Journal of Mathematical Analysis and Applications **250,** 86–97 (2000)
doi:10.1006/jmaa.2000.6958, available online at http://www.idealibrary.com on IDEAL®

Global Solutions for Dissipative Kirchhoff Strings with $m(r) = r^p$ ($p < 1$)

Marina Ghisi

Dipartimento di Matematica, Università degli Studi di Pisa, Via M. Buonarroti 2, 56127 Pisa, Italy
E-mail: ghisi@dm.unipi.it

Submitted by Andrea Donato

Received October 18, 1999

We investigate the evolution problem

$$u'' + \delta u' + m\left(|A^{1/2}u|_H^2\right)Au = 0,$$

$$u(0) = u_0, \qquad u'(0) = u_1,$$

where H is a Hilbert space, A is a self-adjoint non-negative operator on H with domain $D(A)$, $\delta > 0$ is a parameter, and $m(r) = r^p$ with $p < 1$. We prove that this problem has a unique global solution for positive times, provided that the initial data $(u_0, u_1) \in D(A^{\alpha_i/2}) \times D(A^{(\alpha_i - 1)/2})$ satisfy a suitable smallness assumption and the non-degeneracy condition $m(|A^{1/2}u_0|_H^2) > 0$ (where $p \geq 2^{-i}$ and $\alpha_i = 2^i + 1$). Moreover, we prove for this solution decay with a polynomial rate as $t \to +\infty$. These results apply to degenerate hyperbolic PDEs with non-local non-linearities.

Key Words: hyperbolic equations; degenerate hyperbolic equations; dissipative equations; global existence; Kirchhoff equations; asymptotic behavior.

1. INTRODUCTION

Let H be a real Hilbert space, with norm $|\cdot|_H$ and scalar product $\langle \cdot, \cdot \rangle_H$. Let A be a self-adjoint linear non-negative operator on H with dense domain $D(A)$ (i.e., $\langle Au, u\rangle_H \geq 0$ for all $u \in D(A)$). Let us consider the Cauchy problem

$$\begin{gathered} u''(t) + \delta u'(t) + m\left(\left|A^{1/2}u(t)\right|_H^2\right)Au(t) = 0, \qquad t \geq 0, \\ u(0) = u_0, \qquad u'(0) = u_1, \end{gathered} \tag{1.1}$$

where $\delta \geq 0$ and m: $[0, +\infty[\to [0, +\infty[$ is a continuous function.

0022-247X/00 $35.00

Problem (1.1) is an abstract setting of the initial-boundary value problem for the equation

$$u_{tt} + \delta u_t + m\left(\int_\Omega |\nabla u|^2\, dx\right)\Delta u = 0, \qquad \text{in } \Omega \times [0, +\infty[, \quad (1.2)$$

where $\Omega \subseteq \mathbb{R}^n$ is a (not necessarily bounded) open set. This last equation was introduced in the case $n = 1$ by Kirchhoff [9] as a model for the small transversal vibrations of an elastic string with fixed endpoints.

The equations (1.1)–(1.2) have long been studied under various conditions on the function m and on the regularity of the initial data: the interested reader can find appropriate references in the surveys of Arosio [1], Spagnolo [15], and Medeiros *et al.* [10].

In this context we will recall only some results on the existence of global solutions.

When the initial data are A-analytic, Arosio and Spagnolo [2] and later D'Ancona and Spagnolo [3] proved that (1.1) has a global solution if $m, \delta \geq 0$.

In the case of regular small initial data, but not analytic, D'Ancona and Spagnolo [4] showed that (1.2) has a unique global solution if $\Omega = \mathbb{R}^n$, m is locally Lipschitz continuous, $\delta \geq 0$, and $m(r) \geq \nu > 0$, $\forall r \geq 0$. (See also Greenberg and Hu [8] for the one-dimensional case and Yamazaki [16].)

Now let us consider (1.1) with $\delta > 0$ and $(u_0, u_1) \in D(A) \times D(A^{1/2})$ small initial data; and let us assume m to be locally Lipschitz continuous.

