

Kirchhoff equations from quasi-analytic to spectral-gap data

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Abstract

In a celebrated paper (Tokyo J. Math. 1984) K. Nishihara proved global existence for Kirchhoff equations in a special class of initial data which lies in between analytic functions and Gevrey spaces. This class was defined in terms of Fourier components with weights satisfying suitable *convexity* and *integrability* conditions.

In this paper we extend this result by removing the convexity constraint, and by replacing Nishihara's integrability condition with the simpler integrability condition which appears in the usual characterization of quasi-analytic functions.

After the convexity assumptions have been removed, the resulting theory reveals unexpected connections with some recent global existence results for spectral-gap data.

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1 Introduction

Let H be a separable real Hilbert space. For every x and y in H , $|x|$ denotes the norm of x , and $\langle x, y \rangle$ denotes the scalar product of x and y . Let A be a self-adjoint linear operator on H with dense domain $D(A)$. We assume that A is nonnegative, namely $\langle Ax, x \rangle \geq 0$ for every $x \in D(A)$, so that the power $A^\alpha x$ is defined provided that $\alpha \geq 0$ and x lies in a suitable domain $D(A^\alpha)$.

Given $m : [0, +\infty) \rightarrow [0, +\infty)$, we consider the Cauchy problem

$$u''(t) + m(|A^{1/2}u(t)|^2)Au(t) = 0 \quad \forall t \in [0, T), \quad (1.1)$$

$$u(0) = u_0, \quad u'(0) = u_1. \quad (1.2)$$

It is well known that (1.1), (1.2) is the abstract setting of the Cauchy-boundary value problem for the quasilinear hyperbolic integro-differential partial differential equation

$$u_{tt}(x, t) - m\left(\int_{\Omega} |\nabla u(x, t)|^2 dx\right)\Delta u(x, t) = 0 \quad \forall (x, t) \in \Omega \times [0, T), \quad (1.3)$$

where $\Omega \subseteq \mathbb{R}^n$ is an open set, and ∇u and Δu denote the gradient and the Laplacian of u with respect to the space variables.

Throughout this paper we assume that equations (1.1) and (1.3) are strictly hyperbolic, namely

$$\mu := \inf_{\sigma \geq 0} m(\sigma) > 0. \quad (1.4)$$

We also assume that m is locally Lipschitz continuous. We never assume that the operator is coercive or that its inverse is compact.

We refer to the survey [9] and to the references quoted therein for more details on this equation and its history. Here we just recall that, under our assumptions on $m(\sigma)$, problem (1.1), (1.2) has a local solution for all initial data $(u_0, u_1) \in D(A^{3/4}) \times D(A^{1/4})$.

Existence of global solutions is for sure the main open problem in the theory of Kirchhoff equations. A positive answer has been given in five different special cases.

(GE 1) Special forms of the nonlinearity $m(\sigma)$ (see [16]).

(GE 2) Dispersive equations (see [5, 10], and the more recent papers [14, 18]).

(GE 3) Spectral-gap initial data (see [12, 13, 11, 7]).

(GE 4) Analytic initial data (see [2, 4]). In this case the result is actually stronger since it is enough to assume that $m(\sigma)$ is continuous and nonnegative.

(GE 5) ‘‘Quasi-analytic’’ initial data (see [15]).

We refer to the quoted papers for more details on each approach.

In this paper we pursue the path (GE 5), introduced by K. Nishihara in [15]. In that paper he proved global existence for initial data in suitable spaces, defined by imposing the convergence of some series where the Fourier components of data are multiplied by weights satisfying suitable *convexity* and *integrability* conditions (see Section 2.2 for the details).

As remarked in [15], the weights defining Gevrey spaces never fulfill these assumptions, but there are examples of weights satisfying these conditions for which the resulting space contains non-analytic functions. In other words, Nishihara’s spaces are expected to be something in between analytic functions and Gevrey spaces, and for this reason this result is often referred to as a global existence result for quasi-analytic data.

In this paper we prove a similar result *without the convexity assumption*, and we show that Nishihara’s quite strange integrability condition can be replaced with the more standard integrability condition which appears in the usual characterization of quasi-analytic functions.

From the technical point of view, the removal of the convexity condition requires a new proof of the key estimate (Proposition 3.3), which now can no more be established by means of Jensen type inequalities as in [15].

From the point of view of global existence results, apart from providing a cleaner statement for the beauty of the art, the removal of the convexity assumption has a somewhat unexpected impact. Using some weird weights (which of course do not satisfy the previous convexity assumptions) we can indeed exhibit examples of spaces containing functions with low Sobolev regularity where the Kirchhoff equation is well posed (see Theorem 4.2). This reminded us of the spectral-gap global solutions as defined in [12, 13] and then in [7]. The phenomenology is quite similar, but the context and the proof are completely different, and it doesn’t seem so easy to deduce exactly Theorem 4.2 from the known results on spectral-gap solutions.

