Pseudo-Schur complements and their properties

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Schur complements

The notion of **Schur complement** of a partitioned matrix with a square nonsingular block was introduced by **Issai Schur (1874–1941)** in **1917** *

We consider the partitioned matrix

$$\underset{(p+r)\times(q+s)}{M} = \begin{pmatrix} \underset{p\times q}{A} & \underset{p\times s}{B} \\ \underset{r\times q}{C} & \underset{r\times s}{D} \end{pmatrix}$$

* I. Schur, Potenzreihen im Innern des Einheitskreises, J. Reine Angew. Math., 147 (1917) 205–232.

Schur complements

$$\underset{(p+r)\times(q+s)}{M} = \begin{pmatrix} \underset{p\times q}{A} & \underset{p\times s}{B} \\ \underset{r\times q}{C} & \underset{r\times s}{\boxed{D}} \end{pmatrix}$$

If D is square and nonsingular, the Schur complement of D in M is denoted by (M/D) and defined by

$$(M/D) = A - BD^{-1}C$$

Moreover, if A is square, the **Schur determinantal formula** holds

$$\det(M/D) = \frac{\det M}{\det D}.$$

Schur complements

- The term **Schur complement** and the notation (M/D) has been introduced by [Haynsworth, 1968] in two papers.
- Appearences of Schur complement or Schur determinantal formula has been founded in the **1800s** (J.J. Sylvester (1814-1897) and Laplace (1749-1827)).
- They have
  - useful properties in linear algebra and matrix techniques
  - important applications in numerical analysis and applied mathematics (multigrids, preconditioners, statistics, probability, . . .).

Extensive exposition and applications to various branches of mathematics in

F.-Z. Zhang ed., *The Schur Complement and Its Applications*, Springer, in press.

Generalizations

Several generalizations of the Schur complement can be found in the literature.

Here we consider the generalization introduced by [Carlson - Haynsworth - Markam, 1974] and by [Marsiglia - Styan, 1974], but also implicitly considered by [Rohde, 1965] and by [Ben-Israel, 1969]

where the block D is rectangular and/or singular, and so we

will replace its **inverse** by its **pseudo–inverse**.

Pseudo-Schur complements

$$\underset{(p+r)\times(q+s)}{M} = \begin{pmatrix} \underset{p\times q}{A} & \underset{p\times s}{B} \\ \underset{r\times q}{C} & \underset{r\times s}{\boxed{D}} \end{pmatrix}$$

If D is rectangular or square AND singular, we define the **Pseudo-Schur complement** $(M/D)_{\mathcal{P}}$ of D in M by

$$(M/D)_{\mathcal{P}} = A - BD^{\dagger}C$$

where $D^{\dagger}$ is the **pseudo-inverse** (or **Moore-Penrose inverse**) of D.

Remark: We can also define $(M/A)_{\mathcal{P}} = D - CA^{\dagger}B$, $(M/B)_{\mathcal{P}} = C - DB^{\dagger}A$, and $(M/C)_{\mathcal{P}} = B - AC^{\dagger}D$.

Pseudo-Schur compl. - Multiple blocks

Pseudo–Schur complements can also be defined for matrices partitioned into an arbitrary number of blocks.

We consider the $n \times m$ block matrix

$$M = \begin{pmatrix} A_{11} & \cdots & A_{1j} & \cdots & A_{1m} \\ \vdots & & \vdots & & \vdots \\ A_{i1} & \cdots & \boxed{A_{ij}} & \cdots & A_{im} \\ \vdots & & \vdots & & \vdots \\ A_{n1} & \cdots & A_{nj} & \cdots & A_{nm} \end{pmatrix}$$

Pseudo-Schur compl. - Multiple blocks

We denote by

- $A^{(i,j)}$ the $(n-1) \times (m-1)$ block matrix obtained by deleting the ith row of blocks and the jth column of blocks of M

$$
A^{(i,j)} = \begin{pmatrix} A_{11} & \cdots & A_{1j} & \cdots & A_{1m} \\ \vdots & & \vdots & & \vdots \\ A_{i1} & \cdots & A_{ij} & \cdots & A_{im} \\ \vdots & & \vdots & & \vdots \\ A_{n1} & \cdots & A_{nj} & \cdots & A_{nm} \end{pmatrix}
$$

