Calc. Var. 24, 111–129 (2005) **Calculus of Variations**

DOI: 10.1007/s00526-004-0317-2

Marina Ghisi · Massimo Gobbino

The monopolist's problem: existence, relaxation, and approximation

Received: 2 March 2004 / Accepted: 19 October 2004
Published online: 22 December 2004 –

Abstract. We study a variational problem arising from a generalization of an economic model introduced by Rochet and Choné in [5]. In this model a monopolist proposes a set Y of products with price list $p : Y \to \mathbb{R}$. Each rational consumer chooses which product to buy by solving a personal minimum problem, taking into account his/her tastes and economic possibilities. The monopolist looks for the optimal price list which minimizes costs, hence maximizes the profit. This leads to a minimum problem for functionals $\mathcal{F}(p)$ (the "pessimistic cost expectation") and $\mathcal{G}(p)$ (the "optimistic cost expectation"), which are in turn defined through two nested variational problems. We prove that the minimum of $\mathcal{G}$ exists and coincides with the infimum of $\mathcal{F}$. We also provide a variational approximation of $\mathcal{G}$ by smooth functionals defined in finite dimensional Euclidean spaces.

Mathematics Subject Classification (2000): 49J45, 91B

1. Introduction

In this paper we consider variational problems coming from a generalization of an economic model proposed by Rochet and Choné in [5]. This model has also been studied in a general setting by Monteiro and Page [7], and under convexity assumptions by Carlier [1] (to which we refer for a brief survey of the theoretical literature on similar economic problems) who introduced a dual approach leading to a minimum problem for an integral functional under convexity constraints (see also [2,3,6]).

From the mathematical point of view, the main ingredients are: a set X with a finite measure μ; a set Y; a subset $\mathcal{P} \subseteq \{\text{functions}\, p : Y \to \mathbb{R}\}$; a function $s : X \times Y \times \mathbb{R} \to \mathbb{R}$, and a function $c : Y \times \mathbb{R} \to \mathbb{R}$.

For every $(x, p) \in X \times \mathcal{P}$, one defines

$$A_p(x) := \arg\min\{s(x, y, p(y)) : y \in Y\},$$

where "$\arg\min$" denotes the set of minimum points, and then

$$M_p(x) := \max\{c(y, p(y)) :\ y \in A_p(x)\},$$
$$m_p(x) := \min\{c(y, p(y)) :\ y \in A_p(x)\}.$$

M. Ghisi: Università degli Studi di Pisa, Dipartimento di Matematica "Leonida Tonelli", via M. Buonarroti 2, 56127 Pisa, Italy (e-mail: ghisi@dm.unipi.it)

M. Gobbino: Università degli Studi di Pisa, Dipartimento di Matematica Applicata "Ulisse Dini", via Bonanno 25/B, 56126 Pisa, Italy (e-mail: m.gobbino@dma.unipi.it)

Finally, for every $p \in \mathcal{P}$ one defines

$$\mathcal{F}(p) := \int_X M_p(x)\,\mathrm{d}\mu, \qquad \mathcal{G}(p) := \int_X m_p(x)\,\mathrm{d}\mu.$$

Now we describe what all these quantities represent in the model. The set X and the measure μ represent consumers, the set Y represents all the products the monopolist can produce. The subset $\mathcal{P}$ represents the admissible "price functions". The number $s(x, y, \pi)$ is the stress of a consumer x buying y at price π. The number $c(y, \pi)$ is the cost for the monopolist when he produces product y and sells it at price π.

Given an admissible price list $p \in \mathcal{P}$, each consumer x buies one of the products which minimize his stress, hence an element of $A_p(x)$. If $A_p(x)$ has more than one element, this model cannot predict which product the consumer will buy, therefore the exact cost given by x to the monopolist is unknown. However, this cost lies in any case between $m_p(x)$ and $M_p(x)$. Integrating over all consumers, it turns out that $\mathcal{G}(p)$ and $\mathcal{F}(p)$ represent the "cost forecast" for the monopolist in the optimistic and pessimistic situation, respectively. We remark that by definition

$$\mathcal{G}(p) \leq \mathcal{F}(p), \qquad \forall p \in \mathcal{P}. \tag{1.1}$$

We point out that in this model the monopolist looks for the element(s) in $\mathcal{P}$ which minimizes his expected costs: this leads to the minimum problems for $\mathcal{G}(p)$ and $\mathcal{F}(p)$ as p varies in $\mathcal{P}$. From the monopolist's point of view, it is also interesting to consider

$$\mathcal{A}_p := \bigcup_{x \in X} A_p(x), \tag{1.2}$$

where p is an optimal price list for $\mathcal{F}$ or $\mathcal{G}$. This set contains all the products that can be bought by someone.

In the following we list all the assumptions we make on X, μ, Y, $\mathcal{P}$, s and c.

(HP 1) X is a compact metric space with distance d_X;
(HP 2) μ is a Radon measure on X, normalized so that $\mu(X) = 1$;
(HP 3) Y is a compact metric space with distance d_Y;
(HP 4) s is continuous on $X \times Y \times \mathbb{R}$;
(HP 5) c is continuous on $Y \times \mathbb{R}$;
(HP 6) there exists $y_\star \in Y$ such that $c(y_\star, 0) = 0$, and $s(x, y_\star, 0) = 0$ for every $x \in X$;
(HP 7) $\mathcal{P}$ is the set of all lower semicontinuous functions $p : Y \to \mathbb{R}$ such that $p(y_\star) \leq 0$;
(HP 8) $c(y, \pi) \geq 0$ for every $(y, \pi) \in Y \times (-\infty, 0]$;
(HP 9) there exists two constants $c_2 \geq c_1 > 0$ such that

$$c_1(\pi_2 - \pi_1) \leq s(x, y, \pi_2) - s(x, y, \pi_1) \leq c_2(\pi_2 - \pi_1)$$

for every $(x, y, \pi_1, \pi_2) \in X \times Y \times \mathbb{R}^2$ such that $\pi_1 \leq \pi_2$.

Hypotheses (Hp 1) through (Hp 5) are technical, and are trivially satisfied *e.g.* if X and Y are finite sets. With (Hp 6) we assume the existence of a special product $y_\star$, which can be thought as the "nothing". In this way $c(y_\star, 0) = 0$ means that the monopolist has cost zero when producing "nothing" and selling it at price zero; similarly, $s(x, y_\star, 0) = 0$ means that any consumer has stress zero when he buies "nothing" paying zero. In a certain sense, $y_\star$ is the only weapon a consumer has against a monopolist, *i.e.* the possibility of buying nothing. As already pointed out in [5], from the mathematical point of view, it is always possible to fulfill (Hp 6) by adding $y_\star$ as an isolated point to any set Y of products. With (Hp 7) we restrict to lower semicontinuous price functions for technical reasons, and we force the monopolist to sell $y_\star$ at price less than or equal to zero. Assumption (Hp 8) says that selling a product y at a price ≤ 0 has a cost ≥ 0 (as every monopolist knows!). Finally, assumption (Hp 9) says that the stress of a consumer is an increasing function of price (as every consumer knows!), and gives a lower and an upper bound on the growth.

The main tool in our analysis is De Giorgi's Γ-convergence (see [4] for a comprehensive introduction to the subject), which defines a topology on $\mathcal{P}$, with respect to which the direct method of the calculus of variations can be applied.

2. Statements

Let c_1 be the constant of (Hp 9), and let

$$k_1 := \max\{|s(x,y,0)| : \ (x,y) \in X \times Y\}, \qquad K := \frac{2k_1 + 1}{c_1}. \tag{2.1}$$

We consider $\mathcal{P}$ endowed with the topology coming from Γ-convergence. With respect to this topology,

$$\mathcal{P}_K := \{p \in \mathcal{P} : \ |p(y)| \leq K \ \forall y \in Y\}$$

is a compact metric space.