In the non-degenerate case (i.e., $m(r) \geq \nu > 0$ for all $r \geq 0$), when A is a coercive operator, De Brito, Yamada, and Nishihara [5, 6, 11, 14] proved that there exists a unique global solution such that (u, u') decays with an exponential rate as $t \to +\infty$ in $\in D(A^{1/2}) \times H$. The same result, with a polynomial decay of the solution, was afterwards obtained by Nishihara and Yamada [12] if $m(r) = r^\gamma$ $(\gamma \geq 1)$ and $u_0 \neq 0$. When A is only a non-negative operator in [7] it was proved that if $m(r) \geq 0$ for all $r \geq 0$ and $m(|A^{1/2}u_0|_H^2) > 0$ then there exists a global solution and such a solution has a limit u_∞ in $D(A)$ as $t \to +\infty$ such that $m(|A^{1/2}u_\infty|_H^2) = 0$.

At this point it seems natural to wonder what would happen if m were only a continuous function (and not Lipschitz continuous) on the points s such that $m(s) = 0$, always assuming that $m(|A^{1/2}u_0|_H^2) > 0$.

The purpose of this paper is to provide a partial answer to this question in the case when $m(r) = r^p$ $(0 < p < 1)$ and $A^{1/2}u_0 \neq 0$.

Let us give at this point some notation. Let us set for i, $n \in \mathbb{N}$,

$$B := A^{1/2},\ \alpha_i = 2^i + 1,\ a_n = 2 - 2^{1-n},\ \gamma_n = 2^{-n+i} - 1.$$

Moreover, let us define

$$c := \max_{0 \le n \le i} \frac{|B^{\gamma_n} u_1|_H}{|Bu_0|_H^{2p + a_n/2}}, \qquad E_{\alpha_i - 1}(0) := \frac{|B^{\alpha_i - 1} u_1|_H}{|Bu_0|_H^{2p}} + |B^{\alpha_i} u_0|_H,$$

and let us consider the sequences R_n, L_n, defined by

$$R_n = \sqrt{5R_{n-1} + 2\delta c}, \qquad R_1 = \sqrt{5\sqrt{E_{\alpha_i - 1}(0)} + 2\delta c},$$

$$L_n = c + \frac{2}{\delta} R_n, \qquad L_0 = c + \frac{2}{\delta}\sqrt{E_{\alpha_i - 1}(0)}.$$

We shall prove the following result:

THEOREM 1.1. *Let us assume* $\delta > 0$ *and* $m(r) = r^p$ $(0 < p < 1)$, *and let* $i \in \mathbb{N}$ *such that* $p \ge 2^{-i}$. *Let us assume that* $(u_0, u_1) \in D(B^{\alpha_i}) \times D(B^{\alpha_i - 1})$, *with* $Bu_0 \ne 0$ *and*

$$2\left(\frac{|u_1|_H^2}{|Bu_0|_H^{2p}} + |Bu_0|_H^2\right)^{p - 2^{-i}} R_i L_i < \frac{\delta}{2p + 1}. \tag{1.3}$$

Then there exists a unique global solution u *of* (1.1) *such that* $|Bu(t)|_H > 0$ *for all* $t \ge 0$ *and*

$$u \in C^2([0, +\infty[; D(B^{\alpha_i - 2})) \cap C^1([0, \infty[; D(B^{\alpha_i - 1})) \cap C^0([0, \infty[; D(B^{\alpha_i})).$$

Remark 1.2. Since R_i depends with continuity on R_1 and c, and since $R_i = 0$ if $R_1 = c = 0$, then if R_1 and c are small this is also true for R_i (and hence for L_i). Hence there exist initial data verifying (1.3).

Remark 1.3. Theorem 1.1 can be restated also as follows.