Pseudo-Schur compl. - Multiple blocks

We denote by

- $A^{(i,j)}$ the $(n-1)\times(m-1)$ block matrix obtained by deleting the ith row of blocks and the jth column of blocks of M
- $B_j^{(i)}$ the block matrix obtained by deleting the ith block of the jth column of M

$$B_j^{(i)} = \begin{pmatrix} A_{1j} \\ \vdots \\ A_{i-1,j} \\ \cancel{A_{i,j}} \\ A_{i+1,j} \\ \vdots \\ A_{nj} \end{pmatrix}$$

Pseudo-Schur compl. - Multiple blocks

We denote by

- $A^{(i,j)}$ the $(n-1) \times (m-1)$ block matrix obtained by deleting the ith row of blocks and the jth column of blocks of M
- $B_j^{(i)}$ the block matrix obtained by deleting the ith block of the jth column of M
- $C_i^{(j)}$ the block matrix obtained by deleting the jth block of the ith row of M

$$C_i^{(j)} = (A_{i1}, \ldots, A_{i,j-1}, \cancel{A_{i,j}}\, A_{i,j+1}, \ldots, A_{im})$$

Pseudo-Schur compl. - Multiple blocks

We denote by

- $A^{(i,j)}$ the $(n-1) \times (m-1)$ block matrix obtained by deleting the ith row of blocks and the jth column of blocks of M
- $B_j^{(i)}$ the block matrix obtained by deleting the ith block of the jth column of M
- $C_i^{(j)}$ the block matrix obtained by deleting the jth block of the ith row of M

The **pseudo–Schur complement of A_{ij} in M** is defined as

$$(M/A_{ij})_{\mathcal{P}} = A^{(i,j)} - B_j^{(i)} A_{ij}^{\dagger} C_i^{(j)}.$$

Pseudo-inverses

Definition: The **Pseudo-inverse** $A^\dagger$ of a rectangular or square singular matrix A is the **unique matrix** satisfying the four Penrose conditions

$$\begin{aligned} A^\dagger A A^\dagger &= A^\dagger \\ A A^\dagger A &= A \\ (A^\dagger A)^T &= A^\dagger A \\ (A A^\dagger)^T &= A A^\dagger \end{aligned}$$

Remark: If only some of the Penrose conditions are satisfied, the matrix (denoted by A^-) is called a **generalized inverse**.

Pseudo-inverses and linear systems

The Pseudo-inverse notion is related to the least squares solution of systems of linear equations in partitioned form. In fact, it is well known that, if we consider the rectangular system

$$A\mathbf{x} = \mathbf{b}, \quad A \in \mathbb{R}^{p\times q}, \text{ rank } (A) = k \le \min(p,q), \mathbf{x} \in \mathbb{R}^q, \ \mathbf{b} \in \mathbb{R}^p$$

the least square solution of the problem of finding

$$\min_{\mathbf{x}\in V} \|\mathbf{x}\|_2, \quad V = \{\mathbf{x} \in \mathbb{R}^q \mid \|A\mathbf{x} - \mathbf{b}\|_2 = \min\}$$

is given by

$$\mathbf{x} = A^{\dagger}\mathbf{b}$$

Pseudo-inverses

General expression: If rank $(A) = k \leq \min(p, q)$, and if we consider the SVD decomposition

$$A = U\,\Sigma\,V^T$$

where $U \in \mathbb{R}^{p\times p}$ and $V \in \mathbb{R}^{q\times q}$ are orthogonal and

$$\Sigma = \begin{pmatrix} \Sigma_k & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{R}^{p\times q}$$

with $\Sigma_k =$ diag $(\sigma_1, \ldots, \sigma_k)$, $\sigma_1 \geq \sigma_2 \geq \cdots \sigma_k > 0$, then we have

$$A^\dagger = V \begin{pmatrix} \Sigma_k^{-1} & 0 \\ 0 & 0 \end{pmatrix} U^T.$$

Pseudo-inverses

General properties:

$$
\begin{aligned}
(A^{\dagger})^{\dagger} &= A \\
(A^{\dagger})^{T} &= (A^{T})^{\dagger} \\
(A^{T}A)^{\dagger} &= A^{\dagger}(A^{\dagger})^{T}
\end{aligned}
$$

Pseudo-inverses - Particular cases

If we consider particular cases, expression of $A^\dagger$ simplify and additional properties hold.