Theorem 2.1 (Well posedness). *Assume that assumptions (Hp 1) through (Hp 9) are satisfied. Then for every $p \in \mathcal{P}$ we have that*

(1) for every $x \in X$, the function $s(x, y, p(y))$ attains its minimum on Y, hence $A_p(x)$ is well defined and compact;
(2) the set $\mathcal{A}_p$ is a compact subset of Y; moreover, the restriction of p to $\mathcal{A}_p$ is continuous, and $p(y) < K$ for every $y \in \mathcal{A}_p$, where K is defined in (2.1);
(3) for every $x \in X$, the function $c(y, p(y))$ attains its maximum and minimum on $A_p(x)$, hence $M_p(x)$ and $m_p(x)$ are well defined;
(4) $m_p(x)$ is a bounded lower semicontinuous function of x, while $M_p(x)$ is a bounded upper semicontinuous function of x; as a consequence these functions are integrable with respect to the measure μ and therefore $\mathcal{F}(p)$ and $\mathcal{G}(p)$ are well defined.

Theorem 2.2 (Existence of minimizers). *Assume that assumptions (Hp 1) through (Hp 9) are satisfied. Then*

(1) $\mathcal{G}(p)$ is lower semicontinuous on $\mathcal{P}$;
(2) $\mathcal{G}(p)$ attains its minimum in $\mathcal{P}$;
(3) if K is the constant defined in (2.1), then

$$\min\{\mathcal{G}(p) : \ p \in \mathcal{P}\} = \min\{\mathcal{G}(p) : \ p \in \mathcal{P}_K\}. \tag{2.2}$$

Theorem 2.3 (Relaxation). *Assume that assumptions (Hp 1) through (Hp 9) are satisfied.*

Then $\mathcal{G}(p)$ is the relaxation of $\mathcal{F}(p)$. As a consequence

$$\inf\{\mathcal{F}(p) : \ p \in \mathcal{P}\} = \min\{\mathcal{G}(p) : \ p \in \mathcal{P}\}. \tag{2.3}$$

We don't study here regularity and qualitative properties of minimizers. We just remark that statement (2) of Theorem 2.1 implies in particular that any minimizer of $\mathcal{G}$ is continuous in $\mathcal{A}_p$, while (2.2) gives an *a priori* estimate on prices.

There are very simple examples where $\mathcal{F}(p)$ doesn't have minimum in $\mathcal{P}$ (cf. Example 4.1). Therefore in general the monopolist cannot reach the best in the pessimistic situation described by $\mathcal{F}$. However he can arrange prices so that also in the pessimistic situation his costs are as close as he wants to the lowest costs in the optimistic situation! We also remark that (Hp 9) is crucial in the proof of (2.3) (cf. Example 4.2).

The next problem we consider is how to compute the minimum of $\mathcal{G}$, since the expression of $\mathcal{G}(p)$ involves two nested minimum problems. To this end, we approximate $\mathcal{G}$ by a sequence $\{\mathcal{G}_n\}$ of C^∞ functionals defined on sets $P_{n,K}$, which can be identified with one half of an hypercube in $\mathbb{R}^n$.

The construction of the sequence $\{\mathcal{G}_n\}$ is achieved in several steps.

In the sequel, we assume that K is the real number defined in (2.1), and we set

$$M := \max\{|c(y,\pi)| : \ (y,\pi) \in Y \times [-K,K]\}. \tag{2.4}$$

We also use the following notation: given a compact metric space Z, with distance d_Z, and a closed subset $Z' \subseteq Z$, we set

$$r(Z', Z) := \max\{d_Z(w, Z') : \ w \in Z\}.$$

Approximation of X and μ Let $\{X_n\}$ be a sequence of *finite* subsets of X such that $r(X_n, X) \to 0$ as $n \to +\infty$. Now we approximate μ with a sequence $\{\mu_n\}$ of measures such that X_n is the support of μ_n. To this end, let $X_n = \{x_{n,1}, \ldots, x_{n,k_n}\}$ where $k_n = |X_n|$ is the number of elements of X_n. Let us set

$$X_{n,1} := \{x \in X : \ d_X(x, x_{n,1}) \le d_X(x, x_{n,j}), \ j = 1, \ldots, k_n\},$$

and then, for $2 \le i \le k_n$,

$$X_{n,i} := \{x \in X \backslash (X_{n,1} \cup \ldots \cup X_{n,i-1}) : \\ d_X(x, x_{n,i}) \ \le \ d_X(x, x_{n,j}), \ j = i, \ldots, k_n\}.$$

Now we denote by δ_x the Dirac measure with support in x, and we set

$$\mu_n := \sum_{i=1}^{k_n} \mu(X_{n,i}) \delta_{x_{n,i}}.$$

Approximation of Y and $\mathcal{P}$ Let $\{Y_n\}$ be a sequence of subsets of Y such that $|Y_n| = n$ and $y_\star \in Y_n$ for every $n \geq 1$, and $r(Y_n, Y) \to 0$ as $n \to +\infty$.

Then for every $n \geq 1$ we set

$$P_{n,K} := \{p : Y_n \to \mathbb{R} \text{ such that } p(y_\star) \leq 0,\ |p(y)| \leq K\ \forall y \in Y_n\}.$$

A standard extension argument allows to identify $\mathcal{P}_{n,K}$ with a subset of the compact metric space $\mathcal{P}_K$ in such a way that $r(\mathcal{P}_{n,K}, \mathcal{P}_K) \to 0$ as $n \to +\infty$.

Approximation of s and c For every $n \in \mathbb{N}$, let $s_n(x, y, \pi)$ and $c_n(y, \pi)$ be functions which are C^∞ in π for every value of the other variable(s), and such that $s_n \to s$, and $c_n \to c$ uniformly on compact subsets of their domains.

Approximation of a Heaviside function For every $n \in \mathbb{N}$, let $\psi_n : \mathbb{R} \to \mathbb{R}$ be a C^∞ real function such that $\psi_n(r) = 1$ for every $r \leq 0$, $\psi_n(r) = 0$ for every $r \geq 1/n$, and $0 \leq \psi_n(r) \leq 1$ for every $r \in [0, 1/n]$.

Approximation of the minimum of a set Let m be a positive integer, and let $r_1, \ldots, r_m$ be m real numbers. For every real number $q \neq 0$ we define

$$\mathrm{Mean}_q\{r_1, \ldots, r_m\} := \frac{1}{q} \log\left(\frac{1}{m}\sum_{i=1}^{m} e^{q r_i}\right).$$

Approximation of m_p Let M be the constant defined in (2.4). For every $n \in \mathbb{N}$, and for every $(x, y, p) \in X \times Y_n \times \mathcal{P}_{n,K}$, we set

$$\begin{aligned}
\Psi_{n,p}(x,y) &:= 1 - \prod_{z \in Y_n} \psi_n\left[s_n(x, y, p(y)) - s_n(x, z, p(z))\right];\\
f_{n,p}(x,y) &:= c_n(y, p(y)) + 5M\Psi_{n,p}(x,y);\\
m_{n,p}(x) &:= \mathrm{Mean}_{-n}\{f_{n,p}(x,w) :\ w \in Y_n\},
\end{aligned}$$

where, with a little abuse of notation, $\{f_{n,p}(x,w) :\ w \in Y_n\}$ denotes the n real numbers (maybe with repetitions) obtained as w varies among the n elements of Y_n.

Approximation of $\mathcal{G}$ For every $n \in \mathbb{N}$, we define the functional $\mathcal{G}_n : \mathcal{P}_{n,K} \to \mathbb{R}$ by

$$\mathcal{G}_n(p) := \int_X m_{n,p}(x)\, \mathrm{d}\mu_n \qquad \forall p \in \mathcal{P}_{n,K}.$$

We are now ready to state our approximation result.

