STUDYING THE MULTIVARIABLE ALEXANDER POLYNOMIAL BY MEANS OF SEIFERT SURFACES

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ABSTRACT. We show how Seifert surfaces, so useful for the understanding of the Alexander polynomial $\Delta_L(t)$, can be generalized in order to study the multivariable Alexander polynomial $\Delta_L(t_1, \ldots, t_\mu)$. In particular, we give an elementary and geometric proof of the Torres formula.

1. Introduction

The technique of Seifert surfaces, discovered by HERBERT SEIFERT [12] in 1935, enabled him to make great progress in the study of the Alexander polynomial of a knot. In particular, he succeeded in characterizing among all Laurent polynomials $\Delta(t)$ those that can be realized as the Alexander polynomial of a knot. The introduction by RALPH FOX of the multivariable Alexander polynomial $\Delta_L(t_1, \ldots, t_\mu)$ of a μ-component oriented link L naturally gave rise to the corresponding question for this new invariant (see [6, Problem 2]). GUILLERMO TORRES made use of the free differential calculus – developed at that time by Fox – to give several conditions for a polynomial Δ in $\mathbb{Z}[t_1^{\pm 1}, \ldots, t_\mu^{\pm 1}]$ to be the Alexander polynomial of a μ-component link [13, 5]. Since then, very little progress has been made: it is known that the Torres conditions are not sufficient in general [7, 11], but a complete algebraic characterization remains out of reach.

In this paper, we present an original approach to this problem. We show how the technique of Seifert surfaces can be generalized to obtain a new geometric interpretation of $\Delta_L(t^{m_1}, \ldots, t^{m_\mu})$ for any integers $m_1, \ldots, m_\mu$ (see Proposition 2.1 and Corollary 3.4). If an equality holds for $\Delta_L(t^{m_1}, \ldots, t^{m_\mu})$ for any integers $m_1, \ldots, m_\mu$, then it also holds for $\Delta_L(t_1, \ldots, t_\mu)$ (Lemma 2.2); therefore, it is possible to prove properties of Δ_L with this method. As an example, we give an elementary and geometric proof of the celebrated Torres formula, valid for any link in a homology 3-sphere. We also present several properties of Δ_L which turn out to be equivalent to the Torres conditions (Proposition 4.2).

2. Preliminaries

Let us consider an oriented ordered link $L = L_1 \cup \ldots \cup L_\mu$ in a homology 3-sphere Σ, and let X be the exterior of L. If $\widehat{X} \xrightarrow{\hat{p}} X$ denotes the universal abelian covering of X and $\widehat{X}^0$ the inverse image by $\hat{p}$ of a base point X^0 of X,

the homology $\mathrm{H}_1(\widehat{X}, \widehat{X}^0)$ is endowed with a natural structure of a module over the ring $\Lambda_\mu = \mathbb{Z}[t_1^{\pm 1}, \ldots, t_\mu^{\pm 1}]$. Given an $m \times n$ presentation matrix of $\mathrm{H}_1(\widehat{X}, \widehat{X}^0)$ – that is, the matrix $\mathcal{P}$ corresponding to a presentation with n generators and m relations – the $(n-i) \times (n-i)$ minor determinants of $\mathcal{P}$ span an ideal of Λ_μ denoted by $\mathcal{E}_i \mathrm{H}_1(\widehat{X}, \widehat{X}^0)$. The greatest common divisor of these minor determinants is denoted by $\Delta_i \mathrm{H}_1(\widehat{X}, \widehat{X}^0)$; this invariant is well defined up to multiplication by units of Λ_μ, that is, by $\pm t_1^{\nu_1} \cdots t_\mu^{\nu_\mu}$ with $\nu_i \in \mathbb{Z}$. In the sequel, we will write $\Delta \doteq \Delta'$ if two elements Δ, Δ' of a ring R satisfy $\Delta = \varepsilon \Delta'$ for some unit ε of R. The Laurent polynomial $\Delta_1 \mathrm{H}_1(\widehat{X}, \widehat{X}^0)$ is called the *Alexander polynomial* of the link L [1, 4]. It is denoted by $\Delta_L(t_1, \ldots, t_\mu)$.