Using the same weird weights, we can also show that there are special unbounded operators for which Kirchhoff equation is well posed in Sobolev-type spaces such as $D(A^{\alpha+1/2}) \times D(A^\alpha)$ with $\alpha > 1/4$ (see Theorem 4.1). This yields a new item for the list of global existence results:

(GE 6) Special operators.

More important, this adds a new difficulty in the search of a counterexample to the “big problem”, namely the global existence in C^∞ in the concrete case, or in $D(A^\infty)$ for the abstract equation. Now we know indeed that any such counterexample needs to exploit some property of the operator in order to rule out the special operators to which Theorem 4.1 applies.

This paper is organized as follows. In Section 2 we give rigorous definitions of the functional spaces we need, we recall Nishihara’s work, and we state our main result. In Section 3 we prove our main result. In Section 4 we explore the connections with spectral-gap solutions, and we exhibit some strange consequences of this theory.

2 Statements

2.1 Functional spaces

For the sake of simplicity we assume that H admits a countable complete orthonormal system $\{e_k\}$ made by eigenvectors of A . We denote the corresponding eigenvalues by λ_k^2 (with $\lambda_k \geq 0$), so that $Ae_k = \lambda_k^2 e_k$ for every $k \in \mathbb{N}$. Every $u \in H$ can be written in a unique way in the form $u = \sum_{k=0}^{\infty} u_k e_k$, where $u_k = \langle u, e_k \rangle$ are the components of u . In other words, every $u \in H$ can be identified with the sequence $\{u_k\}$ of its components, and under this identification the operator A acts component-wise by multiplication.

We stress that this simplifying assumption is by no means restrictive. Indeed the spectral theorem for self-adjoint unbounded operators on a separable Hilbert space (see [17, Chapter VIII]) states that any such operator is unitary equivalent to a multiplication operator on some L^2 space. More precisely, for every H and A there exist a measure space (M, μ) , a function $a(\xi) \in L^2(M, \mu)$, and a unitary operator $H \rightarrow L^2(M, \mu)$ which associates to every $u \in H$ a function $f(\xi) \in L^2(M, \mu)$ in such a way that Au corresponds to the product $a(\xi)f(\xi)$.

As a consequence, all the definitions we give in terms of u_k and λ_k can be extended to the general case by replacing the sequence of components $\{u_k\}$ of u with the function $f(\xi)$ corresponding to u , the sequence $\{\lambda_k\}$ of eigenvalues of A with the function $a(\xi)$, and summations over k with integrals over M in the variable ξ with respect to the measure μ . Similarly, there is no loss of generality in using components in the proof of the a priori estimate needed for our existence result (Theorem 2.1).

Coming back to functional spaces, using components we have that

$$D(A^\alpha) := \left\{ u \in H : \sum_{k=0}^{\infty} \lambda_k^{4\alpha} u_k^2 < +\infty \right\}.$$

Let now $\varphi : [0, +\infty) \rightarrow [0, +\infty)$ be any function. Then for every $\alpha \geq 0$ and $r > 0$ one can set

$$\|u\|_{\varphi, r, \alpha}^2 := \sum_{k=0}^{\infty} \lambda_k^{4\alpha} u_k^2 \exp(r\varphi(\lambda_k)),$$

and then define the *generalized Gevrey spaces* as

$$\mathcal{G}_{\varphi, r, \alpha}(A) := \{u \in H : \|u\|_{\varphi, r, \alpha} < +\infty\}.$$

These spaces can also be seen as the domain of the operator $A^\alpha \exp((r/2)\varphi(A^{1/2}))$. They are Hilbert spaces with norm $(|u|^2 + \|u\|_{\varphi, r, \alpha}^2)^{1/2}$, and they form a scale of Hilbert spaces with respect to the parameter r . They are a natural generalization of the usual spaces of Sobolev, Gevrey or analytic functions, corresponding to $\varphi(\sigma) = \log(1 + \sigma)$, $\varphi(\sigma) = \sigma^{1/s}$ ($s > 1$), and $\varphi(\sigma) = \sigma$, respectively. In [6] and [9] it is shown that these spaces represent the right setting for Kirchhoff equations.

Spaces of *quasi-analytic functions* fit in this framework. They correspond to weights $\varphi(\sigma)$ which are continuous, strictly increasing, and satisfy

$$\int_1^{+\infty} \frac{\varphi(\sigma)}{\sigma^2} d\sigma = +\infty. \quad (2.1)$$

To be overpedantic, also the continuity and *strict* monotonicity assumptions on φ are not really needed. Indeed, for every nondecreasing function satisfying (2.1), one can always find a smaller function which is continuous, strictly increasing, and still satisfies (2.1).

