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A uniqueness result for Kirchhoff equations with non-Lipschitz nonlinear term

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Abstract

We consider the second order Cauchy problem

$$u'' + m\left(\left|A^{1/2}u\right|^2\right)Au = 0, \qquad u(0) = u_0, \qquad u'(0) = u_1,$$

where $m : [0, +\infty) \to [0, +\infty)$ is a continuous function, and A is a self-adjoint nonnegative operator with dense domain on a Hilbert space.

It is well known that this problem admits local-in-time solutions provided that u_0 and u_1 are regular enough, depending on the continuity modulus of m. It is also well known that the solution is unique when m is locally Lipschitz continuous.

In this paper we prove that if either $\langle Au_0, u_1\rangle \neq 0$, or $|A^{1/2}u_1|^2 \neq m(|A^{1/2}u_0|^2)|Au_0|^2$, then the local solution is unique even if m is not Lipschitz continuous.

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1. Introduction

Let H be a real Hilbert space. For every x and y in H, $|x|$ denotes the norm of x, and $\langle x, y\rangle$ denotes the scalar product of x and y. Let A be a self-adjoint linear operator on H with dense domain $D(A)$. We assume that A is nonnegative, namely $\langle Ax, x\rangle \geqslant 0$ for every $x \in D(A)$, so that for every $\alpha \geqslant 0$ the power $A^\alpha x$ is defined provided that x lies in a suitable domain $D(A^\alpha)$.

Given a continuous function $m : [0, +\infty) \to [0, +\infty)$ we consider the Cauchy problem

$$u''(t) + m\big(\big|A^{1/2}u(t)\big|^2\big)Au(t) = 0, \quad \forall t \in [0, T), \tag{1.1}$$

$$u(0) = u_0, \qquad u'(0) = u_1. \tag{1.2}$$

It is well known that (1.1), (1.2) is the abstract setting of the Cauchy-boundary value problem for the quasilinear hyperbolic integro-differential partial differential equation

$$u_{tt}(t, x) - m\bigg(\int_\Omega \big|\nabla u(t, x)\big|^2\, dx\bigg)\Delta u(t, x) = 0 \quad \forall (x, t) \in \Omega \times [0, T), \tag{1.3}$$

where $\Omega \subseteq \mathbb{R}^n$ is an open set, and ∇u and Δu denote the gradient and the Laplacian of u with respect to the space variables.

From the mathematical point of view, (1.3) is probably the simplest example of quasilinear hyperbolic equation. From the mechanical point of view, this Cauchy boundary value problem is a model for the small transversal vibrations of an elastic string ($n = 1$) or membrane ($n = 2$). In this context it was introduced by G. Kirchhoff in [11].

Eq. (1.1) or (1.3) are called *strictly hyperbolic* when

$$\mu := \inf_{\sigma \geqslant 0} m(\sigma) > 0,$$

and *weakly* (or *degenerate*) *hyperbolic* when $\mu = 0$.

This equation has generated a considerable literature. In this paper we focus on the uniqueness problem for (local) solutions.

1.1. Local existence results

A local-in-time solution to (1.1), (1.2) is known to exist provided that the initial data u_0 and u_1 are regular enough. As in the linear case the required regularity depends on the continuity modulus ω of m, and on the strict or weak hyperbolicity of Eq. (1.1).

In a few words, the weaker the continuity condition on m, the stronger the regularity condition on the initial data. A rough sketch of the situation for the strictly hyperbolic case is provided by the following scheme:

$$\begin{aligned}
\omega(\sigma) = o(1) \quad &\to \quad \text{analytic data,}\\
\omega(\sigma) = \sigma^\alpha\ \big(\text{with } \alpha \in (0,1)\big) \quad &\to \quad \text{Gevrey space } \mathcal{G}_s(A) \text{ with } s = (1-\alpha)^{-1},\\
\omega(\sigma) = \sigma|\log\sigma| \quad &\to \quad D\big(A^\infty\big) \text{ (finite derivative loss),}\\
\omega(\sigma) = \sigma \quad &\to \quad D\big(A^{3/4}\big) \times D\big(A^{1/4}\big) \text{ (no derivative loss).}
\end{aligned}$$

More regularity is required in the weakly hyperbolic case, according to the following scheme:

$$\begin{aligned} \omega(\sigma) = o(1) \quad &\rightarrow \quad \text{analytic data,} \\ \omega(\sigma) = \sigma^{\alpha} \ \big(\text{with } \alpha \in (0,1)\big) \quad &\rightarrow \quad \text{Gevrey space } \mathcal{G}_s(A) \text{ with } s = 1 + \alpha/2, \\ \omega(\sigma) = \sigma \quad &\rightarrow \quad \text{Gevrey space } \mathcal{G}_{3/2}(A). \end{aligned}$$

For the more regular nonlinearity (strictly hyperbolic case, locally Lipschitz continuous m) the more complete local existence result was obtained by A. Arosio and S. Panizzi in [1], where they prove that the Cauchy problem is locally well posed in $D(A^{3/4}) \times D(A^{1/4})$. For the less regular nonlinearity (weakly hyperbolic case, m continuous) existence of at least one local (and actually global) solution was proved by A. Arosio and S. Spagnolo [2] with a technical assumption on m, which was afterwards removed by P. D'Ancona and S. Spagnolo [5,6]. The local existence results under intermediate continuity conditions have been obtained by F. Hirosawa [10] for the concrete equation in $\Omega = \mathbb{R}^n$, and then extended by the authors [8] to the abstract setting. In [8] it is also proved that the given relations between the regularity of m and the regularity of the initial data are sharp.

We refer to Section 2 for a formal statement (Theorem A), and for precise definitions of the functional spaces in the abstract setting.

1.2. Uniqueness results: the Lipschitz case

It is well known that uniqueness holds whenever m is Lipschitz continuous. In the strictly hyperbolic case this result is proved for example in [1], of course with initial data in $D(A^{3/4}) \times D(A^{1/4})$. In the weakly hyperbolic case uniqueness was proved in [2] for analytic initial data. Now we know that in this case we have local existence for initial data in the Gevrey class $\mathcal{G}_{3/2}(A)$, and the uniqueness proof can be easily extended to this larger class. The main argument is indeed always the same, namely a Gronwall type lemma for the difference between two solutions.

The same argument works also in the first author's paper [7]. That paper considers equation (1.1) with the non-Lipschitz nonlinearity $m(\sigma) = \sigma^{\gamma}$ (with $\gamma \in (0,1)$), and an additional dissipative term. The main result is the existence of a unique global solution provided that initial data are small enough and satisfy the mild non-degeneracy assumption $|A^{1/2}u_0| \neq 0$. The key step for the global existence result is showing that $|A^{1/2}u(t)| \neq 0$ for every $t \geqslant 0$. At this point however uniqueness follows from free because the nonlinearity is locally Lipschitz continuous for $\sigma > 0$.

