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# Global-in-time Uniform Convergence for Linear Hyperbolic–Parabolic Singular Perturbations

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**Abstract** We consider the Cauchy problem $\varepsilon u''_\varepsilon + \delta u'_\varepsilon + Au_\varepsilon = 0$, $u_\varepsilon(0) = u_0$, $u'_\varepsilon(0) = u_1$, where $\varepsilon > 0$, $\delta > 0$, $H$ is a Hilbert space, and $A$ is a self-adjoint linear non-negative operator on $H$ with dense domain $D(A)$. We study the convergence of $\{u_\varepsilon\}$ to the solution of the limit problem $\delta u' + Au = 0$, $u(0) = u_0$.

For initial data $(u_0, u_1) \in D(A^{1/2}) \times H$, we prove global-in-time convergence with respect to strong topologies.

Moreover, we estimate the convergence rate in the case where $(u_0, u_1) \in D(A^{3/2}) \times D(A^{1/2})$, and we show that this regularity requirement is sharp for our estimates. We give also an upper bound for $|u'_\varepsilon(t)|$ which does not depend on $\varepsilon$.

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## 1 Introduction

Let $H$ be a real Hilbert space, with norm $|\cdot|$ and scalar product $\langle \cdot, \cdot \rangle$. Let $A$ be a self-adjoint linear operator on $H$ with dense domain $D(A)$. We assume that $A$ is non-negative, i.e. $\langle Au, u\rangle \geq 0$ for all $u \in D(A)$.

For $\varepsilon > 0$, $\delta > 0$, we consider the Cauchy problem

$$\begin{cases} \varepsilon u''_\varepsilon(t) + \delta u'_\varepsilon(t) + Au_\varepsilon(t) = 0, & t \geq 0, \\ u_\varepsilon(0) = u_0, \quad u'_\varepsilon(0) = u_1. \end{cases} \tag{1.1}$$

We study the convergence of $\{u_\varepsilon\}$ to the solution $u$ of the "limit problem"

$$\begin{cases} \delta u'(t) + Au(t) = 0, & t > 0, \\ u(0) = u_0, \end{cases} \tag{1.2}$$

obtained by setting formally $\varepsilon = 0$ in (1), and forgetting the initial datum $u_1$. This "loss of one initial condition" is measured by $w_1 := u'_\varepsilon(0) - u'(0) = u_1 + \frac{Au_0}{\delta}$.

For the convenience of the reader, we recall the classical results on these equations, whose proof can be found in almost every textbook on linear PDEs.

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**Theorem 1.1** *For every $(u_0, u_1) \in D(A^{1/2}) \times H$, Problem* (1.1) *has a unique solution*

$$u \in C^0([0, +\infty[, D(A^{1/2})) \cap C^1([0, +\infty[, H) \cap C^2([0, +\infty[, D(A^{-1/2})).$$

*For every $u_0 \in H$, Problem* (1.2) *has a unique solution $u \in C^0([0, +\infty[, H)$ such that $u \in C^\infty(]0, +\infty[, D(A^k))$ for every $k \geq 0$.*

This singular perturbation problem was considered by Lions (see [1, pp. 491–495]). For initial data $(u_0, u_1) \in D(A^{1/2}) \times H$ he proved that

$$u_\varepsilon \rightharpoonup u \quad \text{weakly * in } L^\infty([0, +\infty[; D(A^{1/2})), \tag{1.3}$$

$$u'_\varepsilon \rightharpoonup u' \quad \text{weakly in } L^2([0, +\infty[; H). \tag{1.4}$$

Moreover, for initial data $(u_0, u_1) \in D(A^2) \times D(A)$, he proved that, for each $T > 0$, there exists a constant $C_T$ such that

$$\|u_\varepsilon - u\|_{L^\infty([0,T]; D(A^{1/2}))} \leq C_T \varepsilon, \tag{1.5}$$

$$\delta \|u'_\varepsilon - u'\|_{L^2([0,T]; H)} \leq |w_1| \sqrt{\varepsilon} + C_T \varepsilon, \tag{1.6}$$

and, for every $t \in [0, T]$,

$$|u'_\varepsilon(t) - u'(t)| \leq |w_1| e^{-\delta t/\varepsilon} + C_T \sqrt{\varepsilon}. \tag{1.7}$$

The constant $C_T$ depends on $u_0$, $u_1$, $\delta$, $T$, but not on $\varepsilon$.

Later on, this theory was generalized to equations with lower order terms (see Benaouda and Madaune–Tort [2]), and to nonlinear equations (see Esham and Weinacht [3, 4], Colli and Grasselli [5]). Singular perturbations have been used also as a tool to find numerical algorithms (see Esham [6]), and in the quasilinear case to prove the existence of solutions of the hyperbolic problem for $\varepsilon$ small (see Milani [7–11]).

However, as far as we know, the convergence results in the literature are generalizations of Lions' ones, i.e.:

- Global-in-time weak convergence for the initial data in the "energy space";
- Local-in-time strong convergence for regular initial data.

In this note we improve both results in the linear case. Indeed we prove:

- Global-in-time convergence with respect to strong topologies for the initial data in the energy space (Theorem 2.1);
- Global-in-time estimates of the convergence rate for the initial data in $D(A^{3/2}) \times D(A^{1/2})$ (Theorem 2.2);
- Optimality of $D(A^{3/2}) \times D(A^{1/2})$ when looking even for local-in-time estimates of the convergence rate of order $O(\varepsilon)$ (Theorem 2.3);
- An estimate for $u'_\varepsilon$ which does not depend on $\varepsilon$ (Theorem 2.4).