Case 1 If $p \geq q$ and rank $(A) = q$, then

$$A^\dagger = (A^T A)^{-1} A^T$$

and we have

$$A^\dagger A = I_q$$

Case 2 If $p \leq q$ and rank $(A) = p$, then

$$A^\dagger = A^T (A A^T)^{-1}$$

and it holds

$$A A^\dagger = I_p$$

Pseudo-inverse of a product

In general,

$$(AB)^\dagger \neq B^\dagger A^\dagger$$

From the two particular cases it follows that, if

$$A \in \mathbb{R}^{p\times q} \text{ and } B \in \mathbb{R}^{q\times m}$$

with $p \geq q$ and $q \leq m$, and rank $(A) =$ rank $(B) = q$ then (Å. Björck, 1996)

$$(AB)^\dagger = B^\dagger A^\dagger = B^T(BB^T)^{-1}(A^TA)^{-1}A^T$$

Remark: Other necessary and sufficient conditions for having $(AB)^\dagger = B^\dagger A^\dagger$ are given by [Greville, 1966].

Pseudo-inverses - Properties

***Properties*:**

Case 1 If $p \geq q$ and rank $(A) = q$, then

$$(AA^{\dagger})^{\dagger} = AA^{\dagger}$$

Case 2 If $p \leq q$ and rank $(A) = p$, then

$$(A^{\dagger}A)^{\dagger} = A^{\dagger}A$$

Schur complements - Gauss

Schur complements are related to **Gaussian factorization** and to the solution of **systems of linear equations**.
Let M a square partitioned matrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

If A is square and nonsingular, we have the factorization

$$M = \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix} \begin{pmatrix} A & B \\ 0 & D - CA^{-1}B \end{pmatrix}$$

from which the **Schur determinantal formula** immediately holds.

Schur complements - linear systems

If both A and D are square and nonsingular, and if we consider the system

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix}$$

the solution is

$$\begin{aligned} \mathbf{x} &= (M/D)^{-1}(\mathbf{u} - BD^{-1}\mathbf{v}) \\ \mathbf{y} &= (M/A)^{-1}(\mathbf{v} - CA^{-1}\mathbf{u}) \end{aligned}$$

Pseudo-Schur complements - Gauss

Similarly **[MRZ, 2004]**, let M a partitioned matrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

Case 1

If $D \in \mathbb{R}^{r \times s}$ with $r \geq s$ and $\text{rank}(D) = s$, then $D^\dagger D = I_s$ and it follows

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I_p & BD^\dagger \\ 0_{r \times p} & I_r \end{pmatrix} \begin{pmatrix} (M/D)_{\mathcal{P}} & 0_{p \times s} \\ C & D \end{pmatrix}$$

Pseudo-Schur compl. - linear systems

So, the system

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix}$$

becomes a block triangular system and,

if $p \geq q$ and $\text{rank}((M/D)_{\mathcal{P}}) = q$,

we have

$$\begin{aligned} \mathbf{x} &= (M/D)_{\mathcal{P}}^{\dagger}(\mathbf{u} - BD^{\dagger}\mathbf{v}) \\ \mathbf{y} &= D^{\dagger}(\mathbf{v} - C\mathbf{x}) \end{aligned}$$

Pseudo-Schur complements - Gauss

Case 2

If $D \in \mathbb{R}^{r\times s}$ with $r \leq s$ and rank$(D) = r$, then $DD^\dagger = I_r$ and it follows

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} (M/D)_{\mathcal{P}} & B \\ 0_{r\times q} & D \end{pmatrix} \begin{pmatrix} I_q & 0_{q\times s} \\ D^\dagger C & I_s \end{pmatrix}$$

