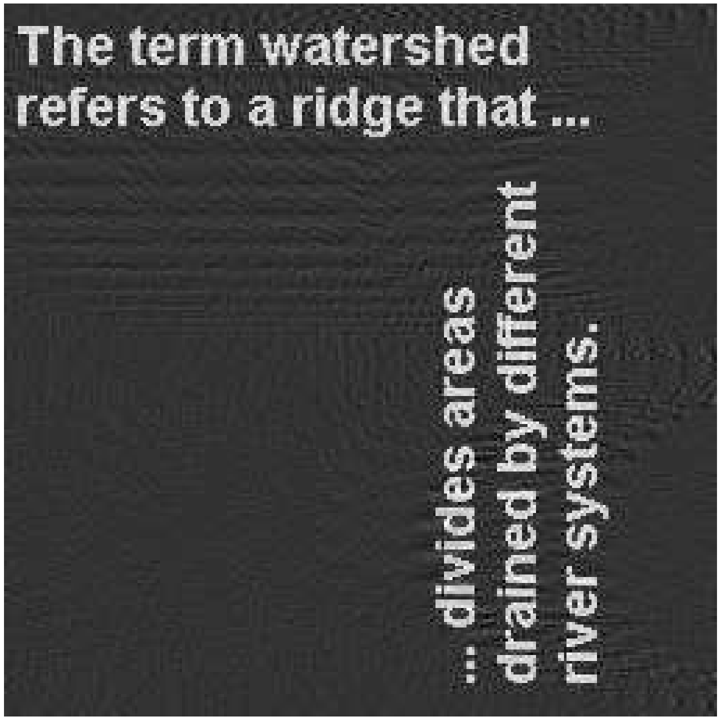
1024 region, 15 iterations



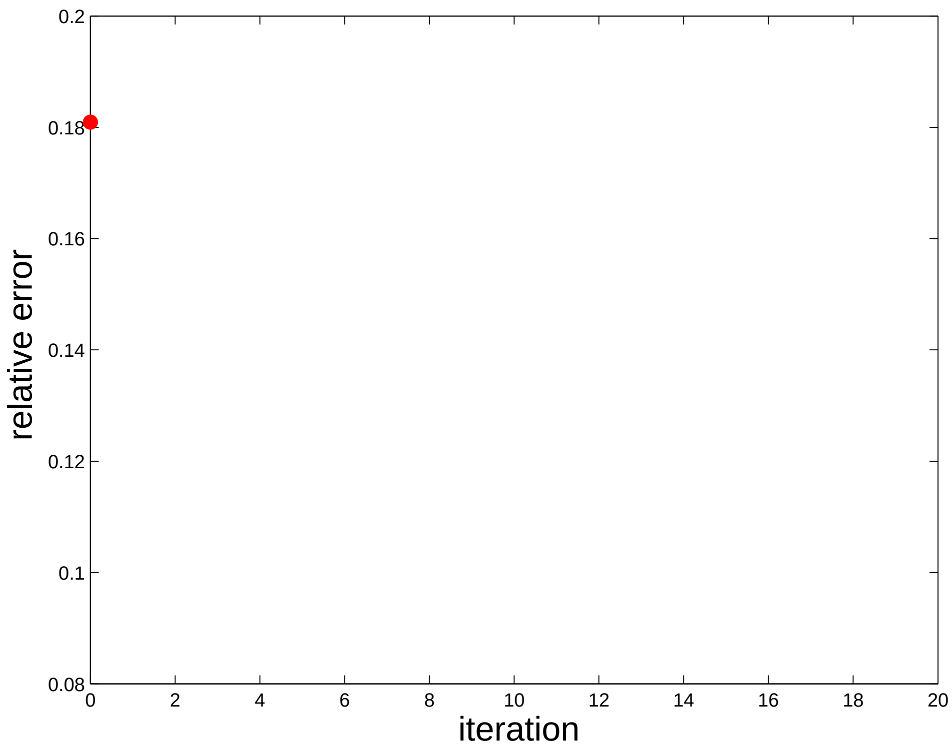
Numerical Examples



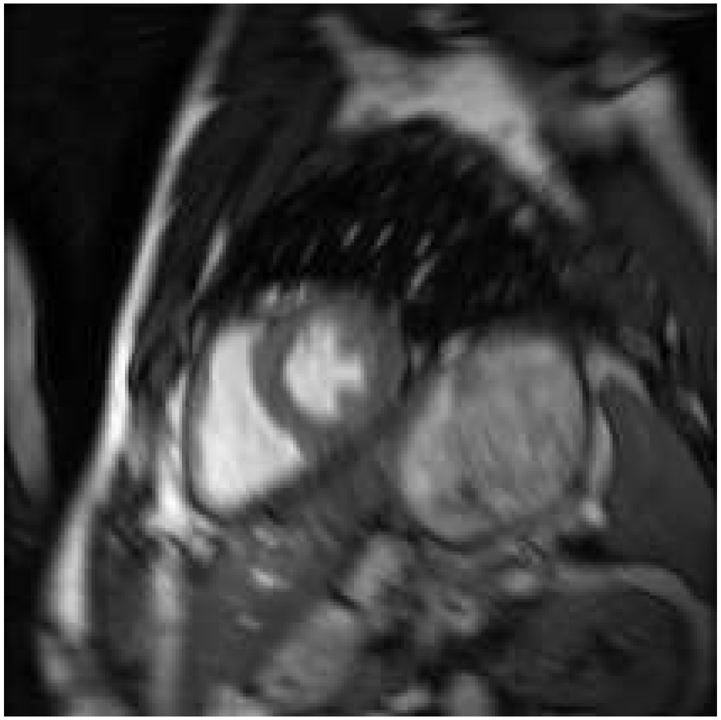
Every pixel, 79 iterations



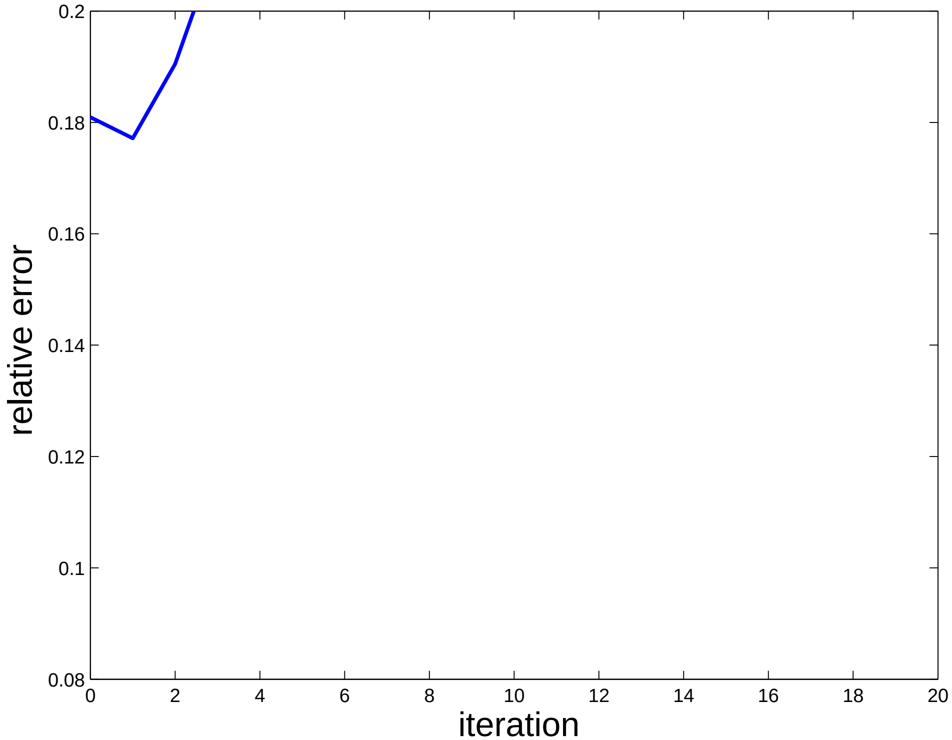
Numerical Examples



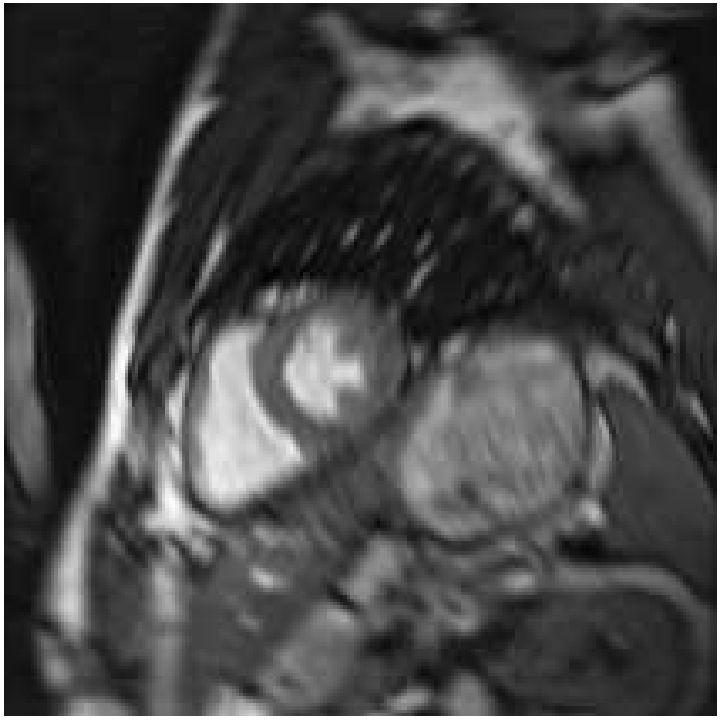