Our results are stated and proved in an abstract Hilbert space setting. The standard application is the possibility to approximate a dissipative wave equation, with a small inertia term, by a heat equation, which was probably the initial motivation of this theory (see Cattaneo [12]).

It is of course possible to prove our results by reducing to ODEs via spectral decomposition. Nevertheless, we prefer to present proofs based on estimates of suitable energies, because they can be generalized to equations with lower order terms and also to non linear PDEs (cf. [13] for a partial extension of Theorem 2.1 to scalar nonlinearities), or to control teory (cf. [14]).

## 2 Statements

In the following we assume, without loss of generality, that $\varepsilon \in ]0, 1]$.

The first result concerns global-in-time strong convergence for the initial data in the energy space.

**Theorem 2.1** *Let $(u_0, u_1) \in D(A^{1/2}) \times H$, and let $u_\varepsilon$ and $u$ be the solutions of* (1.1) *and* (1.2)*, respectively.*

*Then*

$$u_\varepsilon \to u \quad \textit{strongly in } L^\infty([0,+\infty[; D(A^{1/2})), \tag{2.1}$$

$$u'_\varepsilon \to u' \quad \textit{strongly in } L^2([0,+\infty[; H). \tag{2.2}$$

*Moreover, for every* $B > 0$, *we have that*

$$u'_\varepsilon \to u' \qquad \textit{strongly in } L^\infty([B,+\infty[; H). \tag{2.3}$$

We remark that, by the continuity of $u'_\varepsilon - u'$, we cannot expect that $u'_\varepsilon$ converges to $u'$ strongly in $L^\infty([0,+\infty[; H)$ if $w_1 \neq 0$, even for very regular initial data.

The second result concerns global-in-time estimates of the convergence rate for regular initial data; this proves in particular that the constant $C_T$ in Lions' inequalities (1.5), (1.6), and (1.7) can be taken to be independent on $T$.

**Theorem 2.2** *Let* $(u_0, u_1) \in D(A^{3/2}) \times D(A^{1/2})$, *and let* $u_\varepsilon$ *and* $u$ *be the solutions of* (1.1) *and* (1.2), *respectively.*

*Then there exists a constant* $C$ *such that*

$$\|u_\varepsilon - u\|_{L^\infty([0,+\infty[; D(A^{1/2}))} \leq C\varepsilon, \tag{2.4}$$

$$\delta \|u'_\varepsilon - u'\|_{L^2([0,+\infty[; H)} \leq |w_1|\sqrt{\varepsilon} + C\varepsilon, \tag{2.5}$$

*and that, for every* $t \geq 0$,

$$|u'_\varepsilon(t) - u'(t)| \leq |w_1| e^{-\delta t/\varepsilon} + C\sqrt{\varepsilon}. \tag{2.6}$$

*The constant* $C$ *depends on* $u_0$, $u_1$, $\delta$, *but not on* $\varepsilon$ *and* $t$.

At this point the reader may ask whether such estimates for the convergence rate can be extended to less regular initial data. The following result shows that this is not possible, and that the choice of $D(A^{3/2}) \times D(A^{1/2})$ is sharp when looking even for local-in-time estimates of order $O(\varepsilon)$.

**Theorem 2.3** *Let* $H = L^2(]0, 2\pi[)$, *and let* $Au = -u_{xx}$ (*with zero Dirichlet boundary conditions*). *Then there exists* $u_0 \in H$ *such that:*

- $u_0 \in D(A^\alpha)$ *for every* $\alpha < 3/2$;
- *Considering* $(u_0, 0)$ *as initial data of* $u_\varepsilon$, *then for every* $T > 0$, *we have that*

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \|u_\varepsilon - u\|_{L^\infty([0,T]; D(A^{1/2}))} = +\infty.$$

*A similar result holds true for initial data* $(0, u_1)$ *with* $u_1 \in D(A^\beta)$ *for every* $\beta < 1/2$.

**Remark 1** Theorem 2.1 gives uniform convergence for the initial data in $D(A^{1/2}) \times H$, i.e. in the natural setting for problem (1.1). In this case there is a "gap of 1/2" between the spaces where $u_\varepsilon$ and $u'_\varepsilon$ are defined, a situation which is typical of second order evolution problems such as (1.1).

On the contrary, Theorem 2.3 shows that the best space where one can prove estimates such as (2.4) is $D(A^{3/2}) \times D(A^{1/2})$, hence with a "gap of 1". This situation is typical for first order evolution problems such as (1.2).

The proof of Theorem 2.1 relies on Theorem 2.2 and an approximation argument. To this end, the following estimate is crucial in the proof of (2.3):

**Theorem 2.4** *Let* $(u_0, u_1) \in D(A^{1/2}) \times H$, *and let* $u_\varepsilon$ *be the solution of* (1.1). *Then*

$$|u'_\varepsilon(t)| \leq |u_1| + \frac{|A^{1/2} u_0|}{\sqrt{\delta t}}, \qquad \forall\, t > 0. \tag{2.7}$$

We point out that the right-hand side of (2.7) does not depend on $\varepsilon$, while the standard energy estimate on $u'_\varepsilon(t)$ diverges as $\varepsilon \to 0^+$.

## 3 Proofs

In all the proofs we assume, without loss of generality, that $\varepsilon \in ]0, 1]$. To begin with, we assume also for simplicity that the operator $A$ is coercive, i.e. there exists a constant $\nu > 0$ such that

$\langle Au, u\rangle \geq \nu|u|^2$ for every $u \in D(A)$. In Section 3.6 we outline how our proofs should be modified in the non-coercive case. In many estimates we also use the inequality

$$2\,|\langle x, y\rangle| \leq \sigma|x|^2 + \frac{1}{\sigma}|y|^2,$$

which holds true for every $x \in H$, $y \in H$, $\sigma > 0$.