Pseudo-Schur compl. - linear systems

So, the system

$$M^T \begin{pmatrix} \mathbf{x}' \\ \mathbf{y}' \end{pmatrix} = \begin{pmatrix} \mathbf{u}' \\ \mathbf{v}' \end{pmatrix}$$

becomes a block triangular system and,

if $p \leq q$ and $\text{rank}((M/D)^T_{\mathcal{P}}) = p$,

since $(D^T)^\dagger D^T = I_r$, we have

$$\begin{aligned} \mathbf{x}' &= ((M/D)^T_{\mathcal{P}})^\dagger(\mathbf{u}' - (D^\dagger C)^T \mathbf{v}') \\ \mathbf{y}' &= (D^T)^\dagger(\mathbf{v}' - B^T \mathbf{x}') \end{aligned}$$

Schur and Pseudo-Schur complements

Particular cases [MRZ, 2004]:

Case 1 If $r \geq s$ and rank $(D) = s$, then

$$(M/D)_{\mathcal{P}} = A - B(D^T D)^{-1} D^T C$$

So,

$$(M/D)_{\mathcal{P}} = (M'/D^T D)$$

where

$$M' = \begin{pmatrix} A & B \\ D^T C & D^T D \end{pmatrix} \in \mathbb{R}^{(p+s)\times(q+s)}$$

Schur and Pseudo-Schur complements

Case 2 If $r \leq s$ and rank $(D) = r$, then

$$(M/D)_{\mathcal{P}} = A - BD^T(DD^T)^{-1}C$$

So,

$$(M/D)_{\mathcal{P}} = (M''/DD^T)$$

where

$$M'' = \begin{pmatrix} A & BD^T \\ C & DD^T \end{pmatrix} \in \mathbb{R}^{(p+r)\times(q+r)}$$

Bordered matrices

Let $M^\dagger$ be the pseudo-inverse of the bordered matrix M.
We set

$$M^\ddagger = \begin{pmatrix} A^\dagger + A^\dagger B S^\dagger C A^\dagger & -A^\dagger B S^\dagger \\ -S^\dagger C A^\dagger & S^\dagger \end{pmatrix}$$

where $S = D - CA^\dagger B \in \mathbb{R}^{r \times s}$ that is $S = (M/A)_{\mathcal{P}}$.

Formula given by [Ben-Israel, 1969]. Generalization of the block bordering method [Brezinski-MRZ, 1991].

In general $M^\ddagger \neq M^\dagger$

For necessary and sufficient conditions see [Bhimasankaram, 1971], [Burns - Carlson - Haynsworth - Markham, 1974].

Bordered matrices - Properties

Anyway **[MRZ, 2004]**, it holds

$$(M^{\ddagger}/S^{\dagger})_{\mathcal{P}} = A^{\dagger}$$

This formula generalizes Duncan inversion formula (1944).

Moreover we have the decomposition

$$M^{\ddagger} = \begin{pmatrix} I_q & -A^{\dagger}BS^{\dagger}S \\ 0_{s\times q} & I_s \end{pmatrix} \begin{pmatrix} A^{\dagger} & 0_{q\times r} \\ -S^{\dagger}CA^{\dagger} & S^{\dagger} \end{pmatrix}$$

Bordered matrices - Properties

Particular cases [Brezinski - MRZ, 2004]:

Case 1 If $p \geq q$ and rank $(A) = q$, if $r \geq s$ and rank $(S) = s$, then $A^\dagger A = I_q$, $S^\dagger S = I_s$ and it holds

$$M^\ddagger M = I_{q+s}$$

Case 2 If $p \leq q$ and rank $(A) = p$, if $r \leq s$ and rank $(S) = r$, then $AA^\dagger = I_p$, $SS^\dagger = I_r$ and it holds

$$MM^\ddagger = I_{p+r}$$

These properties were used in the construction of new acceleration schemes for vector sequences [Brezinski - MRZ, 2004].

Bordered matrices - Properties

If D is square and non singular (so $(M/D)_{\mathcal{P}} = (M/D)$), and if we set

$$(M/D)^{\ddagger} = A^{\dagger} + A^{\dagger} B S^{\dagger} C A^{\dagger}$$

(expression that generalizes the Sherman-Morrison (1949) and Woodbury (1950) formula) we have **[MRZ, 2004]**

