DIVISORS OF SMALL GENUS ON ALGEBRAIC SURFACES AND PROJECTIVE EMBEDDINGS

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Abstract. In this article we give a numerical criterion, valid in all characteristics, for the very ampleness of a line bundle H on a curve C (possibly reducible and non reduced) lying on a smooth algebraic surface.

We show by the way that this criterion essentially implies the results of Bombieri and Ekedahl on pluricanonical embeddings of surfaces of general type.

We use our numerical criterion to obtain the fine classification of non special rational surfaces in $\mathbf{P}^4$ (with one exception dealt with in a forthcoming article).

We also describe in full detail the embeddings when C is a curve of genus 2 and H has degree 5.

Contents

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1. Introduction

One of the archetypal results in algebraic geometry is that the linear system of plane cubics passing through 6 points yields an embedding of the rational surface S obtained by blowing up the plane at those points, provided that no one of them is infinitely near, no 3 of them lie on a line, and that they do not all lie on a conic; S is embedded in $\mathbf{P}^3$ as a smooth cubic surface, and all smooth cubic surfaces are obtained in this way.

Almost half of the classical book "*Le Superficie Razionali*" by Conforto-Enriques (cf. [Co], the racial laws prohibiting in 1939 the name of Enriques to appear as an author) is devoted to a beautiful analysis of the linear system of plane curves yielding models of rational surfaces of degree up to 5, but when the degree gets higher the intricacy increases tremendously.

A still open problem is the determination of the smooth rational surfaces in $\mathbf{P}^4$: up to now is even unknown the determination of their possible degrees, which are bounded by the result of [E-P] (an effective bound for the degree, which is still not sharp, is given in [B-F]).

J. Alexander in [Al] has classified the possible linear systems of plane curves $|aL - \sum b_i x_i|$ which could give non special surfaces in $\mathbf{P}^4$ (i.e. with $H^1(\mathcal{O}_S(1)) = 0$): he has determined the values of the integers a and of the b_i's, and has indeed shown that for a sufficiently general choice of the points x_i these surfaces do in fact exist (the existence of rational surfaces of degree ≤ 8 had been previously established by classical authors, by P.Ionescu in [Io 1-2] and by C.Okonek in [Ok 1-3]).

For non special rational surfaces the degree is at most 9, and recently examples of special rational surfaces of higher degree has been found by many authors, in particular a novel interesting method (relying also on computer calculations) and a huge list can be found in the article by Decker,Ein and Schreyer ([D-E-S]).

The complete determination of the *Bordiga surfaces* (the non special rational surfaces of degree 6) was done e.g. in the thesis of Weinfurtner (cf. [Wei]) by using Reider's method, which cannot longer apply when the degree is at least 7.

By complete determination we mean, as in the case of cubic surfaces in $\mathbf{P}^3$, the statement of precise necessary and sufficient conditions on the points x_i in order that the sytem $|aL - \sum b_i x_i|$ be very ample.

The motivation for our work was thus the desire to develop more powerful methods in order to analyse the problem whether a linear system is very ample, e.g. on a rational surface.

One of the corollaries of our results (see section 5) is the complete determination (with a simple proof) of the non special rational surfaces of degree ≤ 8, and also the remainig case of degree 9 can be settled by a refinement of the present methods (work in progress by the first author and K.Hulek).

Such a complete determination, with the further assumption that our surface S would be a projection of a smooth surface in $\mathbf{P}^5$ from an inner point, had been previously obtained by I.Bauer in [Ba], and was extended to the general case in the thesis [Fra] of the second author, but with a too long and complicated proof.

Further applications of the present methods to the problem of *k-very-ampleness* have been given in [Fra], and improvements will be given in a forthcoming article by the second author.

Morally, the advantage of rational surfaces is that they have a lot of curves: this on the one hand forces some necessary conditions in order that a divisor H on S may be very ample (for instance, it should have intersection at least 3 with curves of arithmetic genus 1, whereas $H.C \geq 5$ if C has genus 2), on the other hand, as shown by I. Bauer in [Ba], the existence of these curves can be used effectively to show that H is very ample simply by restricting to them.

The more precise form of the criterion used in [Ba] (which holds in any dimension) is as follows: if $H = C + D$, with C effective, $dim|D| \geq 1$, and $|H|$ restricts to very ample linear systems on C and on each Δ in $|D|$, then H is very ample.

The main new contribution of the present paper is an embedding theorem, valid in all characteristic (cf. section 3), which gives a sufficient numerical criterion for very ampleness of a divisor H on a curve C. The criterion applies to curves C lying on a smooth algebraic surface S (C may be reducible and non reduced), or to the irreducible reduced Gorenstein curves, and states that if the degree of H on each subcurve D of C is at least $2p_a(D) + 1$, $p_a(D)$ being the arithmetic genus of D, then necessarily H is very ample on C.

To illustrate the power of this result, we show (cf. section 4) that it immediately implies the results of Bombieri and Ekedahl (cf. [Bo] and [Ek]) on pluricanonical embeddings of surfaces of general type with K_S ample once one knows that the cohomology groups $H^1(mK_S)$ vanish for $m \geq 2$ (this being easy to prove in characteristic zero nowadays, and false in char=p only in a very special case, for $m = 2, p = 2$, and two families with $\chi = 1$, cf. [Ek] and section 4).

It took us perhaps more fatigue to discover the statement of the above embedding theorem for curves than to prove it: in fact for our applications it suffices to know the result in the case where C has genus $p_a = 1$ and H has degree $d = 3, 4$, or when $p_a = 2$, $d = 5$.

Originally we studied intensively the geometry of these curves of genus ≤ 2, which turns out to be particularly beautiful and intricate in the case $p_a = 2, d = 5$, and gave us a lot of fun.

Therefore, after showing in the brief section 6 that the case $p_a = 1$,$d = 3$ yields plane cubics, and that the one $p_a = 1, d = 4$ gives rise to only two cases (complete intersection of 2 quadrics in $\mathbf{P}^3$, or the union of a plane cubic with a line meeting it, possibly infinitely near), we devote the long section 7 to the complete classification and to a detailed study of the geometry of the case $p_a = 2, d = 5$; we show in particular that one obtains only curves in $\mathbf{P}^3$ of degree 5 which are projectively Cohen-Macaulay.

In section 5 one finds, for all non special rational surfaces of degree $\leq$ 8, a table of the sufficient and necessary conditions for very ampleness of the linear system $|aL - \sum b_i x_i|$, and proofs.

Finally, a brief mention of the tools we employ for the embedding theorem: essentially they are deriving from the basic notion of k-*connectedness*, introduced by Franchetta for a curve C on a smooth surface S, and Franchetta-Ramanujam's inequality (cf. [Ram]), through a series of refinements ([Bo], [B-C], [Ca 1], [Ba]).

C is said to be k-*connected* if for all decomposition $C = Y + Z$, then $Y.Z \geq k$, and Ramanujam's inequality says that if a section of a line bundle L vanishes identically exactly on the subcurve Z, then the degree of L on Y is at least $Y.Z$.

With the above notions, non reduced curves can be treated as reduced curves, and the Gorenstein property (cf. [Ser],[Ca 1]) plays also a crucial role.

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2. General results on linear systems on curves

Throughout this paper S will be a smooth algebraic surface over an algebraically closed field of characteristic $p \geq 0$, C a (possibly reducible and non reduced) curve lying on S and H a divisor on C. Therefore C will be written (as a divisor on S) as $\sum m_i C_i$, and a subcurve $C' \leq C$ will mean a curve $\sum n_i C_i$, with $n_i \leq m_i$. If $D = C_i$ for some i, the multiplicity m_D of D in C is simply the integer n_i.

We will denote by $p_a(C)$ the arithmetic genus of C, by $H.C$ the degree of H on C and by K_C the divisor associated to the dualizing sheaf ω_C, so that $K_C.C = 2p_a(C) - 2$.

A general problem is to study the behaviour of the rational map $\varphi_{|H|}$ associated to the linear system $|H|$ of divisors of sections of $H^0(C, \mathcal{O}_C(H))$.

Firstly, one can predict the dimension of $H^0(C, \mathcal{O}_C(H))$ if H has sufficiently "positive multidegree" on C:

Lemma 2.1. *Let C be a curve lying on a smooth algebraic surface S and let H be a divisor on C.*

Then $H^1(C, \mathcal{O}_C(H)) = 0$ if for all subcurves $B \le C$, $H.B \ge (2p_a(B) - 1)$.

Proof. If $h^1(\mathcal{O}_C(H)) = h^0(\mathcal{O}_C(K_C - H)) \neq 0$ there exists a non zero section $\sigma \in H^0(\mathcal{O}_C(K_C - H))$.

Let Z be the maximal curve $\le C$ on which $\sigma \equiv 0$ (i.e. σ maps to 0 in $H^0(Z, \mathcal{O}_Z(K_C - H)))$, and let $Y = C - Z$. Then (cf. [Ram]) by virtue of the following exact sequence introduced by Franchetta and Ramanujam

$$0 \to \mathcal{O}_Y(K_C - H - Z) \to \mathcal{O}_C(K_C - H) \to \mathcal{O}_Z(K_C - H) \to 0,$$

upon dividing σ by a section ζ with $div(\zeta) = Z$, we obtain $\sigma/\zeta = \sigma'$ a section of $H^0(Y, \mathcal{O}_Y(K_C - H - Z)$ vanishing on a finite set.

Whence we have what from now on we will call the standard Ramanujam inequality $(K_C - H).Y \ge Z.Y$. This inequality, since by adjunction $\mathcal{O}_Y(K_C) = \mathcal{O}_Y(K_Y + Z)$, is equivalent to $H.Y \le 2p_a(Y) - 2$, a contradiction. □

In particular the following corollary holds:

Corollary 2.2. *Let C and H be as in lemma 2.1.*

Then $h^0(\mathcal{O}_C(H)) = C.H - p_a(C) + 1$.

Through a conversation of the first author with K.Hulek came out the following proposition which is an improvement of a useful criterion introduced firstly in [Ba], thm.2.15. This criterion gives sufficient conditions in order that a divisor H yields a linear system on C which is basepoint free, respectively very ample.

We shall prove here only part **(I)**. The proof of **(II)** and **(III)** is similar, and the proof of **(III)** can be found in [C-H], whereas for **(II)**, a different argument will be given in the course of proving thm.3.1.

Proposition 2.3. *Let C be a curve lying on a smooth algebraic surface S and let H be a divisor on C.*

(I) *A point $x \in C$ is not a base point for $|H|$ if for every subcurve Y of C we have:*

$$H.Y \geq 2p_a(Y).$$

(II) *Two points $x, y \in C$ such that $x \neq y$ are separated by $|H|$ if for every subcurve Y of C we have:*

$$H.Y \geq 2p_a(Y) + 1.$$

(III) *$|H|$ is a local embedding at $x \in C$ if every subcurve Y of C we have either:*

(i) x is not in Y and $H.Y \geq 2p_a(Y) - 1$ or
(ii) x is in Y, x is smooth for C and $H.Y \geq 2p_a(Y) + 1$ or
(iii) x is in Y, x is singular for C and $H.Y \geq 2p_a(Y) - 1 + mult_x(Y)$.

Now we procede analysing the case where a point $x \in C$ is a base point for a linear system $|H|$ when there are subcurves Y of C such that $H.Y \leq 2p_a(Y) - 1$.

Proposition 2.4. *Let C be a curve lying on a smooth algebraic surface S and let H be a divisor on C such that*

(1)$H.C \geq 2p_a(C) - 2$
(2) *For each proper subcurve Y of C $H.Y \geq 2p_a(Y) - 1$.*

Then:
(a) *x is a base point for $|H|$ if there exists a decomposition $C = Y + Z$ such that $Y.Z = 1$, x is a smooth point of Y and $\mathcal{O}_Y(H) \cong \mathcal{O}_Y(K_Y + x)$.*

(b) *$|H|$ is free from base points if there exists no point x as in* (**a**) *and moreover, either*

$(b.1)$ $H.C \geq 2p_a(C) - 1$ and there exists no smooth point $x \in C$ such that $\mathcal{O}_C(H) \cong \mathcal{O}_C(K_C + x)$ or
$(b.2)$ $\mathcal{O}_C(H) \cong \mathcal{O}_C(K_C)$, $C \neq \mathbf{P}^1$, or
$(b.3)$ $H.C = 2p_a(C) - 2$, there exists no point x of multiplicity 2 for C such that, $\pi : \hat{C} \to C$ being the blow up of C at x, $\mathcal{O}_{\hat{C}}(\pi^*(K_C - H)) \cong \mathcal{O}_{\hat{C}}$, and there does not exist a pair of distinct smooth points x, y of C such that $H = K_C + x - y$.

Recall (cf.[Ram]) that a curve $C \subset S$ is said to be *m-connected* if for every decomposition $C = Z + Y$, then $Z.Y \geq m$.