We refer to [3] for more details on quasi-analytic functions in the concrete case.

2.2 Nishihara's work

The following is the main result of [15], restated and somewhat simplified using the notations we have just introduced.

Theorem A *Let H be a separable Hilbert space, and let A be a nonnegative self-adjoint (unbounded) operator on H with dense domain. Let $m : [0, +\infty) \rightarrow (0, +\infty)$ be a locally Lipschitz continuous function satisfying the nondegeneracy condition (1.4).*

Let $\varphi : [0, +\infty) \rightarrow [0, +\infty)$ be a continuous and strictly increasing function such that, setting $M(\sigma) := e^{\varphi(\sigma)}$, we have that

(φ 1) *the function $\sigma \rightarrow M(\sqrt{\sigma})$ is convex,*

(φ 2) *if $M^{-1}(\sigma)$ denotes the inverse function of $M(\sigma)$, then we have that*

$$\int_1^{+\infty} \frac{1}{\sigma M^{-1}(\sigma)} d\sigma = +\infty.$$

Let us finally assume that

$$(u_0, u_1) \in \mathcal{G}_{\varphi,1,1/2}(A) \times \mathcal{G}_{\varphi,1,0}(A). \quad (2.2)$$

Then problem (1.1), (1.2) admits a unique global solution

$$u \in C^1([0, +\infty); \mathcal{G}_{\varphi,1,1/2}(A)) \cap C^0([0, +\infty); \mathcal{G}_{\varphi,1,0}(A)). \quad (2.3)$$

To be more precise, the original statement involved further assumptions on $M(0)$, and on the spectrum and the inverse of A , which however can be easily removed using arguments that nowadays are quite standard.

We point out that, in contrast with other results for Kirchhoff equations, this solution lies in a fixed Hilbert *space* instead of a Hilbert *scale* (namely in (2.3) the radius $r = 1$ is the same for all times).

We also remark that in general it is not possible to replace $r = 1$ in (2.2) with a smaller value of r . The point is that, when we replace $\varphi(\sigma)$ with $r\varphi(\sigma)$, there is no reason for the new function $M(\sigma)$ to satisfy $(\varphi 1)$.

Let us briefly comment conditions $(\varphi 1)$ and $(\varphi 2)$. It is easy to see that they are satisfied when $\varphi(\sigma) = \sigma$, namely by analytic functions. In this case Theorem A provides an alternative proof of the global existence result for analytic initial data under more restrictive assumptions on the nonlinearity $m(\sigma)$ (the classical result in the analytic case only requires $m(\sigma)$ to be continuous and nonnegative). More important, assumptions $(\varphi 1)$ and $(\varphi 2)$ are satisfied when $\varphi(\sigma) = \sigma/\log(1 + \sigma)$, in which case the corresponding space contains non-analytic functions. Finally, assumption $(\varphi 2)$ is not satisfied when $\varphi(\sigma) = \sigma^{1/s}$ with $s > 1$, which means that Gevrey spaces are never contained in Nishihara's spaces.

2.3 Our result

In this paper we extend Nishihara's result by replacing assumptions $(\varphi 1)$ and $(\varphi 2)$ of Theorem A with the unique assumption (2.1). Our main result is the following.

Theorem 2.1 *Let H be a separable Hilbert space, and let A be a nonnegative self-adjoint (unbounded) operator on H with dense domain. Let $m : [0, +\infty) \rightarrow (0, +\infty)$ be a locally Lipschitz continuous function satisfying the nondegeneracy condition (1.4).*

Let $\varphi : [0, +\infty) \rightarrow [0, +\infty)$ be a continuous and strictly increasing function satisfying (2.1). Let us finally assume that

$$(u_0, u_1) \in \mathcal{G}_{\varphi, r_0, 3/4}(A) \times \mathcal{G}_{\varphi, r_0, 1/4}(A) \quad (2.4)$$

for some $r_0 > 0$.

Then problem (1.1), (1.2) admits a unique global solution

$$u \in C^1([0, +\infty); \mathcal{G}_{\varphi, r_0, 3/4}(A)) \cap C^0([0, +\infty); \mathcal{G}_{\varphi, r_0, 1/4}(A)). \quad (2.5)$$

Let us comment our assumptions on the weight φ , on the initial data, and on the nonlinearity.

Remark 2.2 There do exist strictly increasing continuous functions $\varphi(\sigma)$ satisfying (2.1) but not $(\varphi 1)$. A nontrivial example is provided in section 4. A careful inspection of that example reveals that not only the function $\sigma \rightarrow M(\sqrt{\sigma})$ is not convex, but also its convex envelope is a constant function (due to the fact that $M(\sqrt{\sigma_k}) = \sqrt{\sigma_k}$ on a sequence $\sigma_k \rightarrow +\infty$).