1.3. Uniqueness results: the non-Lipschitz case

As a general fact, uniqueness for a nonlinear evolution equation in much more difficult to establish if the nonlinear term is not locally Lipschitz continuous. Therefore it is hardly surprising that also in the case of Kirchhoff equations this problem seems to be widely unexplored. To our knowledge indeed uniqueness issues have been considered only in Section 4 of [2], where two results are presented.

The first one is a one-dimensional example where problem (1.1), (1.2) admits infinitely many local solutions. The second result is a detailed study of the case where u_0 and u_1 are *eigenvectors* of A relative to the *same eigenvalue*. In this special two-dimensional case the authors proved that uniqueness of the local solution fails if and only if the following three conditions are satisfied:

(AS1) $\langle Au_0, u_1\rangle = 0$;
(AS2) $|A^{1/2}u_1|^2 - m(|A^{1/2}u_0|^2)|Au_0|^2 = 0$;
(AS3) m satisfies a suitable integrability condition in a neighborhood of $|A^{1/2}u_0|^2$.

As a consequence the local solution is unique if at least one of the conditions above is not satisfied.

1.4. Our main result

In this paper we extend the first two parts of this result from the two-dimensional case with equal eigenvalues to the infinite-dimensional case with arbitrary eigenvalues. In Theorem 2.1 we prove indeed that if either condition (AS1) or condition (AS2) is not satisfied, then even in the general case the local solution is always unique.

The proof of this result relies on two main steps.

The first step is what we call *trajectory uniqueness*. We prove indeed that the image of the curve $(u(t), u'(t))$ in the phase space is unique. To this end we parametrize the curve using the variable $s = |A^{1/2}u(t)|^2$ instead of the variable t. In this new variable the trajectory is the image of a curve $(z(s), w(s))$, where $z(s)$ and $w(s)$ are the solutions of a system in which the non-Lipschitz nonlinear term $m(|A^{1/2}u(t)|^2)$ has become a non-Lipschitz coefficient $m(s)$, which doesn't affect uniqueness.

The second step is what we call *parametrization uniqueness*. We prove indeed that the unique trajectory obtained in the first step can be covered by solutions in a unique way. To this end we first show that the parametrization $s(t) = |A^{1/2}u(t)|^2$ satisfies a first order autonomous ordinary differential equation with non-Lipschitz right-hand side. Such an equation may of course have infinitely many solutions with the same initial condition, but it is well known (see Lemma 3.4) that only one solution has the property that $s(t) \neq s(0)$ for $t > 0$. This is the point where the quite mysterious conditions (AS1) and (AS2) play their role. They are indeed equivalent to $s'(0) = 0$ and $s''(0) = 0$. If at least one of them is false, then clearly $s(t) \neq s(0)$ in a right-hand neighborhood of $t = 0$.

We didn't find this approach in the literature. We hope it could be useful to handle also different evolution equations with non-Lipschitz terms.

This paper is organized as follows. In Section 2 we recall the definition of continuity modulus and Gevrey-type functional spaces. Moreover we state the classical local existence result for (1.1), (1.2) (Theorem A) and our uniqueness result (Theorem 2.1). In Section 3 we prove Theorem 2.1. In Section 4 we collect some open problems concerning uniqueness of solutions.

2. Preliminaries and statements

For the sake of simplicity we assume that H admits a countable complete orthonormal system $\{e_k\}_{k\geqslant 1}$ made by eigenvectors of A. We denote the corresponding eigenvalues by λ_k^2 (with $\lambda_k \geqslant 0$), so that $Ae_k = \lambda_k^2 e_k$ for every $k \geqslant 1$. By means of the orthonormal system every $u \in H$ can be written in a unique way in the form $u = \sum_{k=1}^{\infty} u_k e_k$, where $u_k = \langle u, e_k\rangle$ are the components of u. In other words, every $u \in H$ can be identified with the set $\{u_k\}$ of its components, and under this identification the operator A acts component-wise by multiplication.

This simplifying assumption might look restrictive, but it is not. Indeed the spectral theorem for self-adjoint (unbounded) operators on a separable Hilbert space says that any such operator is unitary equivalent to a multiplication operator on some L^2 space.

More precisely, given H and A there exist a measure space (M,μ), a function $a(\xi)\in L^2(M,\mu)$, and a unitary operator which associates to every $u\in H$ a function $f(\xi)\in L^2(M,\mu)$ in such a way that Au corresponds to the product $a(\xi)f(\xi)$. The interested reader is referred to [12, Chapter VIII] for the general theory, and to [9] for an application in a Kirchhoff context.

Therefore every definition, statement or proof given in this paper can be extended to the general case by replacing the sequence of components of u with the function $f(\xi)$ corresponding to u, the sequence of eigenvalues with the function $a(\xi)$, and summations over k with integrals over M with respect to ξ.

Let us define the functional spaces we are interested in. First of all for every $\alpha\geqslant 0$ we have that

$$D\big(A^\alpha\big):=\left\{u\in H\colon \sum_{k=1}^{\infty}\lambda_k^{4\alpha}u_k^2<+\infty\right\}.$$

Let now $\varphi:[0,+\infty)\to[1,+\infty)$ be any function. Then for every $\alpha\geqslant 0$ and $r>0$ one can set

$$\|u\|_{\varphi,r,\alpha}^2:=\sum_{k=1}^{\infty}\lambda_k^{4\alpha}u_k^2\exp\big(r\varphi(\lambda_k)\big), \tag{2.1}$$

and then define the spaces

$$\mathcal{G}_{\varphi,r,\alpha}(A):=\big\{u\in H\colon \|u\|_{\varphi,r,\alpha}<+\infty\big\}.$$

These spaces are a generalization of the usual spaces of Sobolev, Gevrey or analytic functions. They are Hilbert spaces with norm $(|u|^2+\|u\|_{\varphi,r,\alpha}^2)^{1/2}$.

A *continuity modulus* is a continuous increasing function $\omega:[0,+\infty)\to[0,+\infty)$ such that $\omega(0)=0$, and $\omega(a+b)\leqslant\omega(a)+\omega(b)$ for every $a\geqslant 0$ and $b\geqslant 0$.

The function m is said to be ω-continuous if there exists a constant $L\in\mathbb{R}$ such that

$$\big|m(a)-m(b)\big|\leqslant L\omega\big(|a-b|\big)\quad\forall a\geqslant 0,\ \forall b\geqslant 0. \tag{2.2}$$

The following result sums up the state of the art concerning existence of local solutions. For a proof we refer to Appendix A in [8] (see also Theorem 2.1 and Theorem 2.2 in [10]).