We shall also express the *1-connectedness* of C by saying that C is *numerically connected.*

Corollary 2.5. *Let C be a curve of genus $p_a(C) \geq 1$ lying on a smooth algebraic surface S.*

(**i**) *If C is numerically connected then the base points of $|K_C|$ are precisely the points x such that there exists a decomposition $C = Y + Z$ with $Y.Z = 1$, where x is smooth for Y and $\mathcal{O}_Y(x) \cong \mathcal{O}_Y(Z)$.*

(**ii**) *If C is 2-connected then $|\omega_C|$ is free.*

(**iii**) *If C is 2-connected and H is numerically equivalent to K_C then H is free if there exists no point x of multiplicity 2 for C such that, $\pi : \hat{C} \to C$ being the blow up of C at x, $\mathcal{O}_{\hat{C}}(\pi^*(K_C - H)) \cong \mathcal{O}_{\hat{C}}$, and there does not exist a pair of distinct smooth points x, y of C such that $H = K_C + x - y$.*

Remark 2.6. *In* (**i**) *of cor.2.5 it could happen that $Y = Z$ and $\mathcal{O}_Y(x) \cong \mathcal{O}_Y(Z)$.*

Proof of prop.2.4. Consider the exact sequence

$$0 \to \mathcal{M}_x(H) \to \mathcal{O}_C(H) \to \mathcal{O}_C/\mathcal{M}_x \to 0,$$

where $\mathcal{M}_x$ is the maximal ideal of the local ring $\mathcal{O}_{C,x}$.

Then x is not a base point for $|H|$ if $H^1(\mathcal{M}_x(H)) \hookrightarrow H^1(\mathcal{O}_C(H))$.

From our assumptions and lemma 2.1 it follows that either H is non special, i.e., $H^1(\mathcal{O}_C(H)) = 0$, or $H \equiv K_C$.

If H is non special x is not a base point for $|H|$ if and only if $H^1(\mathcal{M}_x(H)) = 0$, or equivalently, by the Serre-Grothendieck duality, if $Hom(\mathcal{M}_x, K_C - H) = 0$.

If instead $H \equiv K_C$, x is not a base point for $|H|$ if and only if $dim(Hom(\mathcal{M}_x, K_C - H)) = 1$.

We shall proceed by contradiction, by choosing a nonzero $\varphi \in Hom(\mathcal{M}_x, (K_C - H))$ in the case $H \not\equiv K_C$, and by considering a particular φ in the case $H \equiv K_C$, where we make the assumption $dim(Hom(\mathcal{M}_x, K_C - H)) = 2$.

Case 1. There exists a nonzero $\varphi \in Hom(\mathcal{M}_x, K_C - H)$ vanishing only on a finite set.

(1.1) If x is smooth for C then φ is a section of $H^0(C, \mathcal{O}_C(K_C - H + x))$.

Whence if $H.C \geq 2p_a(C)$, $\mathcal{O}_C(K_C - H + x)$ has negative degree and $\varphi \equiv 0$, absurd.

If $H.C = 2p_a(C) - 1$, $\mathcal{O}_C(K_C - H + x)$ has degree 0, whence φ yields an isomorphism $\varphi : \mathcal{O}_C(H) \to \mathcal{O}_C(K_C + x)$, and conversely if we have such an isomorphism, x is a base point.

In the case where $H.C = 2p_a(C) - 2$, φ is a section of a line bundle of degree 1, whence it defines a smooth point y.

If $H \not\equiv K_C$ this contradicts our assumptions.

If $H \equiv K_C$, let D be the irreducible component containing x.

If it were $C = D$, we would have $C \cong \mathbf{P}^1$ ($\mathcal{O}_D(K_C - H + x)$ is a line bundle of degree 1 and 2 independent sections), a contradiction.

Thus $D < C$ and, since C is connected by 2), let y be a point of $D \cap (C - D)$; we can choose then φ to be a non zero section of $H^0(C, \mathcal{O}_C(K_C - H + x))$ vanishing at y: then φ vanishes identically on a component of C and so case 2 applies.

(1.2) If x is singular for C, take a, b local parameters on S at x (a, b are a basis of $\mathcal{M}_x/\mathcal{M}_x^2$) such that a, b do not give tangents to any branch of C at x, so that in particular a, b do not restrict to zero divisors of $\mathcal{O}_C$.

Now, if x is singular for C, by lemma 2.1 of [Ca 1] $Hom(\mathcal{M}_x, \mathcal{O}_C)$ embeds in $H^0(\hat{C}, \mathcal{O}_{\hat{C}})$, where $\pi : \hat{C} \to C$ is the blow up of C at x, and φ is given locally by multiplication with $\varphi(b)/b = \varphi(a)/a$.

Let $\hat{\varphi} \in H^0(\hat{C}, \mathcal{O}_{\hat{C}}(\pi^*(K_C - H)))$ be the image of φ.

Since $(\pi^*(K_C - H)).\hat{C} = (K_C - H).C$, if $degK_C < degH$, $\hat{\varphi}$ must be identically zero, a contradiction.

If $degH = degK_C$ then $\hat{\varphi}$ is a never vanishing section of $\mathcal{O}_{\hat{C}}(\pi^*(K_C - H)) \cong \mathcal{O}_{\hat{C}}$.

Let $\hat{\mathcal{C}}$ be the conductor ideal $\mathcal{H}om_{\mathcal{O}_C}(\pi_*\mathcal{O}_{\hat{C}}, \mathcal{O}_C)$.

It is well known (cf. e.g. [O-R], prop.1.8) that, if m is the multiplicity of C at x, $\hat{\mathcal{C}} = \mathcal{M}_x^{m-1}$.

Moreover, since $length(\pi_*\mathcal{O}_{\hat{C}}/\mathcal{O}_C) = length(\mathcal{O}_C/\hat{\mathcal{C}})$, it follows, e.g. from thm.1.5.c) of [Ca 1], that $\pi_*\mathcal{O}_{\hat{C}} = \mathcal{H}om_{\mathcal{O}_C}(\hat{\mathcal{C}}, \mathcal{O}_C)$.

Notice that, by the exact sequence

$$0 \to \mathcal{M}_x \to \mathcal{O}_C \to \mathcal{O}_C/\mathcal{M}_x \to 0,$$

$\mathcal{E}xt^1_{\mathcal{O}_C}(\mathcal{O}_C, \mathcal{M}_x) = 0$.

Therefore, dualizing the exact sequence

$$0 \to \hat{\mathcal{C}} \to \mathcal{M}_x \to \mathcal{M}_x/\hat{\mathcal{C}} \to 0$$

we get an exact sequence

$$0 \to \mathcal{H}om_{\mathcal{O}_C}(\mathcal{M}_x, \mathcal{O}_C) \to \pi_*\mathcal{O}_{\hat{C}} \to \Delta \to 0,$$

where $\Delta = \mathcal{E}xt^1_{\mathcal{O}_C}(\mathcal{O}_C, \mathcal{M}_x/\hat{\mathcal{C}})$ and $length(\Delta) = length(\mathcal{M}_x/\hat{\mathcal{C}}) = (m(m-1)/2) - 1$ (cf. [Ca 1], 1.5.c)).

Therefore, if $m \geq 3$ and $\varphi \in \mathcal{H}om_{\mathcal{O}_C}(\mathcal{M}_x, \mathcal{O}_C(K_C - H))$, $length(\mathcal{O}_{\hat{C}}(\pi^*(K_C - H))/\hat{\varphi} \cdot \mathcal{O}_{\hat{C}}) \geq 1$, absurd.

If instead $m = 2$ the fact that φ is never vanishing contradicts our assumptions if $H \not\equiv K_C$, whereas if $H \equiv K_C$, we have assumed

$dim(Hom(\mathcal{M}_x, K_C - H)) \geq 2$. Therefore we can find a $\varphi \in (Hom(\mathcal{M}_x, K_C - H))$ vanishing at a point $y \neq x$.

But then the corresponding $\hat{\varphi}$ is also vanishing at y and therefore our chosen φ vanishes on a component of C and case 2 applies.

Case 2. There exists a nonzero φ vanishing on a component of C.

Let then Z be the maximal subcurve of C on which φ vanishes identically and let $Y = C - Z$.

Let a, b local parameters at x as in (1.2) and ζ, η be the respective local equations for Z and Y at x.

(2.1) If x is smooth for Y we claim that φ/ζ gives a section of $H^0(\mathcal{O}_Y(K_C - H - Z - x))$ vanishing on a finite set.

This is obvious if x is smooth also for C. Otherwise, notice that φ is given by multiplication by $\varphi(b)/b$. By our assumptions $\varphi(b) = \zeta u$ with u vanishing on a finite set, thus $\varphi/\zeta = u/b$. Since $\mathcal{M}_{x|Y}$ is invertible and spanned by b, we are done.

But now, by adjunction, $\mathcal{O}_Y(K_C - H - Z - x) \cong \mathcal{O}_Y(K_Y - H - x)$, therefore $H^0(\mathcal{O}_Y(K_C - H - Z - x)) = 0$ if $H.Y \geq 2p_a(Y)$.

If instead $H.Y = 2p_a(Y) - 1$ then φ provides an isomorphism $\mathcal{O}_C(H) \cong \mathcal{O}_C(K_C + x)$, and conversely if there exists such an isomorphism x is base point for $|H|$.

(2.2) If x is singular for Y notice that $a\varphi(b) - b\varphi(a)$ gives a local equation of C and recall that, locally, $\varphi(b) = \zeta u$.

Since ζ divides $a\varphi(b) - b\varphi(a) = \zeta a u - b\varphi(a)$ there exists v such that $\zeta v = b\varphi(a)$, and $au - v$ is a local equation of Y at x. Therefore, since x is singular for Y, $u \in \mathcal{M}_x$.

Therefore $\varphi/\zeta = u/b$ yields a section $\hat{\varphi} \in H^0(\hat{Y}, \mathcal{O}_{\hat{Y}}(\pi^*(K_C - H - Z)))$, where $\hat{Y}$ is the blow-up of Y at x.

Since by adjunction $\mathcal{O}_{\hat{Y}}(K_C - H - Z) \cong \mathcal{O}_{\hat{Y}}(K_Y - H)$ and by assumption $H.Y \geq 2p_a(Y) - 1$, $\hat{\varphi}$ is a section of a line bundle of negative degree, a contradiction. $\square$

Proof of prop.2.3(I). By (a) of prop.2.4 $|H|$ is free unless there exists a decomposition $C = Y + Z$ such that $Y.Z = 1$, x is a smooth point of Y and $\mathcal{O}_Y(H) \cong \mathcal{O}_Y(K_Y + x)$: but then $H.Y = 2p_a(Y) - 1$ against our assumptions. $\square$

Proof of cor.2.5 C is 1-connected iff for each decomposition $C = Y + Z$, $Y.Z \geq 1$.

Hence, if H is numerically equivalent to K_C, $H.Y = 2p_a(Y) - 2 + Y.Z \geq 2p_a(Y) - 1$.

Thus (i) follows from (a) and (b.2) of prop.2.4.

If C is 2-connected and H is numerically equivalent to K_C then $\forall Y < C$, $H.Y \geq 2p_a(Y)$, hence a point x as in (a) of 2.4 does not exist, whence (ii) follows from (b.2), (iii) from (b.3). $\square$

In the case where $|H|$ is base point free let us consider the schematic image of C:

Lemma 2.7. *Let C be a curve lying on a smooth algebraic surface S and let H be a divisor on C such that $|H|$ is basepoint free and yields a finite morphism $\varphi = \varphi_{|H|} : C \to \mathbf{P}^n$. Let Γ be the schematic image of C.*

Then Γ is "pure" of dimension 1.

Proof. Let $V = H^0(C, H)$ and let $\mathcal{R} = \mathcal{R}(C, H)$ be the graded ring associated to the invertible sheaf $\mathcal{O}_C(H)$.

Then $\varphi^* : S = Sym(V) \to \mathcal{R}$ and Γ is defined by the graded ideal $\mathcal{I} = ker\varphi^*$.

If l is a linear form such that $dim(\varphi(C) \bigcap \{l = 0\}) = 0$, then we have the vanishing of $s = \varphi^*(l)$ on a finite set $\Sigma \subset C$. We can show that s is a non 0-divisor in $\mathcal{R}$. In fact $s \cdot \sigma = 0$ implies $\sigma \equiv 0$ outside Σ. But C is pure of dimension 1, thus $\sigma \equiv 0$ and then l is a non 0-divisor in every localization of $S/\mathcal{I}$. We must then see that $\forall x \in \Gamma$, there exixts a non 0-divisor in $\mathcal{O}_{\Gamma,x}$. For this it is sufficient to choose l vanishing in x such that $dim(\varphi(C) \bigcap \{l = 0\}) = 0$. $\square$

3. The embedding theorem

In this section we extend to the case of a curve lying on a smooth surface (possibly singular, reducible and non reduced) and of an irreducible (reduced) Gorenstein curve the following well known result: a divisor H on a smooth curve C of genus p is very ample if its degree is $> 2p$.

Theorem 3.1. *Let C be a curve lying on a smooth algebraic surface S and let H be a divisor on C such that*

$$\forall B \leq C, \quad (2p_a(B) + 1) \leq H.B.$$

Then H is very ample on C.

Let us deal first with the case where C is irreducible. We generalize the statement to every Gorenstein curve, i.e., to every curve C such that ω_C is invertible.

Proposition 3.2. *Let C be an irreducible Gorenstein curve and let H be a divisor on C such that $H.C = deg_C H \geq 2p_a(C) + 1$.*

Then H is very ample on C.

