The Local Limit Theorem and the Almost Sure Local Limit Theorem *

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Abstract

We review some classical Local Limit Theorems. We present the latest results in the theory of the Almost Sure Local Limit Theorem.

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1 Introduction

Generally speaking, the local limit theorem describes how the density of a sum of random variables follows the normal curve. Historically the local limit theorem appeared before the celebrated central limit theorem, which supplanted it, especially when it became clear that the CLT could be proved using the fundamental tool of characteristic functions (in the work of the russian school). In later years almost sure versions of the CLT were investigated (the so called "Almost Sure Central Limit Theorem" ASCLT); nowadays the theory of the ASCLT is completely settled; on the contrary, the probem of establishing almost sure versions of the local limit theorem attracted attention only recently. In these notes we review some classical local limit theorems and present the latest results in the theory of the Almost Sure Local Limit Theorem (ASLLT).

2 The Local Limit Theorem

2.1 The DeMoivre – Laplace theorem and the Gnedenko local limit theorem

The following result is perhaps the oldest local limit theorem; it was stated by A. DeMoivre in *Approximatio ad Summam Terminorum Binomii* $(a+b)^n$ *in Seriem expansi* (1733); for a complete DeMoivre's biography see [1].

Theorem 2.1 *Let* $g_n(k)$ *be the probability of getting* k *heads in* n *tosses of a coin which gives a head with probability* p*. Then*

$$\lim_{n\to\infty} \frac{g_n(k)}{\left(\frac{1}{\sqrt{2npq}} e^{-\frac{(k-np)^2}{2npq}}\right)} = 1,$$

uniformly for k *such that* $\left|\frac{k-np}{\sqrt{npq}}\right|$ *remains bounded.*

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Actually, DeMoivre proved this result only for a fair coin ($p = 1/2$). It was Pierre-Simon Laplace who proved it in full generality in *Théorie Analytique des probabilités* (1812).

We refer to [2] for a proof.

Notice that $p = E[X_1]$ and $pq = VarX_1$. Hence we could expect a general result of the form: Let $(X_n)_{n\geqslant 1}$ be a sequence of i.i.d. random variables, with $E[X_1] = \mu$, $VarX_1 = \sigma^2$. Then

$$P(S_n = k) \approx \frac{1}{\sqrt{2\pi n}\sigma} e^{-\frac{(k-n\mu)^2}{2n\sigma^2}}$$

if $\left|\frac{k-n\mu}{\sqrt{npq}}\right|$ is bounded.

But certainly this cannot be true in general: if

$$X_1 = \begin{cases} -1 & \text{with probability}\, q \\ 1 & \text{with probability}\, p \end{cases}$$

and k is odd, then $P(S_{2n} = k) = 0$.

Let's check what happens in the above situation with $p = q = \frac{1}{2}$ and for even k, say $k = 2h$. Since $\mu = 0$ and $\sigma^2 = 1$, we should obtain (with $2n$ in place of n and $2h$ in place of k)

$$P(S_{2n} = 2h) \approx \frac{1}{2\sqrt{\pi n}} e^{-\frac{h^2}{n}},$$

with $\frac{h}{\sqrt{n}}$ bounded. Let's see whether this is true; for the sake of simplicity we shall consider a sequence (h_n) such that $\frac{h_n}{\sqrt{n}} \to_n \frac{x}{\sqrt{2}}$. The above formula becomes

$$P(S_{2n} = 2h_n) \approx \frac{1}{2\sqrt{\pi n}} e^{-\frac{x^2}{2}}. \tag{1}$$

On the contrary, a careful calculation gives

Theorem 2.2 *In the previous situation ($p = q = \frac{1}{2}$, $\frac{h_n}{\sqrt{n}} \to_n \frac{x}{\sqrt{2}}$) we have*

$$\lim_{n\to\infty} \sqrt{n} P(S_n = 2h_n) = \frac{1}{\sqrt{\pi}} e^{-\frac{x^2}{2}}. \tag{2}$$

Proof. Postponed to the next section.

□

Hence a general theorem should roughly state that

$$P(S_n = k) \approx \frac{c}{\sqrt{2\pi n}\sigma} e^{-\frac{(k-n\mu)^2}{2n\sigma^2}}$$

if k is a value of S_n and $\left|\frac{k-n\mu}{\sqrt{npq}}\right|$ is bounded, for a suitable c. But what is c? Comparing (1) and (2), we notice that the main difference is a factor of 2 in the second member of (2); where does it come from? We also notice that in this case the support of S_{2n} is concentrated on even integers, and two successive even integers differ by 2. So one guesses that $c = 2$ in our case, and in general c is maybe connected with the gap between successive values of S_n.

In the sequel we shall see the correct formulation of the general theorem, which, taking into account the possibility of a periodicity in the values of S_n, settles completely the question, not only in the previous example, but also in the case of a general sequence $(X_n)_{n\geqslant 1}$. We need some preliminaries. In what follows we set $\mathcal{L}(a, \lambda) := a + \lambda\mathbb{Z} = \{a + \lambda k, k \in \mathbb{Z}\}$.

Definition 2.3 *A random variable X has a* **lattice distribution** *if there exist two constant a and $\lambda > 0$ such that $P(X \in \mathcal{L}(a,\lambda)) = 1$.*

We shall denote by ϕ the characteristic function of X, i.e. $\phi(t) = \mathbb{E}[e^{itX}]$. The next results links the concept of lattice distribution to the behaviour of the characteristic function.

Theorem 2.4 *There are only three possibilities:*

(i) there exists a $t_0 > 0$ such that $|\phi(t_0)| = 1$ and $|\phi(t)| < 1$ for every $0 < t < t_0$. In this case X has a lattice distribution.

*(ii) $|\phi(t)| < 1$ for every $t \neq 0$ (*non lattice *distribution).*

*(iii) $|\phi(t)| = 1$ for every $t \in \mathbb{R}$: In this case X is constant a.s. (*degenerate *distribution).*

The proof of Theorem 2.4 (see next section) shows in particular that

Corollary 2.5 *In case (i) of Theorem 2.4, we have*

$$\frac{2\pi}{t_0} = \max\{\lambda > 0 : \exists\, a \in \mathbb{R},\, P(X \in \mathcal{L}(a,\lambda)) = 1\}.$$

Corollary 2.5 justifies the following

Definition 2.6 *In case (i) of the preceding Theorem 2.4, the number*

$$\Lambda = \frac{2\pi}{t_0} = \max\{\lambda > 0 : \exists\, a \in \mathbb{R},\, P(X \in \mathcal{L}(a,\lambda)) = 1\}$$

is called the (maximal) span *of the distribution of X.*

Remark 2.7 *In case (i), when it is possible to choose $a = 0$ (hence $P(X \in \Lambda\mathbb{Z}) = 1$), we say that X has* arithmetic *distribution. Moreover ϕ is periodic, with period t_0. In fact, since $\Lambda = \frac{2\pi}{t_0}$, we have*

$$\phi(t+t_0) = \sum_{k\in\mathbb{Z}} P(X = \Lambda k) e^{i(t+t_0)\frac{2\pi}{t_0}k} = \sum_{k\in\mathbb{Z}} P(X = \Lambda k) e^{it\frac{2\pi}{t_0}k} = \phi(t).$$

Some examples. (i) Let

$$X_1 = \begin{cases} -1 & \text{with probability } \frac{1}{2} \\ 1 & \text{with probability } \frac{1}{2}. \end{cases}$$

Then $\phi(t) = \frac{1}{2}\big(e^{it} + e^{-it}\big) = \cos t$, and $|\phi(t)| = 1$ if and only if $t = n\pi$, $n \in \mathbb{Z}$. Hence $t_0 = \pi$, and the maximal span of the distribution is $\frac{2\pi}{t_0} = 2$.

(ii) Let X_1 have standard gaussian law. Then $\phi(t) = e^{-\frac{t^2}{2}}$. In this case $|\phi(t)| = 1$ only for $t = 0$.

(iii) If $X_1 = c$ (c some constant) we have $\phi(t) = e^{itc}$, and $|\phi(t)| = 1$ for every t.

We state the first local theorem, due to B.V. Gnedenko (1948) (see [3]).

Let $(X_n)_{n\geqslant 1}$ be a sequence of i.i.d random variables, with $\mathbf{E}[X_i] = \mu$ and $\mathbf{Var}X_i = \sigma^2$ finite, having lattice distribution with maximal span Λ. Let $S_n = X_1 + \cdots + X_n$. If $P(X_i \in \mathcal{L}(a,\Lambda)) = 1$, then $P(S_n \in \mathcal{L}(na,\Lambda) = 1$.