Blurred image



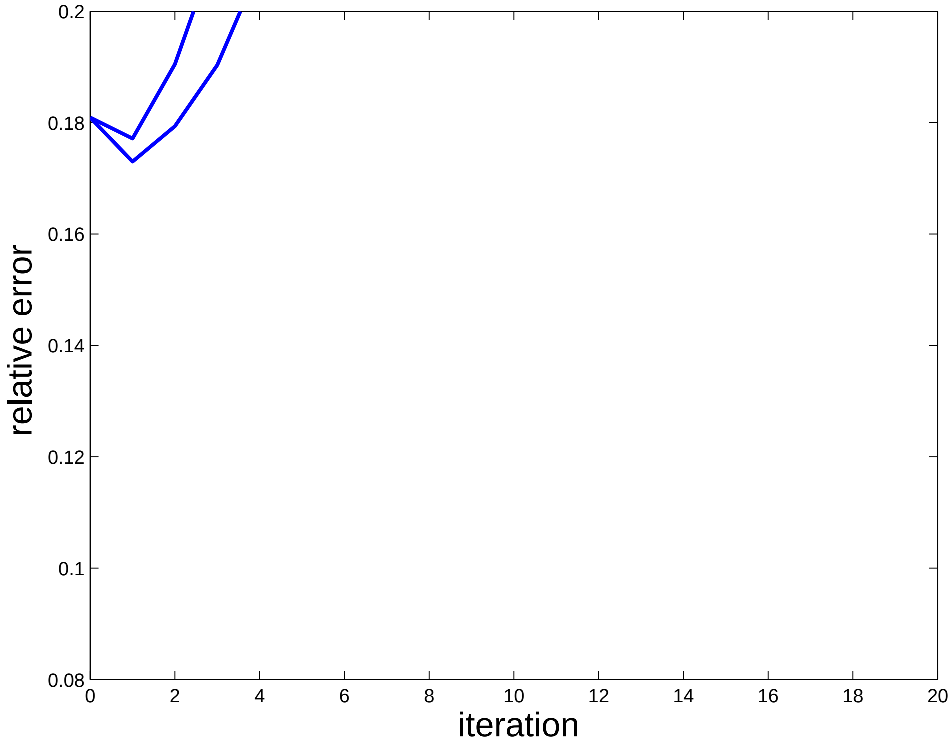
Numerical Examples



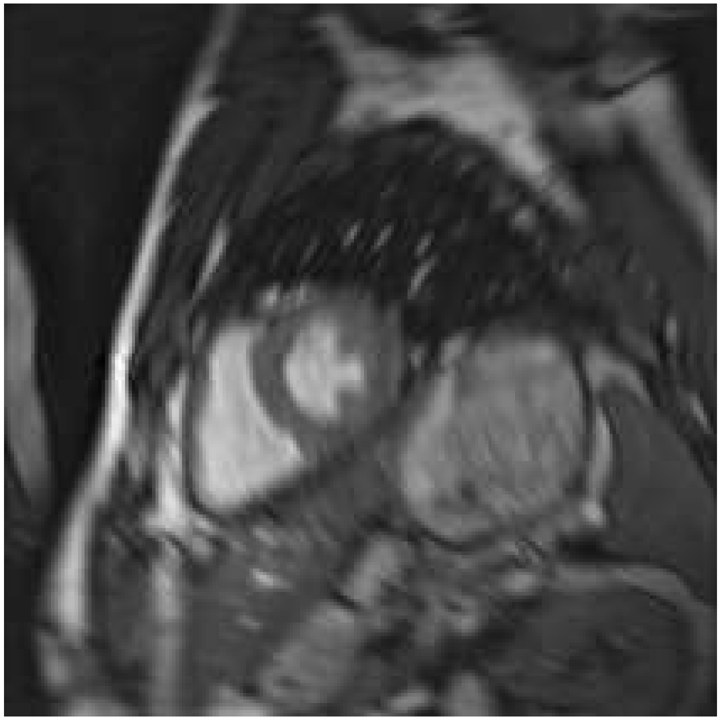
1 region, 1 iterations



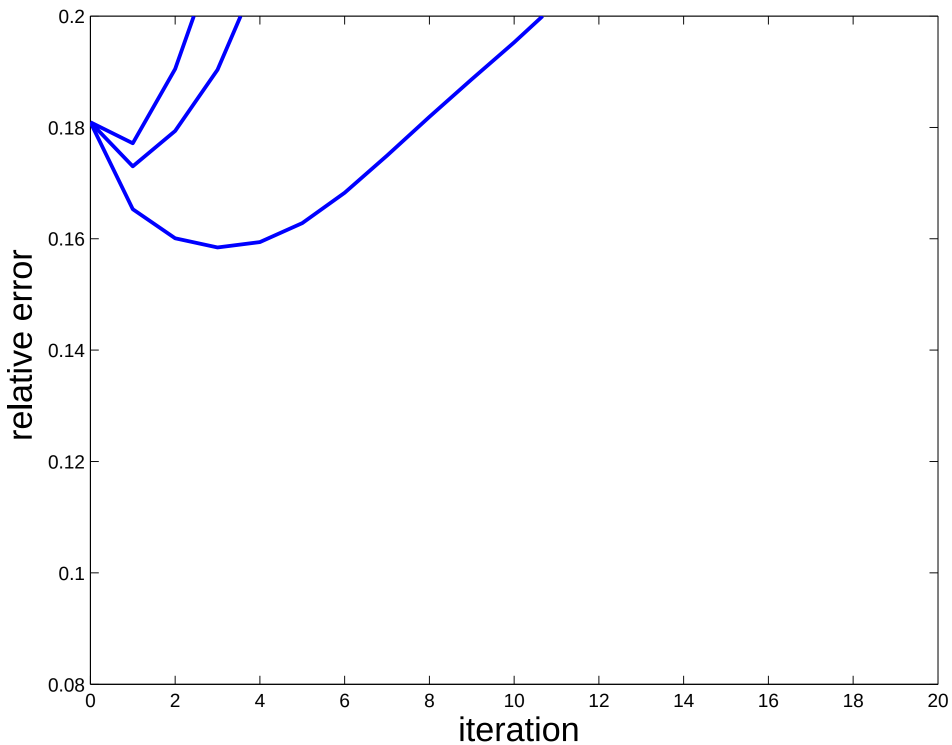
Numerical Examples



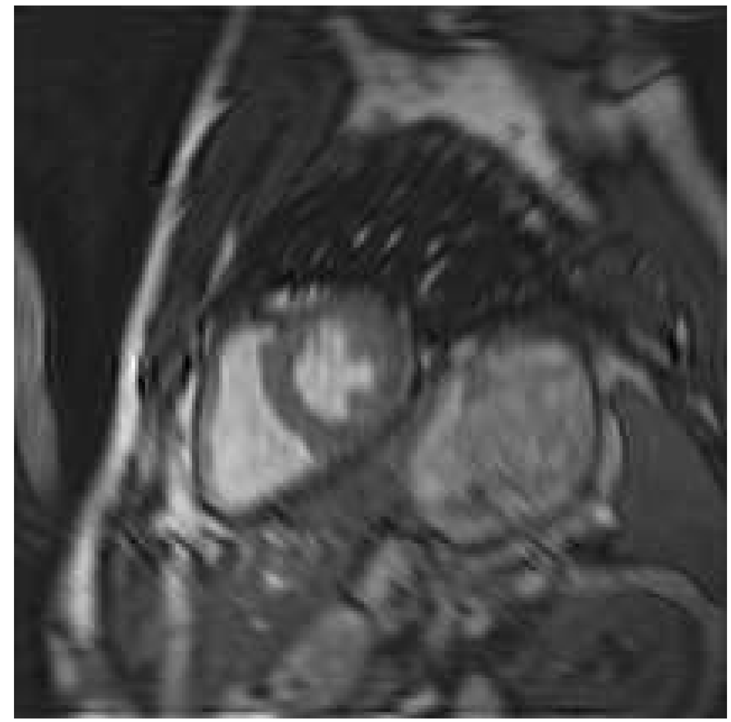
4 regions, 1 iteration