Let us assume that $m(r) = r^p$ $(0 < p < 1)$ *and let* $i \in \mathbb{N}$ *such that* $p \ge 2^{-i}$. *Let us assume that* $(u_0, u_1) \in D(B^{\alpha_i}) \times D(B^{\alpha_i - 1})$, *with* $Bu_0 \ne 0$. *Then there exists* $\delta_0 > 0$, *such that for all* $\delta \ge \delta_0$, (1.1) *has a unique global solution* u, *with* $|Bu(t)|_H > 0$ *for all* $t \ge 0$ *and*

$$u \in C^2([0, +\infty[; D(B^{\alpha_i - 2})) \cap C^1([0, \infty[; D(B^{\alpha_i - 1})) \cap C^0([0, \infty[; D(B^{\alpha_i})).$$

Indeed, it is enough to observe that

$$R_i \le \max\left\{R_1, \frac{5 + \sqrt{25 + 8\delta c}}{2}\right\} := S_\delta,$$

hence $R_i/\delta \to 0$ when $\delta \to +\infty$, and $L_i \le c + (2/\delta) S_\delta$.

Moreover, we are able to obtain the following result on the asymptotic behavior of the solutions of (1.1).

THEOREM 1.4. *Let us assume that all of the conditions of Theorem* 1.1 *are satisfied. Then,*

$$|u'|_H^2 + |Bu|_H^{2p+2} \leq \frac{c_{p,0}}{1+t}, \tag{1.4}$$

$$\frac{|Bu'|_H^2}{|Bu|_H^{2p}} + |B^2u|_H^2 \leq \frac{c_{p,\varepsilon}}{(1+t)^{2/(p+1)-\varepsilon}} \qquad \forall \varepsilon > 0, \tag{1.5}$$

for some constants $c_{p,\varepsilon}$ *depending only on* p, ε, *and the initial data.*

This last result can be improved in the case in which A is a coercive operator (i.e., $\langle B^2u, u\rangle_H \geq c|u|_H^2$ for some constant $c > 0$) as follows.

THEOREM 1.5. *Let us assume that all the conditions of Theorem* 1.1 *are satisfied, and let us suppose that* A *is a coercive operator. Then,*

$$|u'|_H^2 + |Bu|_H^{2p+2} \leq \frac{c_{p,0}}{(1+t)^{(p+1)/p}}, \qquad \frac{|Bu'|_H^2}{|Bu|_H^{2p}} + |B^2u|_H^2 \leq \frac{c_{p,\varepsilon}}{(1+t)^{1/p-\varepsilon}} \qquad \forall \varepsilon > 0,$$

for some constants $c_{p,\varepsilon}$ *depending only on* p, ε, *and the initial data.*

Remark 1.6. By using Theorems 1.4 and 1.5 and Lemma 2.2 it is possible to obtain also some estimate on the asymptotic decay of $|B^k u'|_H$ and $|B^{k+1}u|_H$ for $0 \leq k \leq \alpha_i - 1$.

2. PROOFS

Let us enunciate, first of all, a result of the existence of local solutions for (1.1); the proof of this theorem can be obtained by a simple adaptation of the proof of Theorem 2.1 in [7].

THEOREM 2.1 (Local Existence). *Let us assume* $\delta \geq 0$, $m(r) = r^p$ $(0 < p < 1)$, *and* $(u_0, u_1) \in D(B^{\alpha_i}) \times D(B^{\alpha_i - 1})$ *with* $|Bu_0|_H^{2p} > 0$.

Then there exists $T > 0$ *such that the problem* (1.1) *has a unique solution* u *with* $|Bu(t)|_H > 0$ *in* $[0, T[$ *and*

$$u \in C^2([0,T[; D(B^{\alpha_i - 2})) \cap C^1([0,T[; D(B^{\alpha_i - 1})) \cap C^0([0,T[; D(B^{\alpha_i})).$$

Moreover, u can be uniquely continued to a maximal solution defined in an interval $[0, T_*[$, and at least one of the following statements is valid:

(i) $T_* = +\infty$;

(ii) $\limsup_{t \to T_*^-} |Bu'(t)|^2 + |B^2 u(t)|^2 = +\infty$;

(iii) $\liminf_{t \to T_*^-} |Bu(t)|_H^{2p} = 0$.