Theorem A. *Let H be a separable Hilbert space, and let A be a nonnegative self-adjoint (unbounded) operator on H with dense domain. Let ω be a continuity modulus, let $m:[0,+\infty)\to[0,+\infty)$ be an ω-continuous function, and let $\varphi:[0,+\infty)\to[1,+\infty)$.*

Let us assume that there exists a constant Λ such that

$$\sigma\omega\left(\frac{1}{\sigma}\right)\leqslant\Lambda\varphi(\sigma)\quad\forall\sigma>0, \tag{2.3}$$

in the strictly hyperbolic case, and

$$\sigma \leqslant \Lambda\varphi\left(\frac{\sigma}{\sqrt{\omega(1/\sigma)}}\right) \quad \forall\sigma > 0, \tag{2.4}$$

in the weakly hyperbolic case.

Let us assume that

$$(u_0, u_1) \in \mathcal{G}_{\varphi,r_0,3/4}(A) \times \mathcal{G}_{\varphi,r_0,1/4}(A) \tag{2.5}$$

for some $r_0 > 0$.

Then there exist $T > 0$, *and* $R > 0$ *with* $RT < r_0$ *such that problem* (1.1), (1.2) *admits at least one local solution*

$$u \in C^1\big([0,T]; \mathcal{G}_{\varphi,r_0-Rt,1/4}(A)\big) \cap C^0\big([0,T]; \mathcal{G}_{\varphi,r_0-Rt,3/4}(A)\big). \tag{2.6}$$

The main result of this paper is the following uniqueness result for these solutions.

Theorem 2.1. *Let* H, A, ω, m, φ *be as in Theorem* A. *Let us assume that*

$$(u_0, u_1) \in \mathcal{G}_{\varphi,r_0,3/2}(A) \times \mathcal{G}_{\varphi,r_0,1}(A) \tag{2.7}$$

for some $r_0 > 0$, *and*

$$\big|\langle Au_0, u_1\rangle\big| + \big||A^{1/2}u_1|^2 - m\big(|A^{1/2}u_0|^2\big)|Au_0|^2\big| \neq 0. \tag{2.8}$$

Let us assume that problem (1.1), (1.2) *admits two local solutions* v_1 *and* v_2 *in*

$$C^2\big([0,T]; \mathcal{G}_{\varphi,r_1,1/2}(A)\big) \cap C^1\big([0,T]; \mathcal{G}_{\varphi,r_1,1}(A)\big) \cap C^0\big([0,T]; \mathcal{G}_{\varphi,r_1,3/2}(A)\big) \tag{2.9}$$

for some $T > 0$, *and some* $r_1 \in (0, r_0)$.

Then we have the following conclusions.

(1) *There exists* $T_1 \in (0, T]$ *such that*

$$v_1(t) = v_2(t) \quad \forall t \in [0, T_1]. \tag{2.10}$$

(2) *Let* T_* *denote the supremum of all* $T_1 \in (0, T]$ *for which* (2.10) *holds true. Let* $v(t)$ *denote the common value of* v_1 *and* v_2 *in* $[0, T_*]$.
Then either $T_* = T$ *or*

$$\big|\langle Av(T_*), v'(T_*)\rangle\big| + \big||A^{1/2}v'(T_*)|^2 - m\big(|A^{1/2}v(T_*)|^2\big)|Av(T_*)|^2\big| = 0. \tag{2.11}$$

Example 2.2. Admittedly assumptions (2.3) and (2.4) do not lend themselves to a simple interpretation. Let us give some examples in the simplest concrete situation where $H = L^2((0, 2\pi))$ and $Au = -u_{xx}$ with homogeneous boundary conditions, so that $\lambda_k = k$. In all the examples below, what is relevant is the behavior of $\omega(\sigma)$ for small values of σ, and the behavior of $\varphi(\sigma)$ for large values of σ.

- Assumptions (2.3) and (2.4) are satisfied if $\varphi(\sigma) = \sigma$ and $\omega(\sigma) = o(1)$ as $\sigma \to 0^+$. In this case (2.2) means that m is just continuous, and (2.5) means that u_0 and u_1 are analytic. This is the situation considered in [2].
- Assumption (2.3) is satisfied when $\omega(\sigma) = \sigma^\alpha$, and $\varphi(\sigma) = \sigma^{1-\alpha}$ for some $\alpha \in (0, 1)$. In this case (2.2) means that m is α-Hölder continuous, while (2.5) means that u_0 and u_1 are in the Gevrey space $\mathcal{G}_s$ with $s = (1-\alpha)^{-1}$.
- Assumption (2.4) is satisfied when $\omega(\sigma) = \sigma^\alpha$, and $\varphi(\sigma) = \sigma^{2/(\alpha+2)}$ for some $\alpha \in (0, 1]$. Once again (2.2) means that m is α-Hölder continuous (Lipschitz continuous when $\alpha = 1$), while (2.5) means that u_0 and u_1 are in the Gevrey space $\mathcal{G}_s$ with $s = 1 + \alpha/2$.
- Assumption (2.3) is satisfied when $\omega(\sigma) = \sigma|\log\sigma|$ (which means that m is log-Lipschitz continuous), and $\varphi(\sigma) = \log\sigma$. In this case $\mathcal{G}_{\varphi,r,\beta}(A) = D(A^{\beta+r/4}) = H_0^{2\beta+r/2}((0, 2\pi))$, so that (2.5) means Sobolev type regularity. Moreover (2.6) says that r decreases during the evolution, hence the solution may exhibit a progressive derivative loss.
- Finally assumption (2.3) is satisfied when $\omega(\sigma) = \sigma$ (which means that m is Lipschitz continuous), and $\varphi(\sigma) \equiv 1$. This gives the usual local existence result in $D(A^{3/4}) \times D(A^{1/4})$.

Remark 2.3. The space (2.9) is the natural one when initial data satisfy (2.7). Indeed from the linear theory it follows that any solution $u(t)$ of (1.1) with

$$u \in C^0\big([0, T]; D\big(A^{3/4}\big)\big) \cap C^1\big([0, T]; D\big(A^{1/4}\big)\big)$$

and initial data as in (2.7) lies actually in (2.9).

Remark 2.4. Assumption (2.7) on the initial data is stronger than the corresponding assumption in Theorem A. This is due to a technical point in the proof.

However in most cases the difference is only apparent. For example if $\omega(\sigma) = \sigma^\beta$ for some $\beta \in (0, 1]$, then the following implication

$$u \in \mathcal{G}_{\varphi,r,0}(A) \quad \Longrightarrow \quad u \in \mathcal{G}_{\varphi,r-\varepsilon,\alpha}(A)$$

holds true for every $r > 0$, $\varepsilon \in (0, r)$, $\alpha \geqslant 0$. Therefore in this case every solution satisfying (2.6) fulfills (2.9) with $r_1 = r(T)/2$.