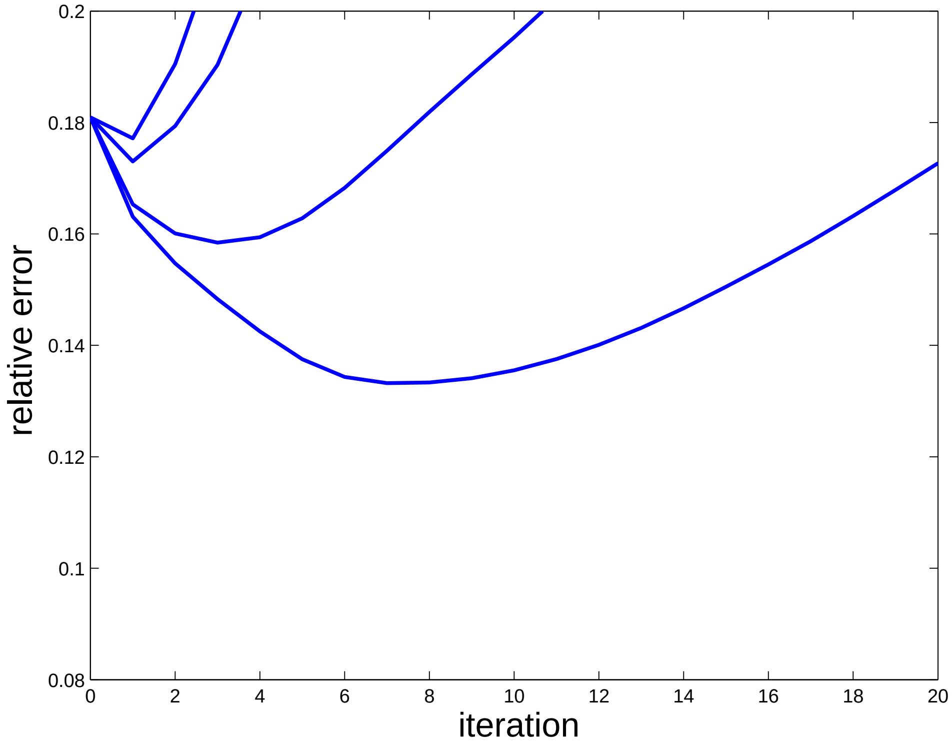
Numerical Examples



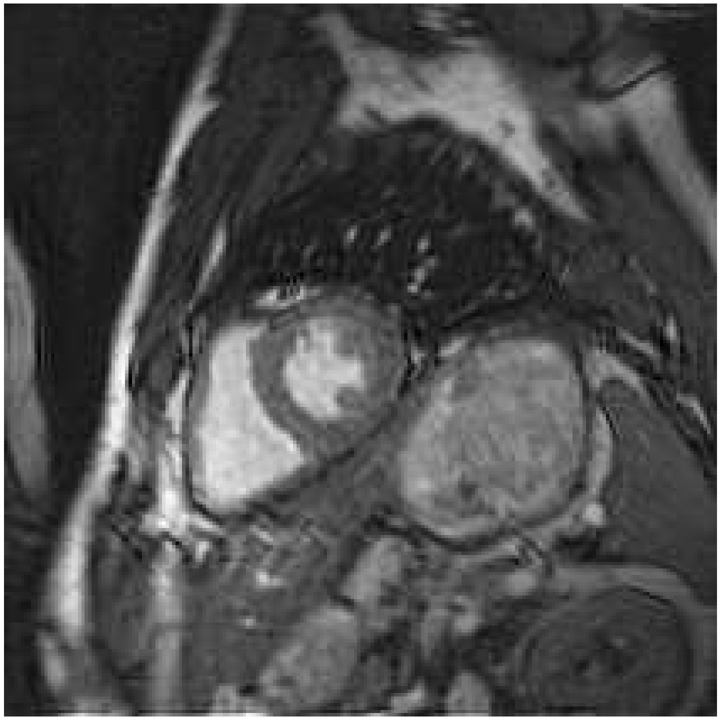
16 regions, 3 iterations



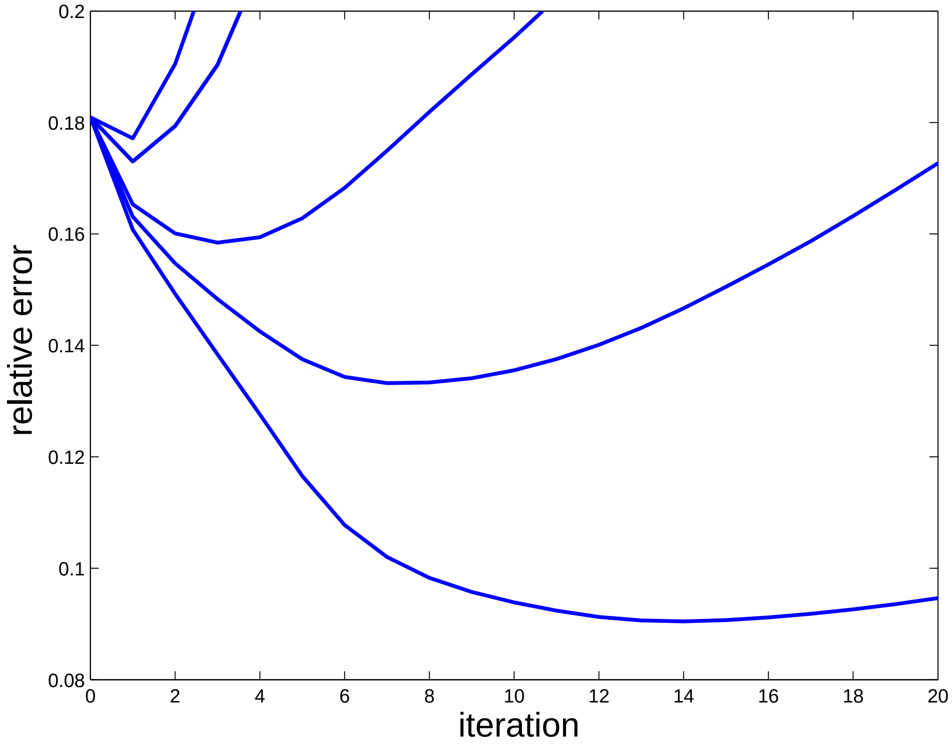
Numerical Examples



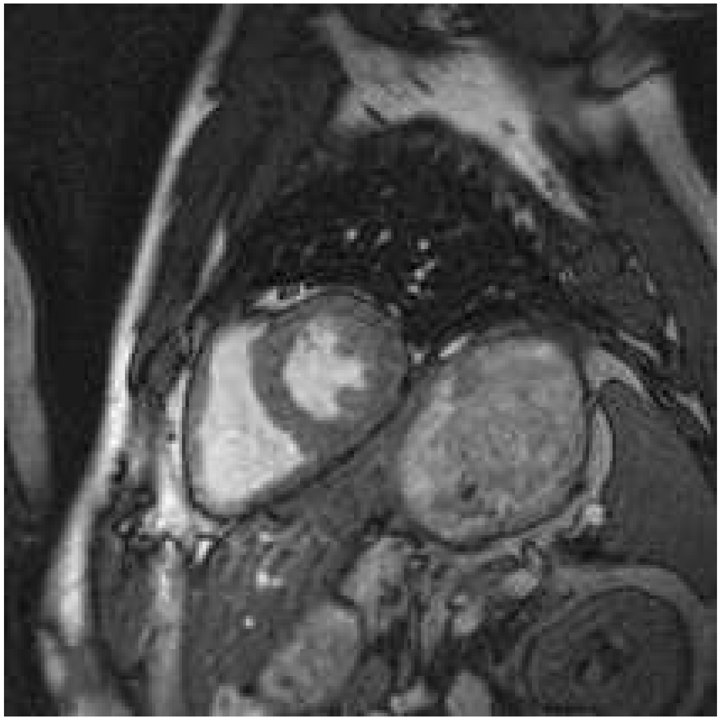
64 regions, 7 iterations



Numerical Examples



Every pixel, 14 iterations



Computational Issues

- Choosing a regularization parameter
- Fast matrix-vector multiplication
  should scale well with number of regions
- Preconditioning
  especially when using a lot of regions, or all pixel information
- How to get motion information?

Preconditioning

- A is ill-conditioned $\Rightarrow$ preconditioner is ill-conditioned
- How to regularize preconditioner?
- Relatively easy for circulant, and other transform based preconditioners.
- But, A is not well approximated by circulant matrix.
- Alternative approach – try:

$$A \approx C \otimes D$$

Preconditioning using Kronecker Products

Using the model: $$A = \sum_{k=1}^{p} I_k A_k$$

- Each A_k can be decomposed as:

$$A_k = \sum_{j=1}^{r} C_j^{(k)} \otimes D_j^{(k)}$$

- Therefore, our model for A is:

$$A = \sum_{k=1}^{p} I_k \left(\sum_{j=1}^{r} C_j^{(k)} \otimes D_j^{(k)} \right)$$

The End!

- It is possible to efficiently implement iterative methods for non-uniform motion blurs.
- More information provides substantially better restorations.
- However, more information results in slower convergence of iterative methods.

www.mathcs.emory.edu/$\sim$nagy