## 3.1 Standard Estimates

Let us recall the standard estimates for solutions of (1.1) and (1.2), which will be used in this paper.

Let $u_\varepsilon$ be the solution of (1.1). Taking the scalar product of the equation by $2u_\varepsilon'(t)$, and integrating in $[0, T]$, we obtain that

$$\varepsilon\,|u_\varepsilon'(T)|^2 + |A^{1/2}u_\varepsilon(T)|^2 + 2\delta\int_0^T |u_\varepsilon'(t)|^2 dt = \varepsilon|u_1|^2 + |A^{1/2}u_0|^2.$$

We refer to this as the standard energy estimate. It provides good estimates for the norm of $u_\varepsilon'$ in $L^2([0, +\infty[; H)$, and the norm of $u_\varepsilon$ in $L^\infty([0, +\infty[; D(A^{1/2}))$. However, if $u_0 \neq 0$, this gives an upper bound for $|u_\varepsilon'(t)|$ which diverges as $\varepsilon \to 0^+$.

For the solution of (1.2), we recall that if $u_0 \in D(A^{1/2})$, then

$$|A^{1/2}u(t)| \leq |A^{1/2}u_0|, \qquad \forall\, t \geq 0, \tag{3.1}$$

and

$$|u'(t)| \leq \frac{|A^{1/2}u_0|}{\sqrt{\delta t}} \qquad \forall\, t > 0. \tag{3.2}$$

We omit the classical proofs.

## 3.2 Proof of Theorem 2.2

Before we enter into the technical details, we introduce some notations. Following Lions, for every $\varepsilon > 0$ we define the corrector $\theta_\varepsilon$ as the solution of $\varepsilon\theta_\varepsilon''(t) + \delta\theta_\varepsilon'(t) = 0$, $\theta_\varepsilon(0) = 0$, $\theta_\varepsilon'(0) = w_1$, so that

$$\theta_\varepsilon(t) = w_1\frac{\varepsilon}{\delta}(1 - e^{-\delta t/\varepsilon}). \tag{3.3}$$

Moreover, we define $r_\varepsilon$ in such a way that $u_\varepsilon(t) = u(t) + \theta_\varepsilon(t) + r_\varepsilon(t)$, $\forall\, t \geq 0$.

With simple calculations, it turns out that $r_\varepsilon$ is the solution of the Cauchy problem

$$\begin{cases} \varepsilon r_\varepsilon''(t) + \delta r_\varepsilon'(t) + Ar_\varepsilon(t) = -\varepsilon u''(t) - A\theta_\varepsilon(t), \\ r_\varepsilon(0) = 0, \quad r_\varepsilon'(0) = 0. \end{cases} \tag{3.4}$$

With these notations, inequalities (2.4), (2.5), and (2.6) are equivalent to proving that there exists a constant $C$ such that

$$|A^{1/2}r_\varepsilon(t)|^2 \leq C\varepsilon^2 \qquad \forall\, t \geq 0, \tag{3.5}$$

$$\delta\int_0^{+\infty} |r_\varepsilon'(t)|^2 dt \leq C\varepsilon^2, \tag{3.6}$$

$$|r_\varepsilon'(t)|^2 \leq C\varepsilon \qquad \forall\, t \geq 0. \tag{3.7}$$

We also set $R_\varepsilon(t) = r_\varepsilon(t) + \theta_\varepsilon(t)$, $\forall\, t \geq 0$, so that $R_\varepsilon$ is the solution of the Cauchy problem

$$\begin{cases} \varepsilon R_\varepsilon''(t) + \delta R_\varepsilon'(t) + AR_\varepsilon(t) = -\varepsilon u''(t), \\ R_\varepsilon(0) = 0, \quad R_\varepsilon'(0) = w_1. \end{cases} \tag{3.8}$$

We conclude this preliminary part of the proof with a remark on the regularity required on the initial data. In the statement of Theorem 2.2, $(u_0, u_1)$ are assumed to be in $D(A^{3/2}) \times D(A^{1/2})$, but in this proof we always work with $A\theta_\varepsilon$, which by (17) depends on $Aw_1$, and therefore can be defined as an element of $H$ only if $(u_0, u_1) \in D(A^2) \times D(A)$. However, the results we prove hold true for $(u_0, u_1) \in D(A^{3/2}) \times D(A^{1/2})$, and this can be rigorously justified in at least two different ways:

- Working in $D(A^2) \times D(A)$ and then, since all the constants depend at most on $|A^{3/2}u_0|$ and $|A^{1/2}w_1|$, passing to the limit;
- Working in $D(A^{3/2}) \times D(A^{1/2})$, and thinking of all the equations involving $A\theta_\varepsilon$ as equalities in $D(A^{-1/2})$ and not in $H$.

### 3.2.1 Estimate on $u''$

We prove that

$$\int_0^{+\infty} |u''(t)|^2 dt \leq \frac{1}{2\delta^3}|A^{3/2}u_0|^2. \tag{3.9}$$

Indeed, taking the scalar product of $2u''(t)$ with the derivative of the equation in (1.2), we obtain that $2\delta|u''(t)|^2 + (|A^{1/2}u'(t)|^2)' = 0$, so that, integrating in $[0, T]$, we find that

$$2\delta \int_0^T |u''(t)|^2 dt + |A^{1/2}u'(T)|^2 = |A^{1/2}u'(0)|^2 = \frac{1}{\delta^2}|A^{3/2}u_0|^2.$$

Taking the limit as $T \to +\infty$, inequality (3.9) is proved.