Our method will be to prove statements on this polynomial in an indirect way, by studying all the infinite cyclic coverings of X. Since these coverings are classified by $\mathrm{Hom}(\mathrm{H}_1(X), \mathbb{Z}) \simeq \mathrm{H}^1(X; \mathbb{Z}) \simeq \mathrm{H}_1(L) = \bigoplus_{i=1}^\mu \mathbb{Z} L_i$, this leads to the following definition [3]. A *multilink* is an oriented link $L = L_1 \cup \ldots \cup L_\mu$ in a homology sphere Σ together with an integer m_i associated with each component L_i, with the convention that a component L_i with multiplicity m_i is the same as $-L_i$ (L_i with reversed orientation) with mutliplicity $-m_i$. Throughout this paper, we will write $\underline{m}$ for the ordered set of integers $m_1, \ldots, m_\mu$, d for their greatest common divisor, and $L(\underline{m})$ for the multilink. Finally, we will also denote by $\underline{m}$ the morphism $\mathrm{H}_1(X) \to \mathbb{Z}$ given by $\underline{m}(\gamma) = \sum_{i=1}^\mu m_i \ell k(L_i, \gamma)$. Let $\widetilde{X} \xrightarrow{\tilde{p}} X$ be the regular $\mathbb{Z}$-covering determined by $\underline{m}$. If $\widetilde{X}^0 = \tilde{p}^{-1}(X^0)$, the homology $\mathrm{H}_1(\widetilde{X}, \widetilde{X}^0)$ can be thought of as a module over the ring $\mathbb{Z}[t^{\pm 1}]$. The Laurent polynomial $\Delta_{L(\underline{m})}(t) = \Delta_1 \mathrm{H}_1(\widetilde{X}, \widetilde{X}^0)$ is called the *Alexander polynomial* of the multilink $L(\underline{m})$. Note that if $\underline{m} \neq \underline{0}$, the exact sequence of the pair $(\widetilde{X}, \widetilde{X}^0)$ implies at once that $\mathcal{E}_1 \mathrm{H}_1(\widetilde{X}, \widetilde{X}^0) = \mathcal{E}_0 \mathrm{H}_1(\widetilde{X})$. Therefore, $\Delta_{L(\underline{m})}(t)$ is also equal to $\Delta_0 \mathrm{H}_1(\widetilde{X})$.

Here is the dictionary between the polynomials Δ_L and $\Delta_{L(\underline{m})}$:

PROPOSITION 2.1 (Eisenbud-Neumann [3]).

$$\Delta_{L(\underline{m})}(t) \doteq \begin{cases} \Delta_L(t^{m_1}) & \text{if } \mu = 1; \\ (t^d - 1)\, \Delta_L(t^{m_1}, \ldots, t^{m_\mu}) & \text{if } \mu \geq 2. \end{cases}$$

Proof. To check this equality, we need the well-known fact that $\mathcal{E}_1 \mathrm{H}_1(\widehat{X}, \widehat{X}^0) = (\Delta_*) \cdot I$, where I is the augmentation ideal $(t_1 - 1, \ldots, t_\mu - 1)$ and Δ_* some polynomial in Λ_μ. This can be proved by purely homological algebraic methods using the fact that the group $\pi_1(X)$ has defect ≥ 1 (see [3, Theorem 6.1]). By considering a finite presentation of $\mathrm{H}_1(\widehat{X}, \widehat{X}^0)$ given by an equivariant cellular decomposition of $\widehat{X}$, it is easy to show that $\mathrm{H}_1(\widehat{X}, \widehat{X}^0) \otimes_{\Lambda_\mu} \mathbb{Z}[t^{\pm 1}] = \mathrm{H}_1(\widetilde{X}, \widetilde{X}^0)$, where $\mathbb{Z}[t^{\pm 1}]$ is endowed with the structure of Λ_μ-algebra given by $t_i \mapsto t^{m_i}$ for $i = 1, \ldots, \mu$. Hence,

$$\begin{aligned} \mathcal{E}_1 \mathrm{H}_1(\widetilde{X}, \widetilde{X}^0) &= \mathcal{E}_1(\mathrm{H}_1(\widehat{X}, \widehat{X}^0) \otimes_{\Lambda_\mu} \mathbb{Z}[t^{\pm 1}]) \\ &= (\Delta_*(t^{m_1}, \ldots, t^{m_\mu})) \cdot (t^{m_1} - 1, \ldots, t^{m_\mu} - 1) \\ &= (\Delta_*(t^{m_1}, \ldots, t^{m_\mu})) \cdot (t^d - 1). \end{aligned}$$

Since $\Delta_L = (t_1 - 1)\Delta_*$ if $\mu = 1$ and $\Delta_L = \Delta_*$ if $\mu \geq 2$, the proposition is proved. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

In order to show that properties of $\Delta_{L(\underline{m})}$ translate directly into properties of Δ_L, we also need the following lemma.