3. Proofs

3.1. Technical lemmata

Lemma 3.1. *Let $\omega : [0, +\infty) \to [0, +\infty)$ be a continuity modulus.*

Then

$$\omega(x) \geqslant \omega(1) \cdot \frac{x}{x+1} \quad \forall x \geqslant 0. \tag{3.1}$$

Proof. Inequality (3.1) is trivial for $x = 0$. From the subadditivity of ω it follows that $\omega(\lambda x) \leqslant (\lambda + 1)\omega(x)$ for every $\lambda \geqslant 0$ and $x \geqslant 0$ (this can be easily proved by induction on the integer part of λ). Applying this inequality with $x > 0$ and $\lambda = 1/x$ we obtain (3.1) for $x > 0$. □

Lemma 3.2. *For $i = 1, 2$ let $\eta_i : (0, T] \to [0, +\infty)$ be a continuous function with finite integral. Let $y \in C^0([0, T]; \mathbb{R}) \cap C^1((0, T]; \mathbb{R})$ be a function such that $y(0) = 0$, and*

$$y'(t) \leqslant \eta_1(t)y(t) + \eta_2(t) \quad \forall t \in (0, T]. \tag{3.2}$$

Then

$$y(t) \leqslant \exp\left(\int_0^t \eta_1(\tau)\,d\tau\right) \cdot \int_0^t \eta_2(\tau)\,d\tau \quad \forall t \in [0, T]. \tag{3.3}$$

Proof. Let us consider the ordinary differential equation

$$v'(t) = \eta_1(t)v(t) + \eta_2(t). \tag{3.4}$$

Assumption (3.2) is equivalent to say that y is a subsolution of (3.4). Since $\eta_1(t)$ and $\eta_2(t)$ are nonnegative it is easy to verify that the right-hand side of (3.3) is a supersolution of (3.4). Therefore estimate (3.3) follows from the standard comparison principle. $\square$

Lemma 3.3. *Let $y : [0, T] \to [0, +\infty)$ be a continuous function. Let us assume that there exists $k \geqslant 0$ such that*

$$y(t) \leqslant k \int_0^t \frac{1}{s\sqrt{s}} \int_0^s y(\sigma)\,d\sigma\,ds.$$

Then $y(t) = 0$ for every $t \in [0, T]$.

Proof. Let us set $M := \max\{y(t):\ t \in [0, T]\}$. Then an easy induction gives

$$y(t) \leqslant \frac{4^n k^n M}{n!} t^{n/2} \quad \forall t \in [0, T],\ \forall n \in \mathbb{N},$$

which implies the conclusion. $\square$

Lemma 3.4. *Let $s_0 > 0$, let $g : [0, s_0] \to \mathbb{R}$ be a continuous function, and let $T > 0$.*

Then there exists at most one function $y : [0, T] \to [0, s_0]$ of class C^1 such that

$$y(0) = 0, \tag{3.5}$$

$$y'(t) > 0 \quad \forall t \in (0, T], \tag{3.6}$$

$$y'(t) = g\big(y(t)\big) \quad \forall t \in (0, T]. \tag{3.7}$$

Proof. Let $y_1(t)$ and $y_2(t)$ be two solutions of (3.5), (3.6), (3.7). Let $s_1 := y_1(T)$, $s_2 := y_2(T)$. By (3.6) the functions $y_1 : [0, T] \to [0, s_1]$ and $y_2 : [0, T] \to [0, s_2]$ are strictly increasing and invertible. Their inverse functions $z_1(s)$ and $z_2(s)$ are defined and continuous in $[0, s_3]$, where $s_3 := \min\{s_1, s_2\} > 0$.

Moreover z_1 and z_2 are of class C^1 in $(0, s_3]$, and by (3.7)

$$z_1'(s) - z_2'(s) = \frac{1}{y_1'(z_1(s))} - \frac{1}{y_2'(z_2(s))} = \frac{1}{g(s)} - \frac{1}{g(s)} = 0 \qquad \forall s \in (0, s_3].$$

Since by (3.5) we have that $z_1(0) = z_2(0) = 0$, it follows that $z_1(s) = z_2(s)$ for every $s \in (0, s_3]$, and in particular $s_1 = s_2 = y_1(T) = y_2(T)$.

Therefore also the inverse functions of z_1 and z_2, namely y_1 and y_2, coincide. □

3.2. *A variable change*

Let $u(t)$ be any solution of (1.1) defined in an interval $[0, T]$. Let us assume that u belongs to the space (2.9), and its initial data (1.2) satisfy (2.8). Let us set

$$\psi(t) := \left|A^{1/2}u(t)\right|^2 - \left|A^{1/2}u_0\right|^2. \tag{3.8}$$

Then $\psi \in C^2([0, T])$, and

$$\psi(0) = 0, \qquad \psi'(0) = 2\langle Au_0, u_1\rangle, \qquad \psi''(0) = 2\left(\left|A^{1/2}u_1\right|^2 - m\left(\left|A^{1/2}u_0\right|^2\right)|Au_0|^2\right).$$

Our assumption (2.8) is equivalent to say that either $\psi'(0) \neq 0$ or $\psi''(0) \neq 0$. In both cases we can conclude that there exists $T_0 \in (0, T]$ such that $\psi'(t)$ has constant sign in the interval $(0, T_0]$.

Let us assume, without loss of generality, that $\psi'(t) > 0$ in $(0, T_0]$. Setting $s_0 = \psi(T_0)$, this implies that $\psi : [0, T_0] \to [0, s_0]$ is strictly increasing and invertible. Its inverse function $\psi^{-1} : [0, s_0] \to [0, T_0]$ belongs to $C^0([0, s_0]) \cap C^2((0, s_0])$, and

$$\left(\psi^{-1}\right)'(s) = \frac{1}{\psi'(\psi^{-1}(s))} = \frac{1}{2\langle Au(\psi^{-1}(s)), u'(\psi^{-1}(s))\rangle} > 0 \quad \forall s \in (0, s_0]. \tag{3.9}$$

Let us set now

$$z(s) := A^{1/2}u\left(\psi^{-1}(s)\right), \qquad w(s) := u'\left(\psi^{-1}(s)\right). \tag{3.10}$$

From the regularity of u and ψ^{-1} it follows that $z(s)$ and $w(s)$ belong to

$$C^0\left([0, s_0], \mathcal{G}_{\varphi, r_1, 1}\right) \cap C^1\left((0, s_0], \mathcal{G}_{\varphi, r_1, 1/2}\right) \tag{3.11}$$

for some $r_1 > 0$. Moreover they satisfy the initial conditions

$$z(0) = A^{1/2}u_0, \qquad w(0) = u_1. \tag{3.12}$$

The derivatives of $z(s)$ and $w(s)$ with respect to the variable s can be easily computed using (1.1) and (3.9). For every $s \in (0, s_0]$ it turns out that

$$z'(s) = \frac{A^{1/2}w(s)}{2\langle A^{1/2}z(s), w(s)\rangle}, \tag{3.13}$$

$$w'(s) = -c(s)\frac{A^{1/2}z(s)}{2\langle A^{1/2}z(s), w(s)\rangle}, \tag{3.14}$$

where $c(s) := m(s + |A^{1/2}u_0|^2)$.