Now we can prove Theorem 1.1.

Step 1. Let $[0, T_*[$ be the maximal interval where the solution exists. Let us set

$$c(t) := |Bu(t)|_H^{2p}$$

and

$$T := \sup\left\{\tau \in [0, T_*[: \left|\frac{c'(t)}{c(t)}\right| \le \frac{p\delta}{2p+1}, c(t) > 0\ \forall t \in [0, \tau]\right\}.$$

Let us now consider for $t \in [0, T[$ the functions

$$E_k(t) := \frac{|B^k u'(t)|_H^2}{c(t)} + |B^{1+k} u(t)|_H^2, \qquad k = 0, \dots, \alpha_i - 1.$$

Since

$$E_k'(t) = -\frac{1}{c(t)}\left(2\delta + \frac{c'(t)}{c(t)}\right)|B^k u'(t)|_H^2 \le -\frac{\delta |B^k u'(t)|_H^2}{c(t)},$$

hence

$$E_k(t) \le E_k(0) \qquad t \in [0, T[, k = 0, \dots, \alpha_i - 1. \tag{2.1}$$

Moreover, for $t \in [0, T[$ we have

$$0 < c(0) e^{-\delta t/2} \le c(t) \le c(0) e^{\delta t/2}. \tag{2.2}$$

Our purpose is to prove that $T = T_* = +\infty$. Indeed, in this case we have a global solution, and by (2.2) $|Bu(t)|_H > 0$ for all $t \ge 0$, hence this solution is also unique.

Now let us assume that $T = T_* < +\infty$; then by (2.2) and (2.1), Statements ii and iii of Theorem 2.1, respectively, are false, and this contradicts Theorem 2.1.

Then we must only prove that $T = T_*$.

Step 2 (Proof of $T = T_*$). Let us assume by contradiction that $T < T_*$. Hence by (2.2) and by the maximality of T we have that necessarily

$$\left|\frac{c'(T)}{c(T)}\right| = \frac{p\delta}{2p+1}. \tag{2.3}$$

Before continuing, let us introduce some notation. For $n \geq 0$ we define

$$h_n := \gamma_n + 2, \qquad H_n := \begin{cases} \delta L_0 + \sqrt{E_{\alpha_i - 1}(0)} & n = 0 \\ R_n + \delta L_n & n \geq 1. \end{cases}$$

We can now enunciate a lemma, the proof of which we put off for later.

LEMMA 2.2. *For* $0 \leq t \leq T$ *we have*

$$(1)_n \quad \frac{|B^{\gamma_n} u'(t)|_H}{c(t)|Bu(t)|_H^{a_n/2}} \leq L_n \qquad \textit{for } 0 \leq n \leq i;$$

$$(2)_n \quad \frac{|B^{\gamma_n} u''(t)|_H}{c(t)|Bu(t)|_H^{a_n/2}} \leq H_n \qquad \textit{for } 0 \leq n \leq i;$$

$$(3)_n \quad \frac{|B^{h_n} u(t)|_H}{|Bu(t)|_H^{a_n/2}} \leq R_n \qquad \textit{for } 1 \leq n \leq i.$$

Since $\gamma_i = 0$ and $h_i = 2$, by Lemma 2.2 we deduce for $0 \leq t \leq T$;

$$\frac{|u'(t)|_H |B^2 u(t)|_H}{|Bu(t)|_H^{2p+2-2^{1-i}}} = \frac{|u'(t)|_H}{|Bu(t)|_H^{2p+a_i/2}} \frac{|B^2 u(t)|_H}{|Bu(t)|_H^{a_i/2}} \leq L_i R_i.$$

Therefore, as $p \geq 2^{-i}$, we obtain

$$\begin{aligned}\left|\frac{c'(T)}{c(T)}\right| &= 2p\left|\frac{\langle u'(T), B^2 u(T)\rangle_H}{|Bu(T)|_H^2}\right| \\ &\leq 2p|Bu(T)|^{2p-2^{1-i}} L_i R_i \\ &\leq 2pE_0(0)^{p-2^{-i}} L_i R_i < \frac{\delta p}{2p+1}.\end{aligned}$$

This last inequality contradicts (2.3).