LEMMA 2.2. *Consider two polynomials Δ and Δ' in Λ_μ such that*

$$\Delta(t^{m_1}, \ldots, t^{m_\mu}) \doteq \Delta'(t^{m_1}, \ldots, t^{m_\mu}) \quad in \quad \mathbb{Z}[t^{\pm 1}]$$

for all $(m_1, \ldots, m_\mu)$ in $\mathbb{Z}^\mu$ except possibly a finite number of them. Then, $\Delta \doteq \Delta'$ in Λ_μ.

Proof. Without loss of generality, it may be assumed that $\Delta = \sum a_{i_1 \cdots i_\mu} t_1^{i_1} \cdots t_\mu^{i_\mu}$ and $\Delta' = \sum b_{j_1 \cdots j_\mu} t_1^{j_1} \cdots t_\mu^{j_\mu}$ with $a_{0\ldots 0} > 0$, $b_{0\ldots 0} > 0$, and only non-negative indices $i_k, j_k \geq 0$. By hypothesis, there are maps $\mathbb{Z}^\mu \xrightarrow{\varepsilon} \{\pm 1\}$ and $\mathbb{Z}^\mu \xrightarrow{\nu} \mathbb{Z}$ such that the equality

$$\sum a_{i_1 \cdots i_\mu} t^{\sum_k m_k i_k} \; = \; \varepsilon(m_1, \ldots, m_\mu) t^{\nu(m_1, \ldots, m_\mu)} \sum b_{j_1 \cdots j_n} t^{\sum_k m_k j_k}$$

holds for all but a finite number of $(m_1, \ldots, m_\mu)$ in $\mathbb{Z}^\mu$. Let us choose an integer N greater than $\max_k \deg_{t_k} \Delta$ and $\max_k \deg_{t_k} \Delta'$, and set $m_1 = 1$, $m_2 = N$, $\ldots, m_\mu = N^{\mu-1}$. By choosing N sufficiently large, it may be assumed that the equality above holds for this ordered set of integers. Since all these integers are positive as well as the coefficients $a_{0\ldots 0}$ and $b_{0\ldots 0}$, it follows that $\varepsilon(1, N, \ldots, N^{\mu-1}) = +1$ and $\nu(1, N, \ldots, N^{\mu-1}) = 0$. This gives

$$\sum a_{i_1 \cdots i_\mu} t^{i_1 + N i_2 + \ldots + N^{\mu-1} i_\mu} = \sum b_{j_1 \cdots j_\mu} t^{j_1 + N j_2 + \ldots + N^{\mu-1} j_\mu}.$$

But the equality $i_1 + N i_2 + \ldots + N^{\mu-1} i_\mu = j_1 + N j_2 + \ldots + N^{\mu-1} j_\mu$ with $0 \leq i_k, j_k < N$ for all k implies that $(i_1, \ldots, i_\mu) = (j_1, \ldots, j_\mu)$. Hence, $a_{i_1, \ldots, i_\mu} = b_{i_1, \ldots, i_\mu}$ for all multi-indices $(i_1, \ldots, i_\mu)$, which proves the result. $\qquad\qquad \square$

3. Generalized Seifert surfaces

One of the advantages of multilinks is that they can be studied via generalized Seifert surfaces [3]. A *Seifert surface* for a multilink $L(\underline{m})$ is an open embedded oriented surface $F \subset \Sigma \setminus L$ such that, if F_0 denotes $F \cap (\Sigma \setminus \mathrm{int}\, \mathcal{N}(L))$, the closure $c\ell(F)$ of F intersects a closed tubular neighborhood $\mathcal{N}(L_i)$ of L_i as follows for each i:

- If $m_i \neq 0$, $c\ell(F) \cap \mathcal{N}(L_i)$ consists of $|m_i|$ sheets meeting along L_i; F is oriented such that $\partial F_0 = m_i L_i$ in $\mathrm{H}_1(\mathcal{N}(L_i))$.

- If $m_i = 0$, $c\ell(F) \cap \mathcal{N}(L_i)$ consists of discs transverse to L_i; F is oriented such that the intersection number of L_i with each of these discs is the same (either always $+1$ or always -1).
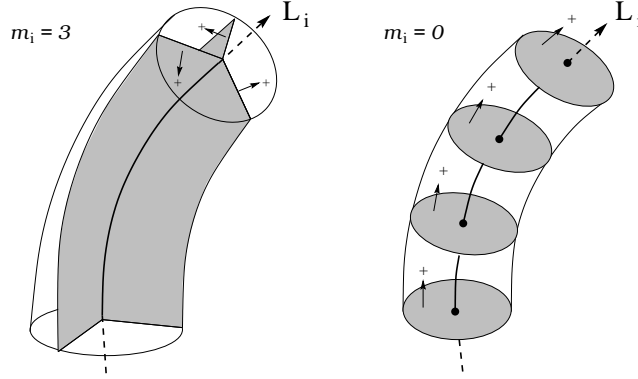
Figure 1: A Seifert surface near the multilink.