This system is singular when denominators vanish for $s=0$, i.e., when $\langle Au_0, u_1\rangle = 0$. However we claim that there exists $s_1 \in (0, s_0]$ such that (γ_1 is the first of a long list of constants)

$$\langle A^{1/2}z(s), w(s)\rangle \geqslant \gamma_1\sqrt{s} \quad \forall s \in (0, s_1]. \tag{3.15}$$

To this end we first remark that

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\langle A^{1/2}z, w\rangle^2\right) = \left|A^{1/2}w(s)\right|^2 - c(s)\left|A^{1/2}z(s)\right|^2, \tag{3.16}$$

hence (we recall that ψ' is assumed to be positive)

$$\langle A^{1/2}z(s), w(s)\rangle = \left[\langle Au_0, u_1\rangle^2 + \int_0^s \left(\left|A^{1/2}w(\sigma)\right|^2 - c(\sigma)\left|A^{1/2}z(\sigma)\right|^2\right) d\sigma\right]^{1/2}. \tag{3.17}$$

If $\langle Au_0, u_1\rangle > 0$, then (3.15) is trivial provided that s_1 is small enough. If $\langle Au_0, u_1\rangle = 0$, then assumption (2.8) implies that $|A^{1/2}u_1|^2 - m(|A^{1/2}u_0|^2)|Au_0|^2 > 0$, hence the right-hand side of (3.16) is larger than a positive constant in a right neighborhood of 0, so that (3.15) follows from (3.17).

3.3. *Trajectory uniqueness*

Let $v_1(t)$ and $v_2(t)$ be two solutions of (1.1), (1.2). Let us define $\psi_1(t)$ and $\psi_2(t)$ according to (3.8), and then $(z_1(s), w_1(s))$ and $(z_2(s), w_2(s))$ according to (3.10). Let $s_1 > 0$ be small enough so that $z_1(s)$, $z_2(s)$, $w_1(s)$, $w_2(s)$ are defined in $[0, s_1]$, and in this interval they are as regular as prescribed by (3.11), and they satisfy system (3.13), (3.14), and estimate (3.15).

We claim that $z_1(s) = z_2(s)$ and $w_1(s) = w_2(s)$ in $[0, s_2]$ for a suitable $s_2 \in (0, s_1]$. To this end we introduce the differences

$$x(s) := z_1(s) - z_2(s), \qquad y(s) := w_1(s) - w_2(s). \tag{3.18}$$

Setting for simplicity

$$d_1(s) := 2\langle A^{1/2}z_1(s), w_1(s)\rangle, \qquad d_2(s) := 2\langle A^{1/2}z_2(s), w_2(s)\rangle,$$

it is easy to see that $x(s)$ and $y(s)$ are solutions in $(0, s_1]$ of the system

$$x'(s) = \frac{A^{1/2}y(s)}{d_1(s)} + \left(\frac{1}{d_1(s)} - \frac{1}{d_2(s)}\right)A^{1/2}w_2(s), \tag{3.19}$$

$$y'(s) = -c(s)\frac{A^{1/2}x(s)}{d_1(s)} - c(s)\left(\frac{1}{d_1(s)} - \frac{1}{d_2(s)}\right)A^{1/2}z_2(s), \tag{3.20}$$

with initial data $x(0) = y(0) = 0$.

Let us introduce the Fourier components $x_k(s)$, $y_k(s)$, $z_{i,k}(s)$, $w_{i,k}(s)$ of $x(s)$, $y(s)$, $z_i(s)$, $w_i(s)$ (with $i = 1, 2$). System (3.19), (3.20) becomes a system of infinitely many ordinary differential equations of the form

$$x_k'(s) = \frac{\lambda_k y_k(s)}{d_1(s)} + \lambda_k\left(\frac{1}{d_1(s)} - \frac{1}{d_2(s)}\right)w_{2,k}(s), \tag{3.21}$$

$$y_k'(s) = -c(s)\frac{\lambda_k x_k(s)}{d_1(s)} - c(s)\lambda_k\left(\frac{1}{d_1(s)} - \frac{1}{d_2(s)}\right)z_{2,k}(s), \tag{3.22}$$

all with initial data $x_k(0) = y_k(0) = 0$.

If $\lambda_k = 0$ it is clear that $x_k(s) = y_k(s) = 0$ in $[0, s_1]$. So let us concentrate on the components corresponding to positive eigenvalues. To this end we consider the *approximated energy estimates* introduced in [3] and [4], which are different in the strictly hyperbolic and in the weakly hyperbolic case.

3.3.1. The strictly hyperbolic case

Let us assume that

$$m(\sigma) \geqslant \gamma_2 > 0 \quad \forall \sigma \geqslant 0. \tag{3.23}$$

In particular the same estimate holds true for $c(s)$. Formally we need $c(s)$ to be defined only for $s \in [0, s_1]$. In order to make convolutions we extend $c(s)$ to the whole real line by setting $c(s) = c(0)$ for every $s \leqslant 0$, and $c(s) = c(s_1)$ for every $s \geqslant s_1$.

Let us fix once for all a function $\rho : \mathbb{R} \to [0, +\infty)$ of class C^∞, with compact support and integral equal to 1. For every $\varepsilon > 0$ let us set

$$c_\varepsilon(s) := \int_{\mathbb{R}} c(s + \varepsilon\sigma)\rho(\sigma)\, d\sigma.$$

From the boundedness and the ω-continuity of $c(s)$ it is easy to deduce that for every $s \in [0, s_1]$ (actually for every $s \in \mathbb{R}$) we have that (from now on all constants are independent on ε)

$$\left|c_\varepsilon(s) - c(s)\right| \leqslant \gamma_3\omega(\varepsilon), \tag{3.24}$$

$$\left|c_\varepsilon'(s)\right| \leqslant \gamma_4\frac{\omega(\varepsilon)}{\varepsilon}, \tag{3.25}$$

$$\gamma_2 \leqslant c_\varepsilon(s) \leqslant \gamma_5. \tag{3.26}$$

Let us consider the energy

$$E_{k,\varepsilon}(s) := |y_k|^2 + c_\varepsilon(s)\left|x_k(s)\right|^2. \tag{3.27}$$

From (3.21) and (3.22) we have that

$$\begin{aligned} E_{k,\varepsilon}'(s) &= c_\varepsilon'(s)|x_k|^2 + 2\bigl(c_\varepsilon(s) - c(s)\bigr)\frac{\lambda_k x_k y_k}{d_1(s)} \\ &\quad + 2\lambda_k\left(\frac{1}{d_1(s)} - \frac{1}{d_2(s)}\right)\bigl(c_\varepsilon(s)x_k w_{2,k} - c(s)y_k z_{2,k}\bigr) \\ &=: I_1(s) + I_2(s) + I_3(s). \end{aligned} \tag{3.28}$$