After all that we provide a proof of Lemma 2.2.

Step 3 (Proof of Lemma 2.2). We shall proceed by finite induction on n. First of all, we show that $(1)_0$, $(2)_0$, and $(3)_1$ are true.

$(1)_0$ Let us consider the function $G_{\gamma_0}(t) := |B^{\gamma_0}u'(t)|_H^2/c^2(t)$. We have

$$G'_{\gamma_0}(t) = -2\left(\delta + \frac{c'(t)}{c(t)}\right)G_{\gamma_0}(t) + \frac{2}{c(t)}\langle B^{\gamma_0}u'(t), B^{\gamma_0+2}u(t)\rangle_H$$
$$\leq -\delta G_{\gamma_0}(t) + 2\sqrt{G_{\gamma_0}(t)}\sqrt{E_{\alpha_i-1}(t)}.$$

Hence, by (2.1), using a classical ODE lemma,

$$\frac{|B^{\gamma_0}u'(t)|_H}{c(t)} \leq \max\left\{\frac{|B^{\gamma_0}u_1|_H}{c(0)}, \frac{2}{\delta}\sqrt{E_{\alpha_i-1}(0)}\right\} \leq L_0.$$

$(2)_0$ By the equation in (1.1), (2.1), and $(1)_0$, we obtain

$$\frac{|B^{\gamma_0}u''(t)|_H}{c(t)} \leq |B^{\gamma_0+2}u(t)|_H + \delta\frac{|B^{\gamma_0}u'(t)|_H}{c(t)} \leq \sqrt{E_{\alpha_i-1}(0)} + \delta L_0 = H_0.$$

$(3)_1$ Taking the scalar product of the equation in (1.1) with $B^{2h_1-2}u(t)/c(t)|Bu(t)|_H^{a_1}$, we get

$$\frac{\langle B^{2h_1-3}u''(t), Bu(t)\rangle_H}{c(t)|Bu(t)|_H^{a_1}} + \frac{|B^{h_1}u(t)|_H^2}{|Bu(t)|_H^{a_1}} + \delta\frac{\langle B^{2h_1-3}u'(t), Bu(t)\rangle_H}{c(t)|Bu(t)|_H^{a_1}} = 0.$$

Moreover, $2h_1 - 3 = \gamma_0$ and $a_1 = 1$, hence using $(1)_0$ and $(2)_0$ we have

$$\frac{|B^{h_1}u(t)|_H^2}{|Bu(t)|_H^{a_1}} \leq H_0 + \delta L_0 = 2\delta c + 5\sqrt{E_{\alpha_i-1}(0)} = R_1^2.$$

Let us now assume that $(1)_n$, $(2)_n$, and $(3)_{n+1}$ are verified, and

- if $n+2 \leq i$ let us prove that $(1)_{n+1}$, $(2)_{n+1}$, and $(3)_{n+2}$ are true;
- if $n+1 = i$ let us prove that $(1)_{n+1}$ and $(2)_{n+1}$ are true.

$(1)_{n+1}$ Let us set

$$G_{\gamma_{n+1}}(t) := \frac{|B^{\gamma_{n+1}}u'(t)|_H^2}{c^2(t)|Bu(t)|_H^{a_{n+1}}}.$$

Then we have

$$G'_{\gamma_{n+1}}(t) \leq -2\delta\left(1 - \frac{p}{2p+1}\right)G_{\gamma_{n+1}}(t) + 2\sqrt{G_{\gamma_{n+1}}(t)}\,\frac{|B^{h_{n+1}}u(t)|_H}{|Bu(t)|_H^{a_{n+1}/2}}$$
$$- a_{n+1}\frac{\langle B^2u(t), u'(t)\rangle_H}{|Bu(t)|_H^2}G_{\gamma_{n+1}}(t).$$