This shows that Theorem 2.1 is a real extension of Theorem A, and cannot be deduced from Theorem A applied with a smaller weight which satisfies $(\varphi 1)$ and $(\varphi 2)$ and generates the same functional space.

Remark 2.3 The ‘‘Sobolev-type’’ indices $3/4$ and $1/4$ of (2.5) are quite usual in the theory of Kirchhoff equations (see for example most of the results stated in [9]). On the other hand, if φ grows fast enough (for example if $\varphi(\sigma) \geq \log^2 \sigma$ for every $\sigma \geq 1$), then the inclusion $\mathcal{G}_{\varphi, r_0, \alpha}(A) \subseteq \mathcal{G}_{\varphi, r_1, \beta}(A)$ holds true for every $0 < r_1 < r_0$ and every $0 \leq \alpha \leq \beta$.

We also point out that we allow any $r_0 > 0$ in (2.4). This is just because condition (2.1) doesn’t change if we replace $\varphi(\sigma)$ with $r_0\varphi(\sigma)$.

Remark 2.4 Concerning the nonlinearity $m(\sigma)$, there is no hope to relax assumption (1.4) to $m(\sigma) \geq 0$, or the Lipschitz continuity assumption to mere continuity. The reason is that some examples presented in [6] show that under these weaker assumptions the Cauchy problem (1.1), (1.2) is not even *locally* well posed in classes of quasi-analytic functions.

3 Proofs

3.1 Technical preliminaries

In this section we collect some estimates which are crucial in the proof of our main result. First of all we remark that assumption (2.1) implies in particular that $\varphi(\sigma)$ is unbounded. Since it is also continuous and strictly increasing, it easily follows that φ , thought as a function $\varphi : [0, +\infty) \rightarrow [\varphi(0), +\infty)$, is invertible. From now on we can therefore consider its inverse function $\varphi^{-1} : [\varphi(0), +\infty) \rightarrow [0, +\infty)$.

In the first result we show that assumption (2.1) implies an integrability condition on $\varphi^{-1}(\sigma)$ similar to Nishihara’s assumption ($\varphi 2$).

Lemma 3.1 *Let $\varphi : [0, +\infty) \rightarrow [0, +\infty)$ be a strictly increasing continuous function satisfying (2.1). Let $\varphi^{-1} : [\varphi(0), +\infty) \rightarrow [0, +\infty)$ be its inverse function.*

Then for every $a > 0$, $b \geq 0$, $c > \varphi(0)$ we have that

$$\int_c^{+\infty} \frac{1}{a\varphi^{-1}(y) + b} dy = +\infty. \quad (3.1)$$

Proof Assumption (2.1) is equivalent to say that

$$\int_d^{+\infty} \frac{\varphi(y)}{(ay + b)^2} dy = +\infty$$

for every $a > 0$, $b \geq 0$, $d > 0$. Let us consider the functions

$$F(x) := \int_c^x \frac{\varphi(y)}{(ay + b)^2} dy, \quad G(x) := \int_c^x \frac{1}{a\varphi^{-1}(y) + b} dy,$$

$$H(x) := G(\varphi(x)) - aF(x) - \frac{\varphi(x)}{ax + b},$$

defined for every $x \geq c > \varphi(0)$. We claim that $H(x)$ is constant. If we prove this claim, then (3.1) easily follows because

$$\begin{aligned} \int_c^{+\infty} \frac{1}{a\varphi^{-1}(y) + b} dy &= \lim_{x \rightarrow +\infty} G(x) = \lim_{x \rightarrow +\infty} G(\varphi(x)) \geq H(c) + a \lim_{x \rightarrow +\infty} F(x) = \\ &= H(c) + a \int_c^{+\infty} \frac{\varphi(y)}{(ay + b)^2} dy = +\infty. \end{aligned}$$

In order to prove the claim, let us assume first that φ is of class C^1 . In this case an elementary computation shows that

$$H'(x) = \frac{\varphi'(x)}{ax + b} - \frac{a\varphi(x)}{(ax + b)^2} - \frac{\varphi'(x)}{ax + b} + \frac{a\varphi(x)}{(ax + b)^2} = 0. \quad (3.2)$$

If φ is not of class C^1 (and not even absolutely continuous), then there are at least two standard ways to obtain the same conclusion. The first one is to approximate $\varphi(x)$ with a sequence of strictly increasing functions of class C^1 and then passing to the limit. The second one is recalling that φ lies in $BV_{loc}((0, +\infty))$. Since G is Lipschitz continuous one can therefore apply the chain rule in BV and obtain (3.2) as an equality between measures instead of functions. \square

The second result is quite classical. Roughly speaking, it says that a solution of the differential inequality (3.4) cannot blow up in finite time when the integrability condition (3.3) is satisfied.