This is illustrated in Figure 1. Note that $F \subset \Sigma \setminus L$ and $F_0 \subset \Sigma \setminus \operatorname{int} \mathcal{N}(L)$ determine each other up to isotopy; to simplify the notation, we will consider both of them as Seifert surfaces, and denote both by F. From now on, we will write $\overline{F}$ for the union of $F \subset \Sigma \setminus L$ and L.

LEMMA 3.1 (Eisenbud-Neumann [3]). *Let F be a Seifert surface for a multilink* $L(\underline{m})$. *Then, for $i = 1 \dots, \mu$, the intersection $F \cap \partial \mathcal{N}(L_i)$ gives a d_i component link which is the $(d_i p_i, d_i q_i)$-cable about L_i, where p_i and q_i are coprime, $d_i p_i = m_i$ and $d_i q_i = - \sum_{j \neq i} m_j \ell k(L_i, L_j)$.*

Proof. Let us denote by (P_i, M_i) a basis of $\mathrm{H}_1(\partial \mathcal{N}(L_i))$ given by a standard parallel and meridian. Since F is a Seifert surface for $L(\underline{m})$, $F \cap \partial \mathcal{N}(L_i) = m_i P_i + n_i M_i$ in $\mathrm{H}_1(\partial \mathcal{N}(L_i))$ for some integer n_i. Furthermore, $\partial F = \sum_{j \neq i} m_j L_j + m_i P_i + n_i M_i$ in $\mathrm{H}_1(\Sigma \setminus \operatorname{int} \mathcal{N}(L_i))$. By Alexander duality, this module is isomorphic to $\mathrm{H}^1(\mathcal{N}(L_i); \mathbb{Z}) = \mathbb{Z}$, and the isomorphism is given by the linking number with L_i. It follows $0 = \ell k(L_i, \partial F) = \sum_{j \neq i} m_j \ell k(L_i, L_j) + n_i$, which gives the result. $\square$

In the usual case of an oriented link, a Seifert surface needs to be connected in order to be useful. In the general case of a multilink, it has to be "as connected as possible". More precisely, a Seifert surface for $L(\underline{m})$ is a *good Seifert surface* if it has $d = \gcd(\underline{m})$ connected components.

LEMMA 3.2. *Given a multilink $L(\underline{m})$, there exists a good Seifert surface for $L(\underline{m})$.*

Proof. One easily shows that there exists a Seifert surface for $L(\underline{m})$ (see [3, Lemma 3.1]). If $d > 1$, a good Seifert surface for $L(\underline{m})$ is given by d parallel copies of a connected Seifert surface for $L(\frac{m}{d})$. Therefore, it may be assumed that $d = 1$. Let F be any Seifert surface for $L(\underline{m})$ without closed component,

and let us denote by i_+ (resp. i_-) the epimorphism $\mathrm{H}_0(F) \to \mathrm{H}_0(\Sigma \setminus \overline{F})$ induced by the push in the positive (resp. negative) normal direction off F. If i_+ and i_- are not isomorphisms, it is possible to reduce the number of connected components of F by handle attachement. So, let us assume that all the possible handle attachement(s) have been performed, yielding $F = F_1 \cup \ldots \cup F_n$ with isomorphisms $i_+, i_- \colon \mathrm{H}_0(F) \to \mathrm{H}_0(\Sigma \setminus \overline{F})$. The automorphism of $\mathrm{H}_0(F)$ given by $h = (i_-)^{-1} \circ i_+$ cyclically permutes the connected components of F. (Indeed, consider a component F_i of F; since $X = (\Sigma \setminus \overline{F}) \cup F$ is path connected and i_+, i_- are isomorphisms, there exists an integer m such that $F_i = h^m(F_1)$.) It easily follows that $\partial F_i = \partial F_j$ in $\mathrm{H}_1(\mathcal{N}(L))$ for $i, j = 1, \ldots, n$. Therefore, the equality $\sum_{i=1}^{\mu} m_i L_i = \partial F = \sum_{j=1}^{n} \partial F_j = n\, \partial F_1$ holds in $\mathrm{H}_1(\mathcal{N}(L)) = \bigoplus_{i=1}^{\mu} \mathbb{Z} L_i$. Hence, n divides m_i for $i = 1, \ldots, \mu$. Since $\gcd(m_1, \ldots, m_\mu) = 1$, F is connected. $\qquad\square$