Since

$$\frac{\langle B^2u(t), u'(t)\rangle_H}{|Bu(t)|_H^2} = \frac{c'(t)}{2pc(t)},$$

using $(3)_{n+1}$ we deduce

$$\begin{aligned} G'_{\gamma_{n+1}}(t) &\leq -\delta\left(2 - \frac{2p}{2p+1} - \frac{a_{n+1}}{2(2p+1)}\right)G_{\gamma_{n+1}}(t) + 2\sqrt{G_{\gamma_{n+1}}(t)}\,R_{n+1} \\ &\leq -\delta G_{\gamma_{n+1}}(t) + 2\sqrt{G_{\gamma_{n+1}}(t)}\,R_{n+1}. \end{aligned}$$

Hence

$$\frac{|B^{\gamma_{n+1}}u'(t)|_H}{c(t)|Bu(t)|_H^{a_{n+1}/2}} \leq \max\left\{\frac{|B^{\gamma_{n+1}}u_1|_H}{c(0)|Bu_0|_H^{a_{n+1}/2}}, \frac{2}{8}R_{n+1}\right\} \leq L_{n+1}.$$

$(2)_{n+1}$ By the equation in (1.1), using $(1)_{n+1}$ and $(3)_{n+1}$ we get

$$\begin{aligned} \frac{|B^{\gamma_{n+1}}u''(t)|_H}{c(t)|Bu(t)|_H^{a_{n+1}/2}} &\leq \frac{|B^{h_{n+1}}u(t)|_H}{|Bu(t)|_H^{a_{n+1}/2}} + \delta\frac{|B^{\gamma_{n+1}}u'(t)|_H}{c(t)|Bu(t)|_H^{a_{n+1}/2}} \\ &\leq R_{n+1} + \delta L_{n+1} = H_{n+1}. \end{aligned}$$

$(3)_{n+2}$ If $n+2 = i+1$ we stop; otherwise, taking the scalar product of the equation in (1.1) with $B^{2h_{n+2}-2}u(t)/c(t)|Bu(t)|_H^{a_{n+2}}$, we get

$$\begin{aligned} &\frac{\langle B^{2h_{n+2}-3}u''(t), Bu(t)\rangle_H}{c(t)|Bu(t)|_H^{a_{n+2}}} + \frac{|B^{h_{n+2}}u(t)|_H^2}{|Bu(t)|_H^{a_{n+2}}} \\ &\quad + \delta\frac{\langle B^{2h_{n+2}-3}u'(t), Bu(t)\rangle_H}{c(t)|Bu(t)|_H^{a_{n+2}}} = 0. \end{aligned}$$

Since $a_{n+2} - a_{n+1}/2 = 1$ and $2h_{n+2} - 3 = \gamma_{n+1}$, using $(1)_{n+1}$ and $(2)_{n+1}$ we obtain

$$\frac{|B^{h_{n+2}}u(t)|_H^2}{|Bu(t)|_H^{a_{n+2}}} \leq H_{n+1} + \delta L_{n+1} = 2\delta c + 5R_{n+1} = R_{n+2}^2.$$

∎

Now we prove Theorem 1.4.

We use the same notation as in Theorem 1.1.

It is enough to prove the theorem for $t \geq t_1 := 4/\delta - 1$.

For the inequality (1.4) we have to show that

$$E(t) = \left(|u'(t)|_H^2 + \frac{|Bu(t)|_H^{2(p+1)}}{p+1} \right)(1+t) \leq \phi_1 \qquad \forall t \geq t_1. \quad (2.4)$$

To show this let us first recall that, by taking the scalar product of the equation (1.1) and u, we obtain

$$\left(\langle u', u \rangle_H + \frac{\delta}{2}|u|^2 \right)' - |u'|_H^2 + |Bu|^{2+2p} = 0,$$

hence

$$E'(t) \leq -\delta |u'|_H^2(1+t) - \left(\langle u'(t), u(t) \rangle_H + \frac{\delta}{2}|u(t)|^2 \right)'.$$

Therefore, for all $t \geq t_1$,

$$\begin{aligned} E(t) &\leq E(t_1) + \langle u'(t_1), u(t_1) \rangle_H + \frac{\delta}{2}|u(t_1)|^2 + \frac{|u'(t)|_H^2}{2\delta} \\ &\leq H_1 + \frac{1}{8}E(t), \end{aligned}$$

where, by (2.1), H_1 depends only on the initial data. We have then the required inequality by taking $\phi_1 = 8H_1/7$.