Proof. Suppose C is singular and let $\pi : \tilde{C} \to C$ be the normalization of C. Moreover let $\mathcal{C}$ be the conductor of $\mathcal{O}_{\tilde{C}}$ in $\mathcal{O}_C$, let A be the divisor on $\tilde{C}$ corresponding to $\mathcal{C}$ and let $2\delta = degA$. Now, identifying H with its pull-back to $\tilde{C}$, we have

$$H.C \geq (2p_a(C)+1) \Leftrightarrow ((H-A).\tilde{C}) \geq (2p_a(\tilde{C})+1),$$

whence $|H-A|$ is very ample on $\tilde{C}$.

Thus we have the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & \mathcal{C}(H) & \stackrel{\sim}{\to} & \pi_*(\mathcal{O}_{\tilde{C}}(H-A)) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & \mathcal{O}_C(H) & \to & \pi_*(\mathcal{O}_{\tilde{C}}(H)) & \to & \pi_*(\mathcal{O}_{\tilde{C}})/\mathcal{O}_C & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & \mathcal{O}_C/\mathcal{C} & \to & \pi_*(\mathcal{O}_{\tilde{C}})/\mathcal{C} & \to & \pi_*(\mathcal{O}_{\tilde{C}})/\mathcal{O}_C & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

The middle rows and columns are exact on global sections since $H^1(\mathcal{O}_C(H)) = H^1(\mathcal{O}_{\tilde{C}}(H-A)) = 0$, whence all rows and columns are exact on global sections.

Let us first show that $|H|$ gives a local embedding. We have seen that the map $H^0(\mathcal{O}_C(H)) \to \mathcal{O}_C/\mathcal{C}$ is onto, but $\mathcal{C} \subset \mathcal{M}_x^2$ unless x is a double point or a smooth point (cf. Remark 3.23 of [Ca 1], proven in [O-R] cor.2.9).

In this last case, though, the tangent dimension of C at x is at most 2 and cor.2.2 applies verbatim.

In order to separate a pair of distinct points x and y:

(i) if x and y are both singular, then $\mathcal{C} \subseteq \mathcal{M}_x\mathcal{M}_y$, therefore the map $H^0(\mathcal{O}_C(H)) \to \mathcal{O}_C/\mathcal{M}_x\mathcal{M}_y$ is onto;

(ii) if x is singular and y is smooth, since $|H-A|$ is base point free on $\tilde{C}$, there is a section in $H^0(\mathcal{C}(H))$ vanishing at x but not at y;

(iii) if both x, y are smooth $H^0(\mathcal{C}(H))$ separates them since $|H-A|$ is very ample on $\tilde{C}$.

□

Observe that the condition $H.C \geq (2p_a(C)+1)$ is equivalent to $(K_C - H).C \leq -3$, whence there exists an irreducible $D \leq C$ such that $(K_C - H).D < 0$.

We define such a D to be *H-positive* and we denote by C' the curve $C - D$.

Such a curve D plays an important role in the proof of theorem 3.1:

Lemma 3.3. *Let C and H be as in theorem 3.1 and let D be H-positive. Then the exact sequence*

$$0 \to \mathcal{O}_D(H - C') \to \mathcal{O}_C(H) \to \mathcal{O}_{C'}(H) \to 0$$

is exact on global sections.

Moreover if the multiplicity m_D of D in C is ≥ 2 then $|H|$ is very ample on C iff it is very ample on C'.

Proof. Since $H^1(\mathcal{O}_D(H - C'))^\vee \cong H^0(\omega_D(C' - H)) \cong H^0(\mathcal{O}_D(K_C - H)) = 0$ by degree consideration, the above exact sequence is exact on global sections.

Suppose then $|H|$ to be very ample on C'. Since the multiplicity m_D of D in C is at least 2, C' and C are the same set theoretically.

Therefore $\varphi_{|H|}$ is 1-1 on C since $|H|$ embeds C'.

Let x be a point $\in D$ and let $\mathcal{M}$, $\mathcal{M}'$ be the maximal ideals of $\mathcal{O}_C$, respectively $\mathcal{O}_{C'}$, at x.

If $m_D \geq 3$ we have a factorization

$$\begin{array}{ccccc} \mathcal{O}_C & \to & \mathcal{O}_{C'} & \cong \mathcal{O}_C/\mathcal{I}_D \\ \downarrow & & \downarrow & \\ \mathcal{O}_C/\mathcal{M}^2 & \cong & \mathcal{O}_{C'}/\mathcal{M}'^2 & \end{array}$$

whence $|H|$ is very ample on C since it is very ample on C'.

If $m_D = 2$, by the same argument, φ_H is a local embedding at every point x which is singular for C'.

But in the case where x is a smooth point for C' we can apply cor.2.2. □

Proof of theorem 3.1. The proof will be given by induction on the number ν of components of C taken with multiplicity, $\nu = \sum m_i$. If $\nu = 1$ we apply prop.3.2.

Let $\nu \geq 2$. By lemma 3.3 and by induction it suffices to deal with the case where every *H-positive* component has multiplicity $= 1$.

Let D be $H - positive$ and notice that by induction we know that $|H|$ is very ample on $C' = C - D$; consider moreover the following exact sequence

$$0 \to H^0(\mathcal{O}_D(H - C')) \to H^0(\mathcal{O}_C(H)) \to H^0(\mathcal{O}_{C'}(H)) \to 0. \tag{1}$$

(I) If there exists an irreducible D such that $D.(K_C - H) \leq -3$ since, by adjunction, $(K_C - H).D = 2p_a(D) - 2 + C'.D - H.D$, $|H - C'|$ is very ample on D by prop 3.2.

Thus φ_H separates each pair x, y. In fact if both points belong to C' or both belongs to $D - C'$, then we consider the restriction $|H|_{|C'}$ or the subseries $|H - C'|_{|D}$, while if $x \in C'$, $y \in D - C'$ it suffices to know that $|H - C'|$ is basepoint free.

Since $|H|$ is very ample on C' and $H - C'$ is very ample on D then φ_H is a local embedding at every point $x \in C' - D$ or $D - C'$. If $x \in C' \cap D$ and x is singular for C', then (being $\mathcal{M}, \mathcal{M}'$ as in lemma.3.3) $\mathcal{O}_C/\mathcal{M}^2 \cong \mathcal{O}_{C'}/\mathcal{M}'^2$, whence $|H|$ is a local embedding, while if x is smooth for C' consider the following exact sequence

$$\mathcal{I}_{C'}/\mathcal{M}^2 \cap \mathcal{I}_{C'} \to \mathcal{O}_C/\mathcal{M}^2 \to \mathcal{O}_{C'}/\mathcal{M}'^2.$$

Since x is smooth for C', then $\mathcal{I}_{C'}/\mathcal{M}^2 \cap \mathcal{I}_{C'} = \mathcal{I}_{C'}/\mathcal{I}_{C'} \cdot \mathcal{M}$.

Therefore we have again a local embedding since the subseries $|H - C'|$ is basepoint free.

Thus $|H|$ is very ample on C.

(II) If $\forall D \leq C$ $D.(K_C - H) > -3$ and there exists an irreducible D such that $D.(K_C - H) = -2$, then $|H - C'|$ is basepoint free on D.

Thus φ_H separates each pair x, y except possibly if both x, $y \in D - C'$, and it is a local embedding at a point $x \in C$ except possibly if $x \in D - C'$.

But now, there is also a $\Delta \leq C$, $\Delta \neq D$ such that $\Delta.(K_C - H) \leq -1$. Consider the decomposition $C = \Delta + C''$: if x, $y \in D - C'$ then x, $y \in C''$, therefore φ_H separates x and y, and it is an embedding at x.

(III) If $\forall D \leq C$, $D.(K_C - H) > -2$ it follows that there exist at least 3 different irreducible curves D_1, D_2, D_3, such that $D_i.(K_C - H) = -1$.

Let $C'_i = C - D_i$ and let $x, y \in C$. Let us show that $\exists i$ such that $x, y \in C'_i$. Else, possibly renumbering the indices, we can assume $x \in C'_1$, $y \in D_1 - C'_1$. But then $y \in C'_2$ and C'_3, whence either $x \in C'_2$ or, if $x \in C'_1 - C'_2$, $x \in D_2 - C'_2$, i.e., $x \in C'_3$.

Thus $|H|$ separates x and y.

Moreover, $|H|$ is an embedding at x if there exists an h such that $x \in C'_h - D_h$. Otherwise $x \in D_1 \cap D_2 \cap D_3$ and we can conclude applying the above argument since then x is singular for every C'_i.

Q.E.D. for thm. 3.1

Remark 3.4. *The result of theorem 3.1 should hold more generally for a Gorenstein curve which is reduced. In fact, the hypothesis that C is Gorenstein is used in the irreducible case but does not guarantee in the* *reducible case that the ideal $\mathcal{I}_{C'}$, appearing in* (**I**)*, is invertible (cf.* [*Ca* 1]*, Lemma 1.12).*

It would be interesting to see what happens by further removing some assumptions, e.g. the Gorenstein assumption in the reduced case, or the assumption that C is reduced in the Gorenstein case. The deadline for the present Proceedings being expired is sufficient motivation for us not to treat the problem here.

4. Pluricanonical maps of surfaces of general type

In this section we are going to give further applications of our curve embedding theorem, deriving from it in particular the results of Bombieri and Ekedahl on pluricanonical embeddings of surfaces of general type (cf. [Bo], [Ek]) in the case where the canonical bundle K_S is ample, that is, S has a smooth canonical model.

Theorem 4.1. *Let S be a minimal model of a surface with K_S ample and $H^1(mK_S) = = 0\ \forall m \geq 2$. Then the linear system $|mK_S|$ is very ample if $m \geq 5$, if $m = 4$ and $K_S^2 \geq 2$, $m = 3$ and $p_g \geq 3, K_S^2 \geq 3$.*

(cf. [Ca 2] for a survey of results of this type).

Proof.

Step I. We have $h^0((m-2)K_S) \geq 3$.

In fact, for $m = 3$, this is just the assumption $p_g \geq 3$.

For $m \geq 4$, if $p_g \geq 2$, then clearly $h^0(2K_S)$ is at least 3 and step I follows, otherwise, by Noether's formula

$$10 + 12p_g = 8h^1(\mathcal{O}_S) + 2\Delta + b_2 + K_S^2$$

where $\Delta = 2h^1(\mathcal{O}_S) - b_1$, $\Delta \geq 0$ and $=0$ if $char = 0$ (cf. [B-M]).

Since all the terms are nonnegative, it follows immediately that if $p_g \leq 1$, then $h^1\mathcal{O}_S) \leq 2$, if $p_g = 0$, then $h^1\mathcal{O}_S) \leq 1$.

Therefore if $p_g \leq 1$, then $\chi \geq 0$, hence for $m \geq 4$, by Riemann-Roch $h^0((m-2)K_S) \geq \chi + 1/2(m-2)(m-3)K_S^2$ which is ≥ 3 for $m \geq 5$, and for $m = 4$ if either $\chi \geq 2$ or $K_S^2 \geq 2$.

Step II. By I, for each pair of points x, y (possibly y infinitely near to x), there is a curve C in $|(m-2)K_S|$ passing through x and y.

Since $H^1(2K_S) = 0$ by our assumptions, it suffices to show that $|mK_S|$ restricted to C is very ample.

Step III. By the embedding theorem it suffices then to show that if $D \leq C$, then $mK_S.D \geq 2p_a(D)+1$. I.e., $mK_S.D \geq 3 + D^2 + D.K_S$, or equivalently

$$(m-1)K_S.D \geq D^2 + 3. \tag{2}$$

(2) obviously holds for $m \geq 3$ if $D^2 \leq -1$, or $D^2 = 0$ (since in this last case $K_S.D$ is even, so ≥ 2).

If $D^2 \geq 1$, write $(m-2)K_S = D + D'$.

We want $K_S.D + D.D' \geq 3$, i.e., $D^2 + 2D.D' \geq 3$.

If $D' = 0$, $D = (m-2)K_S$ and $D^2 \geq 3$ by our assumptions.

Otherwise, since C is 1-connected, as it is well known, our inequality holds unless $K_S.D = 1$. But in this latter case by the index theorem $K^2 = D^2 = 1$. $\square$

Remark 4.2. *Observe that the hypothesis $H^1(mK_S) = 0\ \forall m \geq 2$ is easy to be proven in characteristic zero nowadays, and false in char=$p > 0$ only in a very special case, for $m = 2, p = 2, \chi = 1$ and S is (birationally) an inseparable double cover of a K-3 surface or a rational surface (cf. [Ek], thm. II:1.7).*

The following is a particular case of a result of P.Francia (cf. [Fr]).

Theorem 4.3. *Let S be a minimal surface of genreral type with $q = 0, p_g \geq 1$.*

Then $|2K_S|$ is free from base points.

Proof. Take $C \in |K_S|$. Since $2C \in |2K_S|$ it suffices to show that there are no base points on C.

Since $q = 0$, the map $H^0(S, \mathcal{O}_S(2K_S)) \to H^0(C, \mathcal{O}_C(K_C))$ is surjective. Since C is 2-connected (cf. [Bo]), the result follows from cor.2.5 (ii). $\square$

5. Embeddings of rational surfaces

In this section we shall give (cf. the table at the end of the section, with the exception of the family of degree 9) the complete determination of the non special rational surfaces in $\mathbf{P}^4$ listed by Alexander in [Al]. We remind the reader that Alexander's list is complete in characteristic 0.

We start with the difficult cases of degree 7 and 8, then we simply indicate the proof of the remaining cases, which are extremely easy to deal with by the present method.