Theorem 2.8 *With the assumption stated above, we have*

$$\lim_{n\to\infty} \sup_{N\in\mathcal{L}(na,\Lambda)} \Big| \frac{\sqrt{n}}{\Lambda} P(S_n = N) - \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(N-n\mu)^2}{2n\sigma^2}} \Big| = 0.$$

Proof See next section; here we give some heuristics, assuming $\mu = 0$. By the CLT we can write

$$P\Big(S_n = N\Big) \approx P\Big(N - \frac{\Lambda}{2} \leqslant S_n \leqslant N + \frac{\Lambda}{2}\Big) = P\Big(\frac{N}{\sqrt{n}\sigma} - \frac{\Lambda}{2\sigma\sqrt{n}} \leqslant \frac{S_n}{\sigma\sqrt{n}} \leqslant \frac{N}{\sqrt{n}\sigma} - \frac{\Lambda}{2\sigma\sqrt{n}}\Big)$$

$$\approx \int_{\frac{N}{\sqrt{n}\sigma} - \frac{\Lambda}{2\sigma\sqrt{n}}}^{\frac{N}{\sqrt{n}\sigma} + \frac{\Lambda}{2\sigma\sqrt{n}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}\,dt = \int_{\frac{N}{\sqrt{n}} - \frac{\Lambda}{2\sqrt{n}}}^{\frac{N}{\sqrt{n}} + \frac{\Lambda}{2\sqrt{n}}} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{t^2}{2\sigma^2}}\,dt \approx \frac{\Lambda}{\sqrt{n}} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{N^2}{2n\sigma^2}}.$$

□

Actually, the complete formulation of Gnedenko's result is (see [4], § 43)

Theorem 2.9 *With the same assumptions as in Theorem 2.8, in order that*

$$\lim_{n\to\infty} \sup_{N \in \mathcal{L}(na,\lambda)} \Big| \frac{\sqrt{n}}{\lambda} P(S_n = N) - \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(N-n\mu)^2}{2n\sigma^2}} \Big| = 0,$$

it is necessary and sufficient that $\lambda = \Lambda$.

The following result completes the theory (see [5], Th. 4.5.3):

Theorem 2.10 *With the same assumptions as in Theorem 2.8, in order that*

$$\sup_{N \in \mathcal{L}(na,\lambda)} \Big| \frac{\sqrt{n}}{\lambda} P(S_n = N) - \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(N-n\mu)^2}{2n\sigma^2}} \Big| = O(n^{-\alpha}), \qquad 0 < \alpha < \frac{1}{2}$$

it is necessary and sufficient that the following conditions are satisfied
(i)$\lambda = \Lambda$;
(ii) if F *denotes the distribution function of* X_1*, then, as* $u \to \infty$, $\int_{|x|\geqslant u} x^2 F(dx) = O(u^{-2\alpha})$.

We turn to study the nonlattice case, so we shall consider a sequence $(X_n)_{n\geq 1}$ of i.i.d. random variables with characteristic function ϕ such that $|\phi(t)| < 1$ for every $t \neq 0$.

Remark 2.11 *In the nonlattice case, most characteristic functions verify the so–called* Cramer's condition, *i.e.* $\limsup_{t\to\infty} |\phi(t)| < 1$. *Nevertheless there do exist characteristic functions of non-lattice random variables, that do not verify Cramer's condition. One example (due to A. Wintner, see [6], footnote on p. 27) is as follows. Let*

$$Y = \begin{cases} -1 & \text{with probability } \frac{1}{2} \\ \ \ 1 & \text{with probability } \frac{1}{2}. \end{cases}$$

Let $(Y_n)_{n\geqslant 1}$ *be a sequence of i.i.d random variables with the same law as* Y*; the random series*

$$X = \sum_k \frac{Y_k}{k!}$$

defines an a.s. random variable X *the characteristic function of which is*

$$\phi(t) = \prod_{k=1}^{\infty} \cos\Big(\frac{t}{k!}\Big).$$

Now $|\phi(t)| = 1$ *if and only if* $\frac{t}{k!}$ *is a multiple of* π *for each integer* k*, which is clearly impossible unless* $t = 0$ *(let* $\frac{t}{\pi k!} = r$*, for a suitable non–zero integer* r*, and let* p *be a prime number greater than* k *and of all the prime factors of* r*. Then the number*

$$\frac{t}{\pi p!} = \frac{t}{\pi k!} \cdot \frac{1}{(k+1)\cdot \cdots \cdot p} = \frac{r}{(k+1)\cdot \cdots \cdot p}$$

is not an integer).

On the other hand, it is not difficult to verify that

$$1 - \phi(2\pi N!) \to 0, \qquad N \to \infty.$$

In fact, first recall that

$$\sup_{u \in \mathbb{R}} \frac{1 - \cos u}{u^2} = C_1 < \infty, \qquad \sup_{u \in \mathbb{R}} \frac{1 - e^{-u}}{u} = C_2 < \infty.$$

Moreover, for $u > -\frac{1}{2}$ we have

$$\log(1+u) > u - C_3 u^2$$

for a suitable absolute constant $C_3 > 0$. Hence, if $N > 5$ we obtain

$$1 - \phi(2\pi N!) = 1 - \underbrace{\prod_{k=1}^{N} \cos\Big(\frac{2\pi N!}{k!}\Big)}_{=1} \cdot \prod_{k=N+1}^{\infty} \cos\Big(\frac{2\pi N!}{k!}\Big) = 1 - \exp \sum_{k=N+1}^{\infty} \log\{\cos\Big(\frac{2\pi N!}{k!}\Big) - 1 + 1\}$$

$$\leqslant 1 - \exp\Big\{ - \sum_{k=N+1}^{\infty} \Big[1 - \cos\Big(\frac{2\pi N!}{k!}\Big)\Big] - C_3 \sum_{k=N+1}^{\infty} \Big[1 - \cos\Big(\frac{2\pi N!}{k!}\Big)\Big]^2 \Big\}$$

$$\leqslant C_2 \Big\{ \sum_{k=N+1}^{\infty} \Big[1 - \cos\Big(\frac{2\pi N!}{k!}\Big)\Big] + C_3 \sum_{k=N+1}^{\infty} \Big[1 - \cos\Big(\frac{2\pi N!}{k!}\Big)\Big]^2 \Big\}$$

$$\leqslant C_2 \Big\{ C_1 (2\pi)^2 \sum_{k=N+1}^{\infty} \Big(\frac{N!}{k!}\Big)^2 + C_3 C_1 (2\pi)^4 \sum_{k=N+1}^{\infty} \Big(\frac{N!}{k!}\Big)^4 \Big\}$$

Now

$$\sum_{k=N+1}^{\infty} \Big(\frac{N!}{k!}\Big)^4 = \sum_{k=N+1}^{\infty} \frac{1}{(N+1)^4 \cdot \ldots k^4} \leqslant \sum_{k=N+1}^{\infty} \frac{1}{(N+1)^2 \cdot \ldots k^2} = \sum_{k=N+1}^{\infty} \Big(\frac{N!}{k!}\Big)^2$$

and

$$\sum_{k=N+1}^{\infty} \Big(\frac{N!}{k!}\Big)^2 = \sum_{k=N+1}^{\infty} \frac{1}{(N+1)^2 \cdot \ldots k^2} \leqslant \sum_{k=N+1}^{\infty} \frac{1}{k^2} \to 0, \qquad N \to \infty.$$

The following result holds

Theorem 2.12 *Let $(X_n)_{n \geqslant 1}$ be sequence of i.i.d. nonlattice random variables, with $\mathbf{E}[X_1] = \mu$,* $\mathbf{Var} X_1 = \sigma^2 < \infty$*. If $\frac{x_n}{\sqrt{n}} \to x$ and $a < b$,*

$$\lim_{n \to \infty} \sqrt{n} P(S_n - n\mu \in (x_n + a, x_n + b)) = (b-a) \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}.$$

The proof is postponed to the next section. The heuristics are as for Theorem 2.8.

The preceding theorem can be made more precise if the characteristic function has some further property.

Theorem 2.13 *If $|\phi|$ is integrable, then $\frac{S_n - n\mu}{\sigma\sqrt{n}}$ has a density f_n; moreover f_n tends undiformly to the standard normal density*

$$\eta(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}.$$

For the proof, see [2].

Theorem 2.9 is a particular case of a much wider result concerning random variables in the domain of attraction of a stable law.

We first recall some definition. Consider a sequence of i.i.d. random variables $(X_n)_{n\geqslant 1}$ with (common) distribution F (not necessarily lattice), and form as before the partial sums $S_n = X_1 + \cdots + X_n$. Let G be a distribution.

Definition 2.14 *The* **domain of attraction** *of G is the set of distributions F having the following property: there exists two sequences (a_n) and (b_n) of real numbers, with $b_n \to_n \infty$, such that*

$$\frac{S_n - a_n}{b_n} \xrightarrow{\mathcal{L}} G$$

as $n \to \infty$.

Though not relevant for us, we recall that G possesses a domain of attraction iff G is *stable*, i.e.