Let us now turn to the natural generalization to multilinks of the Seifert form. Given F a good Seifert surface for $L(\underline{m})$, the *Seifert forms* associated to F are the bilinear forms

$$\alpha_+, \alpha_- \colon \mathrm{H}_1(F) \times \mathrm{H}_1(\overline{F}) \longrightarrow \mathbb{Z}$$

given by $\alpha_+(x, y) = \ell k(i_+ x, y)$ and $\alpha_-(x, y) = \ell k(i_- x, y)$, where i_+ (resp. i_-) is the morphism $\mathrm{H}_1(F) \to \mathrm{H}_1(\Sigma \setminus \overline{F})$ induced by the push in the positive (resp. negative) normal direction off F. (Note that we use the same notation for the morphisms $\mathrm{H}_0(i_\pm)$ and $\mathrm{H}_1(i_\pm)$; it will always be clear from the context which dimension is concerned.) Let us denote by A_+ and A_- matrices of these forms, called *Seifert matrices*. Here is the generalization of Seifert's famous theorem.

THEOREM 3.3. *Let F be a good Seifert surface for $L(\underline{m})$, and let A_+, A_- be associated Seifert matrices. Then, $A_+ - tA_-$ is a presentation matrix of the module $\mathrm{H}_1(\widetilde{X})$.*

Proof. Given F a good Seifert surface for $L(\underline{m})$, let us denote $\Sigma \setminus \overline{F}$ by Y. By the proof of Lemma 3.2 it is possible to number the connected components $F = F_1 \cup \ldots \cup F_d$ and $Y = Y_1 \cup \ldots \cup Y_d$ such that $i_+ F_k = Y_k$ and $i_- F_k = Y_{k-1}$ (with the indices modulo d). Let us set $N = F \times (-1; 1)$ an open bicollar of F, $N_+ = F \times (0; 1)$, $N_- = F \times (-1; 0)$ and $\{Y^i\}_{i \in \mathbb{Z}}$ (resp. $\{N^i\}_{i \in \mathbb{Z}}$) copies of Y (resp. N). Define

$$E = \bigsqcup_{i \in \mathbb{Z}} Y^i \sqcup \bigsqcup_{i \in \mathbb{Z}} N^i \Big/ \sim,$$

where $Y^i \supset N_+ \sim N_+ \subset N^i$ and $Y^i \supset N_- \sim N_- \subset N^{i+1}$. The obvious projection $E \overset{p}{\to} X$ is the infinite cyclic covering $\widetilde{X} \to X$ determined by $\underline{m}$. Indeed, a loop γ in X lifts to a loop in E if and only if the intersection number of γ with F is zero, that is, if $0 = \gamma \cdot F = \ell k(L(\underline{m}), \gamma) = \underline{m}(\gamma)$.

Consider the Mayer-Vietoris exact sequence of $\mathbb{Z}[t^{\pm 1}]$-modules associated to the decomposition $\widetilde{X} = \left(\bigcup_i Y^i\right) \cup \left(\bigcup_i N^i\right)$; it gives

$$(\mathrm{H}_1(F) \oplus \mathrm{H}_1(F)) \otimes \mathbb{Z}[t^{\pm 1}] \overset{\phi_1}{\to} (\mathrm{H}_1(Y) \oplus \mathrm{H}_1(F)) \otimes \mathbb{Z}[t^{\pm 1}] \overset{\psi}{\longrightarrow} \mathrm{H}_1(\widetilde{X}) \to$$

$$(\mathrm{H}_0(F) \oplus \mathrm{H}_0(F)) \otimes \mathbb{Z}[t^{\pm 1}] \xrightarrow{\phi_0} (\mathrm{H}_0(Y) \oplus \mathrm{H}_0(F)) \otimes \mathbb{Z}[t^{\pm 1}],$$

where the homomorphism ϕ_0 is given by $(\alpha, \beta) \mapsto (i_+\alpha + t\, i_-\beta, \alpha + \beta)$. Since F is good, the homomorphisms $i_{\pm} \colon \mathrm{H}_0(F) \to \mathrm{H}_0(\Sigma \setminus \overline{F})$ are injective, and so is ϕ_0. Therefore, ψ is surjective and there is an exact sequence