Let $\pi : S = \hat{\mathbf{P}}^2(x_1, \dots, x_l) \to \mathbf{P}^2$ be the blow-up of $\mathbf{P}^2 = \mathbf{P}^2(\mathbf{C})$ in l distinct points, possibly infinitely near.

As usual we denote by E_i the total trasform $\pi^{-1}(x_i)$, by L the pull-back of the divisor class of a line in $\mathbf{P}^2$. Fix $a,\ b_i \in\ \mathbf{N}$ and let $|H| = |aL - \sum b_i E_i|$: $|H|$ is the pull-back of the linear system of the curves of degree a and multiplicity at least b_i in x_i, whence we will also use the notation $|aL - \sum b_i x_i|$.

In the study of surfaces in $\mathbf{P}^4$ it is important to point out for which value of l, a, b_i, and for which position of the x_i's the rational map associated with the linear system $|H|$ is an embedding of S in $\mathbf{P}^4$.

A sufficient condition in order to verify the very ampleness of H was introduced by I.Bauer in [Ba]. This method consists in choosing a particular curve C on S and a positive dimensional linear system $|D|$ such that $H = C + D$. Then we consider the restriction $|H|_{|C}$ of the linear system $|H|$ to the curve C (i.e., the restriction to C of each effective divisor Γ linearly equivalent to H) and, similarly, for all $\Delta \in |D|$ we consider the restriction $|H|_{|\Delta}$.

If $|H|_{|C}$ and $|H|_{|\Delta}$ are very ample we claim that $|H|$ itself is very ample. This is generally true for every projective variety of dimension ≥ 2:

Proposition 5.1. *Let X be a smooth projective variety and let C, D be effective divisors with $dim|D| \geq 1$. Let H be the divisor $H = C + D$.*

If $|H|_{|C}$ is very ample, and for all $\Delta \in |D|$, $|H|_{|\Delta}$ is very ample, then H is very ample on X.

For the reader's benefit we reproduce the proof, appearing in [Ba], claim 2.19.

Proof. We need to prove :

0. $|H|$ is basepoint free;
1. $|H|$ separates two distinct points;
2. $|H|$ is a local embedding at every point $x \in X$.

0. Let $x \in X$.By hypothesis $\exists\ \Delta \in |D|$ s.t. $x \in \Delta$. $|H|_{|\Delta}$ is very ample, in particular it is basepoint free; thus $\exists H$ s.t. $x \notin H$.

1.(i) Assume $x, x' \in X - C$.By hypothesis $\exists \Delta \in |D|$ s.t. $x \in \Delta$.

If $x' \in \Delta$, being H very ample on Δ, $\exists \Gamma \in |H|$ s.t. $x \in \Gamma$ but $x' \notin \Gamma$.

If $x' \notin \Delta$, since $x, x' \notin C$, $\Delta + C$ is a divisor in $|H|$ s.t. $x \in \Delta + C$ but $x' \notin \Delta + C$.

(ii) Assume $x, x' \in C$. Since H is very ample on C by hypothesis, $\exists \Gamma \in |H|$ s.t. $x \in \Gamma$ but $x' \notin \Gamma$.

(iii) Finally assume $x \in X - C$ and $x' \in C$. Then if we find a $\Delta \in |D|$ s.t. $x \notin \Delta$, $x' \in \Delta + C$ but $x \notin \Delta + C$. Otherwise, if $\forall \Delta \in |D|$ $x \in \Delta$, let us choose Δ such that $\Delta \ni x'$. Thus we can conclude since $|H|_{|\Delta}$ is very ample.

2. We need to show that if $x \in X$, $v \in T_x X$ then $\exists\ \Gamma \in |H|$ s.t. $x \in \Gamma$ but $v \notin T_x \Gamma$.

(i) Assume $x \in X - C$. By hypothesis $\exists \Delta \in |D|$ with $x \in \Delta$. If $v \notin T_x\Delta$ then $\Delta + C$ is the desired divisor. If $v \in T_x\Delta$, since $|H|_{|\Delta}$ is very ample $\exists \Gamma \in |H|$ s.t. $x \in \Gamma, v \notin T_x\Gamma$.

(ii) Let $x \in C$. If $v \in T_xC$, being $|H|_{|C}$ very ample, $\exists \Gamma \in |H|$ s.t. $\Gamma \ni x$ but $v \notin T_x\Gamma$.

If $v \notin T_xC$ and x is not a base point for $|D|$, we choose $\Delta \in |D|$ s.t. $x \notin \Delta$. So we have $v \notin T_x(\Delta + C)$, while $x \in \Delta + C$.

Otherwise, if x is a base point for $|D|$, being $dim|D| \geq 1$, $\exists \Delta \in |D|$ s.t. $v \in T_x\Delta$, thus we can conclude by the very ampleness of $|H|$ on Δ. □

We have some obvious necessary conditions in order that φ_H be an embedding.

Proposition 5.2. *Let S be a smooth projective surface and H a very ample divisor on S. Let $C \subset S$ be an effective divisor. Then:*

1) $C.H > 0$
2) $p_a(C) \geq 1 \Rightarrow C.H \geq 3$
3) $p_a(C) \geq 2 \Rightarrow C.H \geq 4$
4) $p_a(C) = 2 \Rightarrow C.H \geq 5$

Proof. If H is very ample we obviously have $C.H > 0 \ \ \forall C > 0$.

Concerning the other inequalities we can observe that $h^0(\mathcal{O}_C(H)) = 2 \Rightarrow (C, \mathcal{O}_C(H)) \cong (\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(1))$, while $h^0(\mathcal{O}_C(H)) = 3 \Rightarrow C$ is isomorphic to a plane curve, and in this case the inequalities are achieved.

Let us assume now $h^0(\mathcal{O}_C(H)) \geq 4$. Since H is very ample, we know that there exists a section with 0-dimensional set of zeroes, so we have the following exact sequence

$$0 \to \mathcal{O}_C \to \mathcal{O}_C(H) \to \Delta \to 0$$

where the length of Δ is $H.C$, and then $h^0(\mathcal{O}_C) + H.C \geq h^0(\mathcal{O}_C(H))$.

If $h^0(\mathcal{O}_C) \geq 2$, then there exists a decomposition $C = C_1 + C_2$ with $C_1.C_2 \leq 0$. Since $p_a(C) \leq p_a(C_1) + p_a(C_2)$, by induction we can reduce to consider the case where C is numerically connected and therefore $h^0(\mathcal{O}_C) = 1$. In this case $C.H \geq 3$ but equality cannot hold, since then

$$H^1(\mathcal{O}_C) \cong H^1(\mathcal{O}_C(H)) \Longleftrightarrow H^0(K_C) \cong H^0(K_C(-H))$$

and then H should be contained in the fixed part of K_C, absurd since H moves.

It remains to be proven that if $p_a(C) = 2$, and C is numerically connected $(h^0(\mathcal{O}_C(H)) \geq 4)$, then $H.C \geq 5$.

Otherwise $H^0(\mathcal{O}_C(K_C - H))$ would have dimension at least 1 and there would be a decomposition $C = A + B$ with $B > 0$ s.t. $(K_C - H).A \geq A.B$, or equivalently s.t. $K_A.A \geq H.A$.

Since we can assume $p_a(A) \leq 1$ (else $A.H \geq 4$, thus $C.H \geq 5$), then the above contradicts the ampleness of H.

□

We are now able to consider the rational surfaces of degree 7 and 8 mentioned in the introduction.

The next theorem deals with the case of degree =7.

These surfaces were constructed by Ionescu in [Io 1] (pag.179, prop.8.1) and by Okonek in [Ok 2].

Theorem 5.3. *Let* $S = \hat{\mathbf{P}}^2(x_1, \dots, x_6, y_0, \dots, y_4)$ *and let* $|H| = |6L - \sum_{i=1}^{6} 2x_i - \sum_{j=0}^{4} y_j|$.

H is very ample if and only if:

0) The only possibility of infinitely near points is a point y_j infinitely near

to a point x_i; given a point x_i, there is at most one such y_j.

1) $h^0(L - \sum_{i\in\Delta} x_i - \sum_{j\in\Lambda} y_j) = 0 \quad \forall \Delta \subseteq \{1, \dots 6\},\ \Lambda \subseteq \{0, \dots, 4\}$ $s.t.\ 2\#\Delta + \#\Lambda \geq 6$

2) $h^0(2L - \sum_{i\in\Delta} x_i - \sum_{j\in\Lambda} y_j) = 0 \quad \forall \Delta \subseteq \{1, \dots, 6\},\ \Lambda \subseteq \{0, \dots, 4\}$ $s.t.\ 2\#\Delta + \#\Lambda \geq 12$

3) $h^0(3L - \sum_{i=1}^{6} x_i - \sum_{j\neq h} y_j) = 0 \quad \forall h \in \{0, \dots, 4\}$

PROOF

Necessity of the above conditions:

The necessity of the above conditions 0),1), 2) follows otherwise we would have an effective divisors with a non positive intersection product with H, against the ampleness of H.

The condition 3) is due to the fact that a divisor $A \in |3L - \sum_{i=1}^{6} E_i - \sum_{j\neq h} F_j|$ has $p_a(C) = 1$,while $H.C = deg\varphi_H(C) = 2$, contradicting prop.5.2.

The above conditions are sufficient:

We choose a particular decomposition $H = C + D$ that verifies the assumption of proposition 5.1. We let

$C = 3L - \sum_{i=1}^{6} E_i - \sum_{j\geq 2} F_j$,

$|D| = |H - C| = |3L - \sum_{i=1}^{6} E_i - F_0 - F_1|$

Observe that $dim|D| \geq 9-8=1$, whence we can apply prop. 5.1. Moreover the following lemmas hold:

Lemma 5.4. $|H_{|C}| = |H|_{|C}$.

Proof. Consider the rational surface $Z = \hat{\mathbf{P}}^2(x_1,\dots,x_6)$: since for all y_j there is no x_i infinitely near to y_j, we have a surjective morphism $f : S \longrightarrow Z$.

$M = 3L - \sum_{i=1}^{6} E_i$ is very ample on Z (the x_i satisfy the standard hypotheses, cf. e.g. theorem 4.6 cap.V in [Ha]) and embeds Z as a cubic surface in $\mathbf{P}^3$, thus $H^1(Z, \mathcal{O}_Z(M)) = 0$.

Considering the Leray spectral sequence relative to f we get also the vanishing of $H^1(S, \mathcal{O}_S(3L - \sum_{i=1}^{6} E_i))$.

Moreover, since M is very ample on Z we have an injective map

$$H^1(S, \mathcal{O}_S(3L - \sum_{i=1}^{6} E_i - F_0 - F_1)) \hookrightarrow H^1(S, \mathcal{O}_S(3L - \sum_{i=1}^{6} E_i))$$

and then also $0 = H^1(S, \mathcal{O}_S(3L - \sum_{i=1}^{6} E_i - F_0 - F_1)) = H^1(H - C)$.
□

Lemma 5.5. $|H_{|\Delta}| = |H|_{|\Delta}$ $\forall \Delta \in |H - C|$.

Proof. By Riemann-Roch $h^1(S, \mathcal{O}_S(H-\Delta)) = h^1(S, \mathcal{O}_S(C)) = 0$,since $h^0(S, \mathcal{O}_S(C)) = 1$ by the assumption 3) and $h^2(S, \mathcal{O}_S(C)) = h^0(S, \mathcal{O}_S(K_S - C)) \leq h^0(S, \mathcal{O}_S(K)) = 0$.

□

To conclude the proof of the theorem we need only to show that $H_{|C}$ is very ample, and $H_{|\Delta}$ is very ample $\forall \Delta \in |D|$.

Observe that $p_a(C) = 1$, $p_a(\Delta) = 1$ while $C.H = 3$ and $\Delta.H = 4$. Our purpose is then to verify the assumptions of the embedding theorem 3.1.

Lemma 5.6. *Let S and H be as in theorem 5.3. Let $C = 3L - \sum_{i=1}^{6} E_i - \sum_{j\geq 2} F_j$ and let $\Delta \in |3L - \sum_{i=1^6} E_i - F_1 - F_0|$. Then:*

$$\begin{array}{ll} (i) & \forall B \leq C, \quad B.H \geq (2p_a(B)+1) \\ (ii) & \forall B' \leq \Delta, \quad B'.H >\geq (2p_a(B')+1) \end{array}$$

Proof. Let $B \equiv aL - \sum b_i E_i - \sum c_j F_j$.

First of all observe that an effective divisor B is such that $C - B \geq 0$ or $\Delta - B \geq 0$ only if $0 \leq a \leq 3$, so it is sufficient to consider the effective divisors B (B as above) such that $b_i, c_j \in \mathbf{Z}$ and $0 \leq a \leq 3$.

Let us show firstly that $|H|$ is ample.

To this purpose it is sufficient to consider the case where B is irreducible. If $a \leq 2$ the ampleness of H follows by conditions 0) for the irreducible curves contracted by π, and conditions 1),2) for the rest.

Let $a = 3$. If $\forall i \in \{1, \dots 6\}$ $b_i \leq 1$ and $\forall j \in \{0, \dots 4\}$ $c_j \leq 1$ we have $B.H \geq 18 - 12 - 5 = 1$. Otherwise, since B is irreducible, there exists exactly one index among the b_h's and the c_k's which equals 2 (the other ones being ≤ 1).