Now we have to prove the inequality (1.5).

By taking the scalar product of the equation (1.1) and $(1+t)^\beta B^2 u / c(t)$, we get

$$\begin{aligned} &\left(\left(\frac{\langle Bu', Bu \rangle_H}{c(t)} + \frac{\delta}{2-2p}|Bu|_H^{2-2p} \right)(1+t)^\beta \right)' \\ &\quad - \left(\frac{|Bu'|_H^2}{c(t)} - \frac{c'(t)}{c(t)} \frac{\langle Bu', Bu \rangle_H}{c(t)} \right)(1+t)^\beta + |B^2 u|_H^2 (1+t)^\beta \\ &\quad - \beta(1+t)^{\beta-1}\left(\frac{\langle Bu', Bu \rangle_H}{c(t)} + \frac{\delta}{2-2p}|Bu|_H^{2-2p} \right) = 0. \end{aligned}$$

Then, integrating over (t_1, t) and taking into account $c'(t)\langle Bu', Bu\rangle_H \geq 0$, we obtain

$$\begin{aligned}
\int_{t_1}^{t} |B^2u(s)|_H^2(1+s)^\beta\, ds
&\leq \frac{|Bu'(t)|_H |Bu(t)|_H}{c(t)}(1+t)^\beta \\
&\quad + \int_{t_1}^{t} \frac{|Bu'(s)|_H^2}{c(s)}(1+s)^\beta\, ds \\
&\quad - \frac{1}{2-2p}\left(\delta - \frac{\beta}{(1+t)}\right)|Bu(t)|_H^{2-2p}(1+t)^\beta \\
&\quad + \frac{\beta\delta}{2-2p}\int_{t_1}^{t}(1+s)^{\beta-1}|Bu(s)|_H^{2-2p}\, ds + \psi \\
&\leq \psi + \frac{|Bu'(t)|_H^2}{\delta c(t)}(1+t)^\beta \\
&\quad + \delta\left(\frac{4-\beta}{8-8p} - \frac{1}{4}\right)|Bu(t)|_H^{2-2p}(1+t)^\beta \\
&\quad + \frac{\beta\delta}{2-2p}\int_{t_1}^{t}(1+s)^{\beta-1}|Bu(s)|_H^{2-2p}\, ds \\
&\quad + \int_{t_1}^{t} \frac{|Bu'(s)|_H^2}{c(s)}(1+s)^\beta\, ds \qquad (2.5)
\end{aligned}$$

where, by (2.1), ψ depends only on the initial data.

If we now choose $\beta = ((1-p)/(1+p)) - \varepsilon$ with $\varepsilon > 0$, by (2.4) we get

$$\int_{t_1}^{t}(1+s)^{\beta-1}|Bu(s)|_H^{2-2p}\, ds \leq \frac{\phi_1^{(1-p)/(1+p)}}{\varepsilon},$$

hence

$$\begin{aligned}
\int_{t_1}^{t} |B^2u(s)|_H^2(1+s)^\beta\, ds \leq \frac{|Bu'(t)|_H^2}{\delta c(t)}(1+t)^\beta& \\
+ \int_{t_1}^{t} \frac{|Bu'(s)|_H^2}{c(s)}(1+s)^\beta\, ds + \chi,& \qquad (2.6)
\end{aligned}$$

where χ depends only on p, ε, and on the initial data.