Lemma 3.2 *Let L_1 be a real number, let $g : [L_1, +\infty) \rightarrow (0, +\infty)$ be a positive continuous function, and let $y_0 \geq L_1$ be such that*

$$\int_{y_0}^{+\infty} \frac{1}{g(y)} dy = +\infty. \quad (3.3)$$

Let $T > 0$, and let $y : [0, T) \rightarrow [L_1, +\infty)$ be a function of class C^1 such that $y(0) = y_0$, and

$$y'(t) \leq g(y(t)) \quad \forall t \in [0, T). \quad (3.4)$$

Then

$$\limsup_{t \rightarrow T^-} y(t) < +\infty. \quad (3.5)$$

Proof Let us consider the function $\Gamma : [L_1, +\infty) \rightarrow \mathbb{R}$ defined by

$$\Gamma(x) := \int_{y_0}^x \frac{1}{g(y)} dy.$$

Assumption (3.4) is equivalent to say that $[\Gamma(y(t))]' \leq 1$, hence

$$\Gamma(y(t)) \leq \Gamma(y(0)) + t = t \quad \forall t \in [0, T].$$

Due to assumption (3.3) we have that Γ , thought as a function $\Gamma : [L_1, +\infty) \rightarrow [\Gamma(L_1), +\infty)$, is strictly increasing and invertible, hence

$$y(t) \leq \Gamma^{-1}(t) \quad \forall t \in [0, T].$$

At this point (3.5) easily follows. \square

The last result is the technical core of this paper. The rough idea is that one can estimate an intermediate norm (in this case the sum of $a_k \lambda_k$) by means of a lower order norm (the sum of a_k), and a higher order norm (the sum of $a_k e^{\varphi(\lambda_k)}$). Usually such estimates follow from Jensen type inequalities, hence they do require convexity assumptions as in Nishihara's paper.

Here we prove a result of this type without using convexity. The resulting estimates are weaker than the corresponding ones of the convex case. Nevertheless they are enough to deduce the a priori estimates needed in the sequel, and the proof is surprisingly simple.

Proposition 3.3 *Let $\varphi : [0, +\infty) \rightarrow [0, +\infty)$ be a strictly increasing continuous function satisfying (2.1). Let $\varphi^{-1} : [\varphi(0), +\infty) \rightarrow [0, +\infty)$ be its inverse function.*

Let $\{a_k\}$ and $\{\lambda_k\}$ be two sequences of nonnegative real numbers such that

$$0 < E := \sum_{k=0}^{\infty} a_k < +\infty, \quad F := \sum_{k=0}^{\infty} a_k \max\{\lambda_k, 1\} e^{\varphi(\lambda_k)} < +\infty.$$

Then

$$\sum_{k=0}^{\infty} a_k \lambda_k \leq E \left\{ 1 + \varphi^{-1} \left(\varphi(0) + \log \frac{F}{E} \right) \right\}. \quad (3.6)$$

Proof First of all we remark that $F \geq E > 0$, hence the right-hand side of (3.6) is well defined. Let us set for simplicity

$$\alpha := \varphi^{-1} \left(\varphi(0) + \log \frac{F}{E} \right),$$

and let

$$A := \{k \in \mathbb{N} : \lambda_k < \alpha\}, \quad B := \{k \in \mathbb{N} : \lambda_k \geq \alpha\}.$$

Let us write

$$\sum_{k=0}^{\infty} a_k \lambda_k = \sum_{k \in A} a_k \lambda_k + \sum_{k \in B} a_k \lambda_k,$$

and let us estimate the two sums separately. In A we have that

$$\sum_{k \in A} a_k \lambda_k \leq \alpha \sum_{k \in A} a_k \leq \alpha E. \quad (3.7)$$

For every $k \in B$ we have that $\varphi(\lambda_k) \geq \varphi(\alpha) = \varphi(0) + \log(F/E)$, hence

$$e^{\varphi(\lambda_k)} \geq \exp\left(\varphi(0) + \log \frac{F}{E}\right) \geq \frac{F}{E},$$

and therefore

$$\sum_{k \in B} a_k \lambda_k \leq \frac{E}{F} \sum_{k \in B} a_k \lambda_k e^{\varphi(\lambda_k)} \leq \frac{E}{F} \sum_{k=0}^{\infty} a_k \max\{\lambda_k, 1\} e^{\varphi(\lambda_k)} = E. \quad (3.8)$$

Summing (3.7) and (3.8) we obtain (3.6). \square

3.2 Proof of the main result

The strategy of the proof is standard for Kirchhoff equations. First of all we know that a local solution exists due to classical results. Then we estimate first order energies using the conserved Hamiltonian. Finally we prove an a priori estimate on a *higher order* energy. This is the key point where Proposition 3.3 plays its role. The a priori estimate excludes blow up, and this is enough to deduce global existence.