Definition 2.15 *A non-degenerate distribution G is* **stable** *if it satisfies the following property: let X_1 and X_2 be independent variables with distribution G; for any constants $a > 0$ and $b > 0$ the random variable $aX_1 + bX_2$ has the same distribution as $cX_1 + d$ for some constants $c > 0$ and d. Alternatively, G is stable if its characteristic function can be written as*

$$\varphi(t; \mu, c, \alpha, \beta) = \exp\left[\, it\mu - |ct|^\alpha \,(1 - i\beta\, sgn(t)\Phi)\,\right]$$

where $\alpha \in (0, 2]$, $\mu \in \mathbb{R}$, $\beta \in [-1, 1]$; $sgn(t)$ is just the sign of t and

$$\Phi = \begin{cases} \tan \frac{\pi\alpha}{2} & \text{if } \alpha \neq 1 \\ -\frac{2}{\pi} \log |t| & \text{if } \alpha = 1. \end{cases}$$

The parameter α is the **exponent** *of the distribution.*

Remark 2.16 *The normal law is stable with exponent $\alpha = 2$.*

The complete Local Limit Theorem reads as follows (see [5], Th. 4.2.1)

Theorem 2.17 *Let X_n have lattice distribution with maximal span Λ. In order that, for some choice of constants a_n and b_n*

$$\lim_{n\to\infty} \sup_{N \in \mathcal{L}(na,\lambda)} \left| \frac{b_n}{\lambda} P(S_n = N) - g\Big(\frac{N - a_n}{b_n}\Big) \right| = 0,$$

where g is the density of some stable distribution G with exponent $0 < \alpha \leq 2$,

it is necessary and sufficient that

(i) the common distribution F of the X_n belongs to the domain of attraction of G;

(ii) $\lambda = \Lambda$ (i.e. maximal).

2.2 Some proofs

The proofs of the present section follow [7], pp. 129–134.

Proof of Theorem 2.2 If $-n \le h \le n$, then $S_{2n} = 2h$ if and only if $n+h$ variables resulted in 1 and $n-h$ in -1, which gives

$$\sqrt{n}P(S_{2n} = 2h) = \sqrt{n}\binom{2n}{n+h}\frac{1}{2^{2n}} = \frac{\sqrt{n}}{2^{2n}}\frac{(2n)!}{(n+h)!(n-h)!}.$$

We have to calculate the second member for $h = h_n \sim \frac{x}{\sqrt{2}}\sqrt{n}$, which implies that both $n-h_n$ and $n+h_n$ go to infinity as $n \to \infty$; thus we can apply Stirling formula and with a little algebra we obtain

$$\frac{\sqrt{n}}{2^{2n}}\frac{(2n)!}{(n+h_n)!(n-h_n)!} \sim \frac{1}{\sqrt{\pi}}\sqrt{\frac{n^2}{n^2-h_n^2}}\Big(1+\frac{h_n^2}{n^2-h_n^2}\Big)^n \cdot \Big(1-\frac{2h_n}{n+h_n}\Big)^{h_n},$$

and now it's an easy exercise to verify that

$$\sqrt{\frac{n^2}{n^2-h_n^2}} \to 1;$$

$$\Big(1+\frac{h_n^2}{n^2-h_n^2}\Big)^n \to e^{\frac{x^2}{2}};$$

$$\Big(1-\frac{2h_n}{n+h_n}\Big)^{h_n} \to e^{-x^2}.$$

□

Proof of Theorem 2.4 We prove first that

$$\{t>0 : |\phi(t)| = 1\} = \Big\{t > 0 : P\Big(X \in \mathcal{L}\big(a, \frac{2\pi}{t}\big)\Big) = 1 \text{ for some } a\Big\} =: \mathcal{E}. \tag{3}$$

Thus the case in (i) means that X has a lattice distribution and moreover

$$\max\{\lambda > 0 : \exists\, a \text{ with } P(X \in \mathcal{L}(a,\lambda)) = 1\} = \max\{\lambda > 0 : \lambda = \frac{2\pi}{t}, t \in \mathcal{E}\} = \frac{2\pi}{t_0}.$$

Let's prove (3). Assume first that $P\big(X \in \mathcal{L}(a, \frac{2\pi}{t})\big) = 1$ for some a. Then

$$\phi(t) = \mathbf{E}[e^{itX}] = \sum_{k\in\mathbb{Z}} P\Big(X = a + \frac{2\pi}{t}k\Big)e^{it(a+\frac{2\pi}{t}k)} = e^{ita}\sum_{k\in\mathbb{Z}} P\Big(X = a+\frac{2\pi}{t}k\Big)e^{i2\pi k} = e^{ita}.$$

Conversely, assume that $|\phi(t)| = 1$ and let $U = \cos tX$, $V = \sin tX$. Then

$$\mathbf{E}[U^2] + \mathbf{E}[V^2] = \mathbf{E}[U^2+V^2] = 1 = |\phi(t)|^2 = \Big|\mathbf{E}[U] + i\mathbf{E}[V]\Big|^2 = \mathbf{E}^2[U] + \mathbf{E}^2[V],$$

which means that

$$\Big(\mathbf{E}[U^2] - \mathbf{E}^2[U]\Big) + \Big(\mathbf{E}[V^2] - \mathbf{E}^2[V]\Big) = 0.$$

By Cauchy–Schwartz inequality, the two summands are non–negative, hence they are both equal to 0. By the second part of the the same Cauchy–Schwartz inequality ($\mathbf{E}^2[ST] = \mathbf{E}[S^2]\mathbf{E}[T^2]$ if and only if $\alpha S + \beta T = 0$ for some α and β), we deduce (taking $S = U$ and $T = 1$) that $U = c_1$ and $V = c_2$, where c_1 and c_2 are two constants such that $c_1^2 + c_2^2 = 1$. So there exists α such that $c_1 = \cos\alpha$ and

$c_2 = \sin\alpha$ and the equalities $U = c_1$ and $V = c_2$ are equivalent to $e^{itX} = e^{i\alpha}$; this means that the value of tX is one of the numbers in $\mathcal{L}(\alpha, 2\pi)$, so that $P\Big(X \in \mathcal{L}(\frac{\alpha}{t}, \frac{2\pi}{t})\Big) = P(tX \in \mathcal{L}(\alpha, 2\pi)) = 1$.

We pass to prove the tricotomy. Assume that (ii) doesn't hold, which means that there exists $t_0 \neq 0$ such that $|\phi(t_0)| = 1$. Since $|\phi(-t_0)| = |\overline{\phi(t_0)}| = |\phi(t_0)| = 1$, without loss of generality we can assume that $t_0 > 0$. Put

$$\mathcal{E} = \{t > 0 : |\phi(t)| = 1\}.$$

Two cases are possible (a) $\tau := \inf \mathcal{E} > 0$. This implies that

$$|\phi(\tau)| = \lim_{t_n \in \mathcal{E} \downarrow \tau} |\phi(t_n)| = 1;$$

moreover, by the infimum property of τ, $|\phi(t)| < 1$ if $0 < t < \tau$. Hence case (a) corresponds to (i).
(b) $\tau := \inf \mathcal{E} = 0$. Then there exists a sequence $(t_n)_{n\geqslant 1}$ such that $t_n \downarrow 0$, and $|\phi(t_n)| = 1$. From the first part of the proof it follows that there exists a sequence a_n such that $P\Big(X \in \mathcal{L}(a_n, \frac{2\pi}{t_n})\Big) = 1$. With no loss of generality we can assume that $-\frac{\pi}{t_n} < a_n \leqslant \frac{\pi}{t_n}$ (if not, just put $\tilde{a}_n = a_n + k_n \frac{2\pi}{t_n}$, where $k_n \in \mathbb{Z}$ is such that $-\frac{\pi}{t_n}(2k_n + 1) < a_n \leqslant -\frac{\pi}{t_n}(2k_n - 1)$. Then $\mathcal{L}(a_n, \frac{2\pi}{t_n}) = \mathcal{L}(\tilde{a}_n, \frac{2\pi}{t_n})$ and $-\frac{\pi}{t_n} < \tilde{a}_n \leqslant \frac{\pi}{t_n}$).

Now observe that $P(X = a_n) = P\Big(X \in \Big(-\frac{\pi}{t_n}, \frac{\pi}{t_n}\Big]\Big) \to P(X \in \mathbb{R}) = 1$ (since $t_n \to 0$); this implies that $P(X = a) = 1$ for some a: in fact, let a be such that $P(X = a) > 0$; then the set $E = \{n \in \mathbb{N} : a_n = a\}$ is infinite (if not, let $N = \max E$; then, for every $n > N$ we have $\{X = a_n\} \subseteq \{X \neq a\}$, hence $1 = \lim_n P(X = a_n) \leqslant P(X \neq a)$). Hence

$$P(X = a) = \lim_{n\to\infty, n\in E} P(X = a_n) = 1.$$

□

Proof of Theorem 2.8. With no loss of generality we can assume that $\mu = 0$ (just consider the centered sums $S_n - n\mu$). We put $N = x\sqrt{n}$ and for the sake of brevity we denote

$$p_n(x) = P(S_n = \sqrt{n}x), \qquad x \in \mathcal{L}_n := \frac{\mathcal{L}(na, \Lambda)}{\sqrt{n}};$$

$$\eta(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}, \qquad x \in \mathbb{R}.$$

Recall the inversion formula for a lattice random variable Y with span θ and characteristic function ψ:

$$P(Y = x) = \frac{\theta}{2\pi} \int_{-\frac{\pi}{\theta}}^{\frac{\pi}{\theta}} e^{-itx}\psi(t)dt.$$