Let us estimate the three terms. By (3.25) and (3.26) we have that

$$I_1(s) \leqslant \frac{|c_\varepsilon'(s)|}{c_\varepsilon(s)} \cdot c_\varepsilon(s)\left|x_k(s)\right|^2 \leqslant \gamma_6 \frac{\omega(\varepsilon)}{\varepsilon} E_{\varepsilon,k}(s) \leqslant \gamma_7 \frac{\omega(\varepsilon)}{\varepsilon} \cdot \frac{E_{\varepsilon,k}(s)}{\sqrt{s}}. \tag{3.29}$$

By (3.15), (3.24), and (3.26) we have that

$$I_2(s) \leqslant 2\lambda_k \frac{|c_\varepsilon(s) - c(s)|}{d_1(s)\sqrt{c_\varepsilon(s)}} \cdot \left|y_k(s)\right| \cdot \sqrt{c_\varepsilon(s)}\left|x_k(s)\right| \leqslant \gamma_8 \lambda_k \frac{\omega(\varepsilon)}{\sqrt{s}} E_{k,\varepsilon}(s). \tag{3.30}$$

It remains to estimate $I_3(s)$. Since the norms $|A^{1/2} z_i(s)|$ and $|A^{1/2} w_i(s)|$ are bounded we have that

$$\left|\left|A^{1/2}z_1\right|^2 - \left|A^{1/2}z_2\right|^2\right| = \left|\left\langle A^{1/2}(z_1+z_2), A^{1/2}(z_1 - z_2)\right\rangle\right| \leqslant \gamma_9\left|A^{1/2}x\right|,$$
$$\left|\left|A^{1/2}w_1\right|^2 - \left|A^{1/2}w_2\right|^2\right| = \left|\left\langle A^{1/2}(w_1+w_2), A^{1/2}(w_1 - w_2)\right\rangle\right| \leqslant \gamma_{10}\left|A^{1/2}y\right|,$$

hence by (3.16) and the boundedness of $c(s)$

$$\left|\frac{\mathrm{d}}{\mathrm{d}s}\left(d_1^2(s) - d_2^2(s)\right)\right| \leqslant \gamma_{11}\left(\left|A^{1/2}x(s)\right| + \left|A^{1/2}y(s)\right|\right).$$

It follows that

$$\left|d_1^2(s) - d_2^2(s)\right| \leqslant \gamma_{11} \int_0^s \left(\left|A^{1/2}x(\sigma)\right| + \left|A^{1/2}y(\sigma)\right|\right) d\sigma =: \psi_{1,2}(s), \tag{3.31}$$

hence by (3.15)

$$\left|\frac{1}{d_1(s)} - \frac{1}{d_2(s)}\right| = \frac{|d_2^2(s) - d_1^2(s)|}{d_1(s)d_2(s)(d_1(s)+d_2(s))} \leqslant \gamma_{12}\frac{1}{s\sqrt{s}}\psi_{1,2}(s).$$

Since $c(s)$ and $c_\varepsilon(s)$ are bounded from above we have that

$$\left|c_\varepsilon(s)x_k w_{2,k} - c(s) y_k z_{2,k}\right| \leqslant \gamma_{13}\left(\sqrt{c_\varepsilon(s)}|x_k| \cdot |w_{2,k}| + |y_k| \cdot |z_{2,k}|\right),$$

hence

$$\begin{aligned}
I_3(s) &\leqslant \frac{\gamma_{14}}{\sqrt{s}}\left(\frac{\psi_{1,2}(s)}{s}\lambda_k |w_{2,k}| \cdot \left|\sqrt{c_\varepsilon(s)}x_k\right| + \frac{\psi_{1,2}(s)}{s}\lambda_k|z_{2,k}| \cdot |y_k|\right) \\
&\leqslant \frac{\gamma_{15}}{\sqrt{s}}\left(\frac{\psi_{1,2}^2(s)}{s^2}\lambda_k^2|w_{2,k}|^2 + c_\varepsilon(s)|x_k|^2 + \frac{\psi_{1,2}^2(s)}{s^2}\lambda_k^2|z_{2,k}|^2 + |y_k|^2\right) \\
&= \frac{\gamma_{15}}{\sqrt{s}}E_{k,\varepsilon} + \gamma_{15}\frac{\psi_{1,2}^2(s)}{s^2\sqrt{s}}\lambda_k^2\left(|w_{2,k}|^2 + |z_{2,k}|^2\right).
\end{aligned} \tag{3.32}$$

From (3.28), (3.29), (3.30), (3.32) we therefore obtain that

$$E'_{k,\varepsilon} \leqslant \gamma_{16}\left(\frac{\omega(\varepsilon)}{\varepsilon} + \lambda_k\omega(\varepsilon) + 1\right)\frac{E_{k,\varepsilon}}{\sqrt{s}} + \gamma_{15}\frac{\psi_{1,2}^2(s)}{s^2\sqrt{s}}\lambda_k^2\big(|w_{2,k}|^2 + |z_{2,k}|^2\big).$$

Let us set now $\varepsilon_k = \lambda_k^{-1}$ (we recall that we can limit ourselves to positive eigenvalues). By assumption (2.3) we have that

$$\frac{\omega(\varepsilon_k)}{\varepsilon_k} = \lambda_k\omega(\varepsilon_k) = \lambda_k\omega\left(\frac{1}{\lambda_k}\right) \leqslant \Lambda\varphi(\lambda_k),$$

hence

$$\begin{aligned} E'_{k,\varepsilon_k}(s) &\leqslant \gamma_{17}\frac{\varphi(\lambda_k)+1}{\sqrt{s}} \cdot E_{k,\varepsilon_k}(s) + \gamma_{15}\frac{\psi_{1,2}^2(s)}{s^2\sqrt{s}}\lambda_k^2\big(\big|w_{2,k}(s)\big|^2 + \big|z_{2,k}(s)\big|^2\big) \\ &=: \eta_1(s)E_{k,\varepsilon_k}(s) + \eta_2(s). \end{aligned}$$

The integral of $\eta_1(s)$ in $[0, s_1]$ is finite. Moreover from definition (3.31) of $\psi_{1,2}$ it is clear that $\psi_{1,2}(s) \leqslant \gamma_{18}s$. It follows that also the integral of $\eta_2(s)$ in $[0, s_1]$ is finite.