Let $\Gamma = \{i | b_i = 1\}$, $\gamma = \#\Gamma$, $\Lambda = \{j | c_j = 1\}$ and $\lambda = \#\Lambda$.
Thus if there is a $b_h = 2$

$$B.H \leq 0 \Leftrightarrow \gamma = 5 \text{ and } \lambda = 4 \text{ or } 5$$

while if there is a $c_k = 2$

$$B.H \leq 0 \Leftrightarrow \gamma = 6 \text{ and } \lambda = 4,$$

and in both cases we have $B \leq 3L - \sum_{i=1}^{6} E_i - \sum_{j \neq k} F_j \quad (k \in \{0, \dots 4\})$ contradicting condition 3).

Now, $p_a(B) < 1$ except if $a = 3$ and $b_i, c_j = 0, 1$ (in which case $p_a(B) = 1$). This follows by the *genus formula*

$$2p_a(B) - 2 = a(a-3) - \sum b_i(b_i - 1) \tag{3}$$

since $b_i(b_i - 1) \geq 0$ and > 0 if b_i is negative.

In this last case we want to show that $B.H \geq 3$. But if $p_a(B) = 1$ and $B.H \leq 2$ condition 3) is violated. $\square$

Q.E.D. for thm. 5.3

We pass now to the case of surfaces of degree 8. These surfaces were constructed by Okonek in [Ok 3]; later their existence was reproven by Alexander in [Al].

Theorem 5.7. *Let* $S = \hat{\mathbf{P}}^2(x_0, \dots, x_{10})$, $|H| = |7L - \sum_{i=1}^{10} 2x_i - x_0|$. *$H$ is very ample if and only if*

0) The point x_0 is the unique point which can be infinitely near to another x_i.

1) $h^0(L - \sum_{i \in \Lambda} x_i) = 0$ $\forall \Lambda \subset \{0, \dots, 10\}$ *s.t.*$\#\Lambda \geq 4$.

2) $h^0(2L - \sum_{i \in \Lambda} x_i) = 0$ $\forall \Lambda \in \{1, \dots, 10\}$ *s.t.*$\#\Lambda \geq 7$.

3) $h^0(3L - \sum_{i \neq h} x_i) = 0$ $\forall h \in \{0, \dots, 10\}$.

PROOF

Necessity of the above conditions:

The necessity of the above conditions 0),1), 2) follows otherwise we would have an effective divisors with a non positive intersection product with H, against the ampleness of H.

For 3) it is enough to notice that $A \in \mid 3L - \sum_{i \neq h} E_i \mid$ has arithmetic genus 1 while $A.H \leq 2$.

The above conditions are sufficient:

Let

$C = 3L - \sum_{i>1} E_i$

$|D| = \mid H - C \mid = \mid 4L - \sum_{i>1} E_i - 2E_1 - E_0|$

Observe that $\dim|H - C| \geq 1$. Our purpose is then to verify the very ampleness of H studying $|H|_{|C}$ and $|H|_{|\Delta}$ for all $\Delta \in |H - C|$.

Lemma 5.8. $|H_{|C}| = |H|_{|C}$.

Proof. Since x_0 is the unique point that could be infinitely near to another x_i , $\exists h \in \{0, 2, \ldots, 10\}$ such that x_0 is not infinitely near to x_h. Without loss of generality we may assume $h = 1$: considering the surface $Z = \hat{\mathbf{P}}^2(x_0, x_2, \ldots, x_n)$ we have a surjective morphism $g : S \to Z$.

Now $M = 4L - \sum_{i \neq 1} E_i$ is well known to be very ample on Z (cf. e.g. [Wei] cor. 1.6a)), and $H^1(Z, \mathcal{O}_Z(M)) = 0$ since $H^1(M, \mathcal{O}_M(M)) = 0$ (because $6 = M^2 > 4 = 2p_a(M) - 2$ and M is smooth).

Considering the spectral sequence relative to g we obtain $H^1(S, \mathcal{O}_S(4L - \sum_{i \neq 1} E_i)) = 0$ and, since M is very ample on Z we have an injective map

$$H^1(S, 4L - \sum_{i \neq 1} E_i - 2E_1) \hookrightarrow H^1(S, 4L - \sum_{i \neq 1} E_i),$$

thus $H^1(S, H - C) = 0$.

□

Lemma 5.9. $|H_{|\Delta}| = |H|_{|\Delta}$ $\forall \Delta \in |H - C|$.

Proof. Since $h^0(S, C) = 1$ by condition 3) and $h^2(S, C) = 0$ because the surface is rational, by Riemann-Roch we get that $h^1(S, C) = 0$.

□

As in the previous theorem we need to show that $|H_{|C}|$ and $|H_{|\Delta}|$ are very ample.

Observe that $p_a(C) = 1$, $p_a(\Delta) = 2$ $\forall \Delta \in |H - C|$, while $C.H = 3$ and $\Delta.H = 5$. So it is enough to prove the assumptions of theorem 3.1.

Lemma 5.10. *Let S and H be as in theorem 5.7. Let $C = 3L - \sum_{i>1} E_i$ and let $\Delta \in |4L - \sum_{i\geq 2} E_i - 2E_1 - E_0|$. Then:*

$$\begin{array}{ll} (i) & \forall\ B \leq C, \quad B.H \geq (2p_a(B)+1) \\ (ii) & \forall\ B' \leq \Delta, \quad B.H \geq (2p_a(B')+1) \end{array}$$

Proof. As before, it is sufficient to consider the effective divisors B such that $B \equiv aL - \sum b_i x_i$ with $b_i \in \mathbf{Z}$ and $0 \leq a \leq 4$.

Let us show firstly that $|H|$ is ample. To this purpose we may assume B to be irreducible.

If $a \leq 2$ conditions 0),1),2) imply that H is ample on B.

If $a = 3$ there exists at most an index $h \in \{0, \ldots, 10\}$ s.t. $b_h = 2$. Thus, setting $\Lambda = \{i > 0 | b_i = 1\}$ and $\lambda = \#\Lambda$ we have $B.H \leq 0$ if and only if $\lambda = 8$, contradicting condition 3).

If $B \equiv 4L - \sum_{i\in\Lambda} b_i E_i$, observe that $B \leq \Delta$ only if $b_1 \geq 2$ and $b_i \geq 1\ \forall i \neq 1$. Thus if $B \leq \Delta$ is irreducible, there is at most an index $h \in \{0, \ldots, 10\}$ such that $b_h = 2$ or $b_1 = 3$ and $b_i = 1\ \forall i \neq 1$ and in both cases we have $B.H \geq 1$.

Now, if $a \leq 3$ we have $p_a(B) \leq 1$ and equality, as before, holds only if $a = 3$ and $b_i = 0, 1$. In this last case we observe that $A.H \leq 2$ if and only if $A \equiv 3L - \sum_{i\neq k} E_i$, a contradiction.

Moreover if $a = 4$, by the *genus formula* we have $p_a(B) = 1$ if $\sum_{i=0}^{10} b_i(b_i - 1) \leq 4$, i.e. there are two indices $h, k \in \{0, \ldots, 10\}$ such that $b_h = b_k = 2$, and $\forall i \neq h, k\ b_i = 0, 1$. Then $B.H \geq 3$.

To conclude it suffices to remark that $p_a(B) \geq 2$ only if $a = 4$ and there is at most an index $h \in \{0, \ldots, 10\}$ such that $b_h = 2$: but since $B \leq \Delta$, then must be $B = \Delta$ and $B.H = 5$ as required. $\square$

Q.E.D. for thm. 5.7

We give now a list of the non special rational surfaces S in $\mathbf{P}^4$ and of the decompositions $H = C + D$ which can be used to show that H is very ample on each S (the list of the surfaces appears already in [Al]).

For degree ≤ 6 the proof is easily dealt with by the already used method; the case $d = 9$ will be treated in a forthcoming article by the first author and K.Hulek. As usual we denote by $\varphi_H(\hat{\mathbf{P}}^2(x_1, ..., x_n))$ the image of $\hat{\mathbf{P}}^2(x_1, ..., x_n)$ by the linear system $|H|$.

d=2 (smooth quadric):	
$S = \varphi_H(\hat{\mathbf{P}}^2(x_1, x_2)) \cong \mathbf{P}^1 \times \mathbf{P}^1$ $(\subset \mathbf{P}^3)$	$\lvert H\rvert = \lvert 2L - x_1 - x_2\rvert$
d=3 (cubic ruled):	
$S = \hat{\mathbf{P}}^2(x_1)$	$\lvert H\rvert = \lvert 2L - x_1\rvert$
d=3 (Del Pezzo):	
$S = \hat{\mathbf{P}}^2(x_1, \dots, x_6)$ $(\subset \mathbf{P}^3)$	$\lvert H\rvert = \lvert 3L - \sum_{i=1}^{6} x_i\rvert$
d=4 (Del Pezzo):	
$S = \hat{\mathbf{P}}^2(x_1, \dots, x_5)$	$\lvert H\rvert = \lvert 3L - \sum_{i=1}^{5} x_i\rvert$
d=5 (Castelnuovo):	
$S = \hat{\mathbf{P}}^2(x_1, y_2 \dots, y_8)$	$\lvert H\rvert = \lvert 4L - 2x_1 - \sum_{i=2}^{8} y_i\rvert$
$\lvert D\rvert = \lvert 3L - x_1 - \sum_{i=2}^{8} y_i\rvert$	$p_a(D) = 1,\ H.D = 3$
$\lvert C\rvert = \lvert L - x_1\rvert$	$p_a(C) = 0,\ H.C = 2$
d=6 (Bordiga):	
$S = \hat{\mathbf{P}}^2(x_1, \dots, x_{10})$	$\lvert H\rvert = \lvert 4L - \sum_{i=1}^{10} x_i\rvert$
$C = 3L - \sum_{i=2}^{10} x_i$	$p_a(C) = 1,\ H.C = 3$
$\lvert D\rvert = \lvert L - x_1\rvert$	$p_a(D) = 0,\ H.D = 3$
d=7 :	
$S = \hat{\mathbf{P}}^2(x_1, \dots, x_6, y_7, \dots, y_{11})$	$\lvert H\rvert = \lvert 6L - \sum_{i=1}^{6} 2x_i - \sum_{j=7}^{11} y_j\rvert$
$C = 3L - \sum_{i=1}^{6} x_i - \sum_{j=7}^{9} y_j$	$p_a(C) = 1,\ H.C = 3$
$\lvert D\rvert = \lvert 3L - \sum_{i=1}^{6} x_i - y_{10} - y_{11}\rvert$	$p_a(D) = 1,\ H.D = 4$
d=8 :	
$S = \hat{\mathbf{P}}^2(x_1, \dots, x_{10}, y_{11})$	$\lvert H\rvert = \lvert 7L - \sum_{i=1}^{6} 2x_i - y_{11}\rvert$
$C = 3L - \sum_{i=2}^{10} x_i$	$p_a(C) = 1,\ H.C = 3$
$\lvert D\rvert = \lvert 4L - 2x_1 - y_{11} - \sum_{i=2}^{10} x_i\rvert$	$p_a(D) = 2,\ H.D = 5$

d=9 :	
$S = \hat{\mathbf{P}}^2(x_1, \dots, x_{10})$	$\lvert H\rvert = \lvert 13L - \sum_{i=1}^{10} 4x_i\rvert$
$C = 3L - \sum_{i=2}^{10} x_i$	$p_a(C) = 1,\ H.C = 3$
$\lvert D\rvert = \lvert 10L - 4x_1 - \sum_{i=2}^{10} 3x_i\rvert$	$p_a(D) = 3,\ H.D = 6$

Remark 5.11. *If $H \equiv C + D$, $dim|D| \geq 1 \Rightarrow C$ is a plane curve, and in the known examples either a line, conic, cubic, 4-ic ($H_{|C} \equiv K_C$).*

Finally we give a list of the necessary conditions of very ampleness for non special rational surfaces of degree ≤ 8.

d=4 (Del Pezzo):
$S = \hat{\mathbf{P}}^2(x_1, \dots, x_5)$ $\qquad |H| = |3L - \sum_{i=1}^5 x_i|$
0) no infinetely near points
1) $h^0(L - x_i - x_j - x_k) = 0 \ \forall i, j, k \in \{1, \dots, 5\}$
d=5 (Castelnuovo):
$S = \hat{\mathbf{P}}^2(x_1, y_2 \dots, y_8)$ $\qquad |H| = |4L - 2x_1 - \sum_{i=2}^8 y_i|$
0) at most one infinitely near point, to x_1
1) $h^0(L - x_1 - y_i - y_j) = 0 \ \forall i, j \in \{2, \dots, 8\}$
2) $h^0(L - \sum_{i \in \Delta} y_i) = 0$ for $\#\Delta \geq 4$
3) $h^0(2L - x_1 - \sum_{i \neq j} y_i) = 0 \ \forall j \in \{2, \dots, 8\}$
d=6 (Bordiga):
$S = \hat{\mathbf{P}}^2(x_1, \dots, x_{10})$ $\qquad |H| = |4L - \sum_{i=1}^{10} x_i|$
0) no infinitely near points
1) $h^0(L - \sum_{i \in \Delta} x_i) = 0$ for $\#\Delta \geq 4$
2) $h^0(2L - \sum_{i \in \Delta} x_i) = 0$ for $\#\Delta \geq 8$
3) $h^0(3L - \sum_1^{10} x_i) = 0$
d=7:
$S = \hat{\mathbf{P}}^2(x_1, \dots, x_6, y_7, \dots, y_{11})$ $\quad |H| = |6L - \sum_{i=1}^6 2x_i - \sum_{j=7}^{11} y_j|$
0) at most one y_j infinitely near to a point x_i
1) $h^0(L - \sum_{i \in \Delta} x_i - \sum_{j \in \Lambda} y_j) = 0$ for $2\#\Delta + \#\Lambda \geq 6$
2) $h^0(2L - \sum_{i \in \Delta} x_i - \sum_{j \in \Lambda} y_j) = 0$ for $2\#\Delta + \#\Lambda \geq 12$
3) $h^0(3L - \sum_{i=1}^6 x_i - \sum_{j \neq h} y_j) = 0 \ \forall h \in \{7, \dots, 11\}$
d=8:
$S = \hat{\mathbf{P}}^2(x_1, \dots, x_{11})$ $\qquad |H| = |7L - \sum_{i=1}^{10} 2x_i - x_{11}|$
0) x_{11} is the unique point which should be infinitely near to another x_j
1) $h^0(L - \sum_{i \in \Lambda} x_i) = 0$ for $\#\Lambda \geq 4$
2) $h^0(2L - \sum_{i \in \Lambda} x_i) = 0$ for $\#\Lambda \geq 7$
3) $h^0(3L - \sum_{i \neq h} x_i) = 0 \ \forall h \in \{1, \dots, 11\}$

6. The geometry of the case $p_a(C) = 1$

In this section we want to describe the geometry of the cases where C is of arithmetic genus 1 while the degree of H equals 3 and 4.