$$(\mathrm{H}_1(F) \oplus \mathrm{H}_1(F)) \otimes \mathbb{Z}[t^{\pm 1}] \xrightarrow{\phi_1} (\mathrm{H}_1(Y) \oplus \mathrm{H}_1(F)) \otimes \mathbb{Z}[t^{\pm 1}] \to \mathrm{H}_1(\widetilde{X}) \to 0,$$

with $\phi_1(\alpha, \beta) = (i_+\alpha + t\, i_-\beta, \alpha + \beta)$. This can be transformed into

$$\mathrm{H}_1(F) \otimes \mathbb{Z}[t^{\pm 1}] \xrightarrow{\tilde{\phi}} \mathrm{H}_1(Y) \otimes \mathbb{Z}[t^{\pm 1}] \to \mathrm{H}_1(\widetilde{X}) \to 0,$$

where $\tilde{\phi}(\alpha) = i_+\alpha - t\, i_-\alpha$. Let us fix basis $\mathcal{B}$ for $\mathrm{H}_1(F)$, $\overline{\mathcal{B}}$ for $\mathrm{H}_1(\overline{F})$, and consider the basis $\overline{\mathcal{B}}^*$ for $\mathrm{H}_1(Y)$ which is dual to $\overline{\mathcal{B}}$ under Alexander duality. The matrix of i_+ (resp. i_-) with respect to $\mathcal{B}$ and $\overline{\mathcal{B}}^*$ is given by A_+^T (resp. A_-^T), where A_+ and A_- are the Seifert matrices with respect to the basis $\mathcal{B}$ and $\overline{\mathcal{B}}$. Therefore, a matrix of $\tilde{\phi}$ is given by $A_+^T - tA_-^T$. This concludes the proof. $\square$

COROLLARY 3.4. *Let $L(\underline{m})$ be a multilink with $\underline{m} \neq \underline{0}$. If $m_i = \sum_{j \neq i} m_j \ell k(L_i, L_j) = 0$ for some index i, then $\Delta_{L(\underline{m})}(t) = 0$. If there is no such index, the matrices A_+ and A_- are square, and $\Delta_{L(\underline{m})}(t) \doteq \det(A_+ - tA_-)$.*

Proof. By the proof of Lemma 3.2, a Seifert surface F is good if and only if $\mathrm{rk}\,\widetilde{\mathrm{H}}_0(F) = \mathrm{rk}\,\widetilde{\mathrm{H}}_0(\Sigma \setminus \overline{F})$ which is equal to $\mathrm{rk}\,\mathrm{H}_2(\overline{F})$ by Alexander duality. It is easy to show that $\mathrm{rk}\,\widetilde{\mathrm{H}}_0(\overline{F}) = r$, the number of indices i with $m_i = \sum_{j \neq i} m_j \ell k(L_i, L_j) = 0$. Since $\chi(F) = \chi(\overline{F})$, it follows that $\mathrm{rk}\,\mathrm{H}_1(\overline{F}) = \mathrm{rk}\,\mathrm{H}_1(F) + r$. So if $r = 0$, $A_+ - tA_-$ is a square presentation matrix of $\mathrm{H}_1(\widetilde{X})$ and if $r > 0$, it has more generators than relations. It follows that $\Delta_0 \mathrm{H}_1(\widetilde{X}) \doteq \det(A_+ - tA_-)$ if $r = 0$, and $\Delta_0 \mathrm{H}_1(\widetilde{X}) = 0$ if $r > 0$. $\square$

4. The Torres conditions

Let us now illustrate how Corollary 3.4, along with Proposition 2.1 and Lemma 2.2, can be used to study the multivariable Alexander polynomial. As an example, we present an elementary proof of the Torres formula [13], quite simpler than the original proof. (On the other hand, it should be mentioned that more perspicuous proofs have since been given, for example in [9]).

Throughout this section, we will denote by ℓ_{ij} the linking number $\ell k(L_i, L_j)$.

LEMMA 4.1. *Let $L(\underline{m}) = L(m_1, \ldots, m_{\mu-1}, 0)$ be a multilink, and let $L'(\underline{m}') = L'(m_1, \ldots, m_{\mu-1})$ be the multilink obtained from $L(\underline{m})$ by removing the last component L_μ. Then,*

$$\Delta_{L(\underline{m})}(t) \doteq \left(t^{\sum_i m_i \ell_{i\mu}} - 1\right) \Delta_{L'(\underline{m}')}(t).$$
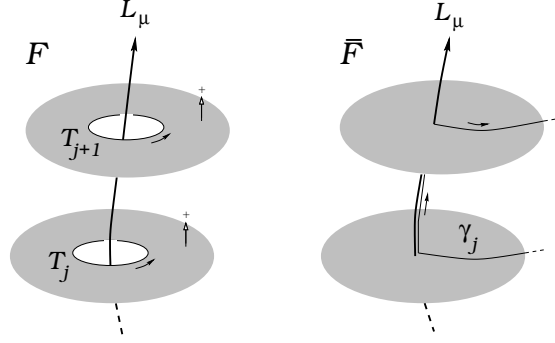
Figure 2: The proof of Lemma 4.1.