Theorem 2.4 (Approximation). *Assume that assumptions (Hp 1) through (Hp 9) are satisfied.*

Then, identifying $\mathcal{P}_{n,K}$ with a subset of $\mathcal{P}_K$, and extending $\mathcal{G}_n$ to $+\infty$ outside $\mathcal{P}_{n,K}$, we have that the sequence $\{\mathcal{G}_n\}$ defined above Γ-converges to $\mathcal{G}$. As a consequence

- *we have that*

$$\lim_{n\to+\infty} \min\{\mathcal{G}_n(p) :\ p \in \mathcal{P}_{n,K}\} = \min\{\mathcal{G}(p) :\ p \in \mathcal{P}\};$$

- *if p_n is a minimizer of $\mathcal{G}_n$ for every $n \in \mathbb{N}$, then the sequence $\{p_n\}$ converges, up to subsequences, to a minimizer of $\mathcal{G}$.*

This is a result of Γ-convergence in the topology induced by Γ-convergence!

3. Proofs

In all our proofs, we repeatedly take subsequences, but we never relabel indices.

3.1. *Proof of Theorem 2.1*

Step 1. We prove statement (1).

Let $(x,p) \in X \times \mathcal{P}$. Since $s(x,y,\pi)$ is continuous in (y,π) and increasing in π (by (Hp 4) and (Hp 9)), it follows that $s(x,y,p(y))$ is a lower semicontinuous function of y, and therefore it attains the minimum on the compact space Y. Moreover, the set $A_p(x)$ of minimum points is a nonempty and closed subset of Y, hence it is compact.

Step 2. We prove the following: "if $x_n \to x_\infty$ in X, $y_n \to y_\infty$ in Y, and $y_n \in A_p(x_n)$ for every $n \in \mathbb{N}$, then $y_\infty \in A_p(x_\infty)$ and $p(y_n) \to p(y_\infty)$".

To this end, let us set

$$\liminf_{n\to+\infty} p(y_n) =: \pi_\infty \geq p(y_\infty).$$

Up to subsequences we can assume that the $\liminf$ is actually a limit. Now let $w \in Y$. Since $y_n \in A_p(x_n)$ we have that

$$\begin{aligned} s(x_\infty, y_\infty, p(y_\infty)) \leq s(x_\infty, y_\infty, \pi_\infty) \ &= \lim_{n\to+\infty} s(x_n, y_n, p(y_n)) \\ &\leq \lim_{n\to+\infty} s(x_n, w, p(w)) \ = \ s(x_\infty, w, p(w)). \end{aligned}$$

Since w is arbitrary, this proves that $y_\infty \in A_p(x_\infty)$.

Now let us assume that $\pi_\infty > p(y_\infty)$. Then setting $w = y_\infty$ in the above inequalities, and remarking that s is strictly increasing in the third variable, we get a contradiction. Therefore the $\liminf$ of $p(y_n)$ is $p(y_\infty)$ for every sequence $y_n \to y_\infty$ (i.e. $p(y_n) \to p(y_\infty)$).

Step 3. We prove that $\mathcal{A}_p$ is closed, hence compact.

To this end, let $\{y_n\} \subseteq \mathcal{A}_p$ be a sequence such that $y_n \to y_\infty$ in Y. Let x_n be such that $y_n \in A_p(x_n)$ for every $n \in \mathbb{N}$. Since X is compact, we can assume that x_n converges to some x_∞, up to subsequences.

By Step 2, we know that $y_\infty \in A_p(x_\infty)$, hence $y_\infty \in \mathcal{A}_p$.

Step 4. We prove that the restriction of p to $\mathcal{A}_p$ is continuous.

Assume that this is not the case. Then there exists $y_n \to y_\infty$ in $\mathcal{A}_p$ such that

$$\lim_{n\to+\infty} p(y_n) > p(y_\infty).$$

Let x_n be such that $y_n \in A_p(x_n)$. We can assume that x_n converges to some x_∞.

We know that $y_\infty \in A_p(x_\infty)$, and $p(y_n) \to p(y_\infty)$, which contradicts our assumption.

Step 5. We prove that if $p \in \mathcal{P}$, and $y \in \mathcal{A}_p$, then $p(y) < K$.

Indeed, let us assume that $p(y) \geq K$. By (Hp 6), (Hp 9) and (2.1), for every $x \in X$ we have that

$$s(x,y,p(y)) \geq s(x,y,K) = s(x,y,K) - s(x,y,0) + s(x,y,0) \geq$$

$$\geq Kc_1 - k_1 = k_1 + 1 > 0 = s(x,y_\star,0) \geq s(x,y_\star,p(y_\star)).$$

This proves that $s(x,y,p(y)) > s(x,y_\star,p(y_\star))$, and therefore $y \notin A_p(x)$ for every $x \in X$. This completes the proof of statement (2).

Step 6. We prove statement (3).

From Step 4 we know that p is continuous if restricted to $\mathcal{A}_p$file, hence also if restricted to $A_p(x)$. It follows that $c(y,p(y))$ is a continuous function of y on the compact $A_p(x)$. This proves that $m_p(x)$ and $M_p(x)$ are well defined.

Step 7. We prove the semicontinuity of $m_p(x)$ and $M_p(x)$.

Let us assume that $x_n \to x_\infty$ in X, and let $y_n \in A_p(x_n)$ be such that $m_p(x_n) = c(y_n, p(y_n))$. Up to subsequences, we can assume that the $\liminf$ of $m_p(x_n)$ is actually a limit, and that y_n converges to some y_∞ in Y.

From Step 2 we know that $y_\infty \in A_p(x_\infty)$ and $p(y_n) \to p(y_\infty)$, hence, since y_∞ is a competitor in the definition of $m_p(x_\infty)$:

$$\lim_{n\to+\infty} m_p(x_n) = \lim_{n\to+\infty} c(y_n, p(y_n)) = c(y_\infty, p(y_\infty)) \geq m_p(x_\infty).$$

This proves that $m_p(x)$ is a lower semicontinuous function of x.

The proof of the upper semicontinuity of $M_p(x)$ is completely analogous.

Step 8. We prove the boundedness of $m_p(x)$ and $M_p(x)$ (i.e. statement (4)).

To this end, using the result of Step 7 and the compactness of X, we have that $m_p(x)$ has a minimum on X, and $M_p(x)$ has a maximum on X. Therefore

$$\min\{m_p(x) :\ x \in X\} \leq m_p(x) \leq M_p(x) \leq \max\{M_p(x) :\ x \in X\}.$$

3.2. Proof of Theorem 2.2

Compactness We prove that

$$\inf\{\mathcal{G}(p) : p \in \mathcal{P}\} = \inf\{\mathcal{G}(p) : p \in \mathcal{P}_K\}. \tag{3.1}$$

Step 1. We prove that the infimum of $\mathcal{G}$ over $\mathcal{P}$ is less or equal than zero.

Indeed among the competitors there is the function $\widehat{p}$ such that $\widehat{p}(y_\star) = 0$ and $\widehat{p}(y) = K$ for every $y \neq y_\star$. By statement (2) of Theorem 2.1, for every $x \in X$ we have that $A_{\widehat{p}}(x) = \{y_\star\}$, hence $m_{\widehat{p}}(x) = c(y_\star, 0) = 0$ (by (Hp 6)), and therefore $\mathcal{G}(\widehat{p}) = 0$.

Step 2. We prove that if $p \in \mathcal{P}$, and $\mathcal{G}(p) < 0$, then $p(y) > -K$ for every $y \in Y$.

Indeed let us assume by contradiction that $\mathcal{G}(p) < 0$, but $p(\overline{y}) \leq -K$ for some $\overline{y} \in Y$. Since $\mathcal{G}(p) < 0$, there exists $x \in X$ such that $m_p(x) < 0$. Let $y \in A_p(x)$ be such that $m_p(x) = c(y, p(y))$. By (Hp 8) we have that $p(y) > 0$. But in this case

$$\begin{aligned} s(x, y, p(y)) &\geq s(x, y, 0) \geq -k_1 > -k_1 - 1 = k_1 - c_1 K \geq \\ &\geq s(x, \overline{y}, 0) - s(x, \overline{y}, 0) + s(x, \overline{y}, -K) \geq s(x, \overline{y}, p(\overline{y})), \end{aligned}$$

and this contradicts the assumption $y \in A_p(x)$.

Step 3. If $p \in \mathcal{P}$, and $\overline{p}(y) = \min\{p(y), K\}$, then $\mathcal{G}(\overline{p}) = \mathcal{G}(p)$.

Indeed, since products with price $\geq K$ cannot in any case be bought, then it is easy to verify that in this case $A_p(x) = A_{\overline{p}}(x)$ for every $x \in X$.

Step 4. If the infimum of $\mathcal{G}(p)$ in $\mathcal{P}$ is zero, then the price function $\widehat{p}$ of Step 1 is a minimizer. Otherwise, we can restrict to price functions p such that $\mathcal{G}(p) < 0$. By Step 2 such functions satisfy $p(y) > -K$ for every $y \in Y$. Moreover, by the truncation argument of Step 3, we can replace any competitor with another one satisfying $p(y) \leq K$ for every $y \in Y$. This completes the proof of (3.1).

Lower semicontinuity. We have to prove that, if $p_n \to p_\infty$ in $\mathcal{P}$, then

$$\liminf_{n\to+\infty} \mathcal{G}(p_n) \geq \mathcal{G}(p_\infty). \tag{3.2}$$

Step 1. Let $x \in X$ be fixed. We prove the following: "if $y_n \to y_\infty$ and $y_n \in A_{p_n}(x)$ for every $n \in \mathbb{N}$, then $y_\infty \in A_{p_\infty}(x)$ and $p_n(y_n) \to p_\infty(y_\infty)$".

The argument is similar to Step 2 of the proof of Theorem 2.1. Let us set

$$\liminf_{n\to+\infty} p_n(y_n) =: \pi_\infty \geq p_\infty(y_\infty).$$

Up to subsequences we can assume that the $\liminf$ is actually a limit. Now let $w \in Y$, and let $\{w_n\} \subseteq Y$ be a recovery sequence for w. Since $y_n \in A_{p_n}(x)$, hence we have that

$$\begin{aligned} s(x, y_\infty, p_\infty(y_\infty)) \leq s(x, y_\infty, \pi_\infty) &= \lim_{n\to+\infty} s(x, y_n, p_n(y_n)) \\ &\leq \lim_{n\to+\infty} s(x, w_n, p_n(w_n)) = s(x, w, p_\infty(w)). \end{aligned}$$

Since w is arbitrary, this proves that $y_\infty \in A_{p_\infty}(x)$.

Now let us assume that $\pi_\infty > p_\infty(y_\infty)$. Then setting $w = y_\infty$ in the above inequalities we immediately get a contradiction. Therefore the $\liminf$ of $p_n(y_n)$ is $p_\infty(y_\infty)$ for every sequence $y_n \to y_\infty$. This proves that $p_n(y_n) \to p_\infty(y_\infty)$.

Step 2. We prove that

$$\liminf_{n\to+\infty} m_{p_n}(x) \geq m_{p_\infty}(x) \qquad \forall x \in X. \tag{3.3}$$

To this end, let us assume that $m_{p_n}(x) = c(y_n, p_n(y_n))$ for every $n \in \mathbb{N}$. We can assume that the $\liminf$ of $m_{p_n}(x)$ is actually a limit, and that y_n converges to some y_∞ in Y. By Step 1, we know that $y_\infty \in A_{p_\infty}(x)$ and $p_n(y_n) \to p_\infty(y_\infty)$, hence, since y_∞ is a competitor in the definition of $m_{p_\infty}(x)$:

$$\lim_{n\to+\infty} m_{p_n}(x) = \lim_{n\to+\infty} c(y_n, p_n(y_n)) = c(y_\infty, p_\infty(y_\infty)) \geq m_{p_\infty}(x).$$

Step 3. We prove that there exists C such that $m_{p_n}(x) \geq C$ for every $n \in \mathbb{N}$.

Since $p_n \to p_\infty$ in the sense of Γ-convergence, then:

$$\lim_{n\to+\infty} \min\{p_n(y) : \ y \in Y\} = \min\{p_\infty(y) : \ y \in Y\}.$$

It follows that there exists γ such that $p_n(y) \geq \gamma$ for every $n \in \mathbb{N}$ and every $y \in Y$.

Moreover, from statement (2) of Theorem 2.1, we know that $p(y) < K$ for every $y \in \mathcal{A}_p$. It follows that, for every $x \in X$,

$$m_{p_n}(x) \geq \min\{c(y,\pi) : \ (y,\pi) \in Y \times [\gamma, K]\} =: C.$$

Step 4. We prove (3.2).

By Step 3 and (Hp 2), we can apply Fatou's lemma, so that by (3.3)

$$\liminf_{n\to+\infty} \mathcal{G}(p_n) \ = \ \liminf_{n\to+\infty} \int_X m_{p_n}(x)\,\mathrm{d}\mu \geq \int_X m_{p_\infty}(x)\,\mathrm{d}\mu \ = \ \mathcal{G}(p_\infty).$$

Conclusion. Since $\mathcal{P}_K$ is compact and $\mathcal{G}$ is lower semicontinuous, it follows that $\mathcal{G}$ has a minimum on $\mathcal{P}_K$. By (3.1) this is also a minimum on $\mathcal{P}$.

3.3. Proof of Theorem 2.3

The liminf inequality. Let us assume that $p_n \to p_\infty$ in $\mathcal{P}$. By (1.1) and the lower semicontinuity of $\mathcal{G}$ we have that

$$\liminf_{n\to+\infty} \mathcal{F}(p_n) \geq \liminf_{n\to+\infty} \mathcal{G}(p_n) \geq \mathcal{G}(p_\infty).$$

The limsup inequality. We have to prove that for every $p \in \mathcal{P}$ there exists a recovery sequence $p_n \to p$ in $\mathcal{P}$ such that

$$\limsup_{n\to+\infty} \mathcal{F}(p_n) \leq \mathcal{G}(p).$$

To this end, we consider the subset $\mathcal{D} \subseteq \mathcal{P}$ defined as follows.

Definition 3.1. Let $p \in \mathcal{P}$. We say that $p \in \mathcal{D}$ if there exists a real number $\alpha > 0$ such that the following implication holds true for every converging sequence $y_n \to y_\infty$:

$$\liminf_{n\to+\infty} p(y_n) > p(y_\infty) \implies \liminf_{n\to+\infty} p(y_n) \geq p(y_\infty) + \alpha.$$

By a standard technique in Γ convergence, the limsup inequality is proved if we show that

- for every $p \in \mathcal{P}$ there exists $\{p_n\} \subseteq \mathcal{D}$ such that $p_n \to p$, and $\mathcal{G}(p_n) \to \mathcal{G}(p)$;
- for every $\eta > 0$, and every $p \in \mathcal{D}$, there exists a sequence $p_n \to p$ such that

$$\limsup_{n\to+\infty} \mathcal{F}(p_n) \leq \mathcal{G}(p) + 2\eta.$$

This is exactly the content of the next two lemmata.

Lemma 3.2. *If assumptions (Hp 1) through (Hp 9) are satisfied, then for every $p \in \mathcal{P}$ there exists $\{p_n\} \subseteq \mathcal{D}$ such that $p_n \to p$, and $\mathcal{G}(p_n) \to \mathcal{G}(p)$.*

Proof. For every $\varepsilon > 0$, let $\overline{p}_\varepsilon : Y \to \mathbb{R}$ be the function defined by

$$\overline{p}_\varepsilon(y) := \varepsilon \cdot \max\{z \in \mathbb{Z} : \ p(y) > \varepsilon z\}.$$

It is easy to prove that, for every $\varepsilon > 0$, we have that

(i) $\overline{p}_\varepsilon(y) \in \varepsilon\mathbb{Z}$ for every $y \in Y$;
(ii) $\overline{p}_\varepsilon$ is lower semicontinuous on Y;
(iii) $p(y) - \varepsilon \leq \overline{p}_\varepsilon(y) \leq p(y)$ for every $y \in Y$.

Now let us set $\mathcal{A}_p^\star = \mathcal{A}_p \cup \{y_\star\}$, and let

$$p_\varepsilon(y) := \begin{cases} p(y) & \text{if } y \in \mathcal{A}_p^\star, \\ \overline{p}_\varepsilon(y) + 2\varepsilon & \text{if } y \in Y \setminus \mathcal{A}_p^\star. \end{cases}$$

If we show that

(1) $p_\varepsilon \in \mathcal{D}$ for every $\varepsilon > 0$,
(2) $p_\varepsilon \to p$ uniformly, hence also in $\mathcal{P}$,
(3) $\mathcal{G}(p_\varepsilon) = \mathcal{G}(p)$ for every $\varepsilon > 0$,

then any sequence $\{p_{\varepsilon_n}\}$ with $\varepsilon_n \to 0^+$ satisfies the thesis of the lemma.

In order to prove Claim (1), let $y_n \to y_\infty$ be any converging sequence, and let

$$\pi_\infty := \liminf_{n\to+\infty} p_\varepsilon(y_n).$$

Up to subsequences we can assume that the $\liminf$ is actually a limit, and that we are in one of the following three cases.

Case 1. $y_n \in \mathcal{A}_p^\star$ for every $n \in \mathbb{N}$. By statement (2) of Theorem 2.1, we have that $y_\infty \in \mathcal{A}_p^\star$, and $p(y_n) \to p(y_\infty)$, hence in this case

$$\pi_\infty = \lim_{n\to+\infty} p_\varepsilon(y_n) = \lim_{n\to+\infty} p(y_n) = p(y_\infty) = p_\varepsilon(y_\infty).$$

Case 2. $y_n \notin \mathcal{A}_p^\star$ for every $n \in \mathbb{N}$, and $y_\infty \in \mathcal{A}_p^\star$. In this case by (iii) we have that $p_\varepsilon(y_n) = \overline{p}_\varepsilon(y_n) + 2\varepsilon \geq p(y_n) + \varepsilon$ for every $n \in \mathbb{N}$, and $p_\varepsilon(y_\infty) = p(y_\infty)$, hence

$$\pi_\infty = \lim_{n\to+\infty} p_\varepsilon(y_n) \geq \varepsilon + \liminf_{n\to+\infty} p(y_n) \geq \varepsilon + p(y_\infty) = \varepsilon + p_\varepsilon(y_\infty).$$

Case 3. $y_n \notin \mathcal{A}_p^\star$ for every $n \in \mathbb{N}$, and $y_\infty \notin \mathcal{A}_p^\star$. In this case p_ε coincides with $2\varepsilon + \overline{p}_\varepsilon$ on every y_n and on y_∞, hence $\pi_\infty = p_\varepsilon(y_\infty) + k\varepsilon$ for some $k \in \mathbb{N}$.

In all the cases we have proved that either $\pi_\infty = p_\varepsilon(y_\infty)$ or $\pi_\infty \geq p_\varepsilon(y_\infty) + \varepsilon$, and therefore p_ε satisfies the condition of Definition 3.1 with $\alpha = \varepsilon$. This proves Claim (1).

In order to prove Claim (2), it is enough to remark that, for every $y \in Y$,

$$|p_\varepsilon(y) - p(y)| \leq |p_\varepsilon(y) - \overline{p}_\varepsilon(y)| + |\overline{p}_\varepsilon(y) - p(y)| \leq 3\varepsilon.$$

Finally, we prove that $A_{p_\varepsilon}(x) = A_p(x)$ for every $x \in X$ and $\varepsilon > 0$, hence $\mathcal{G}(p_\varepsilon) = \mathcal{G}(p)$ for every $\varepsilon > 0$. Indeed, if $y \in A_p(x)$, then

$$s(x, y, p_\varepsilon(y)) = s(x, y, p(y)) \leq s(x, w, p(w)) \leq s(x, w, p_\varepsilon(w))$$

for every $w \in Y$, hence $y \in A_{p_\varepsilon}(x)$. Conversely, if $y \notin A_p(x)$, and $z \in A_p(x)$, then

$$s(x, y, p_\varepsilon(y)) \geq s(x, y, p(y)) > s(x, z, p(z)) = s(x, z, p_\varepsilon(z)),$$

hence $y \notin A_{p_\varepsilon}(x)$. □

Lemma 3.3. *Assume that assumptions (Hp 1) through (Hp 9) are satisfied, and let $\mathcal{D}$ be as in Definition 3.1.*

Then for every $p \in \mathcal{D}$, and every $\eta > 0$, there exists a sequence $\{p_n\} \subseteq \mathcal{P}$ such that

$$p_n \to p, \qquad \limsup_{n\to+\infty} \mathcal{F}(p_n) \leq \mathcal{G}(p) + 2\eta.$$

Proof. Since $p \in \mathcal{D}$, then by definition there exists a constant $\alpha > 0$ such that

$$\liminf_{n\to+\infty} p(y_n) > p(y_\infty) \implies \liminf_{n\to+\infty} p(y_n) \geq p(y_\infty) + \alpha$$

for every converging sequence $y_n \to y_\infty$.

To begin with, let us assume that $p(y) \leq K$ for every $y \in Y$, where K is the constant defined in (2.1), and let us set

$$\begin{aligned} \gamma &:= \min\{p(y) : \ y \in Y\}, \\ \lambda &:= \min\{c(y,\pi) : \ (y,\pi) \in Y \times [\gamma, K]\}, \\ \Lambda &:= \max\{c(y,\pi) : \ (y,\pi) \in Y \times [\gamma, K]\}. \end{aligned}$$

For every $i \in \mathbb{N}$, we set

$$Y_i := \{y \in Y : \ \lambda + \eta(i-1) < c(y, p(y)) \leq \lambda + \eta i\},$$

and, if c_1 and c_2 are the constants in (Hp 9), then we define $\Phi : Y \to \mathbb{R}$ as

$$\Phi(y) := \left(\frac{c_1}{c_2}\right)^i, \qquad \forall y \in Y_i.$$

Finally, for every $\varepsilon > 0$ we set

$$p_\varepsilon(y) := p(y) - \varepsilon\Phi(y)\left[\Lambda - c(y, p(y))\right] \qquad \forall y \in Y.$$

If we prove that

(1) $p_\varepsilon \in \mathcal{P}$ provided that $\varepsilon(\Lambda - \lambda) \leq \alpha$,
(2) $p_\varepsilon \to p$ uniformly in Y, hence in $\mathcal{P}$,
(3) $\mathcal{F}(p_\varepsilon) \leq \mathcal{G}(p) + 2\eta$ for ε small enough,

then any sequence $\{p_{\varepsilon_n}\}$ with $\varepsilon_n \to 0^+$ satisfies the thesis of the lemma for n large.

Step 1. Since $0 \leq \Phi(y) \leq 1$ for every $y \in Y$, then

$$p(y) - \varepsilon(\Lambda - \lambda) \leq p_\varepsilon(y) \leq p(y), \qquad \forall \varepsilon > 0, \ \forall y \in Y. \tag{3.4}$$

This proves Claim (2) provided that $p_\varepsilon \in \mathcal{P}$.

Step 2. We prove that if $y_n \to y_\infty$ in Y, and $p(y_n) \to p(y_\infty)$, then

$$\limsup_{n\to+\infty} \Phi(y_n) \leq \Phi(y_\infty). \tag{3.5}$$

Indeed, let us choose a subsequence such that the $\limsup$ is actually a limit. We can assume (up to subsequences) that every y_n belongs to a fixed Y_{i_0}. This means that

$$\lambda + \eta(i_0 - 1) < c(y_n, p(y_n)) \leq \lambda + \eta i_0, \qquad \Phi(y_n) = \left(\frac{c_1}{c_2}\right)^{i_0}, \qquad \forall n \in \mathbb{N}.$$

Since $p(y_n) \to p(y_\infty)$, and c is continuous, there are only two possibilities:

- $y_\infty \in Y_{i_0}$, and therefore

$$\lim_{n\to+\infty} \Phi(y_n) = \left(\frac{c_1}{c_2}\right)^{i_0} = \Phi(y_\infty);$$

- $y_\infty \in Y_{i_0-1}$, and therefore

$$\lim_{n\to+\infty} \Phi(y_n) = \left(\frac{c_1}{c_2}\right)^{i_0} \le \left(\frac{c_1}{c_2}\right)^{i_0-1} = \Phi(y_\infty).$$

In both cases, inequality (3.5) is proved.

Step 3. We prove Claim (1). Since clearly $p_\varepsilon(y_\star) \le p(y_\star) \le 0$, we have only to prove that for every converging sequence $y_n \to y_\infty$ in Y

$$\liminf_{n\to+\infty} p_\varepsilon(y_n) \ge p_\varepsilon(y_\infty). \tag{3.6}$$

We can assume that the $\liminf$ is actually a limit, and that there exists

$$\pi_\infty := \lim_{n\to+\infty} p(y_n) \ge p(y_\infty).$$

Now we have two cases.

- *Case 1.* $\pi_\infty = p(y_\infty)$, *i.e.* $p(y_n) \to p(y_\infty)$. In this case by Step 2 we have that

$$\begin{aligned}\lim_{n\to+\infty} p_\varepsilon(y_n) &= \lim_{n\to+\infty} \{p(y_n) - \varepsilon\,[\Lambda - c(y_n, p(y_n))]\,\Phi(y_n)\}\\ &\ge p(y_\infty) - \varepsilon\,[\Lambda - c(y_\infty, p(y_\infty))] \limsup_{n\to+\infty} \Phi(y_n)\\ &\ge p(y_\infty) - \varepsilon\,[\Lambda - c(y_\infty, p(y_\infty))]\,\Phi(y_\infty) \;=\; p_\varepsilon(y_\infty),\end{aligned}$$

  which proves (3.6) in this case.
- *Case 2.* $\pi_\infty > p(y_\infty)$. Since $p \in \mathcal{D}$, this means that $\pi_\infty \ge p(y_\infty) + \alpha$. Then by (3.4) it follows that

$$\liminf_{n\to+\infty} p_\varepsilon(y_n) \ge \liminf_{n\to+\infty} p(y_n) - \varepsilon(\Lambda - \lambda) \ge p(y_\infty) + \alpha - \varepsilon(\Lambda - \lambda).$$

  If $\varepsilon(\Lambda - \lambda) \le \alpha$, then inequality (3.6) is proved also in this second case.

Step 4. We prove that if w and y are elements of Y such that

$$c(w, p(w)) > c(y, p(y)) + \eta, \tag{3.7}$$

then for every $\varepsilon > 0$

$$s(x, w, p_\varepsilon(w)) - s(x, w, p(w)) > s(x, y, p_\varepsilon(y)) - s(x, y, p(y)). \tag{3.8}$$

Let us assume that $y \in Y_i$ and $w \in Y_j$. By (3.7) it follows that $j > i$, hence $\Phi(y) \geq (c_2/c_1)\Phi(w)$. Therefore by (Hp 9)

$$\begin{aligned} s(x,y,p(y)) - s(x,y,p_\varepsilon(y)) &\geq c_1(p(y) - p_\varepsilon(y)) = \varepsilon c_1 \Phi(y)\,[\Lambda - c(y,p(y))] \\ &\geq \varepsilon c_2 \Phi(w)\,[\Lambda - c(y,p(y))] \\ &> \varepsilon c_2 \Phi(w)\,[\Lambda - c(w,p(w))] \\ &= c_2(p(w) - p_\varepsilon(w)) \\ &\geq s(x,w,p(w)) - s(x,w,p_\varepsilon(w)). \end{aligned}$$

Step 5. We prove the following: "if $x \in X$, $y \in A_p(x)$, and $w \in A_{p_\varepsilon}(x)$ for some $\varepsilon > 0$, then $c(w,p(w)) \leq c(y,p(y)) + \eta$".

Let indeed $w \in Y$ be such that $c(w,p(w)) > c(y,p(y)) + \eta$. Then, using that $y \in A_p(x)$ and (3.8), we have that

$$\begin{aligned} s(x,w,p_\varepsilon(w)) &= s(x,w,p_\varepsilon(w)) - s(x,w,p(w)) + s(x,w,p(w)) \\ &> s(x,y,p_\varepsilon(y)) - s(x,y,p(y)) + s(x,y,p(y)) \;=\; s(x,y,p_\varepsilon(y)) \end{aligned}$$

which proves that $w \notin A_{p_\varepsilon}(x)$.

Step 6. We prove that for every $x \in X$, and every ε small enough, we have that

$$M_{p_\varepsilon}(x) \leq m_p(x) + 2\eta \tag{3.9}$$

To this end, let us set

$$\omega(\varepsilon) := \sup\{|c(y,\pi_1) - c(y,\pi_2)| : (y,\pi_1,\pi_2) \in Y \times [\gamma, K]^2, |\pi_2 - \pi_1| \leq \varepsilon(\Lambda - \lambda)\}.$$

Since c is uniformly continuous in $Y \times [\gamma, K]$, it follows that $\omega(\varepsilon) \to 0$ as $\varepsilon \to 0^+$, hence $\omega(\varepsilon) < \eta$ if ε is small enough.

Now let $y \in A_p(x)$ and $w \in A_{p_\varepsilon}(x)$ be such that

$$M_{p_\varepsilon}(x) = c(w, p_\varepsilon(w)), \qquad m_p(x) = c(y,p(y)).$$

By Step 5, (3.4), and definition of ω, we have that for ε small enough:

$$\begin{aligned} M_{p_\varepsilon}(x) \;=\; c(w,p_\varepsilon(w)) &= c(w,p(w)) + c(w,p_\varepsilon(w)) - c(w,p(w)) \\ &\leq c(y,p(y)) + \eta + \omega(\varepsilon) \;<\; m_p(x) + 2\eta. \end{aligned}$$

Step 7. Integrating (3.9) with respect to μ, by (Hp 2) we obtain that

$$\mathcal{F}(p_\varepsilon) \leq \mathcal{G}(p) + 2\eta$$

for ε small enough. This completes the proof if $p(y) \leq K$ for every $y \in Y$.

Step 8. If p is not bounded from above by K, we know from Theorem 2.1 that in any case $p(y) \leq K$ for every $y \in \mathcal{A}_p$. Now it is enough to define $p_\varepsilon(y)$ as in the previous case if $p(y) < K$, and to set $p_\varepsilon(y) = p(y)$ if $p(y) \geq K$. Since any $y \in Y$ with $p_\varepsilon(y) \geq K$ cannot belong to $\mathcal{A}_{p_\varepsilon}$, then everything works exactly as before. □

3.4. Proof of Theorem 2.4

Technical lemmata. We first state and prove four lemmata.

Lemma 3.4. *Let X, X_n, μ, μ_n be as in section 2. Let $\gamma_n : X \to \mathbb{R}$ for every $n \in \mathbb{N}$, and let $\gamma_\infty : X \to \mathbb{R}$. Let us assume that there exists a constant C such that $\gamma_n(x) \geq C$ for every $(n,x) \in \mathbb{N} \times X$, and that for every converging sequence $x_n \to x_\infty$ in X:*

$$\liminf_{n\to+\infty} \gamma_n(x_n) \geq \gamma_\infty(x_\infty).$$

Then

$$\liminf_{n\to+\infty} \int_X \gamma_n(x)\,\mathrm{d}\mu_n \geq \int_X \gamma_\infty(x)\,\mathrm{d}\mu.$$

Lemma 3.5. *Let X, X_n, μ, μ_n be as in section 2. Let $\gamma_n : X \to \mathbb{R}$ for every $n \in \mathbb{N}$, and let $\gamma_\infty : X \to \mathbb{R}$. Let us assume that there exists a constant C such that $\gamma_n(x) \leq C$ for every $(n,x) \in \mathbb{N} \times X$, and that for every converging sequence $x_n \to x_\infty$ in X:*

$$\limsup_{n\to+\infty} \gamma_n(x_n) \leq \gamma_\infty(x_\infty).$$

Then

$$\limsup_{n\to+\infty} \int_X \gamma_n(x)\,\mathrm{d}\mu_n \leq \int_X \gamma_\infty(x)\,\mathrm{d}\mu.$$

Lemma 3.6. *Let Y_n and Y be as in section 2. Let $\phi_n : Y_n \to \mathbb{R}$ for every $n \in \mathbb{N}$, and let $\phi_\infty : Y \to \mathbb{R}$. Let us assume that for every sequence $\{y_n\} \subseteq Y$ such that $y_n \in Y_n$, and $y_n \to y_\infty$ in Y we have that $\liminf_{n\to+\infty} \phi_n(y_n) \geq \phi_\infty(y_\infty)$.*

Then

$$\liminf_{n\to+\infty} \mathrm{Mean}_{-n}\{\phi_n(y) :\ y \in Y_n\} \geq \inf\{\phi_\infty(y) :\ y \in Y\}.$$

Lemma 3.7. *Let Y_n and Y be as in section 2. Let $\phi_n : Y_n \to \mathbb{R}$ for every $n \in \mathbb{N}$. Let $\{y_n\} \subseteq Y$ be any sequence such that $y_n \in Y_n$ for every $n \in \mathbb{N}$.*

Then

$$\limsup_{n\to+\infty} \mathrm{Mean}_{-n}\{\phi_n(y) :\ y \in Y_n\} \leq \limsup_{n\to+\infty} \phi_n(y_n).$$

Proof of Lemma 3.4. Let $X_{n,i}$ and $x_{n,i}$ be as in section 2. For every $(n,x) \in \mathbb{N} \times X$, we denote by $T_n(x)$ the element $x_{n,i}$ such that $x \in X_{n,i}$. Since $r(X_n, X) \to 0$, it follows that $T_n(x) \to x$ as $n \to +\infty$. Now let us define $\overline{\gamma}_n(x) := \gamma_n(T_n(x))$ so that by definition of μ_n,

$$\int_X \gamma_n(x)\,\mathrm{d}\mu_n = \int_X \overline{\gamma}_n(x)\,\mathrm{d}\mu.$$

By assumption we have that $\overline{\gamma}_n(x) \geq C$ for every $(n,x) \in \mathbb{N} \times X$, and moreover

$$\liminf_{n\to+\infty} \overline{\gamma}_n(x) = \liminf_{n\to+\infty} \gamma_n(T_n(x)) \geq \gamma_\infty(x) \qquad \forall x \in X.$$

Therefore thesis follows applying Fatou's lemma to the sequence. □

Proof of Lemma 3.5. Analogous to the proof of Lemma 3.4. □

Proof of Lemma 3.6. Since the Mean_q of an n-uple is always greater or equal than the minimum, it is enough to show that

$$\liminf_{n\to+\infty} \min\{\phi_n(y):\ y\in Y_n\} \geq \inf\{\phi_\infty(y):\ y\in Y\}.$$

Let us choose a subsequence such that the $\liminf$ is actually a limit, and then let $y_n \in \arg\min\{\phi_n(y):\ y\in Y_n\}$. We can assume that $\{y_n\}$ converges to some y_∞ in Y.

To complete the proof it is enough to remark that by assumption we have that

$$\begin{aligned}\lim_{n\to+\infty} \min\{\phi_n(y):\ y\in Y_n\} &= \lim_{n\to+\infty} \phi_n(y_n)\\ &\geq \phi_\infty(y_\infty) \geq \inf\{\phi_\infty(y):\ y\in Y\}.\end{aligned}$$ □

Proof of Lemma 3.7. Let $\overline{y}_n \in \arg\min\{\phi_n(y):\ y\in Y_n\}$ for every $n\in\mathbb{N}$. Since $|Y_n| = n$, we have that

$$\begin{aligned}\mathrm{Mean}_{-n}\{\phi_n(y):\ y\in Y_n\} &= -\frac{1}{n}\log\left(\frac{1}{n}\sum_{y\in Y_n} e^{-n\phi_n(y)}\right)\\ &= \phi_n(\overline{y}_n) - \frac{1}{n}\log\left(\frac{1}{n}\sum_{y\in Y_n} e^{-n(\phi_n(y)-\phi_n(\overline{y}_n))}\right)\\ &\leq \phi_n(y_n) - \frac{1}{n}\log\left(\frac{1}{n}\right) \ = \ \phi_n(y_n) + \log\sqrt[n]{n}.\end{aligned}$$

Letting $n\to+\infty$, thesis is proved. □

The liminf inequality. We have to prove that

$$\liminf_{n\to+\infty} \mathcal{G}_n(p_n) \geq \mathcal{G}(p_\infty) \tag{3.10}$$

for every sequence $\{p_n\}$ such that $p_n : Y_n \to [-K,K]$ for every $n\in\mathbb{N}$, and $p_n \to p_\infty$.

Applying Lemma 3.4 with $\gamma_n(x) := m_{n,p_n}(x)$, and $\gamma_\infty(x) := m_{p_\infty}(x)$, we have that inequality (3.10) is proved if we show that for every $x_n \to x_\infty$ in X:

$$\liminf_{n\to+\infty} m_{n,p_n}(x_n) \geq m_{p_\infty}(x_\infty) \tag{3.11}$$

(the existence of C such that $\gamma_n(x) \geq C$ follows from the equi-boundedness of c_n in $Y\times[-K,K]$).

Now let us define $\phi_n : Y_n \to \mathbb{R}$ and $\phi_\infty : Y \to \mathbb{R}$ by

$$\phi_n(y) := f_{n,p_n}(x_n,y) = c_n(y,p_n(y)) + 5M\Psi_{n,p_n}(x_n,y),$$

$$\phi_\infty(y) := \begin{cases} c(y,p_\infty(y)) & \text{if } y\in A_{p_\infty}(x_\infty),\\ 2M & \text{otherwise.}\end{cases}$$

With these notations we have that $m_{n,p_n}(x_n) = \mathrm{Mean}_{-n}\{\phi_n(y) : \ y \in Y_n\}$. Moreover, since $\phi_\infty(y) = c(y, p_\infty(y)) \le M$ for $y \in A_{p_\infty}(x_\infty)$, it follows that

$$m_{p_\infty}(x_\infty) = \min\{\phi_\infty(y) : \ y \in Y\}.$$

Applying Lemma 3.6 we have that inequality (3.11) is proved if we show that for every converging sequence $y_n \to y_\infty$ in Y:

$$\liminf_{n\to+\infty} \phi_n(y_n) \ge \phi_\infty(y_\infty). \tag{3.12}$$

We first choose a subsequence so that the $\liminf$ is actually a limit, and then we choose $\overline{y}_n \in A_{n,p_n}(x_n)$. We can assume to be in one of the following two cases.

Case 1. $s_n(x_n, y_n, p_n(y_n)) \ge s_n(x_n, \overline{y}_n, p_n(\overline{y}_n)) + 1/n$ for every $n \in \mathbb{N}$. In this case, since $\psi_n(r) = 0$ for $r \ge 1/n$, we have that $\Psi_{n,p_n}(x_n, y_n) = 1$, hence (3.12) trivially follows from

$$\phi_n(y_n) = c_n(y_n, p_n(y_n)) + 5M \ge 2M \ge \phi_\infty(y_\infty).$$

Case 2. $s_n(x_n, y_n, p_n(y_n)) < s_n(x_n, \overline{y}_n, p_n(\overline{y}_n)) + 1/n$ for every $n \in \mathbb{N}$. Arguing as in Step 1 of the "lower semicontinuity" part of the proof of Theorem 2.2, we can prove that $y_\infty \in A_{p_\infty}(x_\infty)$, and $p_n(y_n) \to p_\infty(y_\infty)$. Moreover we have that

$$\liminf_{n\to+\infty} \phi_n(y_n) \ge \lim_{n\to+\infty} c_n(y_n, p_n(y_n)) = c(y_\infty, p_\infty(y_\infty)) = \phi_\infty(y_\infty).$$

This completes the the proof of the liminf inequality.

The limsup inequality. Since $\mathcal{G}(p)$ is the relaxation of $\mathcal{F}(p)$, we can limit ourselves to show that

$$\limsup_{n\to+\infty} \mathcal{G}_n(p_n) \le \mathcal{F}(p) \tag{3.13}$$

for a suitable sequence $p_n \to p$, with $p_n \in \mathcal{P}_{n,K}$ for every $n \in \mathbb{N}$. To this end, we set

$$p_n(y) := \min\{p(w) : \ w \in Y, \ d_Y(w, y) \le r(Y_n, Y)\} \qquad \forall y \in Y_n.$$

It is easy to see that $p_n \to p$ in the sense we need. Moreover, from Lemma 3.5 applied with $\gamma_n(x) := m_{n,p_n}(x)$, and $\gamma_\infty(x) := M_{p_\infty}(x)$, it follows that inequality (3.13) is proved if we show that, for every $x_n \to x_\infty$ in X, we have that

$$\limsup_{n\to+\infty} m_{n,p_n}(x_n) \le M_{p_\infty}(x_\infty). \tag{3.14}$$

Now let us define $\phi_n : Y_n \to \mathbb{R}$ by

$$\phi_n(y) := f_{n,p_n}(x_n, y) = c_n(y, p_n(y)) + 5M\Psi_{n,p_n}(x_n, y).$$

With these notations we have that $m_{n,p_n}(x_n) = \mathrm{Mean}_{-n}\{\phi_n(y) : \ y \in Y_n\}$.

Applying Lemma 3.7 we have that inequality (3.14) is proved if we show that there exists a sequence $\{y_n\}$ such that $y_n \in Y_n$ for every $n \in \mathbb{N}$, and

$$\limsup_{n\to+\infty} \phi_n(y_n) \leq M_{p_\infty}(x_\infty). \tag{3.15}$$

For every $n \in \mathbb{N}$, let us choose $y_n \in A_{n,p_n}(x_n)$ (hence $\Psi_{n,p_n}(x_n, y_n) = 0$). In order to prove (3.15), we choose a subsequence so that the $\limsup$ is actually a limit and we assume that $\{y_n\}$ converges to some y_∞ in Y. Arguing as in Step 1 of the "lower semicontinuity" part of the proof of Theorem 2.2, we can prove that $y_\infty \in A_{p_\infty}(x_\infty)$, and $p_n(y_n) \to p_\infty(y_\infty)$, and therefore, since y_∞ is a competitor in the definition of $M_{p_\infty}(x_\infty)$,

$$\lim_{n\to+\infty} \phi_n(y_n) = \lim_{n\to+\infty} c_n(y_n, p_n(y_n)) = c(y_\infty, p_\infty(y_\infty)) \leq M_{p_\infty}(x_\infty).$$

This proves (3.15), and completes the proof of the limsup inequality.

4. Counterexamples

In the first example we show a simple situation where the minimum of $\mathcal{F}$ doesn't exist.

Example 4.1. Let $X = \{x\}$, $Y = \{y_\star, y_1\}$, and for every $\pi \in \mathbb{R}$, let $s(x, y_\star, \pi) = \pi$, $s(x, y_1, \pi) = \pi - 2$, $c(y_\star, \pi) = -\pi$, $c(y_1, \pi) = 1 - \pi$.

All our assumptions (Hp 1) through (Hp 9) are satisfied.

In this case it is easy to see that, when looking for the infimum of $\mathcal{F}$ and $\mathcal{G}$, one can assume that $p(y_\star) = 0$. Under this condition, it turns out that $A_p(x) = y_1$ if $p(y_1) < 2$, $A_p(x) = \{y_\star, y_1\}$ if $p(y_1) = 2$, and $A_p(x) = \{y_\star\}$ if $p(y_1) > 2$, hence

$$\mathcal{F}(p) = M_p(x) = \begin{cases} 1 - p(y_1) & \text{if } p(y_1) < 2, \\ 0 & \text{if } p(y_1) \geq 2. \end{cases}$$

Therefore the infimum of $\mathcal{F}$ is -1, but it is not a minimum.

In the second example we show that, if the stress function $s(x, y, \pi)$ is not strictly increasing in π, then the infimum of $\mathcal{F}$ can be strictly greater than the minimum of $\mathcal{G}$.

Example 4.2. Let $X = \{x\}$, $Y = \{y_\star, y_1\}$, and for every $\pi \in \mathbb{R}$, let $c(y_\star, \pi) = -\pi$, $c(y_1, \pi) = 1 - 2\pi$, $s(x, y_\star, \pi) = \pi$, and

$$s(x, y_1, \pi) = \begin{cases} \pi & \text{if } \pi \leq 0, \\ 0 & \text{if } 0 \leq \pi \leq 1, \\ \pi - 1 & \text{if } \pi \geq 1. \end{cases}$$

In this case assumptions (Hp 1) through (Hp 8) are satisfied.

Let us consider the price function $\overline{p}$ with $\overline{p}(y_\star) = 0$ and $\overline{p}(y_1) = 1$. Then $A_{\overline{p}}(x) = \{y_\star, y_1\}$, hence $\min\{\mathcal{G}(p) : p \in \mathcal{P}\} \leq \mathcal{G}(\overline{p}) = m_{\overline{p}}(x) = -1$. Now let us consider $\mathcal{F}(p)$. If $p(y_1) \leq 0$, then $\mathcal{F}(p) = M_p(x) \geq \min\{-p(y_\star), 1 - 2p(y_1)\} \geq 0$.

If $p(y_1) > 0$, then $s(x, y_\star, p(y_\star)) \leq 0 \leq s(x, y_1, p(y_1))$, and therefore $y_\star \in A_p(x)$, hence $\mathcal{F}(p) = M_p(x) \geq -p(y_\star) \geq 0$.

In any case we have that $\mathcal{F}(p) \geq 0$, hence $\inf\{\mathcal{F}(p) : \; p \in \mathcal{P}\} \geq 0$.

Acknowledgements. We discovered this model during an inspiring conference held by Prof. I. Ekeland at the meeting "Transport me to Toronto!" (Toronto, August 2001). For this reason, we are indebted to him and to the organizers of that workshop. We would like to thank also Prof. G. Carlier for drawing our attention on some references.

References

1. Carlier, G.: A general existence result for the principal-agent problem with adverse selection. J. Math. Econ. **35**, 129–150 (2001)
2. Carlier G., Lachand-Robert, T.: Regularity of solutions for some variational problems subject to a convexity constraint. Comm. Pure Appl. Math. **54**, 583–594 (2001)
3. Carlier, G., Lachand-Robert, T., Maury, B.: A numerical approach to variational problems subject to convexity constraint. Numer. Math. **88**, 299–318 (2001)
4. Dal Maso, G.: An introduction to Γ-convergence. Birkhäuser, Boston 1993
5. Rochet, J.C., Choné, P.: Ironing, sweeping and multidimensional screening. Econometrica **66**, 783–826 (1998)
6. Lions, P.L.: Identification du cône dual des fonctions convexes et applications. C. R. Acad. Sci. Paris **326**, 1385–1390 (2001)
7. Monteiro, P.K., Page Jr., F.H.: Optimal selling mechanism for multiproduct monopolists: incentive compatibility in the presence of budget constraints. J. Math. Econ. **30**, 473–502 (1998)