Proposition 6.1. *Let C be a curve lying on a smooth algebraic surface S such that $p_a(C) = 1$ and let H be a divisor on C of degree 3. Moreover suppose that*

$$\forall B \leq C, \quad (2p_a(B) + 1) \leq H.B.$$

Then $|H|$ yields an isomorphism with a plane cubic.

Proof. By cor.2.2 and thm.3.1 $\varphi = \varphi_H$ yields an embedding

$$\varphi : C \hookrightarrow \mathbf{P}^2.$$

Let $\Gamma \subset \mathbf{P}^2$ be the schematic image of C. By lemma 2.5 Γ is a divisor in $\mathbf{P}^2$ of degree 3 since $H.C = 3$. $\square$

Proposition 6.2. *Let C be a curve lying on a smooth algebraic surface S such that $p_a(C) = 1$ and let H be a divisor on C of degree 4. Moreover suppose that*

$$\forall B \leq C, \quad (2p_a(B) + 1) \leq H.B.$$

Then $|H| = \mathcal{O}_C(H)$ yields an isomorphism of C with a quartic curve in $\mathbf{P}^3$ given either by the complete intersection of two quadric surfaces or by the union of a plane cubic with a line which is not contained in the plane of the cubic but which intersects the cubic.

Proof. By our assumptions φ_H yields an embedding

$$\varphi_H : C \hookrightarrow \mathbf{P}^3.$$

We identify C with the pure subscheme $\subset \mathbf{P}^3$of dimension 1 and of degree 4.

Since $h^1(\mathcal{O}_C(mH)) = 0$ and $h^0(\mathcal{O}_C(mH)) = 4m$ $\forall m \geq 1$ we have

$$h^0(\mathcal{I}_C(mH)) = \begin{cases} 2 & m = 2 \\ 8 & m = 3 \\ 19 & m = 4 \\ 36 & m = 5 \end{cases}$$

whence $H^0(\mathcal{I}_C(2H))$ is generated by the equations q,q' of two quadric surfaces Q, Q'.

(I) If $Q \cap Q'$ has dimension 1 then C and $Q \cap Q'$ have the same degree, whence they coincide.

(II) If $Q \cap Q'$ has dimension 2, then there are linear forms x_0, x, x' such that $Q = = \{x_0 x = 0\}$, $Q' = \{x_0 x' = 0\}$, and x and x' are independent. In this case it is easy to see that the map

$$H^0(\mathcal{I}_C(2)) \otimes H^0(\mathcal{O}_{\mathbf{P}^3}(1)) \to H^0(\mathcal{I}_C(3))$$

has image of dimension =7, whence there exists a g_3 in $H^0(\mathcal{I}_C(3))$ but not in the ideal (q, q').

We shall show that $x_0 | g_3$ gives rise to a contradiction.

Let C' be the subcurve of C which equals, outside a finite set, the subscheme of C defined by the equation $x_0 = 0$. If all components of C are contained in C' we claim that $C \leq 2C'$.

Otherwise $C = 3A + B$, with $A.H = B.H = 1$ and then $p_a(C) = 3(A^2 + A.B - 1)$, contradicting $p_a(C) = 1$.

In particular $H.C' \geq 2$.

If $H.C' = 2$ then $C = 2C'$, whence $p_a(C) = 1$ implies $p_a(C') = 0$ and $(C')^2 = 2$. Thus, from the exact sequence

$$0 \to \mathcal{O}_{C'}(H - C') \to \mathcal{O}_C(H) \to \mathcal{O}_{C'}(H) \to 0$$

it follows easily that x_0 is the unique linear form which is $\equiv 0$ on C'. Since $x_0 x \equiv 0$ on C, x vanishes on C'; similarly $x_0 x' \equiv 0$ implies that $x' \equiv 0$ on C', whence x and x' are linearly dependent, a contradiction.

Thus if $C \leq 2C'$, $H.C' = 3$.

Let D be either

(i) a component of C not in C'

or (in case where such D does not exist) set

(ii) $D = C - C'$.

By what we saw D maps to a line. It is clear that $H^0(\mathcal{I}_C(2))$ is generated by $x_0 l, x_0 l'$, where $\{l = l' = 0\}$ is the line image of D.

In particular we can assume $x = l$,$x' = l'$ and moreover $g_3 \in (x, x')$, $g_3 = xB - x'A$, with $A = A(x_1, x_2, x_3)$, $B = B(x_1, x_2, x_3)$.

Thus $x_0 | g_3$ implies $g_3 \in (q, q')$ which is absurd. Therefore

$$C \subset \Delta = rk \begin{pmatrix} x & x' & 0 \\ A & B & -x_0 \end{pmatrix} = 1$$

and the above determinantal scheme has dimension 1. Since Δ is projectively Cohen-Macaulay of degree 4 it equals C. In particular, the projective ideal of C is generated by

$$q, \quad q', \quad g_3$$

with relations

$$x'q - xq' = 0$$
$$Bq - Aq' - x_0 g_3 = 0$$

$\square$

Remark 6.3. *It is possible also the case $x' = x_0$, where C consists of a conic Γ plus a double structure (not contained in $\{x_0 = 0\}$) on a line in $\{x_0 = 0\}$.*

Remark 6.4. *The above example illustrates the philosophy that the ideal of C is generated by quadrics iff C is 2-connected.*

7. The geometry of the case $p_a(C) = 2$

In this section we will study the geometry of the case $p_a(C) = 2, H.C = 5$. In particular we will show that one obtains curves in $\mathbf{P}^3$ of degree 5 which are projectively Cohen-Macaulay and which lie on each possible type of quadric surface.

Lemma 7.1. *Let (C, H) be as in thm.3.1 with $p_a(C) = 2$ and $H.C = 5$. Then C is 2-connected and $\varphi_H(C)$ is contained in a quadric surface Q.*

Proof. Assume $C = A + B$, with $A.B \leq 1$. Then $2 = p_a(C) = -1 + A.B + p_a(A) + p_a(B)$. The assertion follows since $p_a(A), p_a(B)$ are ≤ 1 and cannot both equal 1, by our assumptions.

By cor.2.2 $h^0(\mathcal{O}_C(2H)) = 9$, whence the other assertion.

□

By lemma 7.1 and cor.2.5 follows immediately the following corollary.

Corollary 7.2. *Let C be as above and let K_C be the canonical divisor of C. Then K_C is nef and $|K_C|$ is a free linear system.*

Lemma 7.3. *Let (C, H) be as in thm.3.1 with $p_a(C) = 2$ and $H.C = 5$. If $D \leq C$, with D irreducible, then $C \not\geq 4D$.*

Moreover, if $C = 3D + A$, then A is reducible, $A = A' + A''$ with $D^2 = 0, A'.A'' = 0$, $D.A' = D.A'' = 1$.

In this case $|H|$ yields an isomorphism with a divisor of type (2,3) on $\mathbf{P}^1 \times \mathbf{P}^1 \subset \mathbf{P}^3$ consisting of a triple vertical line D' and of two horizontal lines (observe that K_C, and $L = H - K_C$ are the respective pull backs of $\mathcal{O}_{\mathbf{P}^1}(1)$ under the two projections).

Proof. We use the key formula

$$p_a(mD) = mp_a(D) - (m-1) + (m(m-1)/2)D^2 \tag{4}$$

Thus, if $C = 5D$, then $p_a(C) = -4 + 10D^2 \neq 2$, since $p_a(D) = 0$.

Let $C = 4D + A$: $p_a(4D) = -3 + 6D^2$, whence $p_a(C) = -4 + 6D^2 + 4D.A \Rightarrow D.A \equiv 0(mod\ 3)$. Since C is connected, $D.A \geq 3$, and $p_a(D + A) \geq 2$, a contradiction.

If $C = 3D + A$, $p_a(3D) = -2 + 3D^2 \leq 1$. Whence $D^2 \leq 1$, but if $D^2 = 1$, then, since there exists an irreducible $B \leq C$, $B \neq D$, such that $B.D \geq 1$, we get $p_a(3D + B) \geq 3$, a contradiction.

Thus $D^2 \leq 0$.

Since $2 = p_a(C) = p_a(A) - 3 + 3D^2 + 3D.A \Rightarrow p_a(A) \equiv -1(mod\ 3)$, and since $p_a(A) \leq 0$, A is reducible, $A = A' + A''$.

Again, $p_a(D+A') \leq 0 \Rightarrow D.A' \leq 1$, similarly $D.A'' \leq 1$.

By the above expression for $2 = p_a(C)$, since $3D.A \leq 6$, we obtain $D^2 = 0, p_a(A) = = -1$, and $D.A = 2$, whence $D.A' = D.A'' = 1$.

Then $-1 = p_a(A) = -1 + A'.A''$, and $A'.A'' = 0$.

To prove the isomorphism with a divisor of type $(2,3)$ it suffices to show that C is contained in a smooth quadric, the rest follows easily since all components of C map to lines.

Now, D maps to a line, and we know, by lemma 7.1, that C is contained in a quadric Q.

Assume that Q is reducible, $Q = \{h \cdot h' = 0\}$. Then it follows that 2D is contained in one of the 2 planes, else $h \cdot h'$ would vanish on 2D, but not on 3D. But then D^2, the degree of the normal bundle of D, would equal 1, and not 0.

Similarly, $D^2 = 1$ if $3D$ is contained in a quadric cone Q.

□

Lemma 7.4. *Let (C,H) be as in thm.3.1 with $p_a(C) = 2$, $H.C = 5$ and assume that there are $A \neq B$ such that $C \geq 2A + 2B$, so that $C = 2A + 2B + D$.*

Assume (by lemma 7.3) that $D \neq A, D \neq B$. Then $A.B = 1$, and we can assume either

$$\begin{array}{rll} i) & A.D = 1, B.D = 1, A^2 = B^2 = -1 & \quad or \\ ii) & A.D = 1, B.D = 0, A^2 = B^2 = 0, & \quad or \\ iii) & A.D = 1, B.D = 0, A^2 = -1, B^2 = 1. & \end{array}$$

Proof. Obviously $A.B, A.D, B.D \leq 1$.

We have $p_a(2A) = -1 + A^2 \leq 0$, whence $A^2 \leq 1$, similarly $B^2 \leq 1$.

Since $1 \geq p_a(2A+2B) = -3 + A^2 + B^2 + 4A.B$, $A^2 + B^2 \leq 0$. Moreover $2 = p_a(C) = -4 + A^2 + B^2 + 4A.B + 2A.D + 2B.D$.

We infer from the above that $A.B = 1$, and that $A^2 + B^2$ is even. Therefore either $A^2 + B^2 = -2$, $A.D = B.D = 1$, or we can assume $A^2 + B^2 = 0, A.D = 1, B.D = 0$.

$1 \geq p_a(2A + B + D) = 1 + A^2 + B.D \Rightarrow A^2 \leq -1$ in the former case, $A^2 \leq 0$ in the latter.

$1 \geq p_a(2B + A + D) = B^2 + 2B.D \Rightarrow B^2 \leq -1$ in the former case.

Thus in the former case $A^2 = B^2 = -1$.

In the latter case there are two possibilities, namely $A^2 = B^2 = 0$, or $A^2 = = -1, B^2 = 1$.

□

Lemma 7.5. *Let C be as above, let K_C be the canonical divisor of C, and set $L = H - K_C$. If L is not nef, then H embeds C in $\mathbf{P}^3$ as a* *subscheme of a reducible quadric. Moreover if L is not nef, there is a unique component D such that $L.D < 0$.*

We have $p_a(D) = 0$, $D.L = -1$, $K_C.D = 2$; if we set $D' = C - D$, then $D.D' = 4$, D' is reducible and for each $B \leq D'$ we have $D.B = H.B$.

Proof. Let D be an irreducible component of C such that $L.D \leq 0$.

Since $H.D \leq K.D \leq 2$ then by our assumptions $p_a(D) = 0$.