Proof. If $m_i = \sum_{j \neq i} m_j \ell_{ij} = 0$ for some index i, the lemma holds by Corollary 3.4. It may therefore be assumed that there is no such index. Let F be a good Seifert surface for $L(\underline{m})$; then, a good Seifert surface for $L'(\underline{m'})$ is given by $F' = F \cup (\overline{F} \cap \mathcal{N}(L_\mu))$. By Lemma 3.1, $\overline{F} \cap \mathcal{N}(L_\mu)$ consists of $d_\mu = \sum_{i=1}^{\mu-1} m_i \ell_{i\mu}$ discs (recall Figure 1). Furthermore, $\overline{F} = \overline{F}' \cup L_\mu$. Therefore, we have the natural isomorphisms

$$\mathrm{H}_1(F) = \mathrm{H}_1(F') \oplus \bigoplus_{j=1}^{d_\mu} \mathbb{Z} T_j \quad \text{and} \quad \mathrm{H}_1(\overline{F}) = \mathrm{H}_1(\overline{F}') \oplus \bigoplus_{j=1}^{d_\mu} \mathbb{Z}\gamma_j,$$

where the cycles T_j correspond to the boundaries of the discs, and the γ_j are the transverse cycles depicted in Figure 2. The associated Seifert matrices $A_\pm$ and $A'_\pm$ are related by

$$A_+ = \begin{pmatrix} A'_+ & & & 0 \\ & 1 & & \\ * & & \ddots & \\ & & & 1 \end{pmatrix} \quad \text{and} \quad A_- = \begin{pmatrix} A'_- & & & 0 & & \\ & & 0 & & & 1 \\ & & 1 & 0 & & \\ * & & & \ddots & \ddots & \\ & & & & 1 & 0 \end{pmatrix}.$$

Corollary 3.4 then gives

$$\Delta_{L(\underline{m})} \;\dot=\; \det(A_+ - tA_-) \;=\; \begin{vmatrix} A'_+ - tA'_- & & & & 0 \\ & 1 & & & -t \\ & -t & 1 & & \\ * & & \ddots & \ddots & \\ & & & -t & 1 \end{vmatrix}$$

$$= \; (t^{d_\mu} - 1) \det(A'_+ - tA'_-) \;\dot=\; (t^{d_\mu} - 1)\,\Delta_{L'(\underline{m'})}(t)$$

and the lemma is proved. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The demonstration of the Torres formula is now a mere translation of Lemma 4.1 via Proposition 2.1.

TORRES FORMULA. [13] *Let $L = L_1 \cup \ldots \cup L_\mu$ be an oriented link with $\mu \geq 2$ components, and let L' be the sublink $L_1 \cup \ldots \cup L_{\mu-1}$. Then,*

$$\Delta_L(t_1, \ldots, t_{\mu-1}, 1) \doteq \begin{cases} \frac{t_1^{\ell_{12}} - 1}{t_1 - 1} \Delta_{L'}(t_1) & \text{if } \mu = 2; \\ (t_1^{\ell_{1\mu}} \cdots t_{\mu-1}^{\ell_{\mu-1,\mu}} - 1) \Delta_{L'}(t_1, \ldots, t_{\mu-1}) & \text{if } \mu > 2. \end{cases}$$

Proof. Let us denote by Δ' the right-hand side of this formula, and let $m_1, \ldots, m_{\mu-1}$ be arbitrary integers with $d = \gcd(m_1, \ldots, m_{\mu-1}) > 0$. We have the equalities

$$
\begin{aligned}
\Delta'(t^{m_1}, \ldots, t^{m_{\mu-1}}) &= \begin{cases} \frac{t^{m_1 \ell_{12}} - 1}{t^{m_1} - 1} \Delta_{L'}(t^{m_1}) & \text{if } \mu = 2; \\ (t^{\sum_i m_i \ell_{i\mu}} - 1) \Delta_{L'}(t^{m_1}, \ldots, t^{m_{\mu-1}}) & \text{if } \mu > 2, \end{cases} \\
\text{(Proposition 2.1)} &\doteq \frac{1}{t^d - 1} (t^{\sum_i m_i \ell_{i\mu}} - 1) \Delta_{L'(\underline{m'})}(t) \\
\text{(Lemma 4.1)} &\doteq \frac{1}{t^d - 1} \Delta_{L(\underline{m})}(t) \\
\text{(Proposition 2.1)} &\doteq \Delta_L(t^{m_1}, \ldots, t^{m_{\mu-1}}, 1)
\end{aligned}
$$

and the proof is settled by Lemma 2.2.   □

Using the same method, it is not hard to show the following result.