Case 1 If $p \geq q$ and rank $(A) = q$, if S is square and non singular, then

$$(M/D)^{\ddagger}(M/D) = I_q$$

Case 2 If $p \leq q$ and rank $(A) = p$, if S is square and non singular, then

$$(M/D)(M/D)^{\ddagger} = I_p$$

Schur complement - Quotient property

Let us consider the matrix

$$M = \begin{pmatrix} A & B & E \\ C & D & F \\ G & H & L \end{pmatrix}$$

and its submatrices

$$A' = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \qquad B' = \begin{pmatrix} B & E \\ D & F \end{pmatrix}$$

$$C' = \begin{pmatrix} C & D \\ G & H \end{pmatrix} \qquad D' = \begin{pmatrix} D & F \\ H & L \end{pmatrix}$$

Quotient property

If (D'/D) is square and non singular then [Crabtree - Haynsworth, 1969]

$$\begin{aligned} (M/D') &= ((M/D)/(D'/D)) && (1) \\ &= (A'/D) - (B'/D)(D'/D)^{-1}(C'/D) && (2) \end{aligned}$$

Different proofs in [Ostrowski, 1971] and in [Brezinski - MRZ, 2003]

(1) was extended to **Pseudo-Schur complement** in [Carlson - Haynsworth - Markham, 1974]

$$(M/D')_{\mathcal{P}} = ((M/D)_{\mathcal{P}}/(D'/D)_{\mathcal{P}})_{\mathcal{P}}$$

Quotient property

In **[MRZ, 2004]**, following the idea given in [Brezinski - MRZ, 2003], we proposed a different proof of $(\mathbf{1})$ and we proved also $(\mathbf{2})$, and the following property holds

Property: If $(D')^{\ddagger} = (D')^{\dagger}$, then

$$
\begin{aligned}
(M/D')_{\mathcal{P}} &= ((M/D)_{\mathcal{P}}/(D'/D)_{\mathcal{P}})_{\mathcal{P}} \\
&= (A'/D)_{\mathcal{P}} - (B'/D)_{\mathcal{P}}(D'/D)_{\mathcal{P}}^{\dagger}(C'/D)_{\mathcal{P}}
\end{aligned}
$$

Quotient property

Proof: From the definition of pseudo-Schur complement of D' in M we have

$$(M/D')_{\mathcal{P}} \;=\; A - (B\ E)\begin{pmatrix} D & F \\ H & L \end{pmatrix}^{\dagger}\begin{pmatrix} C \\ G \end{pmatrix}$$

Setting $S = (D'/D)_{\mathcal{P}}$, since $(D')^{\ddagger} = (D')^{\dagger}$, then

$$(D')^{\ddagger} = \begin{pmatrix} D & F \\ H & L \end{pmatrix}^{\ddagger} = \begin{pmatrix} D^{\dagger} + D^{\dagger}FS^{\dagger}HD^{\dagger} & -D^{\dagger}FS^{\dagger} \\ -S^{\dagger}HD^{\dagger} & S^{\dagger} \end{pmatrix} = (D')^{\dagger}$$

By substituting in the previous formula, we easily obtain $(\mathbf{2})$

$$(M/D')_{\mathcal{P}} \;=\; (A'/D)_{\mathcal{P}} - (B'/D)_{\mathcal{P}}(D'/D)^{\dagger}_{\mathcal{P}}(C'/D)_{\mathcal{P}}$$

Quotient property

We consider the matrix

$$M' = \begin{pmatrix} (A'/D)_{\mathcal{P}} & (B'/D)_{\mathcal{P}} \\ (C'/D)_{\mathcal{P}} & (D'/D)_{\mathcal{P}} \end{pmatrix}$$

and its pseudo-Schur complement

$$\begin{aligned} (M'/(D'/D)_{\mathcal{P}})_{\mathcal{P}} &= (A'/D)_{\mathcal{P}} - (B'/D)_{\mathcal{P}}(D'/D)_{\mathcal{P}}^{\dagger}(C'/D)_{\mathcal{P}} \\ &= (M/D')_{\mathcal{P}} \end{aligned}$$

From the definition of pseudo-Schur complement for matrices partitioned into an arbitrary number of blocks we have

$$(M/D)_{\mathcal{P}} = \begin{pmatrix} A & E \\ G & L \end{pmatrix} - \begin{pmatrix} B \\ H \end{pmatrix} D^{\dagger}(C\ F)$$