Throughout the proof we assume, without loss of generality, that $r_0 = 1$. Indeed the parameter r_0 in the definition of $\mathcal{G}_{\varphi, r_0, \alpha}(A)$ can always be included in $\varphi(\sigma)$ without changing the fundamental assumption (2.1), as previously remarked.

We also assume that

$$|u_1|^2 + |A^{1/2}u_0|^2 \neq 0 \quad (3.9)$$

because otherwise the solution is the constant function $u(t) \equiv u_0$, which is clearly globally defined.

Finally, we assume that m is of class C^1 . Indeed, when m is just locally Lipschitz continuous, we can approximate it with a sequence of smooth functions and then pass all estimates to the limit. We spare the reader from the details of this standard argument.

We also point out that the solution is trivially unique because $m(\sigma)$ is assumed to be Lipschitz continuous (for uniqueness issues the interested reader is referred to [8]).

Maximal local solutions Due to (2.4) we have in particular that $(u_0, u_1) \in D(A^{3/4}) \times D(A^{1/4})$. Therefore the classical local existence theory (see [1, 9]) implies that problem (1.1), (1.2) admits a unique local solution

$$u \in C^1([0, T]; D(A^{1/4})) \cap C^0([0, T]; D(A^{3/4})). \quad (3.10)$$

Moreover, if $[0, T)$ is the maximal interval where this solution is defined, then either $T = +\infty$, or

$$\limsup_{t \rightarrow T^-} |A^{1/4}u'(t)|^2 + |A^{3/4}u(t)|^2 = +\infty. \quad (3.11)$$

So we have only to exclude that (3.11) holds true.

Standard energy estimates Let u be any solution of (1.1), (1.2), with regularity prescribed by (3.10). Let $u_k(t)$ denote the components of $u(t)$ with respect to the orthonormal system e_k (see the simplifying assumptions stated at the beginning of section 2.1). Let us set

$$c(t) := m(|A^{1/2}u(t)|^2), \quad (3.12)$$

and let

$$\begin{aligned} E_k(t) &:= |u'_k(t)|^2 + c(t)\lambda_k^2|u_k(t)|^2, \\ E(t) &:= |u'(t)|^2 + c(t)|A^{1/2}u(t)|^2 = \sum_{k=0}^{\infty} E_k(t). \end{aligned}$$

We claim that there exist positive constants L_1 and L_2 such that

$$L_1 \leq E(t) \leq L_2 \quad \forall t \in [0, T). \quad (3.13)$$

To this end, we consider the usual Hamiltonian

$$\mathcal{H}(t) := |u'(t)|^2 + M(|A^{1/2}u(t)|^2),$$

where

$$M(\sigma) := \int_0^\sigma m(s) ds \quad \forall \sigma \geq 0.$$

It is well known that $\mathcal{H}(t)$ is constant. By (1.4) we have that $M(\sigma) \geq \mu\sigma$ for every $\sigma \geq 0$, hence

$$|A^{1/2}u(t)|^2 \leq \mu^{-1}\mathcal{H}(t) = \mu^{-1}\mathcal{H}(0), \quad (3.14)$$

and therefore

$$\mu \leq c(t) \leq \max\{m(\sigma) : 0 \leq \sigma \leq \mu^{-1}\mathcal{H}(0)\} =: c_1.$$

It follows that

$$E(t) \leq |u'(t)|^2 + c_1|A^{1/2}u(t)|^2 \leq c_2\mathcal{H}(t) = c_2\mathcal{H}(0) =: L_2.$$

Similarly, since $M(|A^{1/2}u(t)|^2) \leq c_1|A^{1/2}u(t)|^2$, we have also that

$$E(t) \geq |u'(t)|^2 + \mu|A^{1/2}u(t)|^2 \geq c_3\mathcal{H}(t) = c_3\mathcal{H}(0) =: L_1,$$

where L_1 is positive due to (3.9). This completes the proof of (3.13).