### 3.2.2 Estimate on $R_\varepsilon$

We prove that there exists a constant $D_1$ such that

$$|A^{1/2}R_\varepsilon(t)|^2 \leq D_1\varepsilon, \qquad \forall\, t \geq 0. \tag{3.10}$$

To this end, we introduce the function $\mathcal{E}(t) := \varepsilon\,|R'_\varepsilon(t)|^2 + |A^{1/2}R_\varepsilon(t)|^2$.

Using (3.8) it follows that

$$\mathcal{E}'(t) = -2\delta\,|R'_\varepsilon(t)|^2 - 2\varepsilon\langle u''(t), R'_\varepsilon(t)\rangle \leq -2\delta\,|R'_\varepsilon(t)|^2 + 2\delta\,|R'_\varepsilon(t)|^2 + \frac{\varepsilon^2}{2\delta}|u''(t)|^2,$$

so that, integrating in $[0, T]$ and using (3.9), we find that

$$\mathcal{E}(T) \leq \mathcal{E}(0) + \frac{\varepsilon^2}{2\delta}\int_0^T |u''(t)|^2 dt \leq \varepsilon|w_1|^2 + \frac{\varepsilon^2}{2\delta}\frac{1}{2\delta^3}|A^{3/2}u_0|^2 \leq \varepsilon\left\{|w_1|^2 + \frac{1}{4\delta^4}|A^{3/2}u_0|^2\right\} =: D_1\varepsilon.$$

By the definition of $\mathcal{E}(t)$, inequality (3.10) is proved.

### 3.2.3 First Estimate on $r_\varepsilon$

There exists a constant $D_2$ such that

$$|A^{1/2}r_\varepsilon(t)|^2 \leq D_2\varepsilon, \qquad \forall\, t \geq 0. \tag{3.11}$$

Indeed, by (3.3) and (3.10) we have that

$$|A^{1/2}r_\varepsilon(t)|^2 = |A^{1/2}(R_\varepsilon(t) - \theta_\varepsilon(t))|^2 \leq 2|A^{1/2}R_\varepsilon(t)|^2 + 2|A^{1/2}\theta_\varepsilon(t)|^2 \leq \varepsilon\left\{2D_1 + \frac{2}{\delta^2}|A^{1/2}w_1|^2\right\} =: D_2\varepsilon.$$

### 3.2.4 Bootstrap Argument

Let us assume that there exist constants $\alpha \in [1, 2]$ and $D > 0$ such that

$$|A^{1/2}r_\varepsilon(t)|^2 \leq D\varepsilon^\alpha, \qquad \forall\ t \geq 0. \tag{3.12}$$

Then, setting $\overline{D} = \frac{1}{2\delta^4}|A^{3/2}u_0|^2 + \frac{4}{\delta}|A^{1/2}w_1|\sqrt{D}$, we have that

$$|A^{1/2}r_\varepsilon(t)|^2 \leq \overline{D}\varepsilon^{\alpha/2+1} \qquad \forall\, t \geq 0, \tag{3.13}$$

$$\delta \int_0^{+\infty} |r'_\varepsilon(t)|^2 dt \leq \overline{D}\varepsilon^{\alpha/2+1}, \tag{3.14}$$

$$|r'_\varepsilon(t)|^2 \leq \overline{D}\varepsilon^{\alpha/2} \qquad \forall\, t \geq 0. \tag{3.15}$$

In order to prove these estimates, let us consider the function $E(t) := \varepsilon\,|r'_\varepsilon(t)|^2 + |A^{1/2}r_\varepsilon(t)|^2$.

Using (3.4) it follows that

$$\begin{aligned} E'(t) &= -2\delta\,|r'_\varepsilon(t)|^2 - 2\varepsilon\langle u''(t), r'_\varepsilon(t)\rangle - 2\langle A\theta_\varepsilon(t), r'_\varepsilon(t)\rangle \\ &\leq -2\delta\,|r'_\varepsilon(t)|^2 + \delta\,|r'_\varepsilon(t)|^2 + \frac{\varepsilon^2}{\delta}|u''(t)|^2 - 2\langle A\theta_\varepsilon(t), r'_\varepsilon(t)\rangle. \end{aligned}$$

Since $E(0)=0$, integrating in $[0,T]$ we find that

$$E(T)+\delta\int_0^T |r_\varepsilon'(t)|^2\,dt \le \frac{\varepsilon^2}{\delta}\int_0^T |u''(t)|^2dt - 2\int_0^T \langle A\theta_\varepsilon(t), r_\varepsilon'(t)\rangle dt. \tag{3.16}$$

The first integral on the right-hand side can be estimated by (3.9). In order to estimate the second one, we use integration by parts and assumption (3.12).

$$\begin{aligned}\left|\int_0^T \langle A\theta_\varepsilon(t), r_\varepsilon'(t)\rangle dt\right| &= \left|\langle A\theta_\varepsilon(T), r_\varepsilon(T)\rangle - \langle A\theta_\varepsilon(0), r_\varepsilon(0)\rangle - \int_0^T \langle A\theta_\varepsilon'(t), r_\varepsilon(t)\rangle dt\right| \\ &\le |\langle A^{1/2}\theta_\varepsilon(T), A^{1/2}r_\varepsilon(T)\rangle| + \int_0^T |A^{1/2}\theta_\varepsilon'(t)|\cdot|A^{1/2}r_\varepsilon(t)|dt \\ &\le |A^{1/2}\theta_\varepsilon(T)|\cdot|A^{1/2}r_\varepsilon(T)| + \sqrt{D}\varepsilon^{\alpha/2}|A^{1/2}w_1|\int_0^{+\infty} e^{-\delta t/\varepsilon}dt. \\ &\le 2|A^{1/2}w_1|\frac{\sqrt{D}}{\delta}\varepsilon^{\alpha/2+1},\end{aligned}$$

where we also used that $r_\varepsilon(0)=0$, and the explicit formula for $\theta_\varepsilon$ found in (3.3).