We can therefore apply Lemma 3.2. Since for every $s \in [0, s_1]$ we have that

$$\exp\left(\int_0^s \eta_1(\sigma)\,d\sigma\right) = \exp\big(2\gamma_{17}\varphi(\lambda_k)\sqrt{s} + 2\gamma_{17}\sqrt{s}\big) \leqslant \gamma_{19}\exp\big(\gamma_{20}\varphi(\lambda_k)\sqrt{s}\big),$$

it follows that

$$E_{k,\varepsilon_k}(s) \leqslant \gamma_{21}\exp\big(\gamma_{20}\varphi(\lambda_k)\sqrt{s}\big)\int_0^s \frac{\psi_{1,2}^2(\sigma)}{\sigma^2\sqrt{\sigma}}\lambda_k^2\big(\big|z_{2,k}(\sigma)\big|^2 + \big|w_{2,k}(\sigma)\big|^2\big)\,d\sigma. \tag{3.33}$$

Let us choose $s_2 \in (0, s_1]$ such that $\gamma_{20}\sqrt{s_2} \leqslant r_1$. By (3.26) and (3.33) we have that

$$\begin{aligned} \big|y_k(s)\big|^2 + \big|x_k(s)\big|^2 &\leqslant \max\left\{1, \frac{1}{\gamma_2}\right\}E_{k,\varepsilon_k}(s) \\ &\leqslant \gamma_{22}\int_0^s \frac{\psi_{1,2}^2(\sigma)}{\sigma^2\sqrt{\sigma}}\lambda_k^2\exp\big(r_1\varphi(\lambda_k)\big)\big(\big|z_{2,k}(\sigma)\big|^2 + \big|w_{2,k}(\sigma)\big|^2\big)\,d\sigma. \end{aligned}$$

Summing over k and recalling that z_2 and w_2 belong to the space (3.11) we find that

$$\begin{aligned} \big|A^{1/2}x(s)\big|^2 + \big|A^{1/2}y(s)\big|^2 &= \sum_{k=1}^{\infty}\lambda_k^2\big(\big|x_k(s)\big|^2 + \big|y_k(s)\big|^2\big) \\ &\leqslant \gamma_{22}\int_0^s \frac{\psi_{1,2}^2(\sigma)}{\sigma^2\sqrt{\sigma}}\big(\big\|z_2(\sigma)\big\|_{\varphi,r_1,1}^2 + \big\|w_2(\sigma)\big\|_{\varphi,r_1,1}^2\big)\,d\sigma \end{aligned}$$

$$\leqslant \gamma_{23} \int_0^s \frac{\psi_{1,2}^2(\sigma)}{\sigma^2\sqrt{\sigma}}\, d\sigma.$$

By definition (3.31) of $\psi_{1,2}$ and Hölder's inequality we obtain that

$$\begin{aligned} \left|A^{1/2}x(s)\right|^2 + \left|A^{1/2}y(s)\right|^2 &\leqslant \gamma_{23} \int_0^s \frac{1}{\sigma^2\sqrt{\sigma}} \left[\int_0^\sigma \left(\left|A^{1/2}x(\tau)\right| + \left|A^{1/2}y(\tau)\right|\right) d\tau \right]^2 d\sigma \\ &\leqslant 2\gamma_{23} \int_0^s \frac{1}{\sigma\sqrt{\sigma}} \int_0^\sigma \left(\left|A^{1/2}x(\tau)\right|^2 + \left|A^{1/2}y(\tau)\right|^2\right) d\tau\, d\sigma. \end{aligned}$$

Applying Lemma 3.3 we conclude that $|A^{1/2}x(s)|^2 = |A^{1/2}y(s)|^2 = 0$ for every $s \in [0, s_2]$, namely $z_1(s) = z_2(s)$ and $w_1(s) = w_2(s)$ in the same interval.

3.3.2. The weakly hyperbolic case

Let us modify $c(s)$ outside the interval $[0, s_1]$ as in the strictly hyperbolic case. Since $c(s)$ is bounded we can also assume that ω is bounded. For every $\varepsilon > 0$ let us set

$$c_\varepsilon(s) := \omega(\varepsilon) + \int_{\mathbb{R}} c(s + \varepsilon\sigma)\rho(\sigma)\, d\sigma.$$

Estimates (3.24) and (3.25) are still true, but (3.26) has to be replaced by the weaker (for the estimate from above we need the boundedness of ω)

$$\omega(\varepsilon) \leqslant c_\varepsilon(s) \leqslant \gamma_{24}. \tag{3.34}$$

Let us define $E_{k,\varepsilon}(s)$ according to (3.27). Its derivative is the same as in the strictly hyperbolic case. So we need to estimates the three summands in (3.28). Using (3.34) instead of (3.26) we find that

$$I_1(s) \leqslant \gamma_{25}\frac{1}{\varepsilon} \cdot \frac{E_{k,\varepsilon}(s)}{\sqrt{s}}, \qquad I_2(s) \leqslant \gamma_{26}\lambda_k\sqrt{\omega(\varepsilon)} \cdot \frac{E_{k,\varepsilon}(s)}{\sqrt{s}}.$$

The estimate on $I_3(s)$ is exactly the same as in (3.32). We finally obtain that

$$E'_{k,\varepsilon}(s) \leqslant \gamma_{27}\left(\frac{1}{\varepsilon} + \lambda_k\sqrt{\omega(\varepsilon)} + 1\right)\frac{E_{k,\varepsilon}(s)}{\sqrt{s}} + \gamma_{28}\frac{\psi_{1,2}^2(s)}{s^2\sqrt{s}}\lambda_k^2\left(\left|w_{2,k}(s)\right|^2 + \left|z_{2,k}(s)\right|^2\right).$$

Now we choose ε as a function of k. We consider the function $h(\varepsilon) := \varepsilon\sqrt{\omega(\varepsilon)}$, which is invertible, and we set $\varepsilon_k := h^{-1}(1/\lambda_k)$.

Applying assumption (2.4) with $\sigma = 1/\varepsilon_k$ we obtain that

$$\frac{1}{\varepsilon_k} \leqslant \Lambda\varphi\left(\frac{1}{\varepsilon_k\sqrt{\omega(\varepsilon_k)}}\right) = \Lambda\varphi\left(\frac{1}{h(\varepsilon_k)}\right) = \Lambda\varphi(\lambda_k), \tag{3.35}$$

and therefore

$$\frac{1}{\varepsilon_k} + \lambda_k\sqrt{\omega(\varepsilon_k)} = \frac{1 + h(\varepsilon_k)\lambda_k}{\varepsilon_k} = \frac{2}{\varepsilon_k} \leqslant 2\Lambda\varphi(\lambda_k) = \gamma_{29}\varphi(\lambda_k),$$

hence

$$E'_{k,\varepsilon_k}(s) \leqslant \gamma_{30}\frac{\varphi(\lambda_k)+1}{\sqrt{s}}E_{k,\varepsilon_k}(s) + \gamma_{28}\frac{\psi_{1,2}^2(s)}{s^2\sqrt{s}}\lambda_k^2\left(\left|w_{2,k}(s)\right|^2 + \left|z_{2,k}(s)\right|^2\right).$$

As in the strictly hyperbolic case we can apply Lemma 3.2 to this differential inequality and obtain that

$$E_{k,\varepsilon_k}(s) \leqslant \gamma_{31}\exp\left(\gamma_{32}\varphi(\lambda_k)\sqrt{s}\right)\int_0^s \frac{\psi_{1,2}^2(\sigma)}{\sigma^2\sqrt{\sigma}}\lambda_k^2\left(\left|z_{2,k}(\sigma)\right|^2 + \left|w_{2,k}(\sigma)\right|^2\right)d\sigma.$$

Let us choose $s_2 \in (0, s_1]$ such that $\gamma_{32}\sqrt{s_2} \leqslant r_1/2$.