If moreover $L.D < 0$, then since H is ample, $K_C.D \geq 2$, whence $K_C.D = 2$ and D is therefore unique.

In particular, $D.H = 1$ and D has no common components with $D' = C - D$.

By the adjunction formula, since $K_C.D = 2$, it follows that $D'.D = 4$. By the same token $p_a(D') = -1$, in particular D' is reducible.

Let B be an irreducible component of D' : it has $p_a(B) = 0$ since $K_C.B = 0$, and, by adjunction and 2-connectedness $K_B.B \leq -2$. Moreover, if $B.D = 2$, then $p_a(D + B) = 1$, whence $H.B \geq 2$.

We conclude therefore (since $D.D' = H.D' = 4$) that $H.B = D.B$ for all $B \leq D'$.

Recall moreover that if B is irreducible and $< D'$, $B.(C - B) = 2$, so that if B is irreducible and $B.D = 2$, B is disjoint from $D' - B$, unless $D' = 2B$.

Case (I) : *if $D' = 2M$, $|H|$ embeds C as a subscheme of a double plane.*

Here, either M is irreducible, or we can apply lemma 7.4, and we are necessarily in case i) since $D.M = 2$. Since $p_a(2M) = -1$, $M^2 = 0$. We have the following exact sequence

$$0 \to \mathcal{O}_M(H - D - M) \to \mathcal{O}_C(H) \to \mathcal{O}_{D+M}(H) \to 0.$$

Observe that $p_a(M + D) = 1$ and $H(M + D) = 3$, whence $|\mathcal{O}_{M+D}(H)|$ embeds $M + D$ as a plane cubic by thm.3.1. Moreover, since divisors on M are classified by their (multi)degree, $\mathcal{O}_M(H - D - M) \cong \mathcal{O}_M$, in particular the above exact sequence yields an exact sequence of global sections and there is exactly one section β vanishing on $D + M$. Since the square of β is zero, $\beta = 0$ provides a plane whose double contains the schematic image of C.

Case (II) : *if $D' \neq 2M$, $|H|$ embeds C as a subscheme of a union of two planes.*

In this case, we claim that there exists a decomposition $D' = B_1 + B_2$ such that $p_a(D + B_i) = 1$, $H.B_i = 2$, $B_1 \cap B_2 = \emptyset$.

If there is an irreducible component B with $B.D = 2$, we can choose $B_1 = B$, and our assertions are clearly verified.

Else, there is a component B with $B.D = H.B = 1$, and such that $2B \not\leq D'$. Since $B.(C - B) = 2$, there is exactly one component B' of D' with $B.B' = 1$, and we choose now B_1 as $B + B'$. Clearly B_2 is disjoint from B_1 and $p_a(D + B_1) = 1$; finally $2 = p_a(C) = p_a(D + B_2) + p_a(B_1) + 1$, and $p_a(D + B_2) \leq 1$, $p_a(B_1) \leq 0$, whence equalities hold throughout. We have the following Mayer-Vietoris sequence

$$0 \to \mathcal{O}_C(H) \to \mathcal{O}_{D+B_1}(H) \oplus \mathcal{O}_{D+B_2}(H) \to \mathcal{O}_D(H) \to 0.$$

Recall that $D \cong \mathbf{P}^1$, and $H.D = 1$, whence by thm.3.1 (since $p_a(D + B_i) = 1$, and $H.(D + B_i) = 3$) we have in particular an exact sequence of global sections and $|H|$ embeds $D + B_i$ as a plane cubic.

By the exact sequence above the two planes are seen to be different and they intersect only in the line which is the image of D.

□

Lemma 7.6. *Let C be as above, let K_C be the canonical divisor of C, and assume that $L = H - K_C$ is nef.*

Then if σ is a section of K_C, and τ is a section of L, and both are not identically zero, while $\sigma\tau$ is identically zero, then $\sigma\tau$ determines a decomposition $C = B + B'$ with $B.B' = 2, p_a(B) = 0, p_a(B') = 1$, and $|H|$ embeds C as a subscheme of a reducible quadric union of two distinct planes.

In this case, moreover, $|L|$ has a base point on B'.

Proof. Let B be a maximal divisor $\leq C$ over which σ vanishes, and write $C = B + B'$. Similarly, τ gives a decomposition $C = A + A'$.

Our assumption is that $B + A \geq C$ (i.e., $A \geq B'$).

We have the standard exact sequences

$$0 \to \mathcal{O}_{B'} \to \mathcal{O}_C(K_C) \to \mathcal{O}_C(K_C)/\sigma\mathcal{O}_{B'} \to 0$$

$$0 \to \mathcal{F} \to \mathcal{O}_C(K_C)/\sigma\mathcal{O}_{B'} \to \mathcal{O}_B(K_C) \to 0,$$

where $dim(\text{supp}\mathcal{F})$=0, and

$$0 \to \mathcal{O}_{A'} \to \mathcal{O}_C(L) \to \mathcal{O}_C(L)/\sigma\mathcal{O}_{A'} \to 0$$

$$0 \to \mathcal{G} \to \mathcal{O}_C(L)/\tau\mathcal{O}_{A'} \to \mathcal{O}_A(L) \to 0,$$

where $dim(\text{supp}\mathcal{G}) = 0$. Taking Chern classes, we get $B.B' + h^0(\mathcal{F}) = K_C.B'$, $A.A' + h^0(\mathcal{G}) = L.A'$ (≤ 3 since L is nef).

By 2-connectedness, $B.B' = 2$, $\mathcal{F} = 0$ and σ defines the Cartier divisor B , whence $0 = K_C.B = B^2$, and $p_a(B) = 0$, $p_a(B') = 1$ by adjunction. In particular, $B'.H \geq 3$ and since $K_C.B' = 2, B'.L \geq 1, B.L \leq 2$.

Therefore $2 \geq H.B = L.B \geq L.A' = 2$, and equalities hold throughout, in particular $L.A' = 2, \mathcal{G} = 0, H.B = 2$. But then $2 = H.B \geq H.A' \geq L.A' = 2$, therefore $B = A'$ and $A = B'$.

Again since $\mathcal{G} = 0$, τ defines the Cartier divisor B'.

We have therefore $K_C = B, L = B', L.B' = 1, K_C.B = 0, B.B' = 2$.

Since we want to apply a Mayer -Vietoris sequence, we want to see firstly whether B, B' can have a common component D.

If such D exists, since $K_C.D = 0$, then $1 \leq L.D \leq L.B'$, whence $L.D = 1 = LB'$, so D is unique and B' is not $\geq 2D$.

If $B \geq 2D$, then $H.B = 2 \Rightarrow B = 2D$, and since then lemma 7.3 applies, we obtain $p_a(B) = p_a(2D) = -1$, a contradiction.

Thus in this case we can write $B = D + E, B' = D + E'$, where E, E' have no common components. Since $p_a(B) = 0$, $D.E = 1$, and since $K_C = B, K_C.D = 0$, we have $D^2 = -D.E = -1$.

In turn $p_a(B') = 1 \Rightarrow D.E' \geq 2$, but $2 = K_C.B' = K_C.E' = D.E' + E.E'$. Therefore $E.E' = 0$ and E, E' are disjoint.

We have therefore a Mayer-Vietoris sequence

$$0 \to \mathcal{O}_C(H) \to \mathcal{O}_B(H) \oplus \mathcal{O}_{B'}(H) \to \mathcal{F} \to 0$$

where $\mathcal{F} = \mathcal{O}_D(H) = \mathcal{O}_{\mathbf{P}^1}(1)$ if B, B' have a common component, otherwise $\mathcal{F}$ is a skyscraper sheaf of length 2.

By thm.3.1, firstly by pure dimension count the sequence of global sections is exact, secondly $|H|$ embeds B as a plane conic and B' as a plane cubic. A similar proof as the one in lemma 7.5 shows that the two planes are distinct.

Notice that if B, B' have a common component, C maps to a double line plus a line $\varphi(E)$ and a conic $\varphi(E')$.

The last assertion follows easily since $L.B' = 1$, $p_a(B') = 1$, L is nef, B' is 2-connected, thus $h^0(\mathcal{O}_{B'}(L)) = 1$.

□

Lemma 7.7. *If $L = H - K_C$, then $h^0(L) = 2$.*

Proof. By Riemann-Roch $h^0(L) = 2 + h^1(L) = 2 + h^0(K_C - L)$.

Let $\zeta \in H^0(K_C - L)$, $\zeta \not\equiv 0$, and let Z be the maximal curve $\leq C$ contained in $\{\zeta = 0\}$. Set $Y = C - Z$: we have then the standard Ramanujam inequality, whence (remembering also that K_C is nef and C is 2-connected)

$$2 - L.Y \geq (K_C - L)Y \geq ZY \geq 2.$$

In particular $L.Y \leq 0$.

If L is not nef, in view of lemma 7.5, D being the unique component with $L.D < 0$, $L.Y \geq -1$, and if $L.Y < 0$, then $L.Y = -1$ and $Y = D$ (if $Y = D + Y'$, since $K_C.Y' = L.Y' = 0$, we have $Y' = 0$ by the ampleness of H).

(a) If $L.Y = 0$, then $K_C.Y = Z.Y = 2$, whence, by adjunction, $p_a(Y) = 1$: since $H.Y = 2$ we have a contradiction.

(b) If $L.Y < 0$, then $D = Y$ and $Y.Z = 4$, which is not $\leq 2 - L.Y$ since $L.Y = -1$: again a contradiction.

$\square$

We shall proceed by analyzing the natural linear map

$$b : H^0(K_C) \otimes H^0(L) \to H^0(K_C + L) = H^0(H)$$

between the above 4-dimensional spaces, and we denote by W the kernel of b, by Σ the quadric $\mathbf{P}(H^0(K_C)) \times \mathbf{P}(H^0(L))$ of rank one tensors inside $\mathbf{P}(H^0(K_C) \otimes H^0(L))$.

Recall that any tensor has rank 1 or 2, therefore it can be written, for a suitable choice of bases, $\{\sigma_0, \sigma_1\}$ of $H^0(K_C)$, $\{\tau_0, \tau_1\}$ of $H^0(L)$ either in the form

$$1)\sigma_0 \otimes \tau_0$$
or
$$2)\sigma_0 \otimes \tau_1 - \sigma_1 \otimes \tau_0.$$

Lemma 7.8. *Ker b=W does not contain subspaces of the form $\sigma \otimes H^0(L)$ or of the form $H^0(K_C) \otimes \tau$; moreover $dim(W) \leq 2$.*

The base locus of $|L|$ has dimension 1 if and only if L is not nef: in this case its curve part equals the irreducible divisor D with $L.D = 1$.

Finally, if L is not nef, then $dim(W) = 2$, $\mathbf{P}(W)$ being a trasversal line to Σ in case (II), and a tangent line in case (I).

Proof. If τ is such that $\sigma \cdot \tau \equiv 0$ for each $\sigma \in H^0(K_C)$, then $\tau \equiv 0$ since $H^0(K_C)$ is free.

If σ is such that $\sigma \cdot \tau \equiv 0$ for each $\tau \in H^0(L)$, then σ is $\equiv 0$ outside the base locus of $|L|$, in particular $\sigma \equiv 0$ unless the base locus of $|L|$ has dimension 1.

Let Z be the curve part of the base locus of $|L|$ (the maximal curve $\leq C$ such that $\tau_{|Z} \equiv 0 \ \forall \tau \in |L|$) and write $C = Z + Y$, so that we have the inequalities

$$2 \leq Z.Y < L.Y \leq 4$$

(the second inequality since $dim|L| \geq 1$).

If $L.Y = 4$, then L is not nef and $H.Y \geq 4$, whence $Z = D$.

If $L.Y = 3$, then $Z.Y = 2$, $Z.L = 0$, whence $Z.H = Z.K_C \leq 2$ implies $p_a(Z) \leq 0$; but by adjunction $p_a(Z) = (1/2)K_C.Z$, whence $K_C.Z = 0$, and $H.Z = 0$, contradicting the ampleness of H.

Assume again σ to be such that $\sigma \cdot \tau \equiv 0$ for each $\tau \in H^0(L)$, so that we are in the case where L is not nef. By the above we can choose τ such that it vanishes identically only on the curve D. Then $\sigma \equiv 0$ on $C - D$.

Since $D.(C-D) = 4 > K_C.D$, the Ramanujam inequality is contradicted and $\sigma \equiv 0$.

Assume that $dim(W) \geq 3$: then $\mathbf{P}(W)$, by the above, intersects Σ in a smooth conic and for each $\sigma \in H^0(K_C)$ there exists a $\tau \in H^0(L)$ such that $\sigma \cdot \tau \equiv 0$, $\tau \not\equiv 0$: but $|K_C|$ being free, we can choose σ to vanish only at a finite set, whence $\tau \equiv 0$.

If L is not nef, finally, we can consider the two cases of lemma 7.5.

Case (I): Since $\mathcal{O}_M(L-D) \cong \mathcal{O}_M \cong \mathcal{O}_M(K_C)$ and $M^2 = 0$, there is a basis $\{\tau_0, \tau_1\}$ of $H^0(L)$ such that D is the subscheme of zeroes of τ_0, $D + M$ the one of τ_1, and a basis $\{\sigma_0, \sigma_1\}$ of $H^0(K_C)$ such that M is the subscheme of zeroes of σ_0, and $\sigma_1 \not\equiv 0$ on M.