Coming back to (3.16) we have that

$$\begin{aligned}E(T)+\delta\int_0^T |r_\varepsilon'(t)|^2dt &\le \frac{\varepsilon^2}{2\delta^4}|A^{3/2}u_0|^2 + \frac{4\sqrt{D}}{\delta}|A^{1/2}w_1|\varepsilon^{\alpha/2+1} \\ &\le \varepsilon^{\alpha/2+1}\left\{\frac{\left|A^{3/2}u_0\right|^2}{2\delta^4} + \frac{4\sqrt{D}}{\delta}|A^{1/2}w_1|\right\} = \overline{D}\varepsilon^{\alpha/2+1}.\end{aligned}$$

By the definition of $E(t)$, estimates (3.13), (3.14), and (3.15) are proved.

#### 3.2.5 Inductive Argument

Let us define the sequence $\{\alpha_n\}$ in the following way: $\alpha_0=1,\ \alpha_{n+1}=\frac{\alpha_n}{2}+1$; and the sequence $\{C_n\}$ in the following way: $C_0=D_2,\ C_{n+1}=\frac{\left|A^{3/2}u_0\right|^2}{2\delta^4}+\frac{4}{\delta}|A^{1/2}w_1|\sqrt{C_n}$, where $D_2$ is the constant introduced in (3.11).

Then for every $n\in\mathbb{N}$ we have that

$$|A^{1/2}r_\varepsilon(t)|^2 \le C_n\varepsilon^{\alpha_n}, \qquad \forall\, t\ge 0. \tag{3.17}$$

Indeed the case $n=0$ is exactly (3.11), while the inductive step follows from the bootstrap argument.

#### 3.2.6 Passing to the Limit

From elementary calculus arguments, we have that $\alpha_n\to 2,\ C_n\to C_\infty$, where $C_\infty$ is the unique real number such that

$$C_\infty = \frac{\left|A^{3/2}u_0\right|^2}{2\delta^4} + \frac{4}{\delta}|A^{1/2}w_1|\sqrt{C_\infty}.$$

Passing to the limit as $n\to+\infty$ in (3.17), we therefore obtain that $|A^{1/2}r_\varepsilon(t)|^2\le C_\infty\varepsilon^2$, $\forall\, t\ge 0$.

This proves estimate (3.5). Using the bootstrap argument with $\alpha=2$ and $D=C_\infty$, (3.6) and (3.7) are also proved. This completes the proof of Theorem 2.2.

### 3.3 Proof of Theorem 2.4

**Lemma 3.1** *Let $v_1\in H$, and let $v$ be the solution of the Cauchy problem*

$$\begin{cases} \varepsilon v''(t)+\delta v'(t)+Av(t)=0, \\ v(0)=0, \quad v'(0)=v_1. \end{cases} \tag{3.18}$$

*Then*

$$t|A^{1/2}v(t)|^2 \le \frac{\varepsilon^2}{\delta}|v_1|^2, \qquad \forall\, t\ge 0. \tag{3.19}$$

*Proof* Let us consider the function $F(t) = \left(\frac{\varepsilon}{\delta} + t\right)\left(\varepsilon|v'(t)|^2 + |A^{1/2}v(t)|^2\right)$.

Using (3.18) we have that

$$\begin{aligned} F'(t) &= \left(\frac{\varepsilon}{\delta} + t\right)\left(-2\delta|v'(t)|^2\right) + \varepsilon\left|v'(t)\right|^2 + |A^{1/2}v(t)|^2 \\ &= -\varepsilon\left|v'(t)\right|^2 - 2\delta t|v'(t)|^2 + |A^{1/2}v(t)|^2. \end{aligned} \tag{3.20}$$

In order to compute the last summand, we take the scalar product of the equation in (3.18) by $v(t)$. We obtain that

$$\varepsilon\langle v''(t), v(t)\rangle + \delta\langle v'(t), v(t)\rangle + \langle Av(t), v(t)\rangle = 0,$$

so that

$$\left(\varepsilon\langle v'(t), v(t)\rangle + \frac{\delta}{2}|v(t)|^2\right)' - \varepsilon|v'(t)|^2 + |A^{1/2}v(t)|^2 = 0.$$

Coming back to (3.20), we have that

$$\begin{aligned} F'(t) &= -\varepsilon\left|v'(t)\right|^2 - 2\delta t|v'(t)|^2 + \varepsilon\left|v'(t)\right|^2 - \left(\varepsilon\langle v'(t), v(t)\rangle + \frac{\delta}{2}|v(t)|^2\right)' \\ &\leq -\left(\varepsilon\langle v'(t), v(t)\rangle + \frac{\delta}{2}|v(t)|^2\right)'. \end{aligned}$$

Integrating in $[0, T]$, and exploiting that $v(0) = 0$, we obtain that

$$F(T) - F(0) \leq -\varepsilon\langle v'(T), v(T)\rangle - \frac{\delta}{2}|v(T)|^2 \leq \frac{\delta}{2}|v(T)|^2 + \frac{1}{2\delta}\varepsilon^2|v'(T)|^2 - \frac{\delta}{2}|v(T)|^2,$$

and therefore, recalling the definition of $F$,

$$\left(\frac{\varepsilon}{2\delta} + T\right)\varepsilon\left|v'(T)\right|^2 + \left(\frac{\varepsilon}{\delta} + T\right)|A^{1/2}v(T)|^2 \leq \frac{\varepsilon^2}{\delta}|v_1|^2.$$

In conclusion, for every $T \geq 0$, we have that

$$T|A^{1/2}v(T)|^2 \leq \left(\frac{\varepsilon}{\delta} + T\right)|A^{1/2}v(T)|^2 \leq \frac{\varepsilon^2}{\delta}|v_1|^2,$$

and this proves (3.19).