We want to apply it to $Y = \frac{S_n}{\sqrt{n}}$, $\theta = \frac{\Lambda}{\sqrt{n}}$, $\psi(t) = \phi^n\Big(\frac{t}{\sqrt{n}}\Big)$. We obtain

$$\frac{\sqrt{n}}{\Lambda} p_n(x) = \frac{\sqrt{n}}{\Lambda} P\Big(\frac{S_n}{\sqrt{n}} = x\Big) = \frac{1}{2\pi} \int_{-\frac{\pi\sqrt{n}}{\Lambda}}^{\frac{\pi\sqrt{n}}{\Lambda}} e^{-itx} \phi^n\Big(\frac{t}{\sqrt{n}}\Big) dt. \tag{4}$$

Recall the inversion formula for absolutely continuous random variable with density f and characteristic function ψ:

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-itx}\psi(t)dt.$$

We apply it to $Y \sim \mathcal{N}(0,\sigma^2)$ and obtain

$$\eta(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-itx} e^{-\frac{t^2\sigma^2}{2}}\, dt. \tag{5}$$

Subtracting (5) from (4) and using the inequality $|e^{-itx}| \leqslant 1$ we get,

$$\Big|\frac{\sqrt{n}}{\Lambda} p_n(x) - \eta(x)\Big| \leqslant \frac{1}{2\pi}\Big\{\int_{-\frac{\pi\sqrt{n}}{\Lambda}}^{\frac{\pi\sqrt{n}}{\Lambda}} \big|\phi^n\Big(\frac{t}{\sqrt{n}}\Big) - e^{-\frac{t^2\sigma^2}{2}}\big|dt + 2\int_{\frac{\pi\sqrt{n}}{\Lambda}}^{\infty} e^{-\frac{t^2\sigma^2}{2}} dt\Big\}, \tag{6}$$

and notice that the second member doesn't depend on x. Moreover, by the integrability of $t \mapsto e^{-\frac{t^2\sigma^2}{2}}$, we have immediately

$$\lim_{n\to\infty} \int_{\frac{\pi\sqrt{n}}{\Lambda}}^{\infty} e^{-\frac{t^2\sigma^2}{2}} dt = 0,$$

hence it remains to prove that also the first integral goes to 0 as $n \to \infty$. First, putting $\Gamma_n = (-\frac{\pi\sqrt{n}}{\Lambda}, \frac{\pi\sqrt{n}}{\Lambda})$ we write

$$\int_{-\frac{\pi\sqrt{n}}{\Lambda}}^{\frac{\pi\sqrt{n}}{\Lambda}} \big|\phi^n\Big(\frac{t}{\sqrt{n}}\Big) - e^{-\frac{t^2\sigma^2}{2}}\big|dt = \int_{\mathbb{R}} 1_{\Gamma_n}(t)\big|\phi^n\Big(\frac{t}{\sqrt{n}}\Big) - e^{-\frac{t^2\sigma^2}{2}}\big|dt = \int_{-A}^{A} + \int_{\mathbb{R}\setminus(-A,A)}, \tag{7}$$

for every constant $A > 0$. For the first integral we have

$$\int_{-A}^{A} 1_{\Gamma_n}(t)\big|\phi^n\Big(\frac{t}{\sqrt{n}}\Big) - e^{-\frac{t^2\sigma^2}{2}}\big|dt \leqslant \int_{-A}^{A} \big|\phi^n\Big(\frac{t}{\sqrt{n}}\Big) - e^{-\frac{t^2\sigma^2}{2}}\big|dt \to 0, \tag{8}$$

since $\phi^n\Big(\frac{t}{\sqrt{n}}\Big) \to e^{-\frac{t^2\sigma^2}{2}}$ as $n \to \infty$ (as in the proof of the CLT) and $|\phi^n\Big(\frac{t}{\sqrt{n}}\Big) - e^{-\frac{t^2\sigma^2}{2}}| \leqslant 2$, so that we can use the dominatd convergence theorem.

We pass to consider the second integral in (7).

Recall that the characteristic function ψ of a random variable Y having n–th moment verifies

$$\Big|\psi(u) - \sum_{k=0}^{n} \mathbf{E}\Big[\frac{(iuY)^k}{k!}\Big]\Big| \leqslant \mathbf{E}\Big[\min\Big\{\frac{|uY|^{n+1}}{(n+1)!}, \frac{2|uY|^n}{n!}\Big\}\Big]$$

(see [7], formula (3.7), p.101). Applying for $\psi = \phi$ and $n = 2$ ($\mathbf{E}[X_1] = 0$ and $\mathbf{E}[X_1^2] = \sigma^2$) we get

$$\Big|\phi(u) - 1 + \frac{u^2\sigma^2}{2}\Big| \leqslant \frac{u^2}{3!}\mathbf{E}[\min\{|u||X_1|^3, 6|X_1|^2\}];$$

thus by the triangular inequality

$$\phi(u)| \leqslant \Big|\phi(u) - 1 + \frac{u^2\sigma^2}{2}\Big| + \Big|1 - \frac{u^2\sigma^2}{2}\Big| \leqslant \Big|1 - \frac{u^2\sigma^2}{2}\Big| + \frac{u^2}{2}\mathbf{E}[\min\{|u||X_1|^3, 6|X_1|^2\}]. \tag{9}$$

Since $\min\{|u||X_1|^3, 6|X_1|^2\} \to 0$ as $u \to 0$ and $\min\{|u||X_1|^3, 6|X_1|^2\} \leqslant 6|X_1|^2$, by the dominated convergence theorem we get $\mathbf{E}[\min\{|u||X_1|^3, 6|X_1|^2\}] \to 0$ as $u \to 0$, so we can pick $\delta > 0$ such that $\mathbf{E}[\min\{|u||X_1|^3, 6|X_1|^2\}] \leqslant \frac{\sigma^2}{2}$ for $|u| < \delta$, which by (9) implies

$$|\phi(u)| \leqslant \Big|1 - \frac{u^2\sigma^2}{2}\Big| + \frac{u^2}{2}\cdot\frac{\sigma^2}{2}.$$

By choosing $\delta < \frac{\sqrt{2}}{\sigma}$ the above becomes

$$|\phi(u)| \leqslant 1 - \frac{u^2\sigma^2}{2} + \frac{\sigma^2 u^2}{4} = 1 - \frac{u^2\sigma^2}{4} \leqslant e^{-\frac{u^2\sigma^2}{4}}, \tag{10}$$

by the elementary inequality $1-z \leqslant e^{-z}$. We apply (10) with $u = \frac{t}{\sqrt{n}}$ and obtain

$$\left|\phi\Big(\frac{t}{\sqrt{n}}\Big)\right|^n \leqslant e^{-\frac{t^2\sigma^2}{4}}, \qquad |t| \leqslant \delta\sqrt{n}.$$

Hence, putting $\Delta_n = (-\delta\sqrt{n}, \delta\sqrt{n})$, we have (second summand in (7))

$$\int_{\{\mathbb{R}\setminus(-A,A)} 1_{\Gamma_n}(t)\Big|\phi^n\Big(\frac{t}{\sqrt{n}}\Big) - e^{-\frac{t^2\sigma^2}{2}}\Big|dt \leqslant$$
$$\int_{\{\mathbb{R}\setminus(-A,A)\}\cap\Delta_n} \Big|\phi^n\Big(\frac{t}{\sqrt{n}}\Big) - e^{-\frac{t^2\sigma^2}{2}}\Big|dt + \int_{\{\mathbb{R}\setminus(-A,A)\}\cap\Delta_n^c} 1_{\Gamma_n}(t)\Big|\phi^n\Big(\frac{t}{\sqrt{n}}\Big) - e^{-\frac{t^2\sigma^2}{2}}\Big|dt, \tag{11}$$

and now

$$\int_{\mathbb{R}\setminus(-A,A)\}\cap\Delta_n} \Big|\phi^n\Big(\frac{t}{\sqrt{n}}\Big) - e^{-\frac{t^2\sigma^2}{2}}\Big|dt \leqslant \int_{\{\mathbb{R}\setminus(-A,A)\}\cap\Delta_n} \Big|\phi\Big(\frac{t}{\sqrt{n}}\Big)\Big|^n dt + \int_{\{\mathbb{R}\setminus(-A,A)\}\cap\Delta_n} e^{-\frac{t^2\sigma^2}{2}}dt$$
$$\leqslant \int_{\{\mathbb{R}\setminus(-A,A)\}\cap\Delta_n} e^{-\frac{t^2\sigma^2}{2}}dt + \int_{\{\mathbb{R}\setminus(-A,A)\}\cap\Delta_n} e^{-\frac{t^2\sigma^2}{2}}dt \leqslant 2\int_A^\infty e^{-\frac{t^2\sigma^2}{2}}dt. \tag{12}$$

We pass to evaluate the second integral in (11). Notice that $\frac{|t|}{\sqrt{n}} \leqslant \frac{\pi}{\Lambda} < \frac{2\pi}{\Lambda}$ for $t \in \Gamma_n$, hence, by Theorem 2.4 (i) and the continuity of $|\phi|$,

$$\sup_{t\in\Gamma_n\cap\Delta_n^c} \Big|\phi(\frac{t}{\sqrt{n}})\Big| \leqslant \sup_{\delta<|t|\leq\frac{\pi}{\Lambda}} |\phi(t)| =: \rho < 1.$$