Applying Lemma 3.1 and (3.35) we have that

$$\max\left\{1, \frac{1}{\omega(\varepsilon_k)}\right\} \leqslant 1 + \frac{1}{\omega(\varepsilon_k)} \leqslant \gamma_{33}\left(1 + \frac{1}{\varepsilon_k}\right) \leqslant \gamma_{34}\left(\varphi(\lambda_k) + 1\right) \leqslant \gamma_{35}\exp\left(r_1\varphi(\lambda_k)/2\right),$$

independently on k, hence

$$\begin{aligned}\left|y_k(s)\right|^2 + \left|x_k(s)\right|^2 &\leqslant \max\left\{1, \frac{1}{\omega(\varepsilon_k)}\right\}E_{k,\varepsilon_k}(s) \\ &\leqslant \gamma_{36}\int_0^s \frac{\psi_{1,2}^2(\sigma)}{\sigma^2\sqrt{\sigma}}\lambda_k^2\exp\left(r_1\varphi(\lambda_k)\right)\left(\left|z_{2,k}(\sigma)\right|^2 + \left|w_{2,k}(\sigma)\right|^2\right)d\sigma.\end{aligned}$$

From now on we proceed exactly as in the strictly hyperbolic case.

3.4. Parametrization uniqueness

Let us come back to the two solutions $v_1(t)$ and $v_2(t)$ of problem (1.1), (1.2). We already defined $\psi_1(t)$ and $\psi_2(t)$ according to (3.8), and then (z_1, w_1), and (z_2, w_2) according to (3.10). For $i = 1, 2$ we have that

$$\begin{aligned}\psi_i'(t) &= 2\left\langle Av_i(t), v_i'(t)\right\rangle = 2\left\langle Av_i\left(\psi_i^{-1}\left(\psi_i(t)\right)\right), v_i'\left(\psi_i^{-1}\left(\psi_i(t)\right)\right)\right\rangle \\ &= 2\left\langle A^{1/2}z_i\left(\psi_i(t)\right), w_i\left(\psi_i(t)\right)\right\rangle\end{aligned}$$

for every small enough t. Since $z_1(s) = z_2(s) =: z(s)$ and $w_1(s) = w_2(s) =: w(s)$ in an interval $[0, s_2]$, we have that in an interval $[0, T_1]$ the functions $\psi_1(t)$ and $\psi_2(t)$ are solutions of the Cauchy problem

$$\psi'(t) = 2\left\langle A^{1/2}z\left(\psi(t)\right), w\left(\psi(t)\right)\right\rangle =: g\left(\psi(t)\right), \qquad \psi(0) = \langle Au_0, u_1\rangle.$$

Since we already know that these solutions are strictly increasing in $[0, T_1]$ we can apply Lemma 3.4 and deduce that $\psi_1(t) = \psi_2(t)$ in $[0, T_1]$. Finally we have that

$$v_1'(t) = v_1'\big(\psi_1^{-1}\big(\psi_1(t)\big)\big) = w_1\big(\psi_1(t)\big) = w_2\big(\psi_2(t)\big) = v_2'\big(\psi_2^{-1}\big(\psi_2(t)\big)\big) = v_2'(t)$$

in $[0, T_1]$, hence also $v_1(t) = v_2(t)$ in the same interval.

3.5. Continuation

Let us prove the second statement of Theorem 2.1. The argument is quite standard. Let us assume by contradiction that two solutions $v_1(t)$ and $v_2(t)$ are defined in an interval $[0, T]$, and coincide in a maximal interval $[0, T_*]$ with $T_* < T$. If (2.11) is not satisfied, then we can apply the first statement with "initial" data in T_*, and deduce that v_1 and v_2 coincide in some interval $[T_*, T_* + \delta]$.

This contradicts the maximality of T_*.

4. Open problems

The uniqueness problem for Kirchhoff equations is quite open. In this section we state four questions in this field.

The first one concerns counterexamples. We don't know any example where uniqueness fails apart from those given in [2]. So we ask whether different counterexamples can be provided.

Open problem 4.1. Let ω, m, φ, u_0, u_1 be as in Theorem 2.1, but without assumption (2.8). Let us assume that problem (1.1), (1.2) admits two local solutions.

Can we conclude that u_0 and u_1 are eigenvectors of A relative to the same eigenvalue?

We point out that this problem is open even in the simple case $H = \mathbb{R}^2$, where ω and φ play non-role, and no regularity is required on initial data.

The second open problem concerns trajectory uniqueness (the key step in our proof).

Open problem 4.2. Let ω, m, φ, u_0, u_1 be as in Theorem 2.1, but without assumption (2.8). Let us consider system (3.13), (3.14), with initial data (3.12).

Does this system admit at most one solution?

Note that in the case where $\langle Au_0, u_1\rangle = 0$ it is by no means clear that a solution always exists, since this implicitly requires that $\langle A^{1/2}z(s), w(s)\rangle \neq 0$ for every $s \in (0, s_0]$. We point out that, even in the non-uniqueness examples of [2], the solution of this system exists and it is unique.

The third open problem concerns the regularity of *initial data*. It may happen indeed that problem (1.1), (1.2) has a solution even for some initial data that do not satisfy the assumptions of Theorem A (see for example the solutions with derivative loss constructed in [8]). Are there uniqueness results for these solutions?

Open problem 4.3. Is it possible to prove the known uniqueness results (namely the Lipschitz case and our Theorem 2.1) with less regularity requirements on initial data?

The last open problem concerns regularity of *solutions*. Both the result in the Lipschitz case, and our result require the a priori assumption that solutions lie in $D(A^{3/4}) \times D(A^{1/4})$ (see Remark 2.3). By the linear theory these solutions automatically belong to the same space (technically to the same scale of spaces) of the initial data. On the other hand, Eq. (1.1) makes perfectly sense in the energy space $D(A^{1/2}) \times H$. Just to give an extreme example, let us consider the strictly hyperbolic case, with a Lipschitz continuous nonlinearity m, and analytic initial data. We know that there is a unique solution in $D(A^{3/4}) \times D(A^{1/4})$, which is actually analytic. However as far as we know no one can exclude that there exists a different solution in $D(A^{1/2}) \times H$ with the same (analytic) initial data.

Open problem 4.4. Is it possible to extend the known uniqueness results to solutions in the energy space?

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