Quotient property

$$
\begin{aligned}
(M/D)_{\mathcal{P}} &= \begin{pmatrix} A - BD^{\dagger}C & E - BD^{\dagger}F \\ G - HD^{\dagger}C & L - HD^{\dagger}F \end{pmatrix} \\
&= M'
\end{aligned}
$$

which proves (1)

$$
(M/D')_{\mathcal{P}} = ((M/D)_{\mathcal{P}}/(D'/D)_{\mathcal{P}})_{\mathcal{P}}
$$

Quotient property - Linear systems

We apply the block Gaussian elimination to the system

$$\begin{pmatrix} A & B & E \\ C & D & F \\ G & H & L \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \\ \mathbf{z} \end{pmatrix} = \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \\ \mathbf{w} \end{pmatrix}$$

If D is square and non singular we obtain

$$\begin{pmatrix} (A'/D) & 0 & (B'/D) \\ C & D & F \\ (C'/D) & 0 & (D'/D) \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \\ \mathbf{z} \end{pmatrix} = \begin{pmatrix} \mathbf{u} - BD^{-1}\mathbf{v} \\ \mathbf{v} \\ \mathbf{w} - HD^{-1}\mathbf{v} \end{pmatrix}$$

Quotient property - Linear systems

In the second step of Gaussian elimination, we suppose that (D'/D) is square and non singular, and we obtain

$$\begin{pmatrix} (M/D') & 0 & 0 \\ C & D & F \\ (C'/D) & 0 & (D'/D) \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \\ \mathbf{z} \end{pmatrix} =$$

$$\begin{pmatrix} (\mathbf{u} - BD^{-1}\mathbf{v}) - (B'/D)(D'/D)^{-1}(\mathbf{w} - HD^{-1}\mathbf{v}) \\ \mathbf{v} \\ \mathbf{w} - HD^{-1}\mathbf{v} \end{pmatrix}$$

so

$$\mathbf{x} = (M/D')^{-1}\left[(\mathbf{u} - BD^{-1}\mathbf{v}) - (B'/D)(D'/D)^{-1}(\mathbf{w} - HD^{-1}\mathbf{v})\right]$$

Quotient property - Linear systems

If D is rectangular or square and singular we can obtain a similar result.

By using D as pivot and if $D^\dagger D = I$ we obtain

$$\begin{pmatrix} (A'/D)_{\mathcal{P}} & 0 & (B'/D)_{\mathcal{P}} \\ C & D & F \\ (C'/D)_{\mathcal{P}} & 0 & (D'/D)_{\mathcal{P}} \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \\ \mathbf{z} \end{pmatrix} = \begin{pmatrix} \mathbf{u} - BD^\dagger \mathbf{v} \\ \mathbf{v} \\ \mathbf{w} - HD^\dagger \mathbf{v} \end{pmatrix}$$

If, in the second step of Gaussian elimination, we suppose that $(D'/D)^\dagger_{\mathcal{P}}(D'/D)_{\mathcal{P}} = I$, it is easy to see that

$$\begin{aligned} &((A'/D)_{\mathcal{P}} - (B'/D)_{\mathcal{P}}(D'/D)^\dagger_{\mathcal{P}}(C'/D)_{\mathcal{P}})\mathbf{x} = \\ &\qquad (\mathbf{u} - BD^\dagger \mathbf{v}) - (B'/D)_{\mathcal{P}}(D'/D)^\dagger_{\mathcal{P}}(\mathbf{w} - HD^\dagger \mathbf{v}) \end{aligned}$$

Quotient property - Linear systems

So, if the Pseudo-Schur quotient property holds, we have

$$(M/D')_{\mathcal{P}}\mathbf{x} = (\mathbf{u} - BD^{\dagger}\mathbf{v}) - (B'/D)_{\mathcal{P}}(D'/D)^{\dagger}_{\mathcal{P}}(\mathbf{w} - HD^{\dagger}\mathbf{v})$$

Reference: M. Redivo Zaglia, Pseudo-Schur complements and their properties, Appl. Numer. Math 50 (2004) 511-519.

Future work: Application of the Pseudo-Schur quotient property to the construction of recursive algoritms for vector sequence transformations proposed in

C. Brezinski, M. Redivo Zaglia, New vector sequence transformations, Linear Algebra Appl. 389 (2004) 189-213.