In order to prove Theorem 2.4, let us write $u_\varepsilon(t) = u_{1\varepsilon}(t) + u_{2\varepsilon}(t)$, where $u_{1\varepsilon}$ and $u_{2\varepsilon}(t)$ are the solutions of the same equation, with the initial data, respectively, $u_{1\varepsilon}(0) = 0,\ u'_{1\varepsilon}(0) = u_1$, and $u_{2\varepsilon}(0) = u_0,\ u'_{2\varepsilon}(0) = 0$.

The standard energy inequality applied to $u_{1\varepsilon}$ gives that

$$\varepsilon\left|u'_{1\varepsilon}(t)\right|^2 + |A^{1/2}u_{1\varepsilon}(t)|^2 \leq \varepsilon\left|u'_{1\varepsilon}(0)\right|^2 + |A^{1/2}u_{1\varepsilon}(0)|^2 = \varepsilon|u_1|^2,$$

and therefore

$$|u'_{1\varepsilon}(t)| \leq |u_1|, \qquad \forall\, t \geq 0. \tag{3.21}$$

On the other hand, setting $v(t) = A^{-1/2}u'_{2\varepsilon}(t)$, we have that $v(t)$ satisfies the assumptions of Lemma 3.1 with

$$v_1 = v'(0) = A^{-1/2}u''_{2\varepsilon}(0) = -\frac{1}{\varepsilon}A^{1/2}u_{2\varepsilon}(0) = -\frac{1}{\varepsilon}A^{1/2}u_0.$$

By (3.19) it follows therefore that

$$t\left|u'_{2\varepsilon}(t)\right|^2 = t|A^{1/2}v(t)|^2 \leq \frac{\varepsilon^2}{\delta}|v_1|^2 = \frac{1}{\delta}|A^{1/2}u_0|^2, \qquad \forall\, t \geq 0. \tag{3.22}$$

In conclusion, by (3.21) and (3.22) we have that

$$|u'_\varepsilon(t)| \leq |u'_{1\varepsilon}(t)| + |u'_{2\varepsilon}(t)| \leq |u_1| + \frac{\left|A^{1/2}u_0\right|}{\sqrt{\delta t}},$$

for every $t > 0$, and this completes the proof of Theorem 2.4.

### 3.4 Proof of Theorem 2.1

In this section, we extend to the energy space $D(A^{1/2}) \times H$ the convergence result proved in Theorem 2.2 for the initial data in $D(A^{3/2}) \times D(A^{1/2})$. The fundamental tool is an approximation technique based on the following well-known result (the completely standard proof is omitted).

**Lemma 3.2** *Let $X$ be a Banach space, and let $f_n, f :]0,1] \to X$ be functions such that*
(i) $\{f_n\} \to f$ *uniformly in* $]0,1]$;
(ii) *for every $n \in \mathbb{N}$ there exists (in $X$) $x_n := \lim_{\varepsilon\to 0^+} f_n(\varepsilon)$.*
*Then the sequence $\{x_n\}$ tends to a limit $x_\infty$ in $X$, and $\lim_{\varepsilon\to 0^+} f(\varepsilon) = x_\infty$.*

Now we are ready to prove Theorem 2.1 for the initial data $(u_0, u_1) \in D(A^{1/2}) \times H$. Let $\{(u_{0n}, u_{1n})\} \subseteq D(A^{3/2}) \times D(A^{1/2})$ be a sequence converging to $(u_0, u_1)$ in $D(A^{1/2}) \times H$, and let $u_{\varepsilon n}$, $u_n$ be the corresponding solutions of (1.1) and (1.2). Let us consider the Banach space

$$X := L^\infty([0,+\infty[; D(A^{1/2})),$$

let $x_n = u_n \in X$, $x_\infty = u \in X$, and let $f_n, f :]0,1] \to X$ be defined in the following way: $f_n(\varepsilon) = u_{\varepsilon n}$, $f(\varepsilon) = u_\varepsilon$, $\forall\, \varepsilon \in ]0,1]$.

With these notations, statement (2.1) is equivalent to showing that $\lim_{\varepsilon\to 0^+} f(\varepsilon) = x_\infty$ in $X$. We prove this convergence using Lemma 3.2. In order to verify assumption (i), we recall that $u_{\varepsilon n} - u_\varepsilon$ is a solution of the equation in (1.1), hence by the standard energy estimate,

$$\begin{aligned} \|f_n(\varepsilon) - f(\varepsilon)\|_X &= \sup\{|A^{1/2}(u_{\varepsilon n}(t) - u_\varepsilon(t))| :\ t \geq 0\} \\ &\leq \{\varepsilon|u_{1n} - u_1|^2 + |A^{1/2}(u_{0n} - u_0)|^2\}^{1/2}. \end{aligned}$$

This proves that $f_n$ converges to $f$ uniformly in $\varepsilon \in ]0,1]$. Assumption (ii) follows immediately from Theorem 2.2. Finally, $x_n$ converges to $x_\infty$ just because $u_n - u$ is a solution of the equation in (1.2), hence, by (3.1),

$$\|x_n - x_\infty\|_X = \sup\{|A^{1/2}(u_n(t) - u(t))| :\ t \geq 0\} \leq |A^{1/2}(u_{0n} - u_0)|.$$

The proof of statement (2.2) is analogous, with the only difference that now the Banach space is $X = L^2([0,+\infty[; H)$.