This implies that

$$\int_{\{\mathbb{R}\setminus(-A,A)\}\cap\Delta_n^c} 1_{\Gamma_n}(t)\Big|\phi^n\Big(\frac{t}{\sqrt{n}}\Big) - e^{-\frac{t^2\sigma^2}{2}}\Big|dt \leqslant \int_{\{\mathbb{R}\setminus(-A,A)\}} 1_{\Gamma_n}(t)\Big|\phi^n\Big(\frac{t}{\sqrt{n}}\Big)\Big|^n dt + \int_{\{\mathbb{R}\setminus(-A,A)\cap\Delta_n^c\}} e^{-\frac{t^2\sigma^2}{2}}dt$$
$$\leqslant \frac{\pi}{\Lambda}\sqrt{n}\rho^n + 2\int_{\delta\sqrt{n}}^\infty e^{-\frac{t^2\sigma^2}{2}}dt \to 0. \tag{13}$$

¿From (7),(8), (11), (12), (13) we conclude that

$$\limsup_{n\to\infty} \int_{-\frac{\pi\sqrt{n}}{\Lambda}}^{\frac{\pi\sqrt{n}}{\Lambda}} \Big|\phi^n\Big(\frac{t}{\sqrt{n}}\Big) - e^{-\frac{t^2\sigma^2}{2}}\Big|dt \leqslant 2\int_A^\infty e^{-\frac{t^2\sigma^2}{2}}dt,$$

for every $A > 0$, and by the arbitrariness of A we deduce

$$\lim_{n\to\infty} \int_{-\frac{\pi\sqrt{n}}{\Lambda}}^{\frac{\pi\sqrt{n}}{\Lambda}} \Big|\phi^n\Big(\frac{t}{\sqrt{n}}\Big) - e^{-\frac{t^2\sigma^2}{2}}\Big|dt = 0.$$

□

Proof of Theorem 2.12. We shall assume $\mu = 0$, and we denote as before $\eta(x) = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{x^2}{2\sigma^2}}$. Let $\delta > 0$ and consider the *Polya's density*

$$h_0(y) = \frac{1}{\pi}\cdot\frac{1-\cos(\delta y)}{\delta y^2}.$$

Its Fourier transform is

$$\hat{h}_0(u) = \int e^{iuy}h_0(y)dy = \begin{cases} 1 - \left|\frac{u}{\delta}\right|, & \text{for } |u| \leq \delta \\ 0, & \text{otherwise .} \end{cases}$$

Now put $h_\theta(y) = e^{i\theta y} h_0(y)$, so that

$$\hat{h}_\theta(u) = \int e^{iuy} h_\theta(y) dy = \int e^{i(u+\theta)y} h_0(y)\, dy = \hat{h}_0(u+\theta). \tag{14}$$

Claim

$$\lim_{n\to\infty} \sqrt{n}\mathbf{E}\big[h_\theta(S_n - x_n)\big] = \eta(x) \int h_\theta(y) dy \qquad \forall \theta.$$

Before proving the claim, we show that it gives our statement. Let μ_n be the law of $S_n - x_n$. Put

$$\alpha_n = \int h_0(y)\mu_n(dy) = \mathbf{E}\big[h_0(S_n - x_n)\big]$$

and define the probability measures

$$\nu_n(B) = \frac{1}{\alpha_n}\int_B h_0(y)\mu_n(dy); \qquad \nu(B) = \int_B h_0(y) dy.$$

Now, by the claim, for all θ,

$$\int e^{i\theta y}\nu_n(dy) = \frac{1}{\alpha_n}\int e^{i\theta y} h_0(y)\mu_n(dy) = \frac{\sqrt{n}}{\sqrt{n}\alpha_n}\mathbf{E}\big[e^{i\theta(S_n - x_n)} h_0(S_n - x_n)\big]$$
$$= \frac{\sqrt{n}\mathbf{E}\big[h_\theta(S_n - x_n)\big]}{\sqrt{n}\mathbf{E}\big[h_0(S_n - x_n)\big]} \to \int h_\theta(y) dy = \int e^{i\theta y} h_0(y)\, dy = \int e^{i\theta y}\nu(dy).$$

This relation says that the characteristic function of ν_n converges to the characteristic function of ν, hence, by the continuity theorem, that $\nu_n \Rightarrow \nu$.

Now, for $|a|$ and $|b| < \frac{2\pi}{\delta}$, consider the function

$$\kappa(y) = \frac{1}{h_0(y)} 1_{(a,b)}(y).$$

Notice that $h_0(y) \neq 0$ for $y < \frac{2\pi}{\delta}$. Moreover the set $\{a, b\}$ of discontinuity points of κ is negligible with respect to ν; hence

$$\frac{P(S_n - x_n \in (a,b))}{\alpha_n} = \frac{1}{\alpha_n}\int 1_{(a,b)}(y)\mu_n(dy) = \frac{1}{\alpha_n}\int \kappa(y) h_0(u)\mu_n(dy)$$
$$= \int \kappa(y)\nu_n(dy) \to \int \kappa(y)\nu(dy) = \int 1_{(a,b)}(y) dy = b - a.$$

By the claim,

$$\sqrt{n}\alpha_n = \sqrt{n}\mathbf{E}\big[h_0(S_n - x_n)\big] \to \eta(x),$$

hence

$$\sqrt{n}P(S_n - x_n \in (a,b)) = \sqrt{n}\alpha_n \cdot \frac{P(S_n - x_n \in (a,b))}{\alpha_n} \to \eta(x)(b-a),$$

which is the conclusion.

So let's prove the claim. The inversion formula for Fourier transform and (14) give

$$h_\theta(x) = e^{i\theta x} h_0(x) = \frac{1}{2\pi}\int e^{-i(u-\theta)x}\hat{h}_0(u)\, du = \frac{1}{2\pi}\int e^{-ivx}\hat{h}_0(v+\theta)\, dv = \frac{1}{2\pi}\int e^{-ivx}\hat{h}_\theta(v)\, dv.$$

Let's integrate this relation with respect to μ_n (law of $S_n - n$). We find

$$\mathbf{E}\big[h_\theta(S_n - x_n)\big] = \frac{1}{2\pi}\int \mu_n(dx)\int e^{-ivx}\hat{h}_\theta(v)\,dv = \frac{1}{2\pi}\int dv\hat{h}_\theta(v)\int e^{-ivx}\mu_n(dx)$$
$$= \frac{1}{2\pi}\int \hat{h}_\theta(v)\phi^n(-v)e^{ivx_n}\,dv; \tag{15}$$

notice that we are allowed to interchange the order of integration since $e^{-ivx}\hat{h}_\theta(v)$ is bounded and has compact support, hence it is integrable with respect to the product measure $\mu_n \otimes \lambda$ (λ = Lebesgue measure). Notice also that $\int e^{-ivx}\mu_n(dx)$ is the characteristic function of $S_n - x_n$ calculated in $-v$, hence it equals $\phi^n(-v)e^{ivx_n}$. Now we pass to the limit with respect to n in (15). Let $[-M, M]$ be an interval containing the support of $\hat{h}_\theta$; let $\delta > 0$ be such that, for $|v| \leqslant \delta$,

$$|\phi(v)| \leqslant e^{-\frac{\sigma^2 v^2}{2}}$$

(we have seen in the course of the preceding proof that such δ exists); split the integral $\int \hat{h}_\theta(v)\phi^n(-v)e^{ivx_n}\,dv$ into the two summands (i) $\int_{\delta<|v|<M}$ and (ii)$\int_{-\delta}^{\delta}$.

(i) Since $|\hat{h}_\theta(v)| \leq 1$ we have

$$\sqrt{n}\int_{\delta<|v|<M} \leqslant \int_{\delta<|v|<M} |\phi^n(-v)|dv \leq (2M)\sqrt{n}\Big(\underbrace{\{\sup_{\delta<|v|<M}|\phi(v)|\}}_{=\gamma<1}\Big)^n = 2M\sqrt{n}\gamma^n \to 0, \qquad n\to\infty.$$

(ii) By the change of variable $v = \frac{t}{\sqrt{n}}$ we obtain

$$\frac{1}{2\pi}\sqrt{n}\int_{-\delta}^{\delta}\hat{h}_\theta(v)\phi^n(-v)e^{ivx_n}\,dv = \frac{1}{2\pi}\int_{-\delta\sqrt{n}}^{\delta\sqrt{n}}\hat{h}_\theta\Big(\frac{t}{\sqrt{n}}\Big)\phi^n\Big(-\frac{t}{\sqrt{n}}\Big)e^{i\frac{tx_n}{\sqrt{n}}}\,dt$$
$$= \frac{1}{2\pi}\int_{\mathbb{R}} 1_{(-\delta\sqrt{n},\delta\sqrt{n})}\hat{h}_\theta\Big(\frac{t}{\sqrt{n}}\Big)\phi^n\Big(-\frac{t}{\sqrt{n}}\Big)e^{i\frac{tx_n}{\sqrt{n}}}\,dt.$$

The integrand is bounded by the integrable function $e^{-\frac{\sigma^2t^2}{4}}$ and, as $n \to \infty$, it converges to $e^{-\frac{\sigma^2t^2}{2}}e^{itx}\hat{h}_\theta(0)$, so that we can use the dominated convergence theorem and find that its limit is

$$\hat{h}_\theta(0)\Big(\frac{1}{2\pi}\int_{\mathbb{R}} e^{-\frac{\sigma^2t^2}{2}}e^{itx}\,dt\Big) = \hat{h}_\theta(0)\eta(x) = \eta(x)\int h_\theta(y)\,dy,$$

by the inversion formula and the definition of $\hat{h}_\theta$ ($\hat{h}_\theta(0) = \int e^{iy0}h_\theta(y)\,dy = \int h_\theta(y)\,dy$). The claim and the Theorem are proved.