Fundamental a priori estimate Let us set

$$F(t) := \sum_{k=0}^{\infty} E_k(t) \max\{1, \lambda_k\} e^{\varphi(\lambda_k)}. \quad (3.15)$$

We claim that $F(t)$ is well defined for every $t \in [0, T)$, and

$$\limsup_{t \rightarrow T^-} F(t) < +\infty. \quad (3.16)$$

To this end, let us first estimate the derivative of $c(t)$. By (3.12) and (3.14) we have that

$$\begin{aligned} |c'(t)| &= |m'(|A^{1/2}u(t)|^2)| \cdot 2 |\langle A^{1/4}u'(t), A^{3/4}u(t) \rangle| \\ &\leq \max\{|m'(\sigma)| : 0 \leq \sigma \leq \mu^{-1}\mathcal{H}(0)\} \cdot (|A^{1/4}u'(t)|^2 + |A^{3/4}u(t)|^2) \\ &\leq c_4 \sum_{k=0}^{\infty} \lambda_k E_k(t), \end{aligned}$$

where the last series converges to a continuous function because we already know that u is at least as regular as prescribed by (3.10). Now we have that

$$E'_k(t) = c'(t) \lambda_k^2 |u_k(t)|^2 \leq \frac{|c'(t)|}{\mu} c(t) \lambda_k^2 |u_k(t)|^2 \leq c_5 E_k(t) \sum_{k=0}^{\infty} \lambda_k E_k(t), \quad (3.17)$$

hence

$$E_k(t) \leq E_k(0) \exp\left(c_5 \int_0^t \sum_{k=0}^{\infty} \lambda_k E_k(\tau) d\tau\right). \quad (3.18)$$

On the other hand, from assumption (2.4) it is easy to deduce that

$$\sum_{k=0}^{\infty} E_k(0) \max\{1, \lambda_k\} e^{\varphi(\lambda_k)} < +\infty.$$

Combining with (3.18) we obtain that the series in (3.15) converges, which proves that $F(t)$ is well defined. Moreover, from (3.17) we deduce also that F is of class C^1 , and its derivative satisfies

$$F'(t) = \sum_{k=0}^{\infty} E'_k(t) \max\{1, \lambda_k\} e^{\varphi(\lambda_k)} \leq c_5 \left(\sum_{k=0}^{\infty} \lambda_k E_k(t) \right) \cdot F(t).$$

Now we apply Proposition 3.3 with $a_k = E_k(t)$. We obtain that

$$\sum_{k=0}^{\infty} \lambda_k E_k(t) \leq E(t) \cdot \left\{ 1 + \varphi^{-1} \left(\varphi(0) + \log \frac{F(t)}{E(t)} \right) \right\},$$

hence by (3.13)

$$\sum_{k=0}^{\infty} \lambda_k E_k(t) \leq L_2 \left\{ 1 + \varphi^{-1} \left(\varphi(0) + \log \frac{F(t)}{L_1} \right) \right\},$$

and in particular

$$F'(t) \leq c_6 F(t) \left\{ 1 + \varphi^{-1} \left(\varphi(0) + \log \frac{F(t)}{L_1} \right) \right\}.$$

Since $F(t) > 0$ for every $t \in [0, T)$, from Lemma 3.2 it follows that $F(t)$ satisfies (3.16) provided that the function

$$g(y) := c_6 y \left\{ 1 + \varphi^{-1} \left(\varphi(0) + \log \frac{y}{L_1} \right) \right\},$$

defined for every $y \geq L_1$, satisfies (3.3). With the variable change $z = \varphi(0) + \log(y/L_1)$ we obtain that

$$\int_{F(0)}^{+\infty} \frac{1}{g(y)} dy = \int_{c_7}^{+\infty} \frac{1}{c_6(\varphi^{-1}(z) + 1)} dz,$$

and the last integral is equal to $+\infty$ due to Lemma 3.1 and our assumption (2.1).

Conclusion For every $t \in [0, T)$ we have that

$$|A^{1/4}u'(t)|^2 + |A^{3/4}u(t)|^2 \leq c_8 \sum_{k=0}^{\infty} \lambda_k E_k(t) \leq c_8 F(t).$$

Therefore (3.11) cannot hold true because of (3.16). This is enough to conclude that the solution is global. From the estimate on $F(t)$ it follows also that u is as regular as required in (2.5). \square

4 Connection with spectral-gap solutions

In this section we present some weird results which can be obtained using weights $\varphi(\sigma)$ without convexity assumptions. The main idea is that a function φ can satisfy assumption (2.1) even if its growth is very slow (for example logarithmic) on a suitable sequence diverging to $+\infty$.