Now let us prove statement (2.3). To this end, we fix $B > 0$, and we consider the Banach space $X_B := L^\infty([B,+\infty[; H)$, the functions $f_n, f :]0,1] \to X_B$ defined by $f_n(\varepsilon) = u'_{\varepsilon n}$, $f(\varepsilon) = u'_\varepsilon$, $\forall\, \varepsilon \in ]0,1]$, and the elements $x_n = u'_n \in X_B$ and $x_\infty = u' \in X_B$.

Now we use Lemma 3.2 once more. In order to verify assumption (i) we apply Theorem 2.4 to $u_{\varepsilon n} - u_\varepsilon$. We obtain that

$$\|f_n(\varepsilon) - f(\varepsilon)\|_{X_B} = \sup\{|u'_{\varepsilon n}(t) - u'_\varepsilon(t)| :\ t \geq B\} \leq |u_{1n} - u_1| + \frac{\left|A^{1/2}(u_{0n} - u_0)\right|}{\sqrt{\delta B}}.$$

This proves that $f_n$ converges to $f$ uniformly in $\varepsilon \in ]0,1]$. Assumption (ii) follows immediately from Theorem 2.2. Finally, $x_n$ converges to $x_\infty$ just because $u_n - u$ is a solution of the equation in (1.2), hence, by (3.2),

$$\|x_n - x_\infty\|_{X_B} = \sup\{|u'_n(t) - u'(t)| :\ t \geq B\} \leq \frac{\left|A^{1/2}(u_{0n} - u_0)\right|}{\sqrt{\delta B}}.$$

This completes the proof of Theorem 2.1.

### 3.5 Proof of Theorem 2.3

In this proof, we use the Fourier series to reduce problems (1.1) and (1.2) to ODEs. For this reason, we begin with an ODE estimate which will be applied to Fourier components.

**Lemma 3.3** *Let $\varepsilon, \delta, \lambda > 0$, let $u_0 \in \mathbb{R}$, let $u_\varepsilon$ be the solution of the ODE*

$$\varepsilon u''_\varepsilon(t) + \delta u'_\varepsilon(t) + \lambda u_\varepsilon(t) = 0, \qquad u_\varepsilon(0) = u_0, \quad u'_\varepsilon(0) = 0,$$

*and let $u$ be the solution of the limit problem $\delta u'(t) + \lambda u(t) = 0$, $u(0) = u_0$.*

*Then, for every $T > 0$, we have that*

$$\liminf_{\varepsilon\to 0^+} \frac{1}{\varepsilon}\|u_\varepsilon - u\|_{L^\infty([0,T];\mathbb{R})} \geq \frac{\lambda|u_0|}{\delta^2}. \tag{3.23}$$

*Proof* In order to prove this inequality, we remark that, for $\varepsilon$ small enough,

$$\|u_\varepsilon - u\|_{L^\infty([0,T];\mathbb{R})} \geq |u_\varepsilon(-\varepsilon\log\varepsilon) - u(-\varepsilon\log\varepsilon)|,$$

and then we compute the right-hand side using the explicit expressions for $u_\varepsilon$ and $u$. In such a way, passing to the limit becomes a (long but standard) calculus exercise.

Let us consider the operator $Au = -u_{xx}$ (with zero Dirichlet boundary conditions) in $H = L^2(]0, 2\pi[)$. Let $\{e_i\}$ be an orthonormal system in $H$ made by eigenvectors of $A$, so that $Ae_i = i^2 e_i$. Let $a_i$ be the component of $u_0$ with respect to $e_i$, so that

$$|A^\alpha u_0|^2 = \sum_{i=1}^{\infty} i^{4\alpha} a_i^2, \tag{3.24}$$

for every $\alpha \geq 0$.

Denoting by $u_{i\varepsilon}(t)$ and $u_i(t)$ the components of $u_\varepsilon(t)$ and $u(t)$ with respect to $e_i$, then for every $n \in \mathbb{N}$, we have that

$$|A^{1/2}(u_\varepsilon(t) - u(t))|^2 = \sum_{i=1}^{\infty} i^2 \left(u_{i\varepsilon}(t) - u_i(t)\right)^2 \geq n^2 \left(u_{n\varepsilon}(t) - u_n(t)\right)^2,$$

so that, applying Lemma 3.3 with $\lambda = n^2$, we have that

$$\liminf_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \|u_\varepsilon - u\|_{L^\infty([0,T];D(A^{1/2}))} \geq n \cdot \liminf_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \|u_{n\varepsilon} - u_n\|_{L^\infty([0,T];\mathbb{R})} \geq \frac{n^3 |a_n|}{\delta^2},$$

for every $T > 0$ and every $n \in \mathbb{N}$. It follows that

$$\liminf_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \|u_\varepsilon - u\|_{L^\infty([0,T];D(A^{1/2}))} \geq \frac{1}{\delta^2} \cdot \sup\left\{n^3 |a_n| : \ n \in \mathbb{N}\right\}. \tag{3.25}$$

Therefore Theorem 2.3 is proved provided we find a sequence $\{a_i\}$ such that

(i) The supremum on the right-hand side of (3.25) is $+\infty$;

(ii) The series in (3.24) converges for every $\alpha < 3/2$.