Since there is exactly one section of $\mathcal{O}_C(H)$ vanishing on $D+M$ (up to a non zero constant), multiplying τ by a constant, we obtain that $\sigma_0\tau_0 \equiv \sigma_1\tau_1$; moreover clearly $\sigma_0\tau_1 \equiv 0$.

Therefore $\mathbf{P}(W)$ is a line meeting Σ in a single point.

Case (II): Let x_i be a point of $B_i - D$, and let τ_i be a section of L vanishing at x_i: since $\mathcal{O}_{B_i}(L-D)$ is a trivial sheaf, $p_a(B_i)$ being =0, τ_i vanishes identically on $B_i + D$, but not on B_j.

Similarly, $\mathcal{O}_{B_i}(K_C) \cong \mathcal{O}_{B_i}$, and we can find $\sigma_i \in H^0(K_C)$, with $\sigma_i \equiv 0$ on B_i, $\sigma_i \not\equiv 0$ on B_j.

We can multiply the above sections by non zero constants so that if δ is a section defining the divisor D, $\tau_i = \delta \cdot \sigma_i$. We clearly have $\sigma_1\tau_2 \equiv 0$, $\sigma_2\tau_1 \equiv 0$ (moreover $(\sigma_1\tau_1)\cdot(\sigma_2\tau_2) \equiv 0$, cf. the fact that C maps to the union of 2 distinct planes), so $\mathbf{P}(W)$ meets Σ in 2 distinct points.

$\square$

Lemma 7.9. *Let L be nef, and assume moreover that there are bases $\{\sigma_0, \sigma_1\}$ of $H^0(K_C)$, $\{\tau_0, \tau_1\}$ of $H^0(L)$ such that $\sigma_0\tau_1 \equiv \sigma_1\tau_0$.*

Then there is an effective Cartier divisor Δ on C which is the base locus of $|L|$, and defines a smooth point of C; moreover $|L| = \Delta + |K_C|$.

Proof. Remember that if L is nef,then the base locus of $|L|$ has dimension 0.

Since $|K_C|$ has no base points, at each x in C either σ_0 or σ_1 does not vanish, thus by the relation $\sigma_0\tau_1 \equiv \sigma_1\tau_0$ the ideal sheaf (τ_1, τ_0) is invertible and defines an effective Cartier divisor Δ on C with 0-dimensional support. Clearly $|L| = \Delta + |K_C|$, and since Δ has degree 1, Δ defines a smooth point of C. □

Corollary 7.10. *$\varphi_H(C)$ is contained in a smooth quadric $\Leftrightarrow W = 0 \Leftrightarrow |L|$ is a free linear system.*

Proof. If $W = 0$, then a basis of $H^0(H)$ is given by the 4 sections $\sigma_i\tau_j$: since $(\sigma_0\tau_0)\cdot(\sigma_1\tau_1) = (\sigma_0\tau_1)\cdot(\sigma_1\tau_0)$, $\varphi_H(C)$ is contained in a smooth quadric.

It is clear that under the last assumption $|L|$ is free.

It suffices thus to verify that if $|L|$ is free, then $W = 0$: but otherwise L being nef, we would have either the assumption of lemma 7.6 or the one of lemma 7.9, whence $|L|$would not be free.

□

Lemma 7.11. *Let C be as in lemma 7.9 ($\sigma_0\tau_1 \equiv \sigma_1\tau_0$ and $|L|$ has base locus a smooth point x).*

Then $dim(W) = 1$ and $|H|$ yields an embedding φ_H of C as a subscheme of a quadric cone Q in such a way that $\varphi_H(x)$ is the vertex v of Q.

The morphism φ_H lifts to a morphism $\psi : C \to \mathbf{F}_2$ ($\mathbf{F}_2$ being as usual the Segre-Hirzebruch surface) which is an isomorphism with a divisor in the linear system $|2H + F|$, F being a fibre of the projection with centre v to $\mathbf{P}^1$.

Proof. Since $\sigma_0\tau_1 \equiv \sigma_1\tau_0$, we have a section Δ whose divisor is x and such that $\tau_i \equiv \Delta\sigma_i$; whence we see easily that we have 3 independent sections of $H^0(H)$ which vanish at x, namely, $\sigma_0\tau_0, \sigma_1\tau_0, \sigma_1\tau_1$, and $\varphi_H(C)$ is contained in a quadric cone Q .

Moreover, φ_H lifts to $\psi : C \to \mathbf{F}_2 \subset \mathbf{P}^3 \times \mathbf{P}^1$, the map ψ being given by the pair (φ_H, φ'), φ' being the morphism associated to $|K_C|$.

Denote by Γ the divisor which is the schematic image of C. If E is the exceptional divisor of $\mathbf{F}_2$, $\psi^*(E)$ is the smooth point x, whence Γ is linearly equivalent to $F + bH$.

But $5 = \Gamma.H$ implies $b = 2$.

□

Remark 7.12. *Conversely, if $\Gamma \in |2H+F|$, $p_a(\Gamma) = 2$, and if E is not a component of Γ, H is very ample on Γ.*

Remark 7.13. *By 7.11, if L is nef and $|L|$ is not free, $dim(W) = 1$. In fact if $dimW = 2$, then there exists a relation of type $\sigma_1\tau_0 \equiv \sigma_1\tau_0$ and lemmas 7.9, 7.11 imply $dim(W) = 1$, a contradiction.*

Finally, if C is as in 7.6, since $\tau \equiv 0$ defines B', the base locus of $|L|$ is contained in B', and it restricts to a smooth point of $B'(L.B' = 1)$.

We can finally conclude the classification of the embedded curves with $p_a = 2$ and degree 5.

Theorem 7.14. *Let (C,H) be as in thm.3.1 with $p_a(C) = 2$ and $H.C = 5$. Then $|H|$ embeds C in $\mathbf{P}^3$, and $\varphi_H(C)$ is contained in a unique quadric surface $Q = \{q = 0\}$.*

Moreover, $\varphi_H(C)$ is projectively Cohen-Macaulay, and we have a length 2 resolution of its homogenous ideal I_C (where $\mathcal{A} \cong \mathbf{C}[x_0,x_1,x_2,x_3]$):

$$0 \to 2\mathcal{A}(-4) \xrightarrow{\alpha} \mathcal{A}(-2) \oplus 2\mathcal{A}(-3) \to I_C \to 0$$

There are 5 different cases, according to the following table (q_1, q_2 are quadratic forms).

$rk(q)$	$\mathbf{P}(W)$	*Base locus of* $\vert L\vert$	L	*Equation of* C
4	$\emptyset$	$\emptyset$	*nef*	$rk\begin{pmatrix} x_0 & x_1 & q_1 \\ x_2 & x_3 & q_2 \end{pmatrix} = 1$
3	*a point* $\notin \Sigma$	*a smooth point*	*nef*	$rk\begin{pmatrix} x_0 & x_1 & q_1 \\ x_1 & x_2 & q_2 \end{pmatrix} = 1$
2	*a point of* Σ	*a point*	*nef*	$rk\begin{pmatrix} x_0 & x_1 & q_1 \\ 0 & x_2 & q_2 \end{pmatrix} = 1$
2	*a line trans-versal to* Σ	*D with* $D.L = -1$	*not nef*	$rk\begin{pmatrix} x_0 & 0 & q_1 \\ 0 & x_1 & q_2 \end{pmatrix} = 1$
1	*a line tan-gent to* Σ	*D with* $D.L = -1$	*not nef*	$rk\begin{pmatrix} x_0 & x_1 & q_1 \\ 0 & x_0 & q_2 \end{pmatrix} = 1$

Each of the above cases occurs and can specialize to each of the cases appearing below it.

Proof. We identify C with $\varphi_H(C)$ and, similarly, its subcurves with their images, which are pure subschemes of $\mathbf{P}^3$ of dimension 1.

That C is contained in a unique quadric Q follows from the previous lemmas, as well as everything appearing in the table, with the exception of the equations defining C.

Notice that, since $h^0(\mathcal{O}_C(3)) = 14$, I_C has dimension at least 6 in degree 3, and a minimal set of generators for I_C must include q and at least two cubics g, g'.

To show that C is projectively Cohen-Macaulay, it would suffices to remark that C is projectively normal and, as the referee points out, this could be proved by Castelnuovo theory.

But we shall follow the strategy of producing a matrix α whose minors q, g, g' vanish on C and define a locus Γ of dimension 1. Then we conclude that the two curves C, Γ coincide since Γ has degree 5: in particular it follows that C is projectively Cohen-Macaulay, by the Hilbert-Burch theorem, and the homogeneous ideal I_C has a resolution as above.

Finally, $\mathrm{rk}(\alpha) = 1$ in the points of C, because if P would be a point where $\alpha(P)$ vanishes, then P would be a point of C with tangent dimension =3.

rk(q)=4: then C is a divisor on $\mathbf{P}^1 \times \mathbf{P}^1$ of bidegree (3,2), $C = \{f_{3,2}(\sigma, \tau) = 0\}$. Then $f_{3,2}\tau_0$ and $f_{3,2}\tau_1$ are induced by two respective cubic forms $g(x), g'(x)$ such that q, g, g' define C.

Clearly if $f_{3,2} = \sigma_0 \tilde{q}_2 - \sigma_1 \tilde{q}_1$, and q_i induces $\tilde{q}_i = \tilde{q}_i(\sigma, \tau)$ on Q, then $g = x_0 q_2 - x_2 q_1$, $g' = x_1 q_2 - x_3 q_1$.

rk(q)=3: let Q be the quadric $\{x_0 x_2 - x_1^2 = 0\}$. Adding to the divisor corresponding to C on $\mathbf{F}_2$ the respective divisors $E + F$, $E + F'$, where F is the proper trasform of the line $\{x_0 = x_1 = 0\}$ (F' respectively of the line $\{x_1 = x_2 = 0\}$) we obtain cubic forms $g(x), g'(x)$ such that q, g, g' define C.

Moreover, $g \in (x_0, x_1)$, $g' \in (x_1, x_2)$, and since g, g' have the same zero locus in Q outside of the two lines, we find that if $g = x_0 q_2 - x_1 q_1$, $g' \equiv x_1 q_2 - x_2 q_1 \pmod{q}$.

rk(q)=2 and L nef: we assume $Q = \{x_0 x_2 = 0\}$, B' to be the scheme of zeroes of x_0, and B to be the curve part of $\{x_2 = 0\}$ (the induced divisor on B' being B plus the base point, cf. lemma 7.6).

Let $q_2(x_0, x_1, x_3)$ be the equation of B in the plane $\{x_2 = 0\}$: thus $g = x_0 q_2$ is a cubic surface containing C, and there is another g' (not divisible by x_2, therefore) intersecting the plane $\{x_2 = 0\}$ in the plane cubic $\{\gamma(x_0, x_1, x_3) = 0\}$ image of B'.

Whence $g' = \gamma + x_0\varrho$ for an appropriate quadratic form ϱ, and q, g, g' define C.

But, modulo x_2, g' is in the ideal (q_2),whence $g' = l(x_0, x_1, x_3)q_2(x_0, x_1, x_3)$ $-x_2q_1$ for a suitable linear form l and a quadratic form q_1.

Consider the matrix $\begin{pmatrix} x_0 & l & q_1 \\ 0 & x_2 & q_2 \end{pmatrix}$ whose top minors give q, g, g'. By a column operation and a change of basis leaving x_0, x_2 fixed, we can assume either $l = x_1$ or $l = 0$.

But if $l = 0$, B' would contain the line $\{x_0 = x_2 = 0\}$, therefore this line would yields a component D with $D.B = 2$ (whence $D.L = -1$ and L not nef, a contradiction) unless $x_0|q_2$, in which case it would be the common component of B, B'. But then $\{q_1 = 0\}$ would cut E' on the plane $\{x_0 = 0\}$, whence $2D$ would be contained in the plane $\{x_2 = 0\}$, contradicting $D^2 = -1$.

Remark 7.15. *In the case where B, B' have a common component D we can assume $q_2 = x_0 \cdot l(x_1, x_3)$. We see that in this case g is divisible by x_2, whence this case occurs iff $x_0|q_2$. The base point of $|L|$ is the point $\{x_0 = x_1 = x_2 = 0\}$, since on B' this is the third collinear point with $B \cap B'$.*

rq(q)=2 and L not nef: let $Q = \{x_0x_1 = 0\}$. Then we find immediately two cubic forms g, g' vanishing on C, namely $x_0 \cdot q_2(x_0, x_2, x_3)$ and $x_0 \cdot q_1(x_1, x_2, x_3)$, since B_1, B_2 maps to conics.

We then proceed as before.

rk(q)=1: let $Q = \{x_0^2 = 0\}$: then, if $q_2(x_1, x_2, x_3)$ is the conic M (cf. lemma 7.5, case (I)), $x_0 \cdot q_2$ vanishes on M.

A cubic form g' not in the ideal (x_0^2, x_0q_2) but vanishing on C must belong to the ideal (x_0, x_1) of the line D and to the ideal (x_0, q_2): therefore there exists a quadratic form q_1 such that $g' = q_2 \cdot x_1 - x_0 \cdot q_1$.

Clearly the zero scheme of $(x_0^2, x_0q_2, q_2x_1 - x_0q_1)$ has the same support as C and contains C: whence it equals C since they are both pure subscheme of degree 5.

The proof of the last assertion is straightforward.

Q.E.D. for thm.7.14

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