Let λ_k be a sequence of nonnegative real numbers such that $\lambda_{k+1} \geq e^{\lambda_k}$ for every $k \in \mathbb{N}$. Let $\tilde{\varphi} : [0, +\infty) \rightarrow [0, +\infty)$ be a piecewise constant function such that

$$\tilde{\varphi}(x) = \log \lambda_{k+1} \quad \forall x \in (\lambda_k, \lambda_{k+1}]$$

for every $k \in \mathbb{N}$. Since $\tilde{\varphi}(x) \geq \lambda_k$ for every $x \in (\lambda_k, \lambda_{k+1}]$ and every $k \geq 1$, we have that

$$\int_1^{+\infty} \frac{\tilde{\varphi}(\sigma)}{\sigma^2} d\sigma \geq \sum_{k=1}^{\infty} \lambda_k \int_{\lambda_k}^{\lambda_{k+1}} \frac{d\sigma}{\sigma^2} = \sum_{k=1}^{\infty} \left(1 - \frac{\lambda_k}{\lambda_{k+1}}\right) = +\infty, \quad (4.1)$$

where the last inequality follows from the fact that $\lambda_{k+1} \geq 2\lambda_k$ for every $k \geq 1$.

Therefore it is quite simple to modify $\tilde{\varphi}(\sigma)$ in order to obtain a function $\varphi(\sigma)$ which is continuous (or even more regular), strictly increasing, satisfies $\tilde{\varphi}(\lambda_k) = \varphi(\lambda_k)$ for all $k \in \mathbb{N}$, and still fulfils assumption (2.1).

Let us assume now that the spectrum of an operator A coincides with the sequence λ_k we have just considered. Since $\varphi(\lambda_k) = \log \lambda_k$ for every $k \geq 1$, it follows that

$$\mathcal{G}_{\varphi,r,\alpha}(A) = D(A^{\alpha+r/4}) \quad \forall \alpha \geq 0, \forall r > 0.$$

As a consequence, for this operator Theorem 2.1 is actually a global existence result in Sobolev-type spaces. We have thus proved the following result.

Theorem 4.1 *Let H be a separable Hilbert space, and let A be a nonnegative self-adjoint operator on H with dense domain. Let us assume that the spectrum of A consists of a sequence λ_k^2 of eigenvalues such that $\lambda_{k+1} \geq e^{\lambda_k}$ for every $k \in \mathbb{N}$.*

Let $m : [0, +\infty) \rightarrow (0, +\infty)$ be a locally Lipschitz continuous function satisfying the nondegeneracy condition (1.4).

Then for every $\alpha > 1/4$, and every pair of initial conditions $(u_0, u_1) \in D(A^{\alpha+1/2}) \times D(A^\alpha)$, problem (1.1), (1.2) admits a (unique) global solution

$$u \in C^1([0, +\infty); D(A^\alpha)) \cap C^0([0, +\infty); D(A^{\alpha+1/2})). \quad (4.2)$$

The above result shows that there do exist special operators for which the Kirchhoff equation is well posed in Sobolev-type spaces. These operators are characterized by a fast growing sequence of eigenvalues, and for this reason we could call them ‘‘spectral-gap operators’’.

The same result can also be seen from a different point of view. The operator is now more general, but initial data have nonzero components only with respect to a sequence of special eigenvectors whose eigenvalues grow fast enough. We obtain the following result.

Theorem 4.2 *Let A be a self-adjoint linear operator on a Hilbert space H . Let us assume that there exist a countable (not necessarily complete) orthonormal system $\{e_k\}$ in H , and a sequence $\{\lambda_k\}$ of nonnegative real numbers such that $\lambda_{k+1} \geq e^{\lambda_k}$ and $Ae_k = \lambda_k^2 e_k$ for every $k \in \mathbb{N}$. Let $S \subseteq H$ be the closure of the subspace generated by $\{e_k\}$.*

Let $m : [0, +\infty) \rightarrow (0, +\infty)$ be a locally Lipschitz continuous function satisfying the nondegeneracy condition (1.4).

Let $\alpha > 1/4$, and let $(u_0, u_1) \in D(A^{\alpha+1/2}) \times D(A^\alpha)$ with $(u_0, u_1) \in S \times S$.

Then problem (1.1), (1.2) admits a (unique) global solution satisfying (4.2).

The proof trivially follows from Theorem 4.1 applied in S . In both cases we could save something on the growth of λ_k by just asking that

$$\sum_{k=1}^{\infty} \left(\frac{1}{\lambda_k} - \frac{1}{\lambda_{k+1}} \right) \log \lambda_{k+1} = +\infty,$$

which is what is really required in (4.1).

Theorem 4.2 reminded us of the recent global existence results for spectral-gap initial data. The connection actually exists because it is possible to show that in most cases elements of S lie in the spaces introduced by R. Manfrin in [12, 13]. On the other hand, the proof given in [13] is based on completely different techniques and requires more regularity both of initial data ($\alpha \geq 1/2$ instead of $\alpha > 1/4$), and of the nonlinear term ($m(\sigma)$ is assumed to be of class C^2). Such technical restrictions have been recently removed in [7], but at the expenses of a faster growth of the sequence of eigenvalues.

In other words Theorem 4.1 and Theorem 4.2, in the form they are stated here, don't follow from the theories developed in [12, 13] and [7].

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