□

3 The Almost Sure Local Limit Theorem

3.1 The motivation

We recall the Classical Almost Sure Central Limit Theorem (originally proved in [8] and [9]). Let $(X_n)_{n\geqslant 1}$ be i.i.d with $\mathbf{E}[X_1] = \mu$, $\mathbf{Var}X_1 = \sigma^2$, and set

$$Z_n = \frac{S_n - n\mu}{\sigma\sqrt{n}}.$$

The **Central Limit Theorem** states that, for every $x \in \mathbb{R}$

$$\mathbf{E}[1_{\{Z_n \leqslant x\}}] = P(Z_n \leqslant x) \xrightarrow[n]{} \Phi(x) := \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}\, dt$$

The **Almost Sure Central Limit Theorem** states that, $P-$a. s., for every $x \in \mathbb{R}$

$$\frac{1}{\log n} \sum_{h=1}^{n} \frac{1}{h} 1_{\{Z_h \leqslant x\}} \xrightarrow[n]{} \Phi(x).$$

Let's proceed by analogy. We treat first the case of Theorems 2.8–2.9 (i.e. in which the variables are in the domain of attraction of the normal law), which is completely settled.

Let $\kappa_n \in \mathcal{L}(na, \Lambda)$ be such that

$$\frac{\kappa_n - n\mu}{\sqrt{n}} \to \kappa.$$

Theorems 2.8–2.9 imply that

$$\mathbf{E}[\sqrt{n} 1_{\{S_n = \kappa_n\}}] = \sqrt{n} P(S_n = \kappa_n) \to \Lambda \eta(\kappa),$$

where as usual

$$\eta(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}.$$

Thus, comparing with the case of the Central Theorem, a tentative **Almost Sure Local Limit Theorem** (**ASLLT** from now on) should state that $P-$a. s.,

$$\frac{1}{\log n} \sum_{h=1}^{n} \frac{1}{h} \big(\sqrt{h} 1_{\{S_h = \kappa_h\}}\big) = \frac{1}{\log n} \sum_{h=1}^{n} \frac{1}{\sqrt{h}} 1_{\{S_h = \kappa_h\}} \xrightarrow[n]{} \Lambda \eta(\kappa).$$

Some history:

- In 1951 Chung and Erdös proved in [14]

Theorem 3.1 *Let $(X_n)_{n \geqslant 1}$ be a centered Bernoulli process with parameter p. Then*

$$\frac{1}{\log n} \sum_{h=1}^{n} \frac{1}{\sqrt{h}} 1_{\{S_h = 0\}} \xrightarrow[n]{} \frac{1}{\sqrt{2\pi p(1-p)}}, \qquad a.s.$$

- In 1993 Csáki, Földes and Révész proved in [15]

Theorem 3.2 *Let $(X_n)_{n \geqslant 1}$ be i.i.d. centered and with finite third moment. Then*

$$\frac{1}{\log n} \sum_{h=1}^{n} \frac{1}{p_h} 1_{\{a_h \leqslant S_h \leqslant b_h\}} \xrightarrow[n]{} 1, \qquad a.s.$$

where $p_n = P(a_n \leqslant S_n \leqslant b_n)$.

We notice that

- Theorem 3.1 is a particular case of our tentative ASLLT: just take $\kappa_n = np$.
- Theorem 3.2 generalizes Theorem 3.1: just take $a_n = b_n = 0$ and recall Theorems 2.8–2.9.

3.2 The ASLLT for random sequences in the domain of attraction of the normal law

Let $(X_n)_{n\geqslant 1}$ be i.i.d. having lattice distribution F with maximal span Λ; assume that $\mathbf{E}[X_1] =: \mu$, $\mathbf{Var}X_1 =: \sigma^2$ are finite. Throughout the following discussion, we shall always assume $\mu = 0$ and $\sigma^2 = 1$. This will cause no loss of generality.

Definition 3.3 *We say that the sequence $(X_n)_{n\geqslant 1}$ satisfies an ASLLT if*

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{h=1}^{n} \frac{1}{\sqrt{h}} 1_{\{S_h=\kappa_h\}} \stackrel{a.s.}{=} \Lambda\eta(\kappa),$$

for any sequence of integers $(\kappa_n)_{n\geqslant 1}$ in $\mathcal{L}(na,\Lambda)$ such that

$$\lim_{n\to\infty} \frac{\kappa_n - n\mu}{\sqrt{n}} = \kappa.$$

The first result concerns the situation of Theorem 2.10. It has been proved in [11]. Here is the statement:

Theorem 3.4 (ASLLT with rate) *Let $\epsilon > 0$ and assume that $\mathbf{E}\big[|X_1^{2+\epsilon}|\big] < \infty$. Then $(X_n)_{n\geqslant 1}$ satisfies an ASLLT. Moreover, if the sequence $(\kappa_n)_{n\geqslant 1}$ verifies the stronger condition*

$$\frac{\kappa_n - n\mu}{\sqrt{n}} = \kappa + O_\delta\big((\log n)^{-1/2+\delta}\big)$$

then

$$\sum_{h=1}^{n} \frac{1}{\sqrt{h}} 1_{\{S_h=\kappa_h\}} = \Lambda\eta(\kappa) + O_\delta\big((\log n)^{-1/2+\delta}\big).$$

Remark 3.5 *If $\mathbf{E}\big[|X_1^{2+\epsilon}|\big] < \infty$ for some positive ϵ, then the condition of Theorem 2.10*

$$\sup_{N\in\mathcal{L}(na,\lambda)} \Big|\frac{\sqrt{n}}{\lambda} P(S_n = N) - \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(N-n\mu)^2}{2n\sigma^2}}\Big| = O(n^{-\alpha}), \qquad 0 < \alpha < \frac{1}{2}$$

is satisfied. In fact, since the span is maximal, by Theorem 2.10 all we have to check is that as $u \to \infty$,

$$\int_{|x|\geqslant u} x^2 F(dx) = O(u^{-2\alpha}),$$

which is true with $\alpha = \epsilon/2$ since

$$\int_{|x|\geqslant u} x^2 F(dx) = \int_{|x|\geqslant u} |x|^{2+\epsilon}|x|^{-\epsilon} F(dx) \leqslant \mathbf{E}[|X_1^{2+\epsilon}|]u^{-\epsilon}.$$

The key ingredients for the proof of Theorem 3.4 are
(i) a suitable correlation inequality, which will be illustrated later;
(ii) Theorem 2.10;
(iii) the notion of quasi orthogonal system.

Definition 3.6 *A sequence of functions $\Psi := (f_n)_{n\geqslant 1}$ defined on a Hilbert space $\mathcal{H}$ is said* **quasi–orthogonal** *if the quadratic form on ℓ^2: $(x_n) \mapsto \sum_{h,k}\langle f_h, f_k\rangle x_h x_k$ is bounded (as a quadratic form).*

A useful criterion for quasi–orthogonality is furnished by the following result (see lemma 7.4.3 in [12]; see also [13], p. 23).

Lemma 3.7 *In order that* $\Psi := (f_n)_{n\geqslant 1}$ *be a quasi–orthogonal system, it is sufficient that*

$$\sup_h \sum_k |\langle f_h, f_k\rangle| < \infty.$$

Remark 3.8 *If* $\mathcal{H} = L^2(T)$, *where* $(T, \mathcal{A}, \mu)$ *is some probability space, then* $\sum_{h,k}\langle f_h, f_k\rangle x_h x_k = \sum_{h,k}\big(\int f_h f_k \, d\mu\big) x_h x_k$. *By Rademacher–Menchov Theorem, it is seen that the series* $\sum_n c_n f_n$ *converges if for instance* $c_n = n^{-\frac{1}{2}}(\log n)^{-b}$ *with* $b > \frac{3}{2}$ *(for more information on this point, see [10]).*

Now we present the basic correlation inequality. Let $Y_h = \sqrt{h}\big(1_{\{S_h=\kappa_h\}} - P(S_h = \kappa_h)\big)$. We put moreover when necessary

$$b_n = \frac{\kappa_n}{\sqrt{2n}}, \qquad M = \sup_n |b_n|.$$

Proposition 3.9 *Assume that the condition of Theorem 2.10 is satisfied, i.e.*

$$r(n) := \sup_{N\in\mathcal{L}(na,\Lambda)} \Big|\frac{\sqrt{n}}{\Lambda}P(S_n = N) - \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(N-n\mu)^2}{2n\sigma^2}}\Big| = O(n^{-\alpha}), \qquad 0 < \alpha < \frac{1}{2} \tag{16}$$