We are at this point able to estimate, for $\varepsilon > 0$ and $t \geq t_1$,

$$F(t) = (1+t)^{2/(p+1)-\varepsilon}\left(\frac{|Bu'(t)|_H^2}{c(t)} + |B^2u(t)|_H^2\right).$$

Indeed, using $|c'(t)/c(t)| \leq p\delta/(2p+1)$ and $t \geq t_1$, we easy get

$$F'(t) \leq -\delta\frac{|Bu'(t)|_H^2}{c(t)}(1+t)^{2/(p+1)-\varepsilon} + \frac{2}{p+1}(1+t)^{2/(p+1)-1-\varepsilon}|B^2u(t)|_H^2.$$

Then by (2.6)–(2.1), we have for some χ_1 depending only on p, ε, and the initial data,

$$F(t) \leq \chi_1 + \tfrac{1}{2}F(t) \qquad \forall t \geq t_1,$$

which gives immediately the inequality (1.5).

Now we prove Theorem 1.5.

For the first inequality we only remark that, since A is coercive, we can use exactly the same proceeding as was used in [13].

To prove the second one we can proceed as in proof of (1.5) in Theorem 1.4, choosing $t_1 := 2/p\delta - 1$: the only difference is that in this case we can now take $\beta = (1-p)/p - \varepsilon$.

REFERENCES

1. A. Arosio, Averaged evolution equations: The Kirchhoff string and its treatment in scales of Banach spaces, *in* "Proceedings of the 2nd Workshop on Functional–Analytic Methods in Complex Analysis, Trieste, 1993," World Scientific, Singapore.
2. A. Arosio and S. Spagnolo, Global existence for abstract evolution equations of weakly hyperbolic type, *J. Math. Pure Appl.* **65** (1986), 263–305.
3. P. D'Ancona and S. Spagnolo, On an abstract weakly hyperbolic equation modelling the nonlinear vibrating string, *in* "Developments in PDE and Applications to Mathematical Physics" (Buttazzo *et al.*, Eds.), Plenum, New York, 1992.
4. P. D'Ancona and S. Spagnolo, Nonlinear perturbations of the Kirchhoff equation, *Comm. Pure Appl. Math.* **47** (1994), 1005–1024.
5. E. H. de Brito, The damped elastic stretched string equation generalized: existence, uniqueness, regularity and stability, *Appl. Anal.* **13** (1982), 219–233.
6. E. H. de Brito, Decay estimates for the generalized damped extensible string and beam equation, *Nonlinear Anal.* **8** (1984), 1489–1496.
7. M. Ghisi and M. Gobbino, Global existence for a mildly degenerate dissipative hyperbolic equation of Kirchhoff type, preprint, Dipt. Mat. Univ. Pisa 2.290.1089, 1997.
8. M. Greenberg and S. C. Hu, The initial value problem for a stretched string, *Quart. Appl. Math.* (1980), 289–311.

9. G. Kirchhoff, “Vorlesungen über Mechanik,” Teubner, Leipzig, 1883.
10. L. A. Medeiros, J. L. Ferrell, and S. B. de Menezes, Vibration of elastic strings, manuscript 1999.
11. K. Nishihara, Global existence and asymptotic behaviour of the solution of some quasilinear hyperbolic equation with linear damping, *Funkcial. Ekvac.* **32** (1989), 343–355.
12. K. Nishihara and Y. Yamada, On global solutions of some degenerate quasilinear hyperbolic equations with dissipative terms, *Funkcial. Ekvac.* **33** (1990), 151–159.
13. K. Ono, Global existence, decay, and blowup of solutions for some mildly degenerate nonlinear Kirchhoff strings, *J. Differential Equations* **137** (1997), 273–301.
14. Y. Yamada, On some quasilinear wave equations with dissipative terms, *Nagoya Math. J.* **87** (1982), 17–39.
15. S. Spagnolo, The Cauchy problem for the Kirchhoff equation, *Rend. Sem. Mat. Fis. Milano* **62** (1992), 17–51.
16. T. Yamazaki, Scattering for a quasilinear hyperbolic equation of Kirchhoff type, *J. Differential Equations* **143** (1998), 1–59.