FOX-TORRES RELATION. [13, 5] *Let $L = L_1 \cup \ldots \cup L_\mu$ be an oriented link with $\mu \geq 2$ components. Then,*

$$\Delta_L(t_1^{-1}, \ldots, t_\mu^{-1}) = (-1)^\mu \, t_1^{\nu_1 - 1} \cdots t_n^{\nu_\mu - 1} \, \Delta_L(t_1, \ldots, t_\mu)$$

with integers ν_i such that $\nu_i \equiv \sum_j \ell_{ij} \pmod 2$ if $\Delta_L \neq 0$.

These results provide necessary conditions for a polynomial Δ in Λ_μ to be the Alexander polynomial of a μ-component link with fixed $\ell k(L_i, L_j) = \ell_{ij}$. They are known as the *Torres conditions* (see [10] for a precise statement). Since these conditions are not sufficient [7, 11], the problem is now to find stronger conditions. By means of a close study of the homology $H_1(F)$ and $H_1(\overline{F})$, it is possible to find necessary conditions for a polynomial Δ in $\mathbb{Z}[t^{\pm 1}]$ to be the Alexander polynomial of a multilink. Via Proposition 2.1, this translates into the following result (see [2] for a proof).

PROPOSITION 4.2. *Let L be an oriented link with $\mu \geq 2$ components. Then, its Alexander polynomial Δ_L satisfies the following conditions. For all integers $\underline{m} = (m_1, \ldots, m_\mu)$ with $d = \gcd(m_1, \ldots, m_\mu)$ and $d_i = \gcd(m_i, \sum_j m_j \ell_{ij})$, there exists some polynomial $\nabla_{L(\underline{m})}(t)$ in $\mathbb{Z}[t^{\pm d}]$ such that:*

  - $\prod_{i=1}^\mu (t^{d_i} - 1) \nabla_{L(\underline{m})}(t) \doteq (t^d - 1)^2 \Delta_L(t^{m_1}, \ldots, t^{m_\mu})$;

- $\nabla_{L(\underline{m})}(t^{-1}) = \nabla_{L(\underline{m})}(t)$;

- $|\nabla_{L(\underline{m})}(1)| = \frac{d^2 \, D}{d_1 \cdots d_\mu m_1 \cdots m_\mu}$, where D is any $(\mu - 1) \times (\mu - 1)$ minor determinant of the matrix

$$
\begin{pmatrix}
-\sum_j m_1 m_j \ell_{1j} & m_1 m_2 \ell_{12} & \dots & m_1 m_\mu \ell_{1\mu} \\
m_1 m_2 \ell_{12} & -\sum_j m_2 m_j \ell_{2j} & \dots & m_2 m_\mu \ell_{2\mu} \\
\vdots & \vdots & \ddots & \vdots \\
m_1 m_\mu \ell_{1\mu} & m_2 m_\mu \ell_{2\mu} & \dots & -\sum_j m_\mu m_j \ell_{\mu j}
\end{pmatrix};
$$

- If $m_i = 0$ for some index i, then $\nabla_{L(\underline{m})} = \nabla_{L'(\underline{m'})}$, where L' denotes the sublink $L \setminus L_i$ and $\underline{m'} = (m_1, \ldots, \widehat{m_i}, \ldots, m_\mu)$. $\qquad\square$

This result easily implies the Torres conditions. It can also be thought of as a generalization of a theorem of Hosokawa [8], which corresponds to the case $m_1 = \ldots = m_\mu = 1$. At first sight, it might therefore seem more general than the Torres conditions. Unfortunately, this is not the case: it can be shown that every polynomial Δ which satisfies the Torres conditions also satisfies the conditions of Proposition 4.2 (see [2]).

By means of a somewhat closer study of the Seifert matrices $A_\pm$, it should be possible to find new properties of $\Delta_{L(\underline{m})}$. They would translate into properties of Δ_L, and provide new conditions, stronger than the ones of Torres.

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