Then there exists a constant C *such that, for all integers* m, n *with* $1 \le m < n$

$$\big|\mathbf{Cov}(Y_m, Y_n)\big| \le C\Big(\frac{1}{\sqrt{\frac{n}{m}} - 1} + \sqrt{\frac{n}{n-m}}\cdot\frac{1}{(n-m)^\alpha}\Big).$$

Proof. By independence and equidistribution,

$$\mathbf{Cov}(Y_m, Y_n) = \mathbf{E}[Y_m \cdot Y_n] = \sqrt{m}P(S_m = \kappa_m)\sqrt{n}\big(P(S_{n-m} = \kappa_n - \kappa_m) - P(S_n = \kappa_n)\big).$$

By the assumption (16), we have

$$\sup_m \sqrt{m}P(S_m = \kappa_m) \le C < \infty.$$

Now, if $A := \sqrt{n}\big(P(S_{n-m} = \kappa_n - \kappa_m) - P(S_n = \kappa_n)\big)$, we have

$$\begin{aligned}
|A| &\leqslant \sqrt{n}\Big|P(S_{n-m} = \kappa_n - \kappa_m) - \frac{\Lambda}{\sqrt{2\pi(n-m)}}e^{-\frac{(\kappa_n-\kappa_m)^2}{2(n-m)}}\Big| \\
&+ \Lambda\sqrt{n}\Big|\frac{1}{\sqrt{2\pi(n-m)}}e^{-\frac{(\kappa_n-\kappa_m)^2}{2(n-m)}} - \frac{1}{\sqrt{2\pi n}}e^{-\frac{\kappa_n^2}{2n}}\Big| \\
&+ \sqrt{n}\Big|\frac{\Lambda}{\sqrt{2\pi n}}e^{-\frac{\kappa_n^2}{2n}} - P(S_n = \kappa_n)\Big| \\
&= \sqrt{\frac{n}{n-m}}\Big|\sqrt{n-m}P(S_{n-m} = \kappa_n - \kappa_m) - \frac{\Lambda}{\sqrt{2\pi}}e^{-\frac{(\kappa_n-\kappa_m)^2}{2(n-m)}}\Big| \\
&+ \frac{\Lambda}{\sqrt{2\pi}}\Big|\sqrt{\frac{n}{n-m}}e^{-\frac{(\kappa_n-\kappa_m)^2}{2(n-m)}} - e^{-\frac{\kappa_n^2}{2n}}\Big| + \Big|\frac{\Lambda}{\sqrt{2\pi}}e^{-\frac{\kappa_n^2}{2n}} - \sqrt{n}P(S_n = \kappa_n)\Big| \\
&=: A_1 + A_2 + A_3.
\end{aligned}$$

By condition (16),

$$\begin{aligned}
A_1 &:= \sqrt{\frac{n}{n-m}}\,\Big|\sqrt{n-m}P(S_{n-m} = \kappa_n - \kappa_m) - \frac{\Lambda}{\sqrt{2\pi}}e^{-\frac{(\kappa_n-\kappa_m)^2}{2(n-m)}}\Big| \\
&\leqslant \Lambda\sqrt{\frac{n}{n-m}}\,r(n-m) \leqslant C\sqrt{\frac{n}{n-m}}\frac{1}{(n-m)^\alpha}.
\end{aligned}$$

Furthermore

$$A_3 := \Big|\frac{\Lambda}{\sqrt{2\pi}}e^{-\frac{\kappa_n^2}{2n}} - \sqrt{n}P(S_n=\kappa_n)\Big| \leqslant \Lambda r(n) \leqslant \frac{C}{\sqrt{n}} \leqslant C\frac{\sqrt{m}}{\sqrt{n}} \leqslant C\frac{\sqrt{m}}{\sqrt{n}-\sqrt{m}} = C\frac{1}{\sqrt{\frac{n}{m}}-1}.$$

It remains to estimate

$$A_2 := \frac{\Lambda}{\sqrt{2\pi}}\Big|\sqrt{\frac{n}{n-m}}e^{-\frac{(\kappa_n-\kappa_m)^2}{2(n-m)}} - e^{-\frac{\kappa_n^2}{2n}}\Big| \leqslant \frac{\Lambda}{\sqrt{2\pi}}\Big|\sqrt{\frac{n}{n-m}}e^{-\frac{(\kappa_n-\kappa_m)^2}{2(n-m)}} - e^{-\frac{(\kappa_n-\kappa_m)^2}{2(n-m)}}\Big|$$
$$+\frac{\Lambda}{\sqrt{2\pi}}\Big|e^{-\frac{(\kappa_n-\kappa_m)^2}{2(n-m)}} - e^{-\frac{\kappa_n^2}{2n}}\Big| = \frac{\Lambda}{\sqrt{2\pi}}e^{-\frac{(\kappa_n-\kappa_m)^2}{2(n-m)}}\Big(\sqrt{\frac{n}{n-m}}-1\Big) + \frac{\Lambda}{\sqrt{2\pi}}\Big|e^{-\frac{(\kappa_n-\kappa_m)^2}{2(n-m)}} - e^{-\frac{\kappa_n^2}{2n}}\Big| =: A_4 + A_5.$$

Now

$$\frac{\sqrt{2\pi}}{\Lambda}A_4 := e^{-\frac{(\kappa_n-\kappa_m)^2}{2(n-m)}}\Big(\sqrt{\frac{n}{n-m}}-1\Big) \leqslant \frac{\sqrt{n}-\sqrt{n-m}}{\sqrt{n-m}} = \frac{1}{\sqrt{n-m}}\cdot\frac{m}{\sqrt{n}+\sqrt{n-m}}$$
$$= \frac{1}{\sqrt{\frac{n}{m}-1}}\cdot\frac{\sqrt{m}}{\sqrt{n}+\sqrt{n-m}} \leqslant \frac{1}{\sqrt{\frac{n}{m}-1}} \leqslant \frac{1}{\sqrt{\frac{n}{m}}-1},$$

since $\sqrt{x}-\sqrt{y} \leqslant \sqrt{x-y}$ if $x \geqslant y \geqslant 0$. On the other hand, by using the inequality $|e^u - e^v| \leqslant |u-v|$ if $u \leqslant 0$ and $v \leqslant 0$, we have

$$\frac{\sqrt{2\pi}}{\Lambda}A_5 := \Big|e^{-\frac{(\kappa_n-\kappa_m)^2}{2(n-m)}} - e^{-\frac{\kappa_n^2}{2n}}\Big| \leqslant \Big| -\frac{(\kappa_n-\kappa_m)^2}{2(n-m)} + \frac{\kappa_n^2}{2n}\Big| =$$
$$\Big| -\frac{(\sqrt{n}b_n-\sqrt{m}b_m)^2}{n-m} + b_n^2\Big| = \Big|\frac{-nb_n^2 - mb_m^2 + 2\sqrt{mn}b_nb_m}{n-m} + b_n^2\Big|$$
$$= \Big|\frac{-nb_n^2 - mb_m^2 + 2\sqrt{mn}b_nb_m + nb_n^2 - mb_n^2}{n-m}\Big| = \Big|\frac{-b_m^2 + 2\sqrt{\frac{n}{m}}b_nb_m - b_n^2}{\frac{n}{m}-1}\Big|$$
$$= \Big|\frac{-(b_n-b_m)^2 + 2b_mb_n\big(\sqrt{\frac{n}{m}}-1\big)}{\frac{n}{m}-1}\Big| \leqslant \frac{2(b_n+b_m)^2 + 2|b_n||b_m|\big(\sqrt{\frac{n}{m}}-1\big)}{\frac{n}{m}-1}$$
$$\leqslant 2M^2\frac{2+\big(\sqrt{\frac{n}{m}}-1\big)}{\frac{n}{m}-1} = \frac{2M^2}{\sqrt{\frac{n}{m}}-1}.$$

□

Now we are ready to sketch the main steps of the proof of Theorem 3.4.

(i) Fix any $\rho > 1$. By means of the basic correlation inequality, we prove the quasi–orthogonality of the sequence

$$Z_j = \sum_{\rho^j \leqslant h < \rho^{j+1}} \frac{Y_h}{h},$$

where as before

$$Y_h = \sqrt{h}\big(1_{\{S_h=\kappa_h\}} - P(S_h=\kappa_h)\big).$$

(ii)By Remark 3.8, we obtain that the series

$$\sum_j \frac{Z_j}{\sqrt{j}(\log j)^b}$$

converges as soon as $b > \frac{3}{2}$.

(iii) By Kronecker's Lemma

$$\frac{1}{\sqrt{n}(\log n)^b}\sum_{j=1}^{n} Z_j = \frac{1}{\sqrt{n}(\log n)^b}\sum_{j=1}^{n}\sum_{\rho^j\leqslant h<\rho^{j+1}}\frac{Y_h}{h} = \frac{1}{\sqrt{n}(\log n)^b}\sum_{1\leqslant h<\rho^{n+1}}\frac{Y_h}{h}\xrightarrow[n]{} 0$$

(iv) The preceding relation yields easily (we omit the details)

$$\frac{\sqrt{\log t}}{(\log\log t)^b}\Big(\frac{1}{\log t}\sum_{h\leqslant t}\frac{Y_h}{h}\Big) = \frac{1}{\sqrt{\log t}(\log\log t)^b}\sum_{h\leqslant t}\frac{Y_h}{h}\xrightarrow[t]{} 0;$$

since $\frac{\sqrt{\log t}}{(\log\log t)^b}\longrightarrow_t \infty$, this implies

$$\frac{1}{\log t}\sum_{h\leqslant t}\frac{Y_h}{h} = \frac{1}{\log t}\sum_{h\leqslant t}\frac{1_{\{S_h=\kappa_h\}}}{\sqrt{h}} - \frac{1}{\log t}\sum_{h\leqslant t}\frac{\sqrt{h}P(S_h=\kappa_h)}{h}\xrightarrow[t]{} 0$$

(v) Last, by Theorems 2.8–2.9

$$\frac{1}{\log t}\sum_{h\leqslant t}\frac{\sqrt{h}P(S_h=\kappa_h)}{h}\xrightarrow[t]{}\Lambda\eta(k),$$

and the result follows.

The second part of the Theorem is proved similarly.

□

Theorem 3.4 concerns the case in which X_1 has a moment $2+\epsilon$. The case in which only second moment exists has been treated in 2002 by M. Denker et S. Koch in [16], but their discussion is incomplete. In particular, they give the following notion of ASLLT:

Definition 3.10 *A stationary sequence of random variables $(X_n)_{n\geqslant 1}$ taking values in $\mathbb{Z}$ or $\mathbb{R}$ with partial sums $S_n = X_1+\cdots+X_n$ satisfies an ASLLT if there exist sequences (a_n) in $\mathbb{R}$ and (b_n) in $\mathbb{R}^+$ with $b_n\to\infty$ such that*

$$\lim_{n\to\infty}\frac{1}{\log n}\sum_{h=1}^{n}\frac{b_n}{n}1_{\{S_n\in\kappa_n+I\}}\overset{a.s.}{=} g(\kappa)|I| \quad\text{as}\quad \frac{\kappa_n-a_n}{b_n}\to\kappa,$$

where g denotes some density and $I\subset\mathbb{R}$ is some bounded interval. Further, $|I|$ denotes the length of the interval I in the case where X_1 is real valued, and the counting measure of I otherwise.

We observe that this definition is incomplete: even in the restricted case of i.i.d. random variables taking values in the lattice $\mathcal{L}(v_0,\Lambda)$, consideration of the span Λ of X_1 is missing. It appears necessary to modify the above formula into

$$\lim_{n\to\infty}\frac{1}{\log n}\sum_{h=1}^{n}\frac{b_n}{n}1_{\{S_n\in\kappa_n+I\}}\overset{a.s.}{=} g(\kappa)\#\{I\cap\mathcal{L}(v_0,\Lambda)\} \quad\text{as}\quad \frac{\kappa_n-a_n}{b_n}\to\kappa.$$

The problem has been solved once and for all by M. Weber in 2011 in [13]. Here is the result

Theorem 3.11 *Let $(X_n)_{n\geqslant 1}$ be square integrable lattice distributed random variables with maximal span Λ. Then $(X_n)_{n\geqslant 1}$ verifies an ASLLT.*

The proof is in two steps:

(i) first one makes the additional assumption (v_k, $k \in \mathbb{Z}$ are the elements of the lattice)

$$P(X = v_k) \wedge P(X = v_{k+1}) > 0 \quad \text{for some } k \in \mathbb{Z}. \tag{17}$$

With this assumption, the author proves a basic correlation inequality (similar to (16)) concerning the (already used) variables

$$Y_h = \sqrt{h}\big(1_{\{S_h=\kappa_h\}} - P(S_h = \kappa_h)\big).$$

More precisely one has

Proposition 3.12 *Assume that* (17) *holds. Then there exists a constant C (depending on the sequence (κ_n)) such that, for all integers m, n with $1 \le m < n$*

$$\mathbf{Cov}(Y_m, Y_n) \le C\Big(\frac{1}{\sqrt{\frac{n}{m}} - 1} + \sqrt{\frac{n}{n-m}} \cdot \frac{1}{n-m}\Big).$$

Then the ingredients for the proof are as before: the just stated correlation inequality, the notion of quasi–orthogonal system and (this time) Theorem 2.8 (it is not possible to use Theorem 2.10 since in this case X_1 has only the second moment).

(ii) In the second step the author uses a clever argument in order to get rid of assumption (17) and obtain the result in full generality.

3.3 The ASLLT for random sequences in the domain of attraction of a stable law with $\boldsymbol{\alpha}$< 2.

In this section we briefly sketch the main ideas of some recent work by R. Giuliano and Z. Szewczak for random sequences in the domain of attraction of a stable law with $\boldsymbol{\alpha}$< **2**. It is work in progress, hence some points are yet missing.

Let $(X_n)_{n\ge1}$ be a sequence of i.i.d. random variables, such that their common distribution F is in the domain of attraction of a stable distribution G (having density g) with exponent α ($0 < \alpha < 2$); see Definitions 2.14 and 2.15; in particular the sequences (a_n) and (b_n) are as in Definition 2.14. It is well known (see [5], p. 46) that $b_n = L(n)n^{1/\alpha}$, where L is slowly varying in Karamata's sense. Assume that X_1 has a lattice distribution with Λ being the maximal span. Since X_1 doesn't possess second moment, the discussion of the preceding section doesn't work in this case. Anyway, we can attempt to use similar ingredients as before.

We shall content ourselves of the particular case in which both F and the limit distribution G are symmetric By Remark 2 p. 402 of [17], we have $a_n = 0$. Put

$$Y_n = b_n\Big(1_{\{S_n=\kappa_n\}} - P(S_n = \kappa_n)\Big).$$

The first ingredient is the correlation inequality

Proposition 3.13 *(i) For every pair (m, n) of integers, with $1 \le m < n$, we have*

$$\Big|\mathbf{Cov}(Y_m, Y_n)\Big| = b_m b_n \Big|P(S_m = \kappa_m, S_n = \kappa_n) - P(S_m = \kappa_m)P(S_n = \kappa_n)\Big|$$
$$\le C\Big\{\Big(\frac{n}{n-m}\Big)^{1/\alpha}\frac{L(n)}{L(n-m)} + 1\Big\}.$$

(ii)For every pair (m,n) of integers, with $1 \le m < n$, we have

$$\big|\mathbf{Cov}(Y_m, Y_n)\big| = b_m b_n \Big| P(S_m = \kappa_m, S_n = \kappa_n) - P(S_m = \kappa_m)P(S_n = \kappa_n)\Big|$$
$$\le C \cdot L(n) \left\{ n^{1/\alpha}\Big(\frac{1}{e^{(n-m)c}} + \frac{1}{e^{nc}}\Big) + \frac{\frac{m}{n}}{\big(1-\frac{m}{n}\big)^{1+1/\alpha}} + \left(\frac{\frac{m}{n}}{\big(1-\frac{m}{2n}\big)^2}\right)^{1/\alpha}\right\}.$$

Corollary 3.14 *For large m and $n \ge 2m$ we have*

$$b_m b_n \Big| P(S_m = \kappa_m, S_n = \kappa_n) - P(S_m = \kappa_m)P(S_n = \kappa_n)\Big| \le C \cdot \Big(\frac{m}{n}\Big)^{\rho},$$

with $\rho := \min(\frac{1}{\alpha}, 1)$.

The second ingredient is Theorem 2.17. Last, instead of quasi–orthogonality, we use the Gaal–Koksma Strong Law of Large Numbers, i.e. (see [18], p. 134); here is the precise statement:

Theorem 3.15 *Let $(Z_n)_{n\geqslant 1}$ be a sequence of centered random variables with finite variance. Suppose that there exists a constant $\beta > 0$ such that, for all integers $m \ge 0$, $n > 0$,*

$$E\Big[\big(\sum_{i=m+1}^{m+n} Z_i\big)^2\Big] \le C\big((m+n)^{\beta} - m^{\beta}\big), \tag{18}$$

for a suitable constant C independent of m and n. Then, for each $\delta > 0$,

$$\sum_{i=1}^{n} Z_i = O(n^{\beta/2}(\log n)^{2+\delta}), \quad P - a.s.$$

We apply the above theorem to the sequence

$$Z_j := \sum_{h=\rho^{j-1}}^{\rho^j - 1} \frac{Y_h}{h}.$$

The main result is as follows:

Theorem 3.16 *Let $\alpha > 1$ and assume that there exists $\gamma \in (0,2)$ such that*

$$\sum_{k=a}^{b} \frac{L(k)}{k} \le C(\log^{\gamma} b - \log^{\gamma} a).$$

Then $(X_n)_{n\geqslant 1}$ satisfies an ASLLT, i.e.

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{b_n}{n} 1_{\{S_n = \kappa_n\}} = \Lambda g(\kappa).$$

Remark 3.17 *The slowly varying sequence $L(k) = \log^{\sigma} k$ with $\sigma < 1$ verifies the assumption of Theorem 3.16.*

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