Let us set

$$a_i := \begin{cases} i^{-3} 2^{\sqrt{\log_2 i}}, & \text{if } i = 2^{k^2} \text{ for some } k \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\sup\left\{n^3 |a_n| : \ n \in \mathbb{N}\right\} = \sup\left\{2^{\sqrt{\log_2 i}} : \ i = 2^{k^2} \text{ for some } k \in \mathbb{N}\right\} = +\infty,$$

which proves (i). On the other hand, $\sum_{i=1}^{\infty} i^{4\alpha} a_i^2 = \sum_{k=0}^{\infty} 2^{k^2(4\alpha - 6)} \cdot 2^{2k}$, which converges for every $\alpha < 3/2$. This proves (ii), and completes the proof of Theorem 2.3.

A similar argument works with initial data $(0, u_1)$.

### 3.6 The Non-coercive Case

When dealing with the non-coercive case, the first step is to derive an estimate for $|u_\varepsilon(t)|$ in terms of the initial data. To this end, multiplying by $2u_\varepsilon(t)$ the equation in (1.1), we obtain that

$$2\varepsilon \langle u_\varepsilon''(t), u_\varepsilon(t)\rangle + 2\delta \langle u_\varepsilon'(t), u_\varepsilon(t)\rangle + 2\langle Au_\varepsilon(t), u_\varepsilon(t)\rangle = 0,$$

so that

$$(2\varepsilon \langle u_\varepsilon'(t), u_\varepsilon(t)\rangle + \delta |u_\varepsilon(t)|^2)' - 2\varepsilon |u_\varepsilon'(t)|^2 + 2|A^{1/2} u_\varepsilon(t)|^2 = 0.$$

Integrating in $[0, T]$, and using the standard energy estimate, we find that

$$\begin{aligned} \delta |u_\varepsilon(T)|^2 &\leq 2\varepsilon \langle u_1, u_0\rangle + \delta |u_0|^2 - 2\varepsilon \langle u_\varepsilon'(T), u_\varepsilon(T)\rangle + 2\varepsilon \int_0^T |u_\varepsilon'(t)|^2 \, dt \\ &\leq 2\varepsilon \langle u_1, u_0\rangle + \delta |u_0|^2 + \frac{\delta}{2} |u_\varepsilon(T)|^2 + \frac{2\varepsilon^2}{\delta} |u_\varepsilon'(T)|^2 + \frac{\varepsilon}{\delta}(\varepsilon |u_1|^2 + |A^{1/2} u_0|^2). \end{aligned}$$

Using the energy estimate once more, after some computations we obtain that, for every $t \geq 0$,

$$\frac{\delta}{2} |u_\varepsilon(t)|^2 \leq 2\delta |u_0|^2 + \frac{4\varepsilon}{\delta}(\varepsilon |u_1|^2 + |A^{1/2} u_0|^2). \tag{3.26}$$

The second step is to decompose $H$. For a fixed $\sigma > 0$, by means of spectral decomposition we can write $H$ as a direct sum of 3 closed $A$-invariant subspaces $H = H_1 \oplus H_2 \oplus H_3$, where $H_1$ is the kernel of $A$, the restriction of $A$ to $H_2$ is a bounded operator with norm $\leq \sigma$, and the restriction of $A$ to $H_3$ is a coercive operator (with $\nu \geq \sigma$). Moreover, the $H_2$-projection of every element of $H$ tends to zero as $\sigma \to 0^+$.

In this way, $u_\varepsilon$ and $u$ can also be written as a sum of their 3 components, each one satisfying the same equation with the corresponding initial data. Moreover, by (3.26) the $H_2$-component of $u_\varepsilon(t)$ tends to zero as $\sigma \to 0^+$, uniformly in $t$. The same holds true for the $H_2$-component of $u(t)$ and $\theta_\varepsilon(t)$.

Now let us begin with Theorem 2.2. In order to complete the proof of Section 3.2 in the non-coercive case, it is enough to show that there exists a constant $C$ such that

$$|r_\varepsilon(t)|^2 \leq C\varepsilon^2, \qquad \forall\, t \geq 0. \tag{3.27}$$

This estimate can be proved on remarking that:

• It is true for the $H_1$-component of $r_\varepsilon$ (in this case $u_\varepsilon$ and $u$ can be explicitly computed by solving ODEs);

• It is true for the $H_3$-component of $r_\varepsilon$, with a constant $C$ independent on $\sigma$: indeed on $H_3$ we can apply estimate (3.5) to the function $A^{-1/2}u_\varepsilon$;

• The $H_2$-component of $r_\varepsilon$ tends to zero as $\sigma \to 0^+$, because the same holds true for the corresponding component of $u_\varepsilon$, $u$ and $\theta_\varepsilon$.

In the proof of Theorem 2.4, we used coerciveness only to prove (3.22), where we estimated $u'_{2\varepsilon}(t)$. In the non-coercive case we can prove the same on remarking that:

• Estimate (3.22) is trivial for the $H_1$-component of $u'_{2\varepsilon}$, which is constantly zero;

• Estimate (3.22) is true for the $H_3$-component of $u'_{2\varepsilon}$, since on $H_3$ the operator $A$ is coercive;

• The $H_2$-component of $u'_{2\varepsilon}$ tends to zero as $\sigma \to 0^+$ (just use the standard energy estimate).

It remains to complete the proof of Theorem 2.1. To this end, we have to show only that $u_\varepsilon$ uniformly converges to $u$. This can be done with the same technique of Section 3.4, using the Banach space $X := L^\infty([0, +\infty[; H)$, and estimates (3.26